Tropical Plücker Functions and Kashiwara Crystals of Types A, B, and C

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1. Introduction

Kashiwara [**K90**] introduced the fundamental notion of a *crystal* in representation theory. This is an edge-colored directed graph in which each connected monochromatic subgraph is a finite path and there are certain interrelations on the lengths of such paths, described in terms of a Cartan matrix M; this matrix characterizes the *type* of a crystal. An important class of crystals is formed by the crystals of representations, or *regular* crystals; these are associated to irreducible highest weight integrable modules (representations) over the quantum enveloping algebra related to M. There are several models to characterize the regular crystals for a variety of types; e.g., via generalized Young tableaux [**KN94**], Lusztig's canonical bases [**Lu90**], Littelmann's path model [**Lt95**], MV-polytopes [**K**].

Here we propose a new model for the Cartan matrix of type A. In this model, the set of vertices of the crystal is the set of integer tropical Plücker functions on the Boolean cube which have zero values on the vertices of the chain $(0,\dots,0)$, $(0,\dots,0,1)$, $(0,\dots,0,1,1),\dots$, $(0,1,\dots,1)$, $(1,\dots,1)$. To decide if a pair of functions f and g is connected by an edge of some color i, we have to restrict f and g to a surface adopted to i (such a surface is obtained as the union of certain 2-faces in the Boolean cube; see for details Section 2).

This model is symmetric on the colors (see Section 5). This allows us to obtain regular crystals for the Cartan matrices of Dynkin B_n- and C_n-types as symmetric extracts from crystals of A_{2n-1}- and A_{2n}-types, respectively. Note that among the above-mentioned models, Littelmann's path model of A-type and Kamnitzer's MV-polytope model are also symmetric, and this property was used in [**NS**] for construction of B- and C-types of Littelmann's path model as symmetric extracts from A-types; a direct combinatorial proof, based on the so-called crossing model, is given in [**DKK12**].

There are some advantages of our TP-method. Firstly, it is not too intricate and provides a new viewpoint on Young tableaux of B and C types. Secondly, using our model, we obtain an explicit description of the principal lattice of crystals of types A, B and C. The principal lattice in A-type crystals was introduced and studied in [**DKK08**].

2010 *Mathematics Subject Classification.* 05E10, 20G42,52C20.

Key words and phrases. Crystals, Plücker relations, tiling diagrams.

In Section 2 we recall some basic facts on tropical Plücker functions (TP-functions). In Section 3 we define a structure of free crystal on the set of TP-functions. In Section 4 we consider 'bounded' subcrystals (intervals) in a connected free crystal. In the last Section 5, we explain how crystals of B_m- and C_m-types can be derived from symmetric TP-functions on $2^{[2m]}$ and $2^{[2m+1]}$, respectively.

2. Tropical Plücker functions

1. For a positive integer n, let $[n]$ denote the ordered set of elements $1, 2, \dots, n$. A real-valued function f on the subsets of $[n]$, or on the Boolean cube $2^{[n]}$, is said to be a *tropical Plücker function*, or a *TP-function*, if it satisfies the *TP3-relation*

$$f(Aik) + f(Aj) = \max\{f(Aij) + f(Ak),\ f(Ai) + f(Ajk)\}, \tag{1}$$

for any triple $i < j < k$ in $[n]$ and subset $A \subseteq [n] - \{i, j, k\}$, where for brevity we write $Ai' \dots j'$ instead of $A \cup \{i'\} \cup \dots \cup \{j'\}$.

The set of integer TP-functions on $2^{[n]}$ is denoted by TP_n, and the set of real TP-functions by $TP_n(\mathbb{R})$.

2. TP_n is a subset of the space $\mathbb{R}^{2^{[n]}}$ of all functions on $2^{[n]}$. The polyhedral conic complex $TP_n(\mathbb{R})$ is stable under multiplication on positive numbers, but not under addition, in general. However, it contains a lineal of dimension $2n$ constituted by *principal* TP-functions. These are affine functions of the form $\alpha + \mu(A)$, where $\alpha \in \mathbb{R}$ and μ is a measure on $[n]$; in other words, such functions depend only on cardinalities of sets (arguments), and their sums.

3. **Definition.** A subset $B \subseteq 2^{[n]}$ is called a *TP-basis*, or simply a *basis*, if the restriction map $res : TP_n(\mathbb{R}) \to \mathbb{R}^B$ is a bijection. In other words, each TP-function is determined by its values on B, and moreover, the values on B can be chosen arbitrarily.

Such bases do exist, and it is convenient for our purposes to assume that a basis is given in the form of the *spectrum* of a *rhombus tiling diagrams*. Let us recall these notions (for details, see [**DKK10**]).

Rhombus tilings live within a zonogon, which is defined as follows. In the upper half-plane, take n non-colinear vectors $\xi_1, \dots, \xi_n$ so that:

(i) $\xi_1, \dots, \xi_n$ follow in this order clockwise around $(0, 0)$, and

(ii) all integer combinations of these vectors are different.

Then the set $Z = Z_n := \{\lambda_1\xi_1 + \dots + \lambda_n\xi_n\colon 0 \le \lambda_i \le 1,\ i = 1, \dots, n\}$ is a $2n$-gone. Moreover, Z is a *zonogon*, as it is the sum of n line-segments $\{\lambda\xi_i\colon 1 \le \lambda \le 1\}$, $i = 1, \dots, n$. Also it is the image by a linear projection π of the solid cube $conv(2^{[n]})$ into the plane $\mathbb{R}^2$, defined by $\pi(x) := x_1\xi_1 + \dots + x_n\xi_n$. The boundary $bd(Z)$ of Z consists of two parts: the *left boundary*, $lbd(Z)$, formed by the points (vertices) $z_i^\ell := \xi_1 + \dots + \xi_i$ $(i = 0, \dots, n)$ connected by the line-segments $z_{i-1}^\ell z_i^\ell := z_{i-1}^\ell + \{\lambda\xi_i\colon 0 \le \lambda \le 1\}$, and the *right boundary*, $rbd(Z)$, formed by the points $z_i^r := \xi_{i+1} + \dots + \xi_n$ $(i = 0, \dots, n)$ connected by the line-segments $z_i^r z_{i-1}^r$. So $z_0^\ell = z_n^r$ is the minimal vertex of Z and $z_n^\ell = z_0^r$ is the maximal vertex. We direct each segment $z_{i-1}^\ell z_i^\ell$ from z_{i-1}^ℓ to z_i^ℓ and direct each segment $z_i^r z_{i-1}^r$ from z_i^r to z_{i-1}^r.

A subset $X \subseteq [n]$ is identified with the corresponding vertex of the n-cube and with the point $\sum_{i\in X}\xi_i$ in the zonogon Z. Due to (ii), all such points in Z are different.

In fact, it does not matter what vectors $\xi_1, \dots, \xi_n$ are chosen subject to (i),(ii). It is convenient for us to assume that these vectors have *unit height*, i.e. each ξ_i is of the form $(a_i, 1)$ (and $a_1 < \dots < a_n$).

By a *tile* we mean a parallelogram τ of the form $X + \{\lambda\xi_i + \lambda'\xi_j : 0 \le \lambda, \lambda' \le 1\}$, where $X \subset [n]$ and $1 \le i < j \le n$; we also call it an *ij-tile* at X and denote by $\tau(X; i, j)$. According to a natural visualization of τ, its vertices X, Xi, Xj, Xij are called the *bottom, left, right, top* vertices of τ and denoted by $b(\tau)$, $\ell(\tau)$, $r(\tau)$, $t(\tau)$, respectively. The edge from $b(\tau)$ to $\ell(\tau)$ is denoted by $b\ell(\tau)$, and the other three edges of τ are denoted as $br(\tau), \ell t(\tau), rt(\tau)$ in a similar way. Also we say that a point (subset) $Y \subseteq [n]$ is of *height* $|Y|$.

4. A *(rhombus) tiling diagram*, or a *tiling* for short, is a certain collection T of tiles $\tau(X; i, j)$ which cover Z_n and such that the corresponding polyhedral complex is homeomorphic to a disc; for a precise definition, see [**DKK10**].

A *vertex* (an *edge*) of a tiling T is a vertex (an edge) of any rhombus in T. Thus, the set of vertices of T defines a collection of corresponding subsets of $[n]$. This collection is called the *spectrum* of T and denoted by $Sp(T)$. Note that boundary vertices of Z_n, namely, $\emptyset$, $\{1\}$, $\{1,2\}$,..., $\{1,2,...,n\}$, $\{2,...,n\}$,..., $\{n\}$, belong to the spectrum of any tiling.

5. It turns out that the spectrum $Sp(T)$ of any tiling T is a TP-basis, and any TP-basis is obtained in this way; see [**DKK10**]. The bijection

$$TP_n(\mathbb{R}) \to \mathbb{R}^{Sp(T)}$$

is a piecewise linear map. One can consider tilings as charts of an atlas for TP_n. The transformation maps between charts take the form of sequences of TP3-relations (1); see [**DKK10**].

6. Let R be a tile $\tau(A; i, j)$. Then the *excess* of a function $f : 2^{[n]} \to \mathbb{R}$ at the rhombus R is defined to be the value

$$\varepsilon(f, R) = f(Ai) + f(Aj) - f(A) - f(Aij).$$

Any function on the vertices of a tiling T is determined by its values on the vertices of the right boundary of Z_n together with the list of tile excesses for T.

3. TP-functions and free crystal of type A

1. A pre-crystal with n colors is a certain digraph K in which each edge is endowed with a *color*, which is an element of $[n]$. In other words, the set of edges of K is partitioned into n subsets: $E(K) = E_1 \sqcup \dots \sqcup E_n$, where the edges in each E_i have color i. In order to be a *pre-crystal*, such a digraph K should satisfy a number of axioms. The first axioms requires that for any color i, the subgraph (K, E_i) is a disjoin union of monochromatic (finite or infinite) directed paths. A move along an edge of a color i is understood as an action of the (partial) operation $\mathbf{i}$ on the set of vertices $V(K)$ of K. Namely, if an edge (v, u) has color i, then $\mathbf{i}v = u$, and we say that the operation $\mathbf{i}$ *acts* at v.

Reversing the edges of K, we can define the reverse operations $\mathbf{i}^{-1}$. That is, if $\mathbf{i}$ acts at v and $u = \mathbf{i}v$, then $\mathbf{i}^{-1}$ acts at u and $v = \mathbf{i}^{-1}u$.

Example. A *commutative pre-crystal* is an Abelian group $\mathbb{Z}^{[n]}$ on which an operation $\mathbf{i}$ sends x to $x + 1_i$, where 1_i is the i-th basis vector.

Let K and K' be two n-colored pre-crystals. Then a *morphism* $K \to K'$ is a mapping $\varphi : V(K) \to V(K')$ which commutes with the actions of operations $\mathbf{i}$; that is if $\mathbf{i}$ acts at v in the pre-crystal K, then $\mathbf{i}$ acts at $\varphi(v)$, and there holds $\varphi(\mathbf{i}v) = \mathbf{i}\varphi(v)$.

The second axiom of pre-crystals requires an existence of a *weight map*, a morphism $wt : K \to \mathbb{Z}^{[n]}$.

2. Crystals are associated to (generalized) Cartan matrices. Let $M = (m_{ij})$, $i, j \in [n]$, be such a matrix, that is $m_{ij} \in \mathbb{Z}$, $m_{ii} = 2$, and $m_{ij} \le 0$ for $i \ne j$. A pre-crystal is a *crystal* if some additional axioms on the weight map and Cartan data are satisfied. However, according to the 2-color reduction theorem in [**KKMMNN**], a 'bounded' pre-crystal is a crystal of an integrable M-module if and only if the restriction of K to any pair of colors i, j is a crystal of the corresponding $M|ij$-module. For simply- and doubly-laced cases, one deals with 2-colored crystals of three types $A_1 + A_1$, A_2 and B_2. Crystals of types A_2 and B_2 were exhaustively studied in [**DKK07, DKK09**].

3. One can obtain a pre-crystal with n colors by considering as the set of vertices the set of integer TP-functions $TP = TP_{n+1}(\mathbb{Z})$ on $2^{[n+1]}$ (as we shall see later, such a pre-crystal is, moreover, an A_n-crystal). We need some definitions.

Let us call a rhombus τ of a tiling T a *left rhombus* if it shares 2 edges with the left boundary of the zonogon Z_{n+1}. Specifically, τ is a left rhombus at height h if $b(\tau) = [h-1]$, $l(\tau) = [h]$, and $t(\tau) = [h+1]$. We denote such a rhombus by LR_h. Analogously, one defines the right rhombus RR_h at height h.

We say that a tiling T in Z_{n+1} *fits* the color i ($i = 1, \dots, n$) if T contains the left rhombus LR_i. For any color i, there exists a tiling which fits i

4. Now all is ready to define a crystal operation $\mathbf{i}$ ($i = 1, \dots, n$) at a TP-function $f \in TP_{n+1}$. Choose a tiling T which fits the color i. Then the function $\mathbf{i}f$ is defined by the rule

$$(\mathbf{i}f)(v) = \begin{cases} f(v) + 1, & \text{if } v = [i], \\ f(v) & \text{otherwise.} \end{cases}$$

In other words, within the chart T, the function $\mathbf{i}f$ differs from f at the only vertex v. Note that the functions f and $\mathbf{i}f$ may differ at many vertices of the Boolean cube. Nevertheless, they coincide on the vertices of $rbd(Z_{n+1})$.

5. **Theorem**. *The operations* $\mathbf{i}$ $(i = 1, ..., n)$ *are well defined and they endow the set* $TP = TP_{n+1}$ *with a structure of a free* A_n*-crystal.*

According to the 2-color reduction theorem of [**KKMMNN**], it suffices to consider the case $n = 3$. In this case, one can establish an explicit bijection with the A_2-crystals described via the crossing model of [**DKK07**].

6. The crystal operations $\mathbf{i}$ commute with the operation of adding any integer principal TP-function. That is, for any TP-function f and any principal TP-function p, there holds

$$\mathbf{i}(f + p) = \mathbf{i}f + p.$$

7. As is said above, the crystal actions preserve values of TP-functions at the vertices of $rbd(Z_{n+1})$. Thus, the values at these vertices are $n + 2$ 'integrals', and any connected component of the crystal $TP_{n+1}(\mathbb{R})$ is specified by $x \in \mathbb{R}^{n+2}$, being the list of values at the vertices of $rbd(Z)$. Denote such a crystal (component) by

$K[x]$. Since the crystals $K[x]$ are isomorphic for all x, we can consider the crystal $K = K[0]$ whose vertices are the integer TP-functions which are zero on the vertices of $rbd(Z_{n+1})$.

8. Consider the set P of principal TP-functions which belong to $K[0]$, i.e. taking value 0 on the vertices of $rbd(Z)$. We can specify the excesses of such functions at the rhombi of each height. This gives an isomorphism between P and the Abelian group $\mathbb{Z}^n$. A natural basis of the lattice P consists of the principal functions $p_1, \dots, p_n$, where i-th function p_i is defined by the following conditions:

(1) p_i is 0 at each vertex of $rbd(Z)$;
(2) for any $j \neq i$ and any rhombus R at the height j, $\varepsilon(p_i, R) = 0$;
(3) for any rhombus R at the height i, $\varepsilon(p_i, R) = 1$.

4. Subcrystals of $K[0]$

1. Since the operations $\mathbf{i}$ and $\mathbf{i}^{-1}$ act at each vertex of $K[0]$, the crystal $K[0]$ is a *free* crystal, that is all maximal monochromatic paths are infinite in both directions. Now we interested in subgraphs of $K[0]$ giving A_n-type crystals.

Introduce the following partial order $\preceq$ on the vertices of $K[0]$: for TP-functions f, g in $K[0]$, we write $f \preceq g$ if there exists a word $\mathbf{w}$ in the alphabet $\{\mathbf{1}, \dots, \mathbf{n}\}$ such that $g = \mathbf{w}f$. In other words, $f \preceq g$ if there exists a directed path in $K[0]$ starting at f and ending at g.

For a principal vertex $p \in K[0]$, we denote by K_p (resp. K^p) the set $\{f \in K[0],\ p \preceq f\}$ (resp. $\{f \in K[0],\ f \preceq p\}$). The vertex p is the unique source in the poset K_p and the unique sink in K^p. We have

$$K^p = p + K^0 \quad \text{and} \quad K_p = p + K_0.$$

Because of this, we are interested in the sets K^0 and K_0.

2. It is useful to give an alternative definition of the set K_0. Recall that a function $f : 2^{[n+1]} \to \mathbb{R}$ is *submodular* if any 'rhombus' R in $2^{[n+1]}$ satisfies $\varepsilon(f, R) \geq 0$. The following property can be shown: a TP-function f is submodular if and only if for any fixed tiling T in Z_{n+1}, $\varepsilon(f, \tau) \geq 0$ holds for each tile $\tau \in T$.

3. **Theorem**. *The set of vertices of K_0 is the set of integer submodular TP-functions which are equal to* 0 *on $rbd(Z_{n+1})$.*

4. The intersection of K_0 and the lattice of principal TP-functions $P \cong \mathbb{Z}^n$ is the semigroup $\mathbb{Z}^n_+$. In fact, every principal function p_i, $i = 1, \dots, n$, is submodular. Moreover, the only nonnegative linear combinations of them are submodular.

5. From Theorem in 4.3 it follows that the crystal $K[0]$ is connected. In other words, any function of $K[0]$ is of the form $\mathbf{w}\,0$, where $\mathbf{w}$ is a word in the alphabet $\{\mathbf{1}^{\pm 1}, \dots, \mathbf{n}^{\pm 1}\}$. Specifically, for any $f \in K[0]$, we can find a principal function $p \in P$ such that the TP-function $f + p$ is submodular. Then, according to Theorem in 4.3, there exists a word $\mathbf{w}$ in the alphabet $\{\mathbf{1}, \dots, \mathbf{n}\}$ such that $f + p = \mathbf{w}0$. Hence $f = \mathbf{w}(-p)$. Since, due to this theorem, there exists a word $\mathbf{v}$ such that $p = \mathbf{v}0$, we have $-p = \mathbf{v}^{-1}0$. This implies $f = \mathbf{w}\mathbf{v}^{-1}0$.

6. Next we give a description of vertices of K^0 using excesses.

Theorem. *A TP-function $f \in K[0]$ belongs to K^0 if and only if each right rhombus RR_i satisfies $\varepsilon(f, RR_i) \leq 0$.*

This theorem has the following generalization:

Let $p \in P \cap K[0]$ be a principal function in K. Then a function $f \in K[0]$ belongs to K^p if and only if for any $i = 1, \dots, n$, there holds $\varepsilon(f, RR_i) \le \varepsilon(p, RR_i)$.

7. The crystals of the form $K_q^p = K^p \cap K_q$ (*intervals*), where $p, q \in P$ and $q \preceq p$, correspond to finite-dimensional integrable modules. W.l.o.g., we may assume that $q = 0$. Then $p = \sum_i c_i p_i$, where $c_i \in \mathbb{Z}_{\ge 0}$. The graph K_0^p is finite and connected, and it has the unique source 0 and the unique sink p.

Due to Theorems in 4.3 and 4.6, a function $f \in K$ belongs to the crystal $K_0^p = K_0 \cap K^p$ (for some $p \in P$) if and only if:

(1) any rhombus R satisfies $\varepsilon(f, R) \ge 0$; and
(2) any right rhombus RR_i, $i = 1, \dots, n$, satisfies $\varepsilon(f, RR_i) \le \varepsilon(p, RR_i)$.

It follows that the intersection of a crystal K_0^p and the principal lattice P consists of the functions of the form $\sum_i c_i p_i$, where $0 \le c_i \le \varepsilon(p, RR_i)$. This means that this intersection is an integer parallelepiped. For a tuple $c = (c_1, ..., c_n) \in \mathbb{Z}^n_{\ge 0}$, we denote by $K(c)$ the crystal K_0^p with $p = \sum_i c_{n+1-i} p_i$.

8. **Proposition**. 1) *c_i is equal to the maximal number α such that the function $\mathbf{i}^\alpha 0$ belongs to $K(c)$.*

2) *c_{n-i+1} is equal to the maximal number β such that the function $\mathbf{i}^{-\beta} p$ belongs to $K(c)$.*

5. Extracts from symmetric A-crystals

1. The inversion $\sigma(i) = n + 2 - i$ of the set $[n + 1]$ can be considered as an inversion of the Dynkin diagram A_n. Consider the inversion γ of the Boolean cube $2^{[n+1]}$, defined by $\gamma(A) = \sigma([n + 1] - A)$.

Consider an extension of the inversion γ to the zonogon $Z = Z_{n+1}$. For this, consider symmetric (w.r.t. σ) vectors $\xi_i = (x_i, 1)$, that is $\xi_{n+2-i} = (-x_i, 1)$. Denote by γ the symmetry of the plane w.r.t. the horizontal line $y = (n + 2)/2$. This symmetry sends Z to itself: if a point $v \in Z$ corresponds to a subset $A \subset [n + 1]$, then the point $\gamma(v)$ corresponds to the subset $\gamma(A)$. This symmetry extends to the space of functions on the Boolean cube. Namely, let $f : 2^{[n+1]} \to \mathbb{R}$ be a function on the Boolean cube. Then the function $\gamma^* f : 2^{[n+1]} \to \mathbb{R}$ sends a set A to $f(\gamma(A))$. Obviously, f is a TP-function if and only if $\gamma^* f$ is a TP-function.

Denote by $\widetilde{TP}$ the set of symmetric TP-functions, $\gamma^* f = f$. We are going to endow the set $\widetilde{TP}$ with a crystal structure. This depends on the parity of n.

2. Let n be odd, $n = 2m - 1$, $m \ge 1$. In this case there are plenty of symmetric tilings. A tiling T is symmetric if its set of vertices and edges is stable under the symmetry γ, i.e. $\gamma T = T$.

In this case, any symmetric TP-function defines a symmetric function on any symmetric tiling, and vice versa. Consider m operations $\widetilde{\mathbf{1}}, \dots, \widetilde{\mathbf{m}}$ on $\widetilde{TP}$, where $\widetilde{\mathbf{1}} = \mathbf{1n} = \mathbf{n1}, \dots, \widetilde{\mathbf{m-1}} = (\mathbf{m-1})(\mathbf{m+1})$, $\widetilde{\mathbf{m}} = \mathbf{m}$, where $\mathbf{1}, \dots, \mathbf{n}$ are the crystal operations on TP. These operations can be defined as follows. For ith operation, we consider a symmetric tiling T which fits the color i. Then, by the symmetry, T also fits the color $n + 1 - i = 2m - i$. The ith operation on the vertices of T is defined by the rule

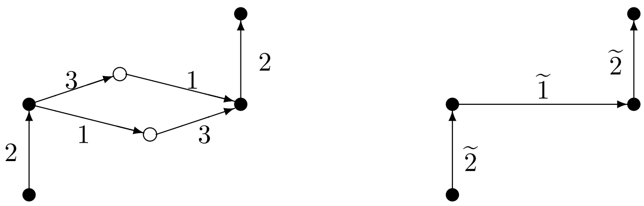
$$(\widetilde{\mathbf{i}} f)(v) = \begin{cases} f(v) + 1, & \text{if } v = [i], \gamma[i] = [2m - i] \\ f(v) & \text{otherwise.} \end{cases}$$

Note that for $i = m$, we have $[m] = [2m - m]$, and ith operation increases by 1 the value of a function at the symmetric vertex $[m]$.

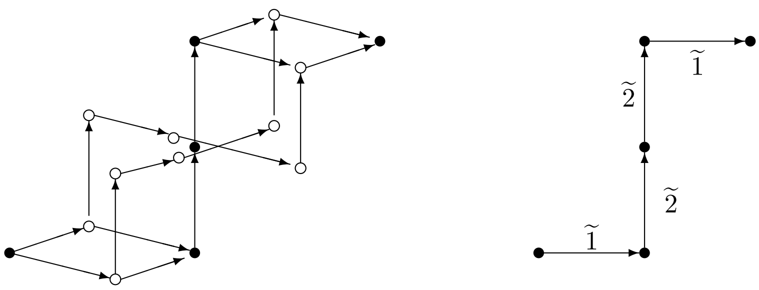
3. **Theorem**. *$\widetilde{TP}$ is a free B_m-crystal.*

4. Considering symmetric functions in the subcrystals K_0, K^0 and $K(c)$, with symmetric c ($c_{\sigma(i)} = c_i$, $i = 1, \dots, m$) of type A_{2m+1}, we obtain B_m-subcrystals in the free B_m-crystal $\widetilde{TP}$. The symmetric part of $\widetilde{K}(c)$ of $K(c)$ is an interval in the poset $\widetilde{K[0]}$ consisting of symmetric TP-functions between the principal vertices 0 and $p = \sum_{i=1}^{2m+1} c_i p_i$.

Let us consider simplest examples for $n = 3$. The A_3-crystal $K(0,1,0)$ is drawn in the left part of the picture below. The symmetric vertices are indicated by bold circles, and the extracted B_2-crystal $\widetilde{K}(0,1)$ is depicted in the right part.

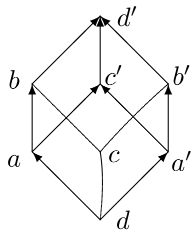


The A_3-crystal $K(1,0,1)$ is drawn in the left part of the next picture, and the extracted B_2-crystal $\widetilde{K}(1,0)$ in the right part.



5. Now let n be even, $n = 2m$. In this case there are no symmetric tilings, but there exist symmetric *hexagonal-rhombus tilings*, or *HR-tilings* for short. Tiles of an HR-tiling are rhombi or hexagons (where a hexagon is the zonogon Z_3). An HR-tiling is symmetric if it contains, for each rhombus R, the symmetric rhombus $\gamma(R)$, and for each hexagon H, the symmetric hexagon $\gamma((H)$.

Consider the case $n = 2$. There is a unique HR-tiling, the zonogon Z_3 itself. Below we illustrate Z_3 and its two rhombus tilings.



A symmetric TP-function is specified by the following conditions: $a = b$, $a' = b'$, and $c = c'$. TP3-relation (1) reads as $c + c' = \max(a + b', b + a')$ and boils down to the equality $2c = a + a'$. Because of this, the values of a symmetric function on the boundary of Z_3 determine the whole function.

In this case, a symmetric TP-function in $K[0]$ is a triple a, $b = a$, $c = a/2$.

Apply the sequence of operations **1221** to a symmetric TP-function corresponding to a, $b = a$ and $c = a/2$. The result is a symmetric function corresponding to $\tilde{a} = a+2$, $\tilde{b} = a+2$, and $\tilde{c} = c+1$. We can apply the sequence of crystal operation to the same symmetric TP-function. The result is again $\tilde{a} = a+2$, $\tilde{b} = a+2$, and $\tilde{c} = c+1$. Thus, the symmetric extraction of A_2 endowed with the operation **1221** is an A_1-crystal.

6. Any symmetric TP-function on the Boolean cube $2^{[2m+1]}$ defines a symmetric function on the vertices of any HR-tiling, and vice versa. As before, we denote by $\widetilde{TP}$ the set of symmetric TP- functions. Define the operations on $\widetilde{TP}$ as follows. The operations $\widehat{\mathbf{1}}, \ldots, \widehat{\mathbf{m-1}}$ are defined as in part 5.2: $\widehat{\mathbf{1}} := \mathbf{1}(\mathbf{2m})$, ..., $\widehat{\mathbf{1}} := (\mathbf{m-1})(\mathbf{m+2})$. The operation $\widehat{\mathbf{m}}$ is defined as the sequence $\mathbf{m}(\mathbf{m+1})(\mathbf{m+1})\mathbf{m} = (\mathbf{m+1})\mathbf{mm}(\mathbf{m+1})$.

In terms of symmetric HR-tilings, these operations are expressed as follows. For $1 \le i < m$, take a symmetric HR-tiling which fits the color i. Then the ith operation is defined on a vertex v of the tiling by the rule

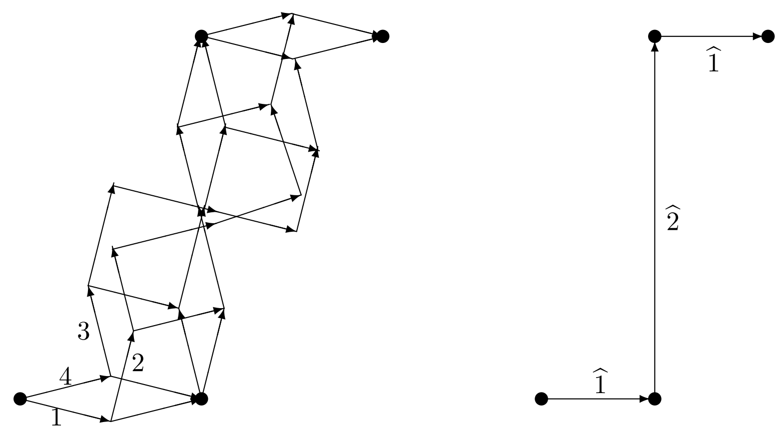
$$(\widetilde{\mathbf{i}}f)(v) = \begin{cases} f(v)+1, & \text{if } v = [i], \gamma[i] = [2m-i] \\ f(v) & \text{otherwise.} \end{cases} \tag{2}$$

For $i = m$, we take a symmetric HR-tiling which has a hexagon containing the vertices $[m-1]$,$[m]$,$[m+1]$,$[m+2]$. Then the mth operation is defined by the rule

$$(\widetilde{\mathbf{m}}f)(v) = \begin{cases} f(v)+2, & \text{if } v = [m], v = [m+1] \\ f(v) & \text{otherwise.} \end{cases} \tag{3}$$

7. **Theorem**. *The set of symmetric function $\widetilde{TP}$ endowed with operations (2) and (3) is a free C_m-crystal.*

8. Analogous to part 5.4, one can define the C_m-subcrystals $\widetilde{K^0}$, $\widetilde{K_0}$, and $\widetilde{K}(c)$. The next picture illustrates the extract $\widetilde{K}(1,0)$ from the crystal $K(1,0,0,1)$.



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