A combinatorial algorithm for the planar multiflow problem with demands located on three holes

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Abstract. We consider an undirected multi(commodity)flow demand problem in which a supply graph is planar, each source-sink pair is located on one of three specified faces of the graph, and the capacities and demands are integer-valued and Eulerian. It is known that such a problem has a solution if the cut and (2,3)-metric conditions hold, and that the solvability implies the existence of an integer solution. We develop a purely combinatorial strongly polynomial solution algorithm.

Keywords: Multi(commodity)flow, planar graph, cut condition, (2,3)-metric condition, strongly polynomial algorithm

AMS Subject Classification: 90C27, 05C10, 05C21, 05C85

1 Introduction

Among a variety of multi(commodity)flow problems, one popular class embraces multiflow demand problems in undirected planar graphs in which the demand pairs are located within specified faces of the graph. More precisely, a problem input consists of: a planar graph $G = (V, E)$ with a fixed embedding in the plane; nonnegative integer *capacities* $c(e) \in \mathbb{Z}_+$ of edges $e \in E$; a subset $\mathcal{H} \subseteq \mathcal{F}_G$ of faces, called *holes* (where $\mathcal{F}_G$ is the set of faces of G); a set D of pairs st of vertices such that both s, t are located on (the boundary of) one of the holes; and *demands* $d(st) \in \mathbb{Z}_+$ for $st \in D$. A *multiflow* for G, D is meant to be a pair $f = (\mathcal{P}, \lambda)$ consisting of a set $\mathcal{P}$ of D-paths P in G and nonnegative real weights $\lambda(P) \in \mathbb{R}_+$. Here a path P is called a *D-path* if $\{s_P, t_P\} = \{s, t\}$ for some $st \in D$, where s_P and t_P are the first and last vertices of P, respectively. We call f *admissible* for c, d if it satisfies the capacity constraints:

$$\sum\Big(\lambda(P)\colon e \in P \in \mathcal{P}\Big) \le c(e), \qquad e \in E, \tag{1.1}$$

and realizes the demands:

$$\sum\Big(\lambda(P)\colon P \in \mathcal{P},\ \{s_P, t_P\} = \{s, t\}\Big) = d(st), \qquad st \in D. \tag{1.2}$$

The (fractional) *demand problem*, denoted as $\mathcal{D}(G,\mathcal{H},D,c,d)$, or $\mathcal{D}(c,d)$ for short, is to find an admissible multiflow for c,d (or to declare that there is none). When the number of holes is "small", this linear program is known to possess nice properties. To recall them, we need some terminology and notation.

For $X\subseteq V$, the set of edges of G with one end in X and the other in $V-X$ is denoted by $\delta(X)=\delta_G(X)$ and called the *cut* in G determined by X. We also denote by $\rho(X)=\rho_D(X)$ the set of pairs $st\in D$ *separated* by X, i.e., such that $|\{s,t\}\cap X|=1$. For a singleton v, we write $\delta(v)$ for $\delta(\{v\})$, and $\rho(v)$ for $\rho(\{v\})$. For a function $g:S\to\mathbb{R}$ and a subset $S'\subseteq S$, $g(S')$ denotes $\sum(g(e)\colon e\in S')$. So $c(\delta(X))$ is the capacity of the cut $\delta(X)$, and $d(\rho(X))$ is the total demand on the elements of D separated by X.

A capacity-demand pair (c,d) is said to be *Eulerian* if $c(\delta(v))-d(\rho(v))$ is even for all vertices $v\in V$.

The simplest sort of necessary conditions for the solvability of the multiflow demand problem with any G,D is the well-known *cut condition*, saying that

$$\Delta_{c,d}(X):=c(\delta(X))-d(\rho(X))\geq 0 \tag{1.3}$$

should hold for all $X\subset V$. It need not be sufficient, and in general the solvability of a multiflow demand problem is provided by metric conditions. In our case the following results have been known.

(A) For $|\mathcal{H}|=1$, Okamura and Seymour [9] showed that the cut condition is sufficient, and that if (c,d) is Eulerian and the problem $\mathcal{D}(c,d)$ has a solution, then it has an *integer* solution, i.e., there exists an admissible multiflow $(\mathcal{P},\lambda)$ with λ integer-valued. Okamura [8] showed that these properties continue to hold for $|\mathcal{H}|=2$.

(B) For $|\mathcal{H}|=3$, the cut condition becomes not sufficient and the solvability criterion involves also the so-called *(2,3)-metric condition.* It is related to a map $\sigma:V\to V(K_{2,3})$, where $K_{p,q}$ is the complete bipartite graph with parts of p and q vertices. Such a σ defines the *metric* $m=m^\sigma$ on V by $m(u,v):=\operatorname{dist}(\sigma(u),\sigma(v))$, $u,v\in V$, where dist denotes the distance (the shortest path length) between vertices in $K_{2,3}$. It gives a partition of V into five sets, with distances 1 or 2 between them, and m is said to be a *(2,3)-metric* on V. (When speaking of a metric, we admit zero distances between different points, i.e., consider a *semimetric* in essence.) We denote $\sum(c(e)m(e)\colon e\in E)$ by $c(m)$, and $\sum(d(st)m(st)\colon st\in D)$ by $d(m)$. Karzanov showed the following

Theorem 1 ([4]). *Let $|\mathcal{H}|=3$. Then $\mathcal{D}(c,d)$ has a solution if and only if cut condition (1.3) holds, and*

$$\Delta_{c,d}(m):=c(m)-d(m)\geq 0 \tag{1.4}$$

holds for all (2,3)-metrics m on V (the (2,3)-metric condition*). Furthermore, if (c,d) is Eulerian and the problem $\mathcal{D}(c,d)$ has a solution, then it has an integer solution.*

We call $\varDelta_{c,d}(X)$ in (1.3) (resp. $\varDelta_{c,d}(m)$ in (1.4)) the *excess* of a set X (resp. a (2,3)-metric m) w.r.t. c, d. One easily shows that $\varDelta_{c,d}(X)$ and $\varDelta_{c,d}(m)$ are even if (c, d) is Eulerian.

(C) When $|\mathcal{H}| = 4$, the situation becomes more involved. As is shown in [5], the solvability criterion for $\mathcal{D}(c, d)$ involves, besides cuts and (2,3)-metrics, metrics $m = m^\sigma$ on V induced by maps $\sigma : V \to V(\varGamma)$ with $\varGamma$ running over a set of planar graphs with four faces (called *4f-metrics*), and merely the existence of a *half-integer* solution is guaranteed in a solvable Eulerian case. When $|\mathcal{H}| = 5$, the set of unavoidable metrics in the solvability criterion becomes ugly (see [3, Sec. 4]), and the fractionality status is unknown so far.

In this paper we focus on algorithmic aspects. The first combinatorial strongly polynomial algorithm (having complexity $O(n^3 \log n)$) to find an integer solution in the Eulerian case with $|\mathcal{H}| = 1$ is due to Frank [1], and subsequently a number of faster algorithms have been devised; a linear-time algorithm is given in [11]. Hereinafter n stands for the number $|V|$ of vertices of the graph. Efficient algorithms for $|\mathcal{H}| = 2$ are known as well. For a survey and references in cases $|\mathcal{H}| = 1, 2$, see, e.g., [10].

Our aim is to give an algorithm to solve problem $\mathcal{D}(c, d)$ with $|\mathcal{H}| = 3$, which checks the solvability and finds an integer admissible multiflow in the Eulerian case. Our algorithm uses merely combinatorial means and is strongly polynomial (though having a high polynomial degree). Its core is a subroutine for a certain planar analogue of the (2,3)-metric minimization problem. We are able to fulfil this task efficiently and in a combinatorial fashion, by reducing it to a series of shortest paths problems in a dual planar graph.

Remark 1. The (2,3)-metric minimization problem in a general edge-weighted graph with a specified set of five terminals can be solved in strongly polynomial time (by use of the ellipsoid method) [2] or by a combinatorial weakly polynomial algorithm [6].

This paper is organized as follows. Section 2 reviews needed facts from [4], which refine the structure of cuts and (2,3)-metrics that are essential for the solvability of our 3-hole demand problem. Using these refinements, Sections 3 and 4 develop efficient combinatorial procedures to verify the cut and (2,3)-metric conditions for problem $\mathcal{D}(c, d)$ with initial or current c, d; moreover, these procedures determine or duly estimate the minimum excesses of regular cuts and (2,3)-metrics, which is important for the efficiency of our algorithm for $\mathcal{D}(c, d)$. This algorithm is described in Section 5.

To slightly simplify the further description, we will assume, w.l.o.g., that the boundary of any hole H contains no isthmus. For if $b(H)$ has an isthmus e, we can examine the cut $\{e\}$. If it violates the cut condition, the problem $\mathcal{D}(c, d)$ has no solution. Otherwise $\mathcal{D}(c, d)$ is reduced to two smaller demand problems, with at most 3 holes and with Eulerian data each, by deleting e and properly modifying demands concerning H.

2 Preliminaries

Throughout the rest of the paper, we deal with $G = (V, E), \mathcal{H}, D, c, d$ as above such that $|\mathcal{H}| = 3$ and (c, d) is Eulerian. Let $\mathcal{H} = \{H_1, H_2, H_3\}$.

One may assume that the graph $G = (V, E)$ is connected and its outer (unbounded) face is a hole (say, H_3). We identify objects in G, such as edges, paths, subgraphs, and etc., with their images in the plane. A face $F \in \mathcal{F}_G$ is regarded as an open region in the plane. Since G is connected, the boundary $b(F)$ of F is connected, and we identify it with the corresponding cycle (closed path) considered up to reversing and shifting cyclically. Note that this cycle may contain repeated vertices or edges (an edge of G may be passed by $b(F)$ twice, in different directions). A subpath in this cycle is called a *segment* in $b(F)$.

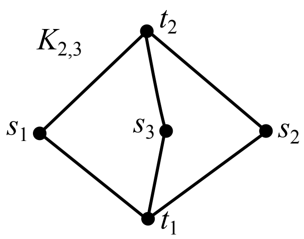
We denote the subgraph of G induced by a subset $X \subseteq V$ by $[X] = [X]_G$, the set of faces of G whose boundary is entirely contained in $[X]$ by $\mathcal{F}(X)$, and the region in the plane that is the union of $[X]$ and all faces in $\mathcal{F}(X)$ by $\mathcal{R}(X)$. We also need additional terminology and notation.

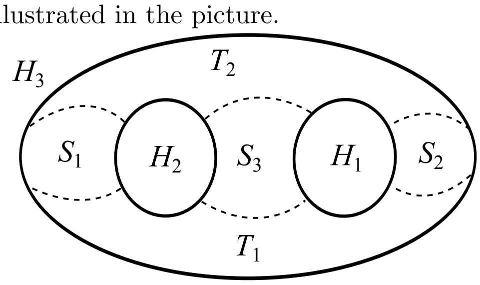
A subset $X \subset V$ (as well as the cut $\delta(X)$) is called *regular* if the region $\mathcal{R}(X)$ is simply connected (i.e., it is connected and any closed curve in it can be continuously deformed into a point), and for each $i = 1, 2, 3$, $[X] \cap b(H_i)$ forms a (possibly empty) segment of $b(H_i)$. In particular, the graph $[X]$ is connected.

Let $\{t_1, t_2\}$ and $\{s_1, s_2, s_3\}$ be the parts (color classes) in $K_{2,3}$. Given $\sigma : V \to V(K_{2,3})$, we denote the set $\sigma^{-1}(t_i)$ by $T_i = T_i^\sigma$, and $\sigma^{-1}(s_j)$ by $S_j = S_j^\sigma$. Then $\Xi^\sigma = (T_1, T_2, S_1, S_2, S_3)$ is a partition of V. The (2,3)-metric m^σ is called *regular* if:

(2.1) (i) all sets T_1, T_2, S_1, S_2, S_3 in Ξ^σ are nonempty;
(ii) for $i = 1, 2, 3$, the region $\mathcal{R}(S_i)$ is simply connected;
(iii) for $i, j \in \{1, 2, 3\}$, $S_i \cap b(H_j) = \emptyset$ holds if and only if $i = j$; and for $i \neq j$, $[S_i] \cap b(H_j)$ forms a segment of $b(H_j)$.

Then the complement to $\mathbb{R}^2$ of $H_1 \cup H_2 \cup H_3 \cup \mathcal{R}(S_1) \cup \mathcal{R}(S_2) \cup \mathcal{R}(S_3)$ consists of two connected components, one containing T_1 and the other containing T_2. The structure described in (2.1) is illustrated in the picture.





The notions of regular sets (cuts) and (2,3)-metric are justified by the following important strengthening of the first assertion in Theorem 1 (cf. [4]).

Theorem 2. *$\mathcal{D}(c, d)$ has a solution if and only if cut condition (1.3) holds for all regular subsets $X \subset V$, and (2,3)-metric condition (1.4) holds for all regular (2,3)-metrics on V.*

Remark 2. In fact, the refined solvability criterion for $\mathcal{D}(c,d)$ given in [4, Stat. 2.1] involves a slightly sharper set of (2,3)-metrics compared with that defined by (2.1); at the same time it does not restrict the set of cuts. Note, however, that if $X \subset V$ is not regular, then there are nonempty sets $X', X'' \subset V$ such that $\delta(X') \cap \delta(X'') = \emptyset$, $\delta(X') \cup \delta(X'') \subseteq \delta(X)$, and $\rho(X) \subseteq \rho(X') \cup \rho(X'')$. Then X is redundant (it can be excluded from verification of (1.3)).

3 Verifying the cut condition

In this section and the next one we describe efficient procedures for checking the solvability of $\mathcal{D}(G, \mathcal{H}, D, c, d)$ (considering the initial or current data). By Theorem 2, it suffices to verify validity of cut condition (1.3) for regular sets and (2,3)-metric condition (1.4) for regular (2,3)-metrics.

A check-up of the cut condition is rather straightforward. Moreover, we can duly estimate from below the minimum excess $\Delta_{c,d}(X)$ among the regular sets $X \subset V$. In fact, we will compute the minimum excess in a somewhat larger collection of sets.

Definition. We say that a subset $X \subset V$ is *semi-regular* if $|\delta(X) \cap b(H_i)| \le 2$ for each $i = 1, 2, 3$.

One can see that any regular set X is semi-regular. Also for each i, the fact that $b(H_i)$ has no isthmus (as mentioned in the Introduction) implies that $|\delta(X) \cap b(H_i)|$ is 0 or 2.

Based on Theorems 1 and 2, we are going to compute the minimum excess $\Delta_{c,d}(X)$ among the semi-regular sets X; denote this minimum by $\mu^{\rm cut}_{c,d}$. In particular, if $\mu^{\rm cut}_{c,d} < 0$, then the problem $\mathcal{D}(c,d)$ has no solution.

To compute $\mu^{\rm cut}_{c,d}$, we fix a nonempty $I \subseteq \{1,2,3\}$ and scan the possible collections $\mathcal{A} = \{A_i \colon i \in I\}$, where each A_i consists of two edges in $b(H_i)$. We say that a semi-regular set X is *consistent* with $\mathcal{A}$ (or with $(I, \mathcal{A})$) if $\delta(X) \cap b(H_i) = A_i$ for each $i \in I$, and $\delta(X) \cap b(H_i) = \emptyset$ for $i \notin I$. Also for $i \in I$, we denote the set of demand pairs $st \in D$ located on $b(H_i)$ and spanning different components (segments) in $b(H_i) - A_i$ by $D(A_i)$. Then for all semi-regular sets X consistent with $\mathcal{A}$, the right hand side value in (1.3) is the same, namely, $d(\rho(X)) = \sum(d(D(A_i) \colon i \in I)$.

Using this, for each $(I, \mathcal{A})$, we compute the minimum excess among the semi-regular sets consistent with $\mathcal{A}$ in a natural way, by solving $2^{|I|-1}$ minimum s–t cut problems. Here each problem arises by choosing one component S_i in $b(H_i) - A_i$, for each $i \in I$. We transform G by shrinking $\cup(S_i \colon i \in I)$ into a new vertex s, shrinking the rest of $b(H_i) - A_i$, $i \in I$, into a new vertex t, and shrinking each cycle $b(H_j)$, $j \notin I$, into a vertex. Solving the corresponding min cut problem in the arising graph (with the induced edge capacities), we obtain the desired minimum excess among those X satisfying $\delta(X) \cap b(H_i) = A_i$, $i \in I$.

Thus, by applying the above procedure to all possible combinations $(I, \mathcal{A})$ (whose number is $O(n^6)$), we can conclude with the following

Proposition 3 *The task of computing $\mu^{\rm cut}_{c,d}$ reduces to finding $O(n^6)$ minimum cuts in graphs with $O(n)$ vertices and edges. In particular, this enables us to efficiently verify cut condition (1.3) for $\mathcal{D}(c,d)$.*

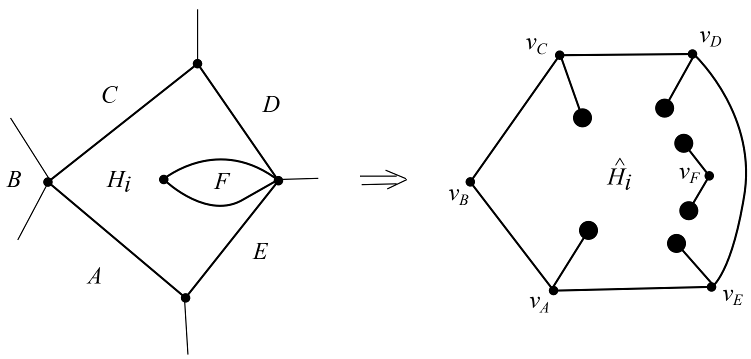
4 Verifying the (2,3)-metric condition

In this section we develop a procedure of verifying the (2,3)-metric condition for $\mathcal{D}(G,\mathcal{H},D,c,d)$. Moreover, the procedure duly estimates from below the minimum excess of a regular (2,3)-metric, which is crucial for our algorithm. We use a shortest paths technique in a modified dual graph.

This graph is constructed as follows. First we take the standard planar dual graph $G^*=(V^*,E^*)$ of G, i.e., V^* is bijective to $\mathcal{F}_G$ and E^* is bijective to E, defined by $F\in\mathcal{F}_G\mapsto v_F\in V^*$ and $e\in E\mapsto e^*\in E^*$. Here a dual edge e^* connects vertices v_F and $v_{F'}$ if F,F' are the faces whose boundaries share e (possibly $F=F'$). (Usually v_F is visualized as a point in F, and e^* as a line crossing e.)

Next we slightly modify G^* as follows. For $i=1,2,3$, let E_i denote the sequence of edges of the cycle $b(H_i)$. (Recall that $b(H_i)$ has no isthmus, hence all edges in E_i are different.) Let z_i denote the vertex of G^* corresponding to the hole H_i. Then z_i has degree $|E_i|$ and is incident with the dual edges e^* for $e\in E_i$. We split z_i into $|E_i|$ vertices $z_{i,e}$ of degree 1 each, where $e\in E_i$, making $z_{i,e}$ be the end of e^* instead of z_i. These pendant vertices are called *terminals*. They belong to the boundary of the same face, denoted as $\widehat{H}_i$, and the set of terminals ordered clockwise around $\widehat{H}_i$ is denoted by Z_i.

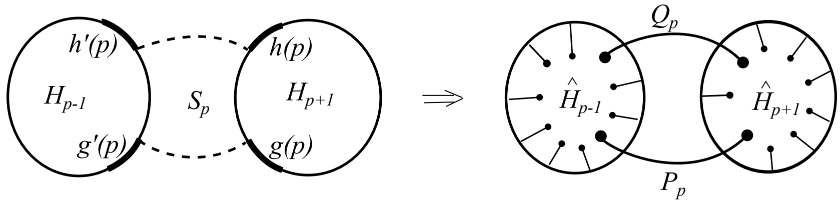
This gives the desired dual graph for $(G,\mathcal{H})$, denoted as $\widehat{G}^*$. An example of transforming G into $\widehat{G}^*$ in a neighborhood of a hole H_i is illustrated in the picture, where $A,\dots,F$ are faces in G, and the terminals in $b(\widehat{H}_i)$ are indicated by bold circles.



The edges of $\widehat{G}^*$ are endowed with *lengths* c inherited from the capacities in G; namely, we assign $c(e^*):=c(e)$ for $e\in E$.

Consider a regular (2,3)-metric $m=m^\sigma$ and its corresponding partition (T_1,T_2,S_1,S_2,S_3) (cf. (2.1)). By the regularity of m, for $i=1,2,3$, the cycle $b(H_i)$ shares two edges with the cut $\delta(S_{i-1})$, say, $g(i-1)$ and $h(i-1)$, and two edges with $\delta(S_{i+1})$, say, $g'(i+1)$ and $h'(i+1)$; let for definiteness $g(i-1), h(i-1), h'(i+1), g'(i+1)$ follow in this order clockwise in $b(H_i)$ (taking indices modulo 3). Note that, although the segments $[S_{i-1}]\cap b(H_i)$ and $[S_{i+1}]\cap b(H_i)$ are disjoint, the edges $g(i-1)$ and $g'(i+1)$ may coincide, and similarly for $h(i-1)$ and $h'(i+1)$.

So, for $p=1,2,3$, the cut $\delta(S_p)$ meets $b(H_{p+1})$ by $\{g(p), h(p)\}$, meets $b(H_{p-1})$ by $\{g'(p), h'(p)\}$, and does not meet $b(H_p)$. Since the region $\mathcal{R}(S_p)$ is simply connected, the cut $\delta(S_p)$ corresponds to a simple cycle $C(S_p)$ in G^*; it passes the elements $g(p)^*, z_{p+1}, h(p)^*, h'(p)^*, z_{p-1}, g'(p)^*$ (in the counterclockwise order). The cycle $C(S_p)$ turns into two disjoint paths in $\widehat{G}^*$: path P_p connecting the terminals $z_{p+1,g(p)}$ and $z_{p-1,g'(p)}$, and path Q_p connecting $z_{p+1,h(p)}$ and $z_{p-1,h'(p)}$. See the picture.



This correspondence gives $c(\delta(S_p)) = c(P_p) + c(Q_p)$, implying

$$c(m) = \sum\Big(c(\delta(S_p))\colon p=1,2,3\Big) = \sum\Big(c(P_p)+c(Q_p)\colon p=1,2,3\Big),$$

taking into account the evident fact that no edge of G connects T_1 and T_2.

In order to express the "demand value" $d(m)$, consider arbitrary edges b_1, b_2, b_3, b_4 occurring in this order in a cycle $b(H_i)$, possibly with $b_q = b_{q+1}$ for some q (letting $b_5 := b_1$). Removal of these edges from the cycle produces four segments $\omega_1, \omega_2, \omega_3, \omega_4$, where ω_q is the (possibly empty) segment between b_q and b_{q+1}. Let $d_i(b_1, b_2, b_3, b_4)$ denote the sum of demands $d(st)$ over the pairs st spanning neighboring segments ω_q, ω_{q+1} plus twice the sum of demands $d(st)$ over st spanning either ω_1 and ω_3, or ω_2 and ω_4.

Now for $i=1,2,3$, take as b_1, b_2, b_3, b_4 the edges $g(i-1), h(i-1), h'(i+1), g'(i+1)$, respectively. Then the contribution to $d(m)$ from the demand pairs on $b(H_i)$ is just $d_i(g(i-1), h(i-1), h'(i+1), g'(i+1))$. Hence

$$d(m) = \sum\Big(d_i(g(i-1), h(i-1), h'(i+1), g'(i+1))\colon i=1,2,3\Big).$$

This prompts the idea to minimize $c(m)$ over a class of (2,3)-metrics m which, for each $i=1,2,3$, deal with the same quadruple of edges in $b(H_i)$, and therefore have equal values $d(m)$. (In reality, we will be forced to include in this class certain non-regular (2,3)-metrics as well.)

On this way we come to the following task, which is solved by comparing $O(1)$ combinations of the lengths of c-shortest paths in $\widehat{G}^*$:

(4.1) Given, for each $i=1,2,3$, a quadruple $\widetilde{Z}_i = (z_i^1, z_i^2, z_i^3, z_i^4 = z_i^0)$ of terminals in Z_i (with possible coincidences), find a set $\mathcal{P}$ of six (simple) paths in $\widehat{G}^*$ minimizing their total c-length, provided that:

(∗) each path in $\mathcal{P}$ connects terminals z_i^p and z_j^q with $i \neq j$, and the set of endvertices of the paths in $\mathcal{P}$ is exactly $\widetilde{Z}_1 \cup \widetilde{Z}_2 \cup \widetilde{Z}_3$ (respecting the possible multiplicities).

Next we need some terminology and notation. For $i = 1, 2, 3$, let A_i be the quadruple of edges in the cycle $b(H_i)$ of G that corresponds to $\widetilde{Z}_i$ (respecting the possible multiplicities). Let $\mathcal{A} := (A_1, A_2, A_3)$. Define $\zeta(\mathcal{A})$ to be the minimum c-length of a path system in (4.1), and define $d(\mathcal{A})$ to be the sum of corresponding demand values $d(A_i)$. Then $d(\mathcal{A}) = d(m)$ for any $m \in \mathcal{M}(\mathcal{A})$, and

$$\zeta(\mathcal{A}) \leq \min\{c(m) \colon m \in \mathcal{M}(\mathcal{A})\}, \qquad (4.2)$$

where $\mathcal{M}(\mathcal{A})$ denote the set of regular (2,3)-metrics $m = m^\sigma$ in G *agreeable to* $\mathcal{A}$, i.e., such that for the partition $\Xi^\sigma = (T_1, T_2, S_1, S_2, S_3)$ and for each $i = 1, 2, 3$, $\delta(S_{i-1}) \sqcup \delta(S_{i+1})$ meets $b(H_i)$ by A_i.

In general, inequality (4.2) may be strong. Nevertheless, we can get a converse inequality by extending $\mathcal{M}(\mathcal{A})$ to a larger class of (2,3)-metrics.

Definition. Let us say that a (2,3)-metric $m = m^\sigma$ is *semi-regular* if the sets S_1, S_2, S_3 in Ξ^σ are nonempty and satisfy (iii) in (2.1).

(Whereas T_1, T_2 may be empty and (ii) of (2.1) need not hold; in particular, subgraphs $[S_i]$ need not be connected.) We show the following

Proposition 4 *$\zeta(\mathcal{A})$ is equal to $c(m)$ for some semi-regular (2,3)-metric m agreeable to $\mathcal{A}$.*

(When a (2,3)-metric m is semi-regular but not regular, it is "dominated by two cuts", in the sense that there are $X, Y \subset V$ such that $\Delta_{c,d}(m) \geq \Delta_{c,d}(X) + \Delta_{c,d}(Y)$, cf. [3, Sec. 3].)

Proof. We use the observation that problem $\mathcal{D}(c, d)$ remains equivalent when an edge e is subdivided into several edges in series, say, $e_1, \ldots, e_k$ $(k \geq 1)$ with the same capacity: $c(e_i) = c(e)$. In particular, we can subdivide edges in the boundaries of holes, due to which we may assume that each quadruple A_i consists of different edges. Then all terminals in each $\widetilde{Z}_i$ become different.

Another advantage is that when considering an optimal path system $\mathcal{P}$ in (4.1), we may assume that the paths in $\mathcal{P}$ are pairwise edge-disjoint. Indeed, if some edge e^* of $\widehat{G}^*$ is used by $k > 1$ paths in $\mathcal{P}$, we can subdivide the corresponding edge e of G into k edges in series. This leads to replacing e^* by a tuple of k parallel edges (of the same length $c(e)$) and we assign each edge to be passed by exactly one of those paths.

We need to improve $\mathcal{P}$ so as to get rid of "crossings". More precisely, consider two paths $P, P' \in \mathcal{P}$, suppose that they meet at a vertex v, let e, e' be the edges of P incident to v, and let g, g' be similar edges of P'. We say that P and P' *cross* (each other) at v if e, g, e', g' occur in this order (clockwise or counterclockwise) around v, and *touch* otherwise.

For an inner (nonterminal) vertex v, let $\mathcal{P}(v)$ be the set of paths in $\mathcal{P}$ passing v, and $\mathcal{E}(v)$ the clockwise ordered set of edges incident to v and occurring in $\mathcal{P}(v)$. We assign to the edges in $\mathcal{E}(v)$ labels 1, 2 or 3, where an edge e is labeled i if for the path $P \in \mathcal{P}(v)$ containing e, P begins or ends at a terminal z in $\widetilde{Z}_i$ and e belongs to the part of P between v and z. (So if P connects $\widetilde{Z}_i$ and $\widetilde{Z}_j$ and e' is the other edge of P incident to v, then e' has label j.)

We iteratively apply the following *uncrossing operation*. Choose a vertex v with $|\mathcal{E}(v)| \geq 4$. Split each path of $\mathcal{P}(v)$ at v. This gives, for each edge $e \in \mathcal{E}(v)$ with label i, a path containing e and connecting v with a terminal in $\widetilde{Z}_i$; denote this path by $Q(e)$. These paths are regarded up to reversing. Now we recombine these paths into pairs as follows, using the obvious fact that for each $i = 1, 2, 3$, the number of edges in $\mathcal{E}(v)$ with label i is at most $|\mathcal{E}(v)|/2$.

Choose two consecutive edges e, e' in $\mathcal{E}(v)$ by the following rule: e, e' have different labels, say, i, j, and the number of edges in $\mathcal{E}(v)$ having the third label k (where $\{i, j, k\} = \{1, 2, 3\}$) is strictly less than $|\mathcal{E}(v)|/2$. (Clearly such e, e' exist.) We concatenate $Q(e)$ and $Q(e)$, obtaining a path connecting $\widetilde{Z}_i$ and $\widetilde{Z}_j$, update $\mathcal{E}(v) := \mathcal{E}(v) - \{e, e'\}$, apply a similar procedure to the updated $\mathcal{E}(v)$, and so on until $\mathcal{E}(v)$ becomes empty.

One can see that the resulting path system $\mathcal{P}'$ satisfies property $(*)$ in (4.1) and has the same total c-length as before (thus yielding an optimal solution to (4.1)), and now no two paths in $\mathcal{P}'$ cross at v. Note that for some vertices $w \neq v$, edge labels in $\mathcal{E}(w)$ may become incorrect (this may happen with those vertices w that belong to paths in $\mathcal{P}'(v)$). For this reason, we finish the procedure of handling v by checking such vertices w and correcting their labels where needed. In addition, if we reveal that one or another path in $\mathcal{P}'(v)$ is not simple, we cancel the corresponding cycle in it (which has zero c-length since $\mathcal{P}'$ is optimal).

At the next iteration we apply a similar uncrossing operation to another vertex v', and so on. Upon termination of the process (taking $< n$ iterations) we obtain a path system $\widetilde{\mathcal{P}}$ such that

(4.3) $\widetilde{\mathcal{P}}$ is optimal to (4.1) and admits no crossings.

Property $(*)$ in (4.1) implies that for each $p = 1, 2, 3$, the sets $\widetilde{Z}_{p-1}$ and $\widetilde{Z}_{p+1}$ are connected by exactly two paths in $\widetilde{\mathcal{P}}$. We denote them by P_p, Q_p and assume that both paths go from $\widetilde{Z}_{p-1}$ to $\widetilde{Z}_{p+1}$ (reversing paths in $\widetilde{\mathcal{P}}$ if needed). Since P_p, Q_p nowhere cross, we can subdivide the space $\mathbb{R}^2 - (\widehat{H}_{p-1} \cup \widehat{H}_{p+1})$ into two closed regions $\mathcal{R}, \mathcal{R}'$ such that $\mathcal{R} \cap \mathcal{R}' = P_p \cup Q_p$, $\mathcal{R}$ lies "on the right from P_p" and "on the left from Q_p", while $\mathcal{R}'$ behaves conversely. (Here we give informal, but intuitively clear, definitions of $\mathcal{R}, \mathcal{R}'$, omitting a precise topological description.) One of $\mathcal{R}, \mathcal{R}'$ does not contain the hole $\widehat{H}_p$; denote it by $\mathcal{R}_p$. We observe the following:

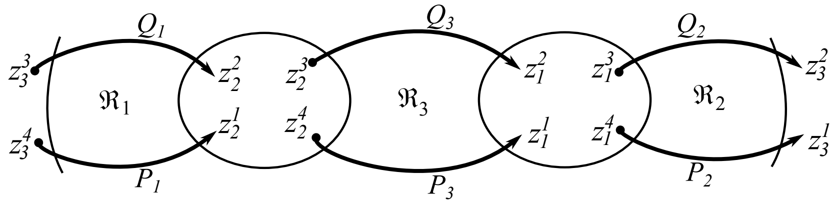
(4.4) no path in $\widetilde{\mathcal{P}}$ meets the interior $\text{int}(\mathcal{R}_p)$ of $\mathcal{R}_p$.

Indeed, if $P \in \widetilde{\mathcal{P}}$ goes across $\mathrm{int}(\mathcal{R}_p)$, then P is different from P_p and Q_p; hence P has one endvertex in $\widetilde{Z}_p$. Since $\widetilde{Z}_p \cap \mathcal{R}_p = \emptyset$, P must cross the boundary of $\mathcal{R}_p$. This implies that P crosses some of P_p, Q_p, contrary to (4.3).

From (4.4) it follows that the interiors of $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are pairwise disjoint and that for $p = 1, 2, 3$, the paths P_p, Q_p begin at consecutive terminals in $\widetilde{Z}_{p-1}$ and end at consecutive terminals in $\widetilde{Z}_{p+1}$ (thinking of both paths as going from $\widetilde{Z}_{p-1}$ to $\widetilde{Z}_{p+1}$). So we may assume for definiteness that

(4.5) for $i = 1, 2, 3$, the terminals $z_i^1, z_i^2, z_i^3, z_i^4$ of $\widetilde{Z}_i$ are, respectively, the end of P_{i-1}, the end of Q_{i-1}, the beginning of Q_{i+1}, and the beginning of P_{i+1};

see the picture, where for simplicity all paths are vertex disjoint.



Then the space $\mathbb{R}^2 - (\widehat{H}_1 \cup \widehat{H}_2 \cup \widehat{H}_3 \cup \mathrm{int}(\mathcal{R}_1) \cup \mathrm{int}(\mathcal{R}_2) \cup \mathrm{int}(\mathcal{R}_3))$ can be subdivided into two closed regions $\mathcal{L}_1$ and $\mathcal{L}_2$, where the former lies "on the right from P_1, P_2, P_3" and the latter lies "on the left from Q_1, Q_2, Q_3". One can see that

(4.6) each edge of P_p is shared by the regions $\mathcal{R}_p$ and $\mathcal{L}_1$, and each edge of Q_p is shared by $\mathcal{R}_p$ and $\mathcal{L}_2$.

Now the sets of faces in (the natural extensions to G^* of) the regions $\mathcal{L}_1, \mathcal{L}_2, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ induce vertex sets T_1, T_2, S_1, S_2, S_3 in G, respectively, giving a partition of V. Let m be the (2,3)-metric determined by this partition. Then (4.5) implies that m is semi-regular and agreeable to $\mathcal{A}$. By (4.6), for $p = 1, 2, 3$, each edge of $\delta(S_p)$ connects S_p with one of T_1, T_2 (whereas no edge of G connects T_1 and T_2, or connects S_i and S_j for $i \neq j$). Therefore,

$$\zeta(\mathcal{A}) = \sum (c(P_p) + c(Q_p) \colon p = 1, 2, 3) = c(m),$$

yielding the proposition. ∎

Remark 4. Strictly speaking, the metric m in the above proof concerns the modified graph, obtained by replacing some edges $e = uv$ of the original graph G by paths L_e connecting u and v. When returning to the original G, those elements of S_p or T_q that are intermediate vertices of such paths L_e disappear, and as a result, there may appear (original) edge connecting T_1 and T_2, or S_i and S_j, $i \neq j$. One can see, however, that this does not affect the value $c(m)$ for the corresponding m.

Finally, define $\mu^{23}_{c,d}(\mathcal{A}) := \zeta(\mathcal{A}) - d(\mathcal{A})$. We conclude with the following

Corollary 1. (i) *Let* $\mathcal{A} = (A_1, A_2, A_3)$*, where* A_i *is a quadruple of edges in* $b(H_i)$*. Then* $\Delta_{c,d}(m) \geq \mu^{23}_{c,d}(\mathcal{A})$ *for any regular (2,3)-metric* m *agreeable to* $\mathcal{A}$*, and there exists a semi-regular (2,3)-metric* m' *agreeable to* $\mathcal{A}$ *such that* $\Delta_{c,d}(m') = \mu^{23}_{c,d}(\mathcal{A})$*. In particular, if* $\mu^{23}_{c,d}(\mathcal{A}) < 0$*, then problem* $\mathcal{D}(c,d)$ *has no solution.*

(ii) *The minimum* $\mu^{23}_{c,d}$ *of excesses* $\Delta_{c,d}(m)$ *over all semi-regular (2,3)-metrics* m *can be found in* $O(n^{12} + n \cdot SP(n))$ *time, where* $SP(n')$ *is the complexity of a shortest paths algorithm in a planar graph with* n' *nodes.*

5 Algorithm

As before, we consider a 3-hole demand problem $\mathcal{D}(G = (V,E), \mathcal{H}, D, c, d)$ in which the capacity-demand pair (c,d) is Eulerian.

The algorithm to solve this problem uses efficient procedures of Sections 3,4 which find, for a current (c,d), the minimum excess $\mu^{\rm cut}_{c,d}$ among the semi-regular sets and the minimum excess $\mu^{23}_{c,d}$ among the semi-regular (2,3)-metrics. Let $\mu_{c,d}$ denote $\min\{\mu^{\rm cut}_{c,d}, \mu^{23}_{c,d}\}$ As mentioned above, Theorems 1 and 2 imply the following

Proposition 5 *Problem* $\mathcal{D}(c,d)$ *has a solution if and only if* $\mu_{c,d} \geq 0$.

The algorithm starts with verifying the solvability of the problem, by finding $\mu_{c,d}$ for the initial (c,d). If $\mu_{c,d} < 0$, it declares that the problem has no solution. Otherwise the algorithm recursively constructs an integer admissible multiflow. We may assume, w.l.o.g., that all current capacities and demands are nonzero (for edges e with $c(e) = 0$ can be immediately deleted from G, and similarly for pairs $st \in D$ with $d(st) = 0$), and that the boundary $b(H_i)$ of each hole H_i is connected and isthmusless, regarding it as a cycle.

An *iteration* of the algorithm applied to current $G, \mathcal{H}, D, c, d$ (with (c,d) Eulerian) chooses arbitrarily $i \in \{1,2,3\}$, an edge $e = uv$ in $b(H_i)$, and a pair $st \in D_i$, where D_i denotes the set of demand pairs for H_i.

Let for definiteness s, u, v, t follow in this order in $b(H_i)$. Suppose that we take a *nonnegative integer* $\varepsilon \leq \min\{c(e), d(st)\}$ and transform (c,d) into the capacity-demand pair (c', d') by

$$c'(e) := c(e) - \varepsilon, \quad d'(st) := d(st) - \varepsilon, \qquad (5.1)$$
$$d'(su) := d(su) + \varepsilon, \quad \text{and} \quad d'(vt) := d(vt) + \varepsilon.$$

(Note that we add to D the demand pair su with $d(su) := 0$ if it does not exist there, and similarly for vt. When $s = u$ $(v = t)$, the pair su (resp. vt) vanishes.) Clearly (c', d') is Eulerian as well. We say that (c', d') is obtained by the (e, st, ε)-*reduction* of (c,d). We call ε a *feasible reduction number* for (c, d, e, st), or, simply, *feasible*, if the problem $\mathcal{D}(c', d')$ is still solvable (and therefore it has an integer solution). The goal of the iteration is to find the maximum (integer) feasible ε and then update c, d accordingly.

Here we rely on an evident transformation of an integer admissible multiflow f' for (c', d') into an integer admissible multiflow f for (c, d): extract from f' an integer subflow g of value ε from s to u and an integer subflow h of value ε from v to t, and increase the flow between s and t by concatenating g, h and the flow of value ε through the edge e.

The maximum feasible ε for (c, d, e, st) is computed in at most three steps, as follows.

First we try to take as ε the maximum possible value, namely, $\varepsilon_1 := \min\{c(e), d(st)\}$; let c_1, d_1 be defined as in (5.1) for this ε_1. Compute the value $\nu_1 := \mu_{c_1,d_1}$ (step 1). If $\nu_1 \geq 0$ then $\varepsilon := \varepsilon_1$ is as required (relying on Proposition 5).

Next, if $\nu_1 < 0$, we take $\varepsilon_2 := \varepsilon_1 + \lfloor \nu_1/4 \rfloor$, define c_2, d_2 as in (5.1) for this ε_2 and for c, d as before. Compute $\nu_2 := \mu_{c_2,d_2}$ (step 2). Again, if $\nu_2 \geq 0$ then ε_2 is just the desired ε.

Finally, if $\nu_2 < 0$, we take as ε the number $\varepsilon_3 := \varepsilon_2 + \nu_2/2$ (step 3).

Lemma 1. *The ε determined in this way is indeed the maximum feasible reduction number for c, d, e, st.*

Proof. We argue in a similar spirit as for an integer splitting in [2]. For a semi-regular set $X \subset V$, define

$$\beta(X) := \omega_X(s, u) + \omega_X(u, v) + \omega_X(v, t) - \omega_X(s, t),$$

where we set $\omega_X(x, y) := 1$ if X separates vertices x and y, and 0 otherwise. Then $\beta(X) \geq 0$ (since ω_X is a metric). Also the fact that $|\delta(X) \cap b(H_i)| \leq 2$ (as X is semi-regular) implies that $\beta(X) \in \{0, 2\}$.

For a semi-regular (2,3)-metric m, define

$$\gamma(m) := m(su) + m(uv) + m(vt) - m(st).$$

Then $\gamma(m) \geq 0$. Also the semi-regularity of m (cf. (iii) in (2.1)) implies that $\gamma(m) \in \{0, 2, 4\}$.

One can check that if (c'', d'') is obtained by the (e, st, ε')-reduction of a pair (c', d') with an arbitrary ε', then

$$\Delta_{c'',d''}(X) = \Delta_{c',d'}(X) - \varepsilon'\beta(X) \quad \text{and} \quad \Delta_{c'',d''}(m) = \Delta_{c',d'}(m) - \varepsilon'\gamma(m). \tag{5.2}$$

Let $\overline{\varepsilon}$ be the maximum feasible reduction number for c, d, e, st. When $\nu_1 \geq 0$, the equality $\overline{\varepsilon} = \varepsilon_1$ is obvious, so suppose that $\nu_1 < 0$. If ν_1 is achieved by the excess $\Delta_{c_1,d_1}(m)$ of a semi-regular (2,3)-metric m and if $\gamma(m) = 4$, then using the second expression in (5.2) and the equality $\varepsilon_2 = \varepsilon_1 + \lfloor \nu_1/4 \rfloor$, we have

$$\begin{aligned}\Delta_{c_2,d_2}(m) = \Delta_{c,d}(m) - \varepsilon_2\gamma(m) &= \Delta_{c,d}(m) - \varepsilon_1\gamma(m) - \lfloor \widetilde{\nu}_1/4 \rfloor \cdot 4 \\ &= \Delta_{c_1,d_1}(m) - \lfloor \widetilde{\nu}_1/4 \rfloor \cdot 4 = \widetilde{\nu}_1 - \lfloor \widetilde{\nu}_1/4 \rfloor \cdot 4 = \tau,\end{aligned}$$

where τ equals 0 if ν_1 is divided by 4, and equals 2 otherwise. (Recall that the excess of any (2,3)-metric is even when the capacity-demand pair is Eulerian.)

In this case we have $\overline{\varepsilon} \le \varepsilon_2$. Indeed for $\varepsilon' := \varepsilon_2 + 1$, the pair (c', d') obtained by the (e, st, ε')-reduction of (c, d) would give $\Delta_{c',d'}(m) = \Delta_{c_2,d_2}(m) - 4 < 0$; so ε' is infeasible.

As a consequence, in case $\nu_2 \ge 0$ we obtain $\overline{\varepsilon} = \varepsilon_2$.

Now let $\nu_2 < 0$. Note that for any semi-regular metric m' with $\gamma(m') = 4$, the facts that $\gamma(m') = \gamma(m)$ and $\Delta_{c_1,d_1}(m') \ge \nu_1 = \Delta_{c_1,d_1}(m)$ imply that $\Delta_{c',d'}(m') \ge \Delta_{c',d'}(m) \ge 0$ for any (c', d') obtained by the (e, st, ε')-reduction of (c, d) with $\varepsilon' \le \varepsilon_2$. Therefore, ν_2 is achieved by the excess of either a semi-regular set X with $\beta(X) = 2$ or a semi-regular (2,3)-metric m'' with $\gamma(m'') = 2$. This implies $\overline{\varepsilon} = \varepsilon_2 + \nu_2/2$. ▮

The above procedure of computing ε together with the complexity results in Sections 3 and 4 gives the following

Corollary 2. *Each iteration (finding the corresponding maximum reduction number and reducing c, d accordingly) takes $O(n^{12})$ time.*

Next, considering (5.2) and using the facts that $\beta(X), \gamma(m) \ge 0$, we can conclude that under a reduction as above the excess of any set or (2,3)-metric does not increase. This implies that

(5.3) if an iteration handles c, d, e, st, then for any capacity-demands (c', d') arising on subsequent iterations, the maximum reduction number for (c', d', e, st) is zero.

Therefore, it suffices to choose each pair (e, st) at most once during the process.

Now we finish our description as follows. Suppose that, at an iteration with i, e, st, the capacity of e becomes zero and the deletion of e from G causes merging H_i with another hole H_j. Then we can proceed with an efficient procedure for solving the corresponding Eulerian 2-hole demand problem. Similarly, if the demand on st becomes zero and if the deletion of st makes D_i empty, then we can withdraw the hole H_i, again obtaining the Eulerian 2-hole case.

Finally, suppose that we have the situation when for some (c, d), the holes H_1, H_2, H_3 are different (and the capacities of all edges are positive), each D_1, D_2, D_3 is nonempty, but the maximum feasible reduction number for any corresponding pair e, st is zero. We assert that this is not the case.

Indeed, suppose such a (c, d) exists. The problem $\mathcal{D}(c, d)$ is solvable, and one easily shows that there exists an integer solution $f = (\mathcal{P}, \lambda)$ to $\mathcal{D}(c, d)$ such that: for some path $P \in \mathcal{P}$ with $\lambda(P) > 0$ and for the hole H_i whose boundary contains s_P, t_P, some edge e of P belongs to $b(H_i)$. But this implies that $s_P t_P \in D_i$ and that $\varepsilon = 1$ is feasible for $(c, d, e, s_P t_P)$; a contradiction.

Thus, we obtain the following

Theorem 6. *The above algorithm terminates in $O(n^3)$ iterations and finds an integer solution to $\mathcal{D}(G, \mathcal{H}, D, c, d)$ with $|\mathcal{H}| = 3$ and (c, d) Eulerian.*

Further algorithmic results (to be presented in a forthcoming paper). (i) Recall that when $|\mathcal{H}| = 4$ and (c, d) is Eulerian, the solvability of $\mathcal{D}(c, d)$ implies the existence of a *half-integer* solution, as is shown in [5] (see (C) in the Introduction). We can find a half-integer solution in strongly polynomial time by using a fast generic LP method; the existence of a combinatorial (weakly or strongly) polynomial algorithm for this problem is still open.

(ii) By a sort of polar duality, the demand problem $\mathcal{D} = \mathcal{D}(G, \mathcal{H}, D, c, d)$ with $|\mathcal{H}| \in \{3, 4\}$ is interrelated to a certain problem on packing cuts and metrics so as to realize the distances within each hole. More precisely, let $\ell : E \to \mathbb{Z}_+$ be a function of *lengths* of edges of G. The solvability criteria for $\mathcal{D}$ with $|\mathcal{H}| = 3, 4$ imply (via the polar duality specified to our objects) that there exist metrics $m_1, \ldots, m_k$ on V and nonnegative reals $\lambda_1, \ldots, \lambda_k$ such that

$$\begin{aligned} \lambda_1 m_1(e) + \ldots + \lambda_k m_k(e) &\le \ell(e) && \text{for each } e \in E; \\ \lambda_1 m_1(st) + \ldots + \lambda_k m_k(st) &= \mathrm{dist}_\ell(st) && \text{for all } s, t \in V \cap b(H),\ H \in \mathcal{H}. \end{aligned}$$

Here: dist_ℓ is the distance of vertices in (G, ℓ); and each m_i is a cut metric or a (2,3)-metric if $|\mathcal{H}| = 3$, and is a cut metric or a (2,3)-metric or a 4f-metric if $|\mathcal{H}| = 4$. Moreover, [3] shows the sharper property: if the lengths of all cycles in (G, ℓ) are even, then in both cases there exists an integer solution (i.e., with λ integer-valued). We develop a purely combinatorial strongly polynomial algorithm to find such solutions.

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