Generalized tilings and Plücker cluster algebras

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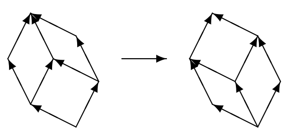
1 Introduction

There is a standard non-commutative deformation of the coordinate ring of the flag variety; in particular, it comes from consideration in theoretical physics. Leclerc and Zelevinsky [8] considered rational coordinate systems in which all elements quasi-commute with each other, and gave a purely combinatorial characterization for a pair of elements to be quasi-commuting, in terms of the so-called *weak separation* of the corresponding index sets. Also they proved that in the n-dimensional case a collection of (pairwise) quasi-commuting Plücker coordinates has cardinality at most $\binom{n+1}{2}+1$, and conjectured that any (inclusion-wise) *maximal quasi-commuting collection* has exactly this cardinality. In [6] we affirmatively answered this conjecture, essentially relying on results in [5] where so-called *generalized tilings* were introduced and studied and their close relation to weakly separated collections was demonstrated.

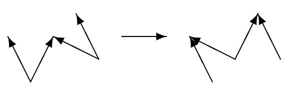
Roughly speaking, a generalized tiling, or a *g-tiling* for short, is a certain generalization of the notion of a *rhombus tiling*. While the latter is a subdivision of an n-zonogon Z in the plane into rhombi, the former is a cover of Z with rhombi that may overlap in a certain way.

Rhombus tilings have been well studied; for a wider discussion and related topics, see, e.g., [1, 4, 7, 11, 12]. An especial role is played by a rhombus tiling associated to the set of all intervals of the ordered set $[n]$ of elements $1, 2, \dots, n$; it is called the *standard* tiling. An important known fact is that any rhombus tilings can be transformed into the standard one by a sequence of *normal flips*, which are viewed locally as follows:



$\curlywedge$-hexagon $\curlyvee$-hexagon

On the other hand, it is shown in [5] that any g-tiling can be reduced to the standard tiling by making a sequence of *semi-normal* flips, as illustrated in the picture:



W-configuration M-configuration

The purpose of this paper is to show that the semi-normal flips of g-tilings can be associated with cluster mutations in the cluster algebra of the coordinate ring of the

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flag variety. (The notion of a cluster algebra was introduced in [3] and has proved its importance in representation theory.) Namely, we associate to a g-tiling T a planar directed graph $\Sigma(T)$ so that any semi-normal flip for T corresponds to a cluster mutation for $\Sigma(T)$. As a consequence of this result and the main theorem in [6], we obtain that any maximal quasi-commuting collection of quantum minors gives rise to a seed in that quantum cluster algebra; this proves a conjecture in [9], see also [2]. Note that in [10] a cluster algebra structure was established on the class of Postnikov's diagrams. In fact, we obtain a generalization of that result, using the transformation of Postnikov's diagrams to special g-tilings as described in the Appendix of [5].

2 Generalized tilings and weakly separated collections

2.1 Weakly separated collections.

We deal with two binary relations on subsets of $[n]$. For $A, B \subseteq [n]$, we write:

(i) $A \lessdot B$ if $B - A$ is nonempty and $i < j$ holds for any $i \in A - B$ and $j \in B - A$ (where $A' - B'$ stands for the set difference $\{i' \colon A' \ni i' \notin B'\}$);

(ii) $A \rhd B$ if both $A - B$ and $B - A$ are nonempty and $B - A$ can be (uniquely) expressed as a disjoint union $B' \sqcup B''$ of nonempty subsets so that $B' \lessdot A - B \lessdot B''$.

Note that these relations need not be transitive in general.

Definition Sets $A, B \subseteq [n]$ are called *weakly separated* (from each other) if either $A \lessdot B$, or $B \lessdot A$, or $A \rhd B$ and $|A| \geq B$, or $B \rhd A$ and $|B| \geq |A|$, or $A = B$. A collection $\mathcal{C} \subseteq 2^{[n]}$ is called weakly separated if any two of its members are weakly separated. We will usually abbreviate the term "weakly separated collection" to "ws-collection".

These notions were introduced by Leclerc and Zelevinsky in [8] where their importance is demonstrated, in particular, in connection with the problem of characterizing quasi-commuting quantum flag minors.

Recall that an $n \times n$-matrix X of indeterminates x_{ab} is meant to be a *quantum matrix* if there is an additional variable (quantum parameter) q and the following relations hold:

$x_{il}x_{ik} = qx_{ik}x_{il} \quad \forall\, i, \forall\, k < l;$

$x_{jk}x_{ik} = qx_{ik}x_{jk} \quad \forall\, i < j, \forall\, k;$

$x_{jk}x_{il} = x_{il}x_{jk} \quad \forall\, i < j,\ \forall\, k < l;$

$x_{jl}x_{ik} = x_{ik}x_{jl} + (q - q^{-1})x_{il}x_{jk} \quad \forall\, i < j, \forall\, k < l.$

It was proved in [8] that, whenever X is lower triangular, quantum flag minors $X_{[i]\times I}$ and $X_{[j]\times J}$, where $I,\ J \subset [n]$, $i = |I|$ and $j = |J|$, are quasi-commuting (which means that $X_{[i]\times I} \times X_{[j]\times J} = q^{c(I,J)} X_{[j]\times J} \times X_{[i]\times I}$) if and only if the sets I and J are weakly separated.

2.2 Generalized tilings.

Tiling diagrams live within a zonogon, which is defined as follows. In the upper half-plane $\mathbb{R} \times \mathbb{R}_+$, take n non-colinear vectors $\xi_1, \ldots, \xi_n$ so that:

(i) $\xi_1, \ldots, \xi_n$ follow in this order clockwise around $(0,0)$, and

(ii) all integer combinations of these vectors are different.

Then the set $Z = Z_n := \{\lambda_1\xi_1 + \ldots + \lambda_n\xi_n \colon \lambda_i \in \mathbb{R},\ 0 \le \lambda_i \le 1,\ i = 1, \ldots, n\}$ is a $2n$-gone. Moreover, Z is a *zonogon*, as it is the sum of n line-segments $\{\lambda\xi_i \colon 1 \le \lambda \le 1\}$, $i = 1, \ldots, n$. Also it is the image by a linear projection π of the solid cube $conv(2^{[n]})$ into the plane $\mathbb{R}^2$, defined by $\pi(x) := x_1\xi_1 + \ldots + x_n\xi_n$. The boundary $bd(Z)$ of Z consists of two parts: the *left boundary* formed by the points (vertices) $z_i^\ell := \xi_1 + \ldots + \xi_i$ ($i = 0, \ldots, n$) connected by the line-segments $z_{i-1}^\ell z_i^\ell := z_{i-1}^\ell + \{\lambda\xi_i \colon 0 \le \lambda \le 1\}$, and the *right boundary* formed by the points $z_i^r := \xi_{i+1} + \ldots + \xi_n$ ($i = 0, \ldots, n$) connected by the line-segments $z_i^r z_{i-1}^r$. So $z_0^\ell = z_n^r$ is the minimal vertex of Z and $z_n^\ell = z_0^r$ is the maximal vertex. We direct each segment $z_{i-1}^\ell z_i^\ell$ from z_{i-1}^ℓ to z_i^ℓ and direct each segment $z_i^r z_{i-1}^r$ from z_i^r to z_{i-1}^r.

A subset $X \subseteq [n]$ is identified with the corresponding vertex of the n-cube and with the point $\sum_{i\in X} \xi_i$ in the zonogon Z. Due to (ii), all such points in Z are different.

In fact, it does not matter what vectors $\xi_1, \ldots, \xi_n$ are chosen subject to (i),(ii). It is convenient for us to assume that these vectors have *unit height*, i.e. each ξ_i is of the form $(a_i, 1)$ (and $a_1 < \ldots < a_n$).

By a *tile* we mean a parallelogram τ of the form $X + \{\lambda\xi_i + \lambda'\xi_j \colon 0 \le \lambda, \lambda' \le 1\}$, where $X \subset [n]$ and $1 \le i < j \le n$; we also call it an *ij-tile* at X and denote by $\tau(X; i, j)$. According to a natural visualization of τ, its vertices X, Xi, Xj, Xij are called the *bottom, left, right, top* vertices of τ and denoted by $b(\tau)$, $\ell(\tau)$, $r(\tau)$, $t(\tau)$, respectively. (We write $Xi' \ldots j'$ for $X \cup \{i'\} \cup \ldots \cup \{j'\}$.) The edge from $b(\tau)$ to $\ell(\tau)$ is denoted by $b\ell(\tau)$, and the other three edges of τ are denoted as $br(\tau), \ell t(\tau), rt(\tau)$ in a similar way. Also we say that a point (subset) $Y \subseteq [n]$ is of *height* $|Y|$.

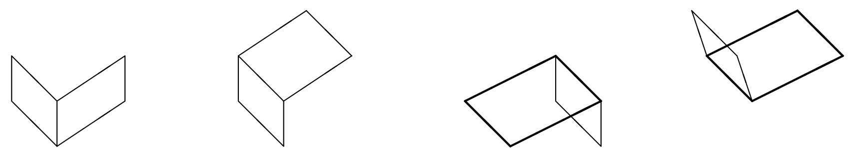
A *generalized tiling*, or a *g-tiling* for short, is a collection T of tiles $\tau(X; i, j)$ which is partitioned into two subcollections T^w and T^b, of *white* and *black* tiles, respectively, obeying axioms (T1)–(T4) below.

We associate to T the directed graph $G_T = (V_T, E_T)$, where V_T and E_T are the sets of vertices and edges, respectively, occurring in tiles of T. For a vertex $v \in V_T$, the set of edges incident with v is denoted by $E_T(v)$, and the set of tiles having a vertex at v is denoted by $F_T(v)$.

(T1) Each boundary edge of Z belongs to exactly one tile. Each edge in E_T not contained in $bd(Z)$ belongs to exactly two tiles. All tiles in T are different, in the sense that no two coincide in the plane.

(T2) Any two white tiles having a common edge do not overlap, i.e. they have no common interior point. If a white tile and a black tile share an edge, then these tiles do overlap. No two black tiles share an edge.

See the picture; here all edges are directed up and the black tiles are drawn in bold.



(T3) Let τ be a black tile. None of $b(\tau), t(\tau)$ is a vertex of another black tile. All edges in $E_T(b(\tau))$ *leave* $b(\tau)$, i.e. they are directed from $b(\tau)$. All edges in $E_T(t(\tau))$ *enter* $t(\tau)$, i.e. they are directed to $t(\tau)$.

We refer to a vertex $v \in V_T$ as *terminal* if v is the bottom or top vertex of some black tile. A nonterminal vertex v is called *ordinary* if all tiles in $F_T(v)$ are white, and *mixed* otherwise (i.e. v is the left or right vertex of some black tile). Note that a mixed vertex may belong, as the left or right vertex, to several black tiles.

Each tile $\tau \in T$ corresponds to a square in the solid cube $conv(2^{[n]})$, denoted by $\sigma(\tau)$: if $\tau = \tau(X; i, j)$ then $\sigma(\tau)$ is the convex hull of the points X, Xi, Xj, Xij in the cube (so $\pi(\sigma(\tau)) = \tau$). (T1) implies that the interiors of these squares are pairwise disjoint and that $\cup(\sigma(\tau) \colon \tau \in T)$ forms a 2-dimensional surface, denoted by D_T, whose boundary is the preimage by π of the boundary of Z. The last axiom is:

(T4) D_T is a disc, in the sense that it is homeomorphic to $\{x \in \mathbb{R}^2 \colon x_1^2 + x_2^2 \leq 1\}$.

When no black tile exists (i.e. $T^b = \emptyset$), T turns into a *pure tiling*; in this case the tiles do not overlap and form a *subdivision* of Z (a pure tiling becomes a *rhombus tiling* if the vectors ξ_i have equal euclidean norms).

The *spectrum* of a g-tiling T is the collection $\mathfrak{S}_T$ of (the subsets of $[n]$ represented by) *nonterminal* vertices in G_T. The following result on g-tilings is of importance.

Theorem 2.1 [6] *The spectrum $\mathfrak{S}_T$ of any generalized tiling T forms an (inclusion-wise) maximal ws-collection. Conversely, for any maximal ws-collection $\mathcal{C} \subseteq 2^{[n]}$, there exists a generalized tiling T on Z_n such that $\mathfrak{S}_T = \mathcal{C}$. (Moreover, such a T is unique and there is an efficient procedure that constructs T from $\mathcal{C}$.)*

2.3 Flips in g-tilings.

Let T be a g-tiling. By an *M-configuration* in T we mean a quintuple of vertices of the form Xi, Xj, Xk, Xij, Xjk with $i < j < k$ (as it resembles the letter "M"), which is briefly denoted as $CM(X; i, j, k)$. By a *W-configuration* in T we mean a quintuple of vertices Xi, Xk, Xij, Xjk, Xik with $i < j < k$ (as resembling "W"), briefly denoted as $CW(X; i, j, k)$. A configuration is called *feasible* if all five vertices are non-terminal, i.e. they belong to the spectrum $\mathfrak{S}_T$.

Proposition 2.2 [5] *Let the spectrum of a g-tiling T contain five non-terminal vertices Xi, Xk, Xij, Xjk, Y, where $i < j < k$ and $Y \in \{Xik, Xj\}$. Then there exists a g-tiling T' such that $\mathfrak{S}_{T'}$ is obtained from $\mathfrak{S}_T$ by replacing Y by the other member of $\{Xik, Xj\}$.*

For such a pair of tilings, we say that T' *covers* T if $Xj = Y \in \mathfrak{S}_T$.

Theorem 2.3 [5] *The set of g-tilings on Z_n forms a poset w.r.t. the cover relation; this poset has a unique minimal and a unique maximal elements.*

3 Generalized tilings and the cluster algebra of the coordinate ring of full flags

In this section we explain how to associate to a generalized tiling T on the zonogon Z a planar directed graph $\Sigma(T)$ (different from G_T) in such a way that the semi-normal flips between g-tilings correspond to cluster mutations between the associated graphs (representing seeds in the related Plücker cluster algebra).

3.1 Construction of a planar digraph $\Sigma(T)$.

Given a g-tiling T, the set $V(\Sigma(T))$ of vertices of the digraph $\Sigma(T)$ is formed by the spectrum $\mathfrak{S}_T$ of T.

The set $E(\Sigma(T))$ of edges of $\Sigma(T)$ consists of some white edges of the graph G_T, some reversed white edges, and "horizontal" diagonals of tiles of T. Here, following terminology from [5], an edge of G_T is called (fully) *white* if both of its end vertices are non-terminal.

Specifically, for each white tile $\tau \in T^w$, the edge set of $\Sigma(T)$ contains the diagonal e_τ going from $\ell(\tau)$ to $r(\tau)$, and for each black tile $\tau' \in T^b$, it contains the diagonal $e_{\tau'}$ going from $r(\tau')$ to $\ell(\tau')$.

For a white edge e of G_T, the edge set $E(\Sigma(T))$ contains either e or its reverse edge $-e$ or none of $e, -e$. This is assigned by the following rules.

Suppose e is an internal edge (i.e. it is not contained in the boundary of Z). Then e is a common edge of two white tiles, say, τ and τ'. There are four possible cases:

a) if e is the edge $rt(\tau)$ of τ and the edge $b\ell(\tau')$ of τ', then we add e to $E(\Sigma(T))$;

b) if $e = br(\tau) = \ell t(\tau')$, then we add $-e$ to $E(\Sigma(T))$;

c) if $e = rt(\tau) = \ell t(\tau')$, then none of $e, -e$ is added to $E(\Sigma(T))$;

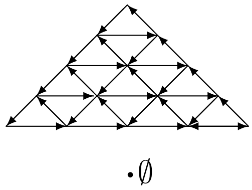
d) if $e = br(\tau) = b\ell(\tau')$, then none of $e, -e$ is added to $E(\Sigma(T))$.

Now suppose that e lies in the left boundary of Z, and let τ be the white tile containing e. If $e = \ell t(\tau)$, then we add $-e$ to $E(\Sigma(T))$. And if $e = b\ell(\tau)$, then neither e nor $-e$ is added to $E(\Sigma(T))$.

Finally, suppose that e lies in the right boundary of Z and belongs to a (white) tile τ'. If $e = rt(\tau')$, then we add e to $E(\Sigma(T))$. And if $e = br(\tau')$, then nether e nor $-e$ is added to $E(\Sigma(T))$.

This gives the desired digraph $\Sigma(T) = (V(\Sigma(T)), E(\Sigma(T)))$.

The picture below illustrates the graph $\Sigma(T)$ for the standard tiling T (in case $n = 5$). Recall that the vertices of such a T are the intervals in $[n]$ (the sets $[i..j] := \{i, i+1, \ldots, j\}$ for $1 \le i \le j \le n$ plus the empty set) and the tiles of T are white and span all quadruples of intervals of the form $[i..j], [i-1..j], [i..j+1], [i-1..j+1]$ or $\emptyset, \{i\}, \{i+1\}, \{i, i+1\}$).



3.2 Cluster algebras.

Let $G = (V(G), E(G))$ be a directed multigraph in which the vertex set $V(G)$ is partitioned into two subsets: a set V_1 of *frozen* vertices, and a set V_2 of *mutable* vertices. The (integer) edge multiplicity function is regarded as being skew-symmetric: if vertices u, v are connected by α edges going from u to v (which are members of $E(G)$), we simultaneously think of these vertices as being connected by $-\alpha$ edges going from v to u. To each vertex v of G one associates a variable x_v so that $\{x_v \colon v \in V(G)\}$ is a transcendence basis of a field of rational functions. Such a pair consisting of a digraph and a transcendence basis indexed by its vertices is said to be a *cluster seed*; it generates a skew-symmetric cluster algebra [3].

The digraph and variables are modified by applying the following operations called cluster mutations. A *cluster mutation* μ_v applied at a mutable vertex $v \in V_2$ changes one variable, namely, x_v, and modifies the digraph G, as follows. For a vertex v, denote $In(v) := \{v' \in V(G) : (v', v) \in E(G)\}$ and $Out(v) := \{v'' \in V(G) : (v, v'') \in E(G)\}$.

The digraph $\mu_v(G)$ has the same vertex set as G, $V(\mu_v(G)) = V(G)$, partitioned into frozen and mutable vertices in the same way as before. The edges $E(\mu_v(G))$ are obtained from edges $E(G)$ by the following rule:

(i) the edges in $E(\mu_v(G))$ incident to the vertex v are exactly the edges in $E(G)$ incident to v but taken with the reverse direction;

(ii) for each pair $v' \in In(v)$ and $v'' \in Out(v)$, form the edge (v', v'') in $E(\mu_v(G))$ whose multiplicity is defined to be $\gamma - \alpha \cdot \beta$, where $\alpha \geq 1$ is multiplicity of the edge (v', v) in $E(G)$, $\beta \geq 1$ is that for (v, v''), and $\gamma \in \mathbb{Z}$ is that for (v', v'');

(iii) the other edges of $\mu_v(G)$ are those of G that neither are incident to v nor connect pairs v', v'' as in (ii).

For $u \neq v$, we put $\mu_v(x_u) := x_u$ and define $\mu_v(x_v) = x_v^{new}$ by the following rule:

$$x_v^{new} \cdot x_v = \prod_{v' \in In(v)} x_{v'} + \prod_{v'' \in Out(v)} x_{v''}.$$

This gives the new digraph $\mu_v(G)$ and variables $\mu_v(x_u)$, $u \in V(\mu_v(G)) = V(G)$.

3.3 Main result.

Let T be a g-tiling, and $\Sigma(T)$ the planar digraph as above. We associate to each vertex $v \in \mathfrak{S}_T$ the Plücker coordinate, that is, the flag minor with the column set indexed by the subset S of $[n]$ corresponding to v (and the row set $[|S|]$). We define the frozen vertices in $\Sigma(T)$ to be the boundary vertices of G_T.

Theorem 3.1 *Let a g-tiling T' cover a g-tiling T. Then $\Sigma(T')$ is obtained from $\Sigma(T)$ by applying a cluster mutation.*

Corollary 3.2 *For any g-tiling T, the pair $(\Sigma(T), \{X_{[|S|]\times S} : S \in \mathfrak{S}_T\})$ represents a cluster seed in in the cluster algebra of the coordinate ring of the flag variety (the Plücker cluster algebra).*

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