Planar Flows and Quadratic Relations over Semirings

Vladimir I. Danilov, Alexander V. Karzanov, and Gleb A. Koshevoy

1. Introduction

In this work, acting in spirit of Lindström's construction [**7**], we consider a wide class of functions which take values in an arbitrary commutative semiring and are generated by flows (systems of paths) in a planar acyclic directed graph. Functions of this sort satisfy plenty of "stable" (or "universal") quadratic relations, extending well-known quadratic relations for minors of matrices (in particular, Plücker's and Dodgson's ones) and their tropical analogues. We develop a combinatorial method to completely characterize the set of such "stable" relations. In particular, applying this method to Gessel–Viennot's model, one can describe quadratic relations on Schur functions (related to semi-standard Young tableaux). The full version of this work is to appear in J. Algebraic Combinatorics (DOI 10.1007/s10801-012-0344-6); see also Arxiv:1102.2578v2[math.CO].

We start with specifying terminology and notation, and with backgrounds.

1.1. Commutative semirings. In order to embrace both algebraic and tropical cases (and more), we will deal with functions taking values in an arbitrary *commutative semiring* (briefly, *CS*), a set $\mathfrak{S}$ equipped with two associative and commutative binary operations $\oplus$ (addition) and $\odot$ (multiplication) satisfying the distributive law $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$. When needed, we additionally assume that $\mathfrak{S}$ contains neutral elements $\underline{0}$ (for addition) and/or $\underline{1}$ (for multiplication). Important special cases are:

(i) a commutative ring (when $\underline{0} \in \mathfrak{S}$ and each element has an additive inverse);

(ii) a CS with division (when $\underline{1} \in \mathfrak{S}$ and each element has a multiplicative inverse); e.g., the set $\mathbb{R}_{>0}$ of positive reals (with $\oplus = +$ and $\odot = \cdot$), and the tropicalization $\mathfrak{L}_{\max}$ of a totally ordered abelian group $\mathfrak{L}$ (with $\oplus = \max$ and $\odot = +$); the most popular case of the latter is the real tropical semiring $\mathbb{R}_{\max}$.

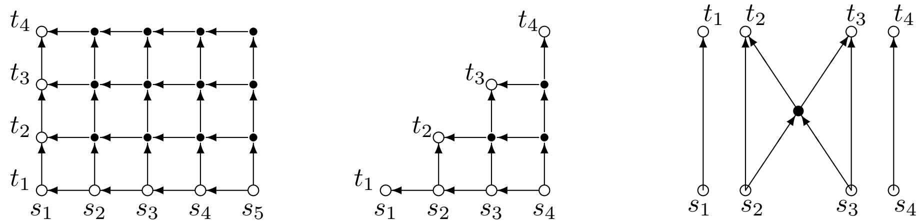
1.2. Planar flows. By a *planar network* we mean a finite directed planar *acyclic* graph $G = (V, E)$ in which two subsets $S = \{s_1, \dots, s_n\}$ and $T = \{t_1, \dots, t_{n'}\}$ of vertices are distinguished, called *sources* and *sinks*, respectively. We assume that the sources and sinks, also called *terminals*, lie on the *boundary* O of a compact convex region in the plane, and the remaining part of G lies inside O.

2000 *Mathematics Subject Classification.* 05C75, 05E99.

Key words and phrases. Plücker relation, Dodgson condensation, tropicalization, semiring, planar graph, network flow, Lindström's lemma, Schur function.

Supported by RFBR grant 10-01-9311-CNRSL_a.

The terminals appear in O in the *clockwise cyclic order* $s_n, \dots, s_1, t_1, \dots, t_{n'}$ (with possibly $s_1 = t_1$ or $s_n = t_{n'}$). Three examples are illustrated in the picture.



Let $\mathcal{E}^{n,n'}$ denote the set of pairs ($I \subseteq [n], I' \subseteq [n']$) with equal sizes: $|I| = |I'|$ (where $[k] := \{1, 2, \dots, k\}$). By an $(I|I')$-*flow* we mean a collection ϕ of $|I|$ *pairwise (vertex) disjoint directed paths* in G going from the source set $S_I := \{s_i : i \in I\}$ to the sink set $T_{I'} := \{t_j : j \in I'\}$. The set of $(I|I')$-flows is denoted by $\Phi_{I|I'} = \Phi^G_{I|I'}$.

Each vertex $v \in V$ is endowed with a *weight* $w(v) \in \mathfrak{S}$, where $\mathfrak{S}$ is a CS (alternatively, one can consider a weighting on the edges, which does not affect our results in essence). This gives rise to the function $f = f_{G,w}$ on $\mathcal{E}^{n,n'}$ defined by

$$f(I|I') := \bigoplus\nolimits_{\phi \in \Phi_{I|I'}} w(\phi), \qquad (I, I') \in \mathcal{E}^{n,n'}, \tag{1.1}$$

where $w(\phi)$ denotes the weight $\odot(w(v) : v \in V_\phi)$ of a flow ϕ, and V_ϕ is the set of vertices occurring in ϕ. We call f a *flow-generated function*, or an *FG-function* for short, and say that f is determined by G, w. The set of such functions over all corresponding G and w (with $n, n', \mathfrak{S}$ fixed) is denoted by $\mathbf{FG}_{n,n'}(\mathfrak{S})$.

Remark 1. When $\mathfrak{S} = \mathbb{R}$, (1.1) is specified as $f(I|I') := \sum_{\phi \in \Phi_{I|I'}} (\prod_{v \in V_\phi} w(v))$, and when $\mathfrak{S} = \mathbb{R}_{\max}$, (1.1) turns into $f(I|I') := \max_{\phi \in \Phi_{I|I'}} (\sum_{v \in V_\phi} w(v))$. In the former (latter) case, we refer to f as an *algebraic* (resp. *tropical*) *FG-function.*

Remark 2. Note that an $(I|I')$-flow in G may not exist, making $f(I|I')$ undefined if $\mathfrak{S}$ does not contain $\underline{0}$. To overcome this trouble, we formally extend $\mathfrak{S}$, when needed, by adding an "extra neutral" element $*$, setting $* \oplus a = a$ and $* \odot a = *$ for all $a \in \mathfrak{S}$. In the extended semiring, one defines $f(I|I') := *$ in case $\Phi_{I|I'} = \emptyset$.

When an $(I|I')$-flow ϕ enters the first $|I| =: k$ sinks (i.e. $I' = [k]$), we say that ϕ is a *flag flow* for I. Accordingly, notation $\Phi_{I|\,[k]}$ is abbreviated to Φ_I, and $f(I|\,[k])$ to $f(I)$. When we are interested in the flag case only, f is regarded as a function on the set $2^{[n]}$ of subsets of $[n]$.

1.3. Lindström's lemma. Assume that weights w of vertices of G belong to a commutative ring and consider the $n' \times n$ matrix M whose entries m_{ji} are defined as $\sum_{\phi \in \Phi_{\{i\}|\{j\}}} (\prod_{v \in V_\phi} w(v))$ (cf. Remark 1). For $(I, I') \in \mathcal{E}^{n,n'}$, let $f_M(I|I')$ denote the minor of M with the column set I and the row set I'. A remarkable property shown by Lindström [**7**] is that $f_M = f_{G,w}$.

(Note that the class of matrices whose minor functions are flow-generated is large. In particular, it has been shown that any totally nonnegative matrix (a real matrix whose all minors are nonnegative) is such; see [**1**]. The question whether this class contains all matrices over any commutative ring is still open, but we can show that it contains any matrix over a *field*; see Arxiv:1102.2578v2[math.CO].)

1.4. Quadratic relations. Minors of (real or complex) matrices obey many quadratic relations. Most popular among them are quadratic relations on flag minors, or *Plücker relations* (which, in particular, describe flag manifolds and Grassmannians embedded in corresponding projective spaces). Therefore, by Lindström's lemma, similar relations should be valid for any FG-function $f = f_{G,w}$ when $\mathfrak{S} = \mathbb{R}$ or $\mathbb{C}$ (or even an arbitrary commutative ring). Below are two examples.

(i) The simplest example of Plücker relations (in the flag case) involve triples: for any three elements $i < j < k$ in $[n]$ and any subset $X \subseteq [n] - \{i, j, k\}$,

$$f(Xik)f(Xj) = f(Xij)f(Xk) + f(Xjk)f(Xi), \tag{1.2}$$

where for brevity we write $Xi' \dots j'$ for $X \cup \{i', \dots, j'\}$ (and as before, $f(I)$ stands for $f(I|\,[|I|]))$. This is called the *AP3-relation* (abbreviating "algebraic Plücker relation with triples").

(ii) The simplest relation in the non-flag case arises from Dodgson's condensation formula for matrices [**3**]: for elements $i < k$ of $[n]$ and elements $i' < k'$ of $[n']$ and for $X \subseteq [n] - \{i, k\}$ and $X' \subseteq [n'] - \{i', k'\}$,

$$f(iX|\,i'X')\,f(Xk|X'k') = f(iXk|\,i'X'k')\,f(X|X') + f(iX|X'k')\,f(Xk|\,i'X'). \tag{1.3}$$

The "tropical counterpart" of (1.2) is the *TP3-relation*, viewed as

$$f(Xik) + f(Xj) = \max\{f(Xij) + f(Xk), f(Xjk) + f(Xi)\}. \tag{1.4}$$

This is valid for any tropical FG-function f; see [**2**] (where the case $\mathfrak{S} = \mathbb{R}_{\max}$ is considered, but the argument is extended straightforwardly to any $\mathcal{L}_{\max}$).

In general, the quadratic relations of our interest involve FG-functions on $\mathcal{E}^{n,n'}$ over an arbitrary CS $\mathfrak{S}$ and can be expressed as

$$\bigoplus\nolimits_{(A,A')\in\mathcal{A}} \left(f(XA|X'A') \odot f(X\overline{A}|X'\overline{A}\,')\right) = \bigoplus\nolimits_{(B,B')\in\mathcal{B}} \left(f(XB|X'B') \odot f(X\overline{B}|X'\overline{B}\,')\right). \tag{1.5}$$

Here: X, Y (resp. X', Y') are disjoint subsets of $[n]$ (resp. $[n']$); we write KL for $K \cup L$; the complement $Y - C$ of $C \subseteq Y$ is denoted by $\overline{C}$, and the complement $Y' - C'$ of $C' \subseteq Y'$ by $\overline{C}\,'$. The families $\mathcal{A}, \mathcal{B}$ consist of certain pairs $(C \subseteq Y, C' \subseteq Y')$, admitting multiple ones. (The sizes of sets above are assumed to be *agreeable*: they should satisfy $|X| + |C| = |X'| + |C'|$ and $|X| + |\overline{C}| = |X'| + |\overline{C}'|$, or, equivalently, $2|X| + |Y| = 2|X'| + |Y'|$ and $|Y| - 2|C| = |Y'| - 2|C'|$.)

In fact, an instance of (1.5) represents a variety of relations of "the same type", which does not depend on X, Y, X', Y' and is specified by two patterns $\mathcal{A}_0$ and $\mathcal{B}_0$. More precisely, letting $m := |Y|$ and $m' := |Y'|$, take the order preserving maps $\gamma : [m] \to Y$ and $\gamma' : [m'] \to Y'$ (i.e. $\gamma(i) < \gamma(j)$ for $i < j$, and similarly for γ'). Then the *pattern* $\mathcal{A}_0$ (inducing $\mathcal{A}$) consists of pairs $(A_0 \subseteq [m],\, A_0' \subseteq [m'])$ so that $\mathcal{A} = \{(\gamma(A_0), \gamma'(A_0'))\colon (A_0, A_0') \in \mathcal{A}_0)$, and the pattern $\mathcal{B}_0$ (inducing $\mathcal{B}$) is defined similarly. We write $\mathcal{A} = \gamma_{Y,Y'}(\mathcal{A}_0)$ and $\mathcal{B} = \gamma_{Y,Y'}(\mathcal{B}_0)$.

It should be noted that in the flag case, the sets X', Y', as well as A', B' in (1.5), are determined uniquely. For this reason, we omit them in the above expressions and think of $\mathcal{A}, \mathcal{B}$ (resp. $\mathcal{A}_0, \mathcal{B}_0$) as consisting of subsets of Y (resp. $[m]$).

Examples. Relation (1.3) deals with $Y = \{i, k\}$, $Y' = \{i', k'\}$, $[m] = \{1, 2\}$, $[m'] = \{1', 2'\}$, $\mathcal{A} = \{i|i'\}$, $\mathcal{B} = \{ik|i'k',\, i|k'\}$, $\mathcal{A}_0 = \{1|1'\}$, and $\mathcal{B}_0 = \{12|1'2',\, 1|2'\}$.

In turn, Plücker's type relations (1.2) and (1.4) concern $Y = \{i, j, k\}$, $m = 3$, $\mathcal{A} = \{ik\}$, $\mathcal{B} = \{ij,\ jk\}$, $\mathcal{A}_0 = \{13\}$, and $\mathcal{B}_0 = \{12,\ 23\}$.

Definition. When (1.5) holds for fixed $\mathcal{A}_0, \mathcal{B}_0$ as above and any corresponding $\mathfrak{S}, G, w, X, Y, X', Y'$ and the families $\mathcal{A} := \gamma_{Y,Y'}(\mathcal{A}_0)$ and $\mathcal{B} := \gamma_{Y,Y'}(\mathcal{B}_0)$, we call (1.5) a *stable quadratic relation*, or an *SQ-relation*, and say that this relation is induced by the patterns $\mathcal{A}_0, \mathcal{B}_0$.

Our goal is to give a relatively simple combinatorial method of characterizing the patterns $\mathcal{A}_0, \mathcal{B}_0$ inducing SQ-relations. In fact, our method generalizes a flow rearranging approach used in [**2**] for proving the TP3-relation for tropical FG-functions. It consists in reducing the task to a certain combinatorial problem on *matchings*, and as a consequence, provides an "efficient" procedure to recognize whether or not a given pair $\mathcal{A}, \mathcal{B}$ yields an SQ-relation. It should be noted that our approach is close in essence to a lattice paths method elaborated in Fulmek and Kleber [**5**] and Fulmek [**4**] to generate quadratic identities on Schur functions.

2. Balanced families and the main result

Consider (agreeable) $X, Y, X', Y', \mathcal{A}, \mathcal{B}, \mathcal{A}_0, \mathcal{B}_0$ as above. It will be convenient for us to think that the elements of Y and Y' are placed, respectively, on the lower half and on the upper half of a circumference O in the plane, in the increasing order from left to right. Also, considering a member (C, C') of $\mathcal{A} \cup \mathcal{B}$, we call the elements of C and C' *white*, and the elements of their complements $\overline{C} = Y - C$ and $\overline{C}' = Y - C'$ *black*. For members of patterns $\mathcal{A}_0$ and $\mathcal{B}_0$, white/black colorings on $[m] \sqcup [m']$ are defined similarly (where $\sqcup$ denotes the disjoint union).

Let M be a *perfect matching* on $Y \sqcup Y'$, i.e. M is a partition of $Y \sqcup Y'$ into 2-element subsets, or *couples*. We say that M is *feasible* for (C, C') (as above) if:

(2.1) (i) For a couple $\pi \in M$, if either $\pi \subseteq Y$ or $\pi \subseteq Y'$, then the elements of π have different colors;
(ii) If one element of $\pi \in M$ belongs to Y and the other to Y', then these elements have the same color;
(iii) M is planar, in the sense that the chords of O connecting the couples in M are pairwise not intersecting.

Let $\mathcal{M}(C, C')$ denote the set of feasible matchings for (C, C'). We define $\mathcal{M}(\mathcal{A})$ to be the family being the union of sets $\mathcal{M}(C, C')$ (respecting multiplicities) over all $(C, C') \in \mathcal{A}$. Analogous families are defined for $\mathcal{B}$ and for $\mathcal{A}_0, \mathcal{B}_0$ (concerning matchings on $[m] \sqcup [m']$).

Definition. Families $\mathcal{A}, \mathcal{B}$ are called *balanced* if $\mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{B})$ (regarding $\mathcal{M}(\cdot)$ as a multi-set).

(Clearly $\mathcal{A}, \mathcal{B}$ are balanced if and only if so are the patterns $\mathcal{A}_0, \mathcal{B}_0$.)

Our main result is the following

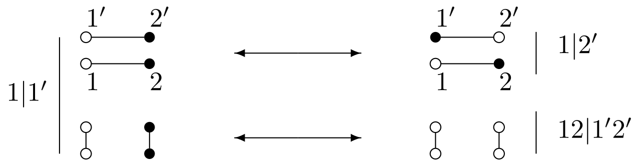
THEOREM 2.1. *(1.5) is an SQ-relation if and only if $\mathcal{A}, \mathcal{B}$ are balanced.*

A sketch of the proof of this theorem will be outlined in Sections 4 and 5.

3. Examples of stable quadratic relations

In this section we illustrate Theorem 2.1 with several simple examples (for more examples, see Arxiv:1102.2578v2[math.CO]). According to this theorem, once we are able to show that one or another pair of patterns $\mathcal{A}_0, \mathcal{B}_0$ is balanced, we can declare that (1.5) holds for any corresponding $X, Y, X', Y', \mathcal{A}, \mathcal{B}$.

3.1. Let $m = m' = 2$. Consider the patterns $\mathcal{A}_0 = \{1|1'\}$ and $\mathcal{B}_0 = \{1|2', 12|1'2'\}$ for the intervals $[m] = \{1, 2\}$ and $[m'] = \{1', 2'\}$. One can see that the only member $1|1'$ of $\mathcal{A}_0$ admits two feasible matchings, namely, $\mathcal{M}(1|1') = \{\{12, 1'2'\}, \{11', 22'\}\}$, whereas each member of $\mathcal{B}_0$ has exactly one feasible matching, namely, $\mathcal{M}(1|2') = \{\{12, 1'2'\}\}$ and $\mathcal{M}(12|1'2') = \{\{\{11', 22'\}\}$. This implies that $\mathcal{A}_0, \mathcal{B}_0$ are balanced. The corresponding feasible matchings and bijection are illustrated in the picture (where the white/black partitions and matchings on $[m] \sqcup [m']$ are indicated by using two-level diagrams).



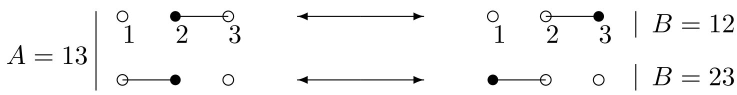
This gives rise to the SQ-relation extending Dodgson's condensation formula (1.3) (by taking $Y = (i < k)$, $Y' = (i' < k')$, $\mathcal{A} = \{i|i'\}$, and $\mathcal{B} = \{i|k', ik|i'k'\}$).

The next two examples concern SQ-relations of Plücker's type (the flag case). Here all members of patterns $\mathcal{A}_0, \mathcal{B}_0$ are subsets C of $[m]$ (as before, we say that the elements of C are *white*, and the ones of $\overline{C} := [m] - C$ are *black*). One can check that these subsets have the same cardinality p; one may assume, w.l.o.g., that $p \ge m - p =: q$. Furthermore, instead of perfect matchings on $[m] \sqcup [m']$ occurring in the general case, we now should consider matchings M of cardinality q on $[m]$. Such an M is called *feasible* for a (white) subset $C \subseteq [m]$ of size p if

(i) the elements of each couple in M have different colors; and

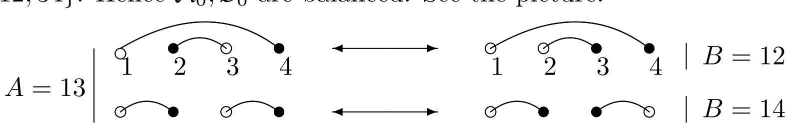
(ii) there are no $i < j < k < \ell$ such that $ik, j\ell \in M$ (i.e. M is nested), and there are no $i < j < k$ such that $ik \in M$ and $j \in C - \cup(\pi \in M)$.

3.2. When $m = 3$ and $p = 2$, there are three p-element subsets in $[m]$, namely, $12, 13, 23$. Each of 12 and 23 admits only one feasible matching, namely, $\mathcal{M}(12) = \{\{23\}\}$ and $\mathcal{M}(23) = \{\{12\}\}$, whereas 13 has two feasible matchings: $\mathcal{M}(13) = \{\{12\}, \{23\}\}$. Hence the patterns $\mathcal{A}_0 := \{13\}$ and $\mathcal{B}_0 := \{12, 23\}$ are balanced. The feasible matchings and bijection are illustrated in the picture.



This gives rise to the SQ-relation "on triples" extending (1.2) and (1.4) (by taking $Y = (i < j < k)$, $\mathcal{A} = \{ik\}$, and $\mathcal{B} = \{ij, jk\}$).

3.3. For $m = 4$ and $p = 2$, take $\mathcal{A}_0 := \{13\}$ and $\mathcal{B}_0 := \{12, 14\}$. Each of 12 and 14 admits a unique feasible matching: $\mathcal{M}(12) = \{\{14, 23\}\}$ and $\mathcal{M}(14) = \{\{12, 34\}\}$, whereas $\mathcal{M}(13)$ consists of two feasible matchings: just those $\{14, 23\}$ and $\{12, 34\}$. Hence $\mathcal{A}_0, \mathcal{B}_0$ are balanced. See the picture.



This implies the following SQ-relation: for $i<j<k<\ell$ and $X\subseteq[n]-\{i,j,k,\ell\}$,

$$f(Xik)\odot f(Xj\ell)=(f(Xij)\odot f(Xk\ell))\oplus(f(Xi\ell)\odot f(Xjk)).$$

4. A sketch of proof of "if" part in the theorem.

Consider corresponding $G,w,\mathfrak{S},X,Y,X',Y',\mathcal{A},\mathcal{B}$. We have to show that if $\mathcal{A},\mathcal{B}$ are balanced, then (1.5) is valid.

First of all one easily shows that it suffices to examine only those planar networks G (with n sources and n' sinks) that satisfy the following condition:

(C) the source set S and sink set T are disjoint, and each vertex has either at most one entering edge, or at most one leaving edge, or both.

Below we refer to an arbitrary, not necessarily directed, path P in G as a *route*, referring to its edges as *forward* and *backward* ones, depending on their orientation in P. A route P is called *simple* if all vertices in it are distinct. A closed route with distinct vertices is called a *circuit*.

Our approach is based on examining certain pairs of flows in G and rearranging them to form other pairs. Fix $(A,A')\in\mathcal{A}$ and consider an $(XA|X'A')$-flow ϕ and a $(X\overline{A}|X\overline{A}')$-flow ϕ' in G. The pair (ϕ,ϕ') is called a *double flow* for (A,A'), and the set of such double flows is denoted by $\mathcal{D}(A,A')$. We use two lemmas; their proofs are rather simple and rely on condition (C). Here we write $C\triangle D$ for the symmetric difference $(C-D)\cup(D-C)$ of sets C,D, and regard a flow as edge set.

Lemma 4.1. *$\phi\triangle\phi'$ is partitioned into (the edge sets of) pairwise disjoint circuits $C_1,\dots,C_d$ and simple routes $P_1,\dots,P_p$, where $p=\frac{1}{2}(m+m')$, and each P_i connects either S_A and $S_{\overline{A}}$, or S_A and $T_{A'}$, or $S_{\overline{A}}$ and $T_{\overline{A}'}$, or $T_{A'}$ and $T_{\overline{A}'}$. In each circuit or route, the edges of ϕ and the edges of ϕ' have opposite directions.*

The next lemma explains how to rearrange a double flow (ϕ,ϕ') for (A,A') so as to obtain a double flow for another (useful) pair $(B\subseteq Y,\ B'\subseteq Y')$. Define $\mathcal{P}(\phi,\phi'):=\{P_1,\dots,P_p\}$. For a route P in $\mathcal{P}(\phi,\phi')$, let $\pi(P)$ denote the pair of elements in $Y\sqcup Y'$ corresponding to the end vertices of P. By Lemma 4.1, $\pi(P)$ belongs to one of $A\times\overline{A}$, $A\times A'$, $A'\times\overline{A}'$, $\overline{A}\times\overline{A}'$. Moreover, the set

$$M(\phi,\phi'):=\{\pi(P):P\in\mathcal{P}(\phi,\phi')\}$$

is a perfect matching on $Y\sqcup Y'$.

Lemma 4.2. *Choose an arbitrary subset $M_0\subseteq M(\phi,\phi')$. Define $Z:=\cup(\pi\in M_0)$, $B:=A\triangle(Z\cap Y)$, and $B':=A'\triangle(Z\cap Y')$. Let U be the set of edges of routes $P\in\mathcal{P}(\phi,\phi')$ with $\pi(P)\in M_0$. Then $\psi:=\phi\triangle U$ gives an $(XB|X'B')$-flow, and $\psi':=\phi'\triangle U$ gives an $(X\overline{B}|X'\overline{B}')$-flow. Also $\psi\sqcup\psi'=\phi\sqcup\phi'$.*

Obviously, $M(\psi,\psi')=M(\phi,\phi')$ and $\mathcal{P}(\psi,\psi')=\mathcal{P}(\phi,\phi')$, and the transformation of ψ,ψ' by use of the routes $P\in\mathcal{P}(\psi,\psi')$ with $\pi(P)\in M_0$ returns ϕ,ϕ'.

Now consider the FG-function $f=f_{G,w}$ on $\mathcal{E}^{n,n'}$. The summand concerning $(A,A')\in\mathcal{A}$ in the L.H.S. of (1.5) can be expressed via double flows as follows:

$$\begin{aligned}(4.1)\quad & f(XA|X'A')\odot f(X\overline{A}|X'\overline{A}')\\ &=\Big(\bigoplus\nolimits_{\phi\in\Phi_{XA|X'A'}}w(\phi)\Big)\odot\Big(\bigoplus\nolimits_{\phi'\in\Phi_{X\overline{A}|X'\overline{A}'}}w(\phi')\Big)\end{aligned}$$

$$= \bigoplus\nolimits_{(\phi,\phi')\in\mathcal{D}(A,A')} w(\phi)\odot w(\phi')$$

$$= \bigoplus\nolimits_{M\in\mathcal{M}(A,A')} \bigoplus\nolimits_{(\phi,\phi')\in\mathcal{D}(A,A')\,:\,M(\phi,\phi')=M} w(\phi)\odot w(\phi').$$

The summand concerning $(B,B')\in\mathcal{B}$ in the L.H.S. of (1.5) is expressed similarly.

Finally, for $(A,A')\in\mathcal{A}$ and $M\in\mathcal{M}(A,A')$, consider $(\phi,\phi')\in\mathcal{D}(A,A')$ such that $M(\phi,\phi')=M$ (if it exists). Since $\mathcal{A},\mathcal{B}$ are balanced, (A,A',M) is bijective to some (B,B',M) such that $(B,B')\in\mathcal{B}$ and $M\in\mathcal{M}(B,B')$. Since M is a feasible matching for both (A,A') and (B,B'), it follows from (2.1)(i),(ii) that (B,B') is obtained from (A,A') by "recoloring" w.r.t. some $M_0\subseteq M$. Then the transformation of (ϕ,ϕ') by use of the routes $P\in\mathcal{P}(\phi,\phi')$ with $\pi(P)\in M_0$ (as described in Lemma 4.2), results in a double flow (ψ,ψ') for (B,B') such that $\psi\sqcup\psi'=\phi\sqcup\phi'$, implying $w(\psi)\odot w(\psi')=w(\phi)\odot w(\phi')$. Moreover, $(\phi,\phi')\mapsto(\psi,\psi')$ gives a bijection between all double flows for (A,A',M) and those for (B,B',M). Now (1.5) follows by considering the last term in (4.1).

5. Necessity of the balancedness

Part "only if" of Theorem 2.1 says that if patterns $\mathcal{A}_0,\mathcal{B}_0$ are not balanced, then there exist corresponding $G,w,\mathfrak{S},X,Y,X',Y'$ for which (1.5) with $\mathcal{A}=\gamma_{Y,Y'}(\mathcal{A}_0)$ and $\mathcal{B}=\gamma_{Y,Y'}(\mathcal{B}_0)$ is violated. (Hereinafter X,Y are disjoint subsets of $[n]$, X',Y' are disjoint subsets of $[n']$, and $X,Y,X',Y',\mathcal{A}_0,\mathcal{B}_0$ should be *agreeable*, i.e. there hold $m+2|X|=m'+2|X'|$ and $m-2|C|=m'-2|C'|$ for all $(C,C')\in\mathcal{A}_0\cup\mathcal{B}_0$, where $m:=|Y|$, $m':=|Y'|$.) We can show a sharper result, saying that if the patterns are not balanced, then (1.5) is violated for *any* choice of X,Y,X',Y' and for $\mathfrak{S}:=\mathbb{Z}_+$.

THEOREM 5.1. *Suppose that patterns $\mathcal{A}_0,\mathcal{B}_0$ are not balanced. Fix (agreeable) X,Y,X',Y'. Then there exists, and can be explicitly constructed, a planar network $G=(V,E)$ such that (1.5) is false for $f=f_{G,w}$, where $w(v)=1$ for all $v\in V$.*

The idea of the proof is roughly as follows. Since $\mathcal{A}_0,\mathcal{B}_0$ are not balanced, there exists a planar perfect matching M on $Y\sqcup Y'$ such that

$$|\mathcal{A}_M|\ne|\mathcal{B}_M|,$$

where $\mathcal{A}_M$ is the set of members of $\mathcal{A}$ having M as a feasible matching, and similarly for $\mathcal{B}$. We succeed to construct a planar network G (depending on X,Y,X',Y',M) with the following properties: for any pair $(C\subseteq Y,\ C'\subseteq Y')$,

(P1) If $M\in\mathcal{M}(C,C')$, then G has a unique $(XC|X'C')$-flow and a unique $(X\overline{C}|X'\overline{C}')$-flow, i.e. $|\Phi_{XC|X'C'}|=|\Phi_{X\overline{C}|X'\overline{C}'}|=1$;

(P2) If $M\notin\mathcal{M}(C,C')$, then at least one of $\Phi_{XC|X'C'}$ and $\Phi_{X\overline{C}|X'\overline{C}'}$ is empty.

Take the function $f=f_{G,w}$ for $w\equiv 1$. By (P1) and (P2), for a pair (C,C'), each of the values $f(XC|X'C')$ and $f(X\overline{C}|X'\overline{C}')$ is equal to 1 if $M\in\mathcal{M}(C,C')$, and at least one of them is 0 otherwise. This implies that the values in the L.H.S. and R.H.S. of (1.5) are exactly $|\mathcal{A}_M|$ and $|\mathcal{B}_M|$, respectively. Thus, these values are different and (1.5) is violated.

6. Applications to Schur functions

It is known that Schur functions (polynomials) are expressed as minors of a certain matrix, by Jacobi–Trudi's formula. Therefore, these functions satisfy many quadratic relations. [**4, 5**] and some other works (see a discussion in [**4**]) explain how to obtain quadratic relations for ordinary and skew Schur functions by use of a lattice paths method based on the Gessel–Viennot interpretation of semistandard Young tableaux [**6**]. This lattice path method is, in fact, a specialization to a particular planar network of the flow approach that we described in Sections 1,2. Below we give a brief discussion on this subject.

Recall that a *partition* of length r is an r-tuple λ of weakly decreasing non-negative integers $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_r$. The *Ferrers diagram* of λ is meant to be the array F_λ of cells with r left-aligned rows containing λ_i cells in ith row. (We assume that the row indices grow from the bottom to the top.) For $N \in \mathbb{N}$, an N-*semistandard Young tableau* of shape λ is a filling T of F_λ with natural numbers not exceeding N so that the numbers weakly increase in each row and strictly increase in each column. We associate to T the monomial x^T to be the product of variables $x_1, \ldots, x_N$, each x_k being taken with the degree equal to the number of occurrences of k in T. Then the *Schur function* for λ and N is the polynomial

$$s_\lambda = s_\lambda(x_1, \ldots, x_N) := \sum\nolimits_T x^T,$$

where the sum is over all N-semistandard Young tableaux T of shape λ. Besides, one often considers a *skew* Schur function $s_{\lambda/\mu}$, where μ is a partition of length r with $\mu_i \le \lambda_i$; it is defined in a similar way w.r.t. the skew Ferrers diagram $F_{\lambda/\mu}$ obtained by removing from F_λ the cells of F_μ. When needed, an "ordinary" diagram F_λ is regarded as $F_{\lambda/\mu}$ with $\mu = (0, \ldots, 0)$, and similarly for tableaux.

There is a one-to-one correspondence between the partitions λ of length r and the r-element subsets A_λ of the set $\mathbb{Z}_{>0}$ of positive integers, namely:

$$\text{(6.1)} \qquad \lambda = (\lambda_1 \ge \ldots \ge \lambda_r) \Longleftrightarrow A_\lambda := \{\lambda_r + 1, \lambda_{r-1} + 2, \ldots, \lambda_1 + r\}.$$

The graph of our interest is the directed square grid $\Gamma = \Gamma(N)$ whose vertices are the points (i,j) for $i \in \mathbb{Z}_{>0}$ and $j \in [N]$ and whose edges e are directed up or to the right, i.e. $e = ((i,j),(i,j+1))$ or $((i,j),(i+1,j))$ (it suffices to take a finite truncation of this grid). The vertices $s_i := (i,1)$ and $t_i := (i,N)$ are regarded as the sources and sinks in Γ, respectively, and we assign to each horizontal edge e at level h the weight to be the indeterminate x_h:

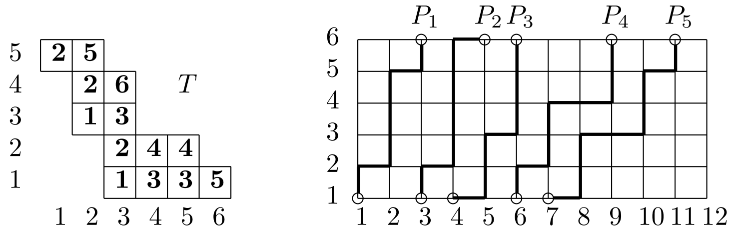
$$\text{(6.2)} \qquad w(e) := x_h \qquad \text{for } e = ((i,h),(i+1,h)),\ i \in \mathbb{Z}_{>0},\ h = 1, \ldots, N,$$

and assign weight 1 to each vertical edge. Now using the Gessel–Viennot model [**6**] (in a slightly different form), one can associate to an N-semistandard skew Young tableau T with shape λ/μ the system $\mathcal{P}_T = (P_1, \ldots, P_r)$ of directed paths in Γ, where for $k = 1, \ldots, r$:

(6.3) P_k is related to $(r+1-k)$th row of T: it goes from the source $s_{k+\mu_{r+1-k}}$ to the sink $t_{k+\lambda_{r+1-k}}$, and for $h = 1, \ldots, N$, the number of horizontal edges of P_k at level h equals the number of occurrences of h in kth row of T.

So the sources occurring in $\mathcal{P}_T$ are the s_i for $i \in A_\mu$, and the sinks are the t_j for $j \in A_\lambda$. Observe that the semistandardness of T implies that these paths are pairwise disjoint, i.e. $\mathcal{P}_T$ is an $(A_\mu|A_\lambda)$-flow in Γ. One can see the converse as well: if $\mathcal{P}$ is an $(A_\mu|A_\lambda)$-flow in Γ, then the filling T of $F_{\lambda/\mu}$ determined, in a due way, by the horizontal edges of paths in $\mathcal{P}$ is just a semistandard skew Young tableau, and one has $\mathcal{P}_T = \mathcal{P}$. This gives a nice bijection between corresponding flows and tableaux. The next picture illustrates an example of a semistandard Young tableau

T with $N = 6$, $r = 5$, $\lambda = (6, 5, 3, 3, 2)$ and $\mu = (2, 2, 1, 1, 0)$, and its corresponding flow $\mathcal{P}_T = (P_1, \dots, P_5)$.



Note that when T is "ordinary" (i.e. $\mu = \mathbf{0}$), the sources used in $\mathcal{P}_T$ are $s_1, s_2, \dots, s_r$; in other words, $\mathcal{P}_T$ is a *co-flag flow* (it becomes a flag flow if we reverse the edges of Γ and swap the sources and sinks).

The above bijection between the N-semistandard skew Young tableaux with shape λ/μ and the $(A_\mu|A_\lambda)$-flows in $\Gamma = \Gamma(N)$ implies that (ordinary or skew) Schur functions are "values" of the flow-generated function $f_{\Gamma,w}$ for the weighting w as in (6.2). (It leads to no confusion that the weights are given on the horizontal edges of Γ and belong to a polynomial ring.) This enables us to exhibit quadratic relations on Schur functions, by properly translating SQ-relations on FG-functions.

References

[1] F. Brenti, Combinatorics and total positivity, *J. Combin. Theory*, Ser. A, **71** (1995) 175–218.

[2] V. Danilov, A. Karzanov, and G. Koshevoy, Tropical Plücker functions and their bases, in: *Tropical and Idempotent Mathematics* (ed. G.L. Litvinov and S.N. Sergeev), *Contemporary Mathematics* **495** (2009) 127–158.

[3] C.L. Dodgson, Condensation of determinants, *Proc. of the Royal Soc. of London* **15** (1866) 150–155.

[4] M. Fulmek, Bijective proofs for Schur function identities, *ArXiv*:0909.5334v1[math.CO], 2009.

[5] M. Fulmek and M. Kleber, Bijective proofs for Schur function identities which imply Dodgson's condensation formula and Plücker relations, *Electron. J. Combin.* **8** (1): Research Paper 16, 2001, 22 pp.

[6] I.M. Gessel and X. Viennot, Determinants, paths, and plane partitions, *Preprint*, 1989.

[7] B. Lindström, On the vector representation of induced matroids, *Bull. London Math. Soc.* **5** (1973) 85–90.

V.I. Danilov and G.A. Koshevoy: Central Institute of Economics and Mathematics of the RAS, 47, Nakhimovskii Prospect, 117418 Moscow, Russia. Emails: danilov@cemi.rssi.ru (V.I. Danilov), and koshevoy@cemi.rssi.ru (G.A. Koshevoy).

A.V. Karzanov: Institute for System Analysis of the RAS, 9, Prospect 60 Let Oktyabrya, 117312 Moscow, Russia. Email: sasha@cs.isa.ru. Corresponding author.