# On universal quadratic identities for minors of quantum matrices (extended abstract)

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**Abstract**

We give a complete combinatorial characterization of homogeneous quadratic identities of "universal character" valid for minors of quantum matrices over a field. This is obtained as a consequence of a study of quantized minors of the so-called *path matrices* associated with certain planar graphs generalizing Cauchon graphs.

*Keywords*: quantum matrix, planar graph, Cauchon diagram, path matrix, Lindström Lemma

## 1 Introduction

The idea of quantization has proved its importance to bridge commutative and non-commutative versions of certain algebraic structures and promote better understanding various aspects of the latter versions. One popular structure is the quantized coordinate ring $\mathcal{R} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ of $m \times n$ matrices over a field $\mathbb{K}$, where $q$ is a nonzero element of $\mathbb{K}$, usually called the *algebra of* $m \times n$ *quantum matrices*. Here $\mathcal{R}$ is the $\mathbb{K}$-algebra generated by entries (indeterminates) of an $m \times n$ matrix $X$ subject to Manin's relations [11]: for $i < \ell \le m$ and $j < k \le n$,

$$
\begin{gathered}
x_{ij}x_{ik} = qx_{ik}x_{ij}, \qquad x_{ij}x_{\ell j} = qx_{\ell j}x_{ij}, \\
x_{ik}x_{\ell j} = x_{\ell j}x_{ik} \quad \text{and} \quad x_{ij}x_{\ell k} - x_{\ell k}x_{ij} = (q - q^{-1})x_{ik}x_{\ell j}.
\end{gathered} \tag{1.1}
$$

We study quadratic identities for minors of quantum matrices, or *quantum minors*. For a discussion on aspects and applications of such identities, see e.g., [6, 7, 8, 9, 12] (where the list is incomplete). We present a novel, and rather transparent, combinatorial method which enables us to completely characterize and efficiently verify homogeneous quadratic identities of universal character that are valid for quantum minors. The identities of our interest can be written as

$$
\sum (\text{sign}_i q^{\delta_i} [I_i|J_i]_q \, [I'_i|J'_i]_q \colon i = 1, \ldots, N) = 0, \tag{1.2}
$$

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where $\delta_i \in \mathbb{Z}$, $\text{sign}_i \in \{+,-\}$, and $[I|J]_q$ denotes the quantum minor whose rows and columns are indexed by $I \subseteq [m]$ and $J \subseteq [n]$, respectively. (Hereinafter, for a positive integer $n'$, we write $[n']$ for $\{1,2,\ldots,n'\}$.) The homogeneity means that each of the sets $I_i \cup I'_i$, $I_i \cap I'_i$, $J_i \cup J'_i$, $J_i \cap J'_i$ is invariant of $i$, and the term "universal" means that (1.2) should be valid independently of $\mathbb{K}, q$ and a $q$-matrix (a matrix whose entries obey Manin's relations and, possibly, additional ones). Note that any cortege $(I|J, I'|J')$ may be repeated in (1.2) many times.

Our approach has two sources. The first one is the *flow-matching* method elaborated in [5] to characterize quadratic identities for usual minors (viz. for $q=1$). In that case the identities are viewed as

$$\sum(\text{sign}_i[I_i|J_i]\,[I'_i|J'_i] \colon i=1,\ldots,N) = 0. \tag{1.3}$$

In the method of [5], each cortege $S = (I|J, I'|J')$ determines a certain set $\mathcal{M}(S)$ of so-called *feasible matchings*. The main theorem in [5] asserts that (1.3) is valid (universally) if and only if the families $\mathcal{I}^+$ and $\mathcal{I}^-$ of corteges $S_i$ with $\text{sign}_i = +$ and $\text{sign}_i = -$, respectively, are *balanced*, in the sense that the total families of feasible matchings for corteges occurring in $\mathcal{I}^+$ and in $\mathcal{I}^-$ are equal.

The second source is the path method due to Casteels [2, 3]. He associated with each Cauchon diagram $C$ of size $m\times n$ (see [1]) a certain directed planar graph $G = G_C$ with $m+n$ distinguished vertices $r_1,\ldots,r_m, c_1,\ldots,c_n$, and considered the $m\times n$ *path matrix* $P_G = (p_{ij})$ of $G$. This matrix possesses three important properties. (i) It is a $q$-matrix, and therefore, $x_{ij} \mapsto p_{ij}$ gives a homomorphism of $\mathcal{R}$ to the corresponding algebra generated by the $p_{ij}$. (ii) It admits an analog of Lindström's Lemma [10]: for any $I \subseteq [m]$ and $J \subseteq [n]$ with $|I| = |J|$, the minor $[I|J]_q$ of $P_G$ can be expressed via systems of *disjoint paths* from $\{r_i \colon i\in I\}$ to $\{c_j \colon j \in J\}$ in $G$. (iii) Using Cauchon's Algorithm [1] interpreted in graph terms in [2, 3], one shows that if the diagram $C$ is maximal (i.e., has no black cells), then $\text{Path}_G$ becomes a *generic $q$-matrix* (see Corollary 3.2.5 in [3]).

In this work we consider a more general class of planar graphs, called *SE-graphs*; they possess the above properties (i),(ii) as well. Our goal is to characterize quadratic identities just for the class of path matrices of SE-graphs. Since this class contains a generic $q$-matrix, the identities are automatically valid in $\mathcal{R}$. As a result, we obtain necessary and sufficient conditions for the quantum version (in Theorems 4.3 and 4.1), namely: (1.2) is valid (universally) if and only if the families of corteges $\mathcal{I}^+$ and $\mathcal{I}^-$ along with the function $\delta$ are *$q$-balanced*, which now means the existence of a bijection between the families of feasible matchings for $\mathcal{I}^+$ and $\mathcal{I}^-$ that is agreeable with $\delta$ in a certain sense. Note also that our method of establishing or verifying one or another universal identity admits a rather transparent implementation.

The paper is organized as follows. Section 2 contains basic definitions and statements. Section 3 describes important ingredients and tools in our method: *double flows* (pairs of path systems related to corteges $(I|J, I'|J')$), feasible matchings, and transformations of double flows by use of *exchange operations*. The crucial working tool exhibited here is Corollary 3.5 which follows from a result on exchange operations proved in [4] (stated in Theorem 3.4). Based on these, Section 4 outlines a proof of the sufficiency: (1.2) is valid if the corresponding $\mathcal{I}^+, \mathcal{I}^-, \delta$ are $q$-balanced (Theorem 4.1). Also Section 4 contains an algorithm of recognizing the $q$-balancedness and finishes with Theorem 4.3 (without a proof) concerning the necessity of the $q$-balancedness.

## 2 Basic definitions and statements

**Paths in graphs.** Throughout, by a *graph* we mean a directed graph. A *path* in a graph $G = (V, E)$ is a sequence $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ such that each $e_i$ is an edge connecting the vertices $v_{i-1}, v_i$. An edge $e_i$ is called *forward* if it is directed from $v_{i-1}$ to $v_i$, denoted as $e_i = (v_{i-1}, v_i)$, and *backward* otherwise (when $e_i = (v_i, v_{i-1})$). The path $P$ is called *directed* if it has no backward edge, and *simple* if all vertices $v_i$ are different. When $k > 0$ and $v_0 = v_k$, $P$ is called a *cycle*, and called a *simple cycle* if, in addition, $v_1, \dots, v_k$ are different.

**SE-graphs.** A graph $G = (V, E)$ of this sort (also denoted as $(V, E; R, C)$) satisfies the following conditions:

(SE1) $G$ is planar (with a fixed layout in the plane);

(SE2) $G$ has edges of two types: *horizontal* edges, or *H-edges*, which are directed to the right, and *vertical* edges, or *V-edges*, which are directed downwards (so each edge points to either *south* or *east*, justifying the term "SE-graph");

(SE3) $G$ has two distinguished subsets of vertices: set $R = \{r_1, \dots, r_m\}$ of *sources* and set $C = \{c_1, \dots, c_n\}$ of *sinks*; moreover, $r_1, \dots, r_m$ are disposed on a vertical line, in this order upwards, and $c_1, \dots, c_n$ are disposed on a horizontal line, in this order from left to right; each vertex of $G$ belongs to a directed path from $R$ to $C$.

We denote by $W = W_G$ the set $V - (R \cup C)$ if *inner* vertices of $G$. We also say that $G$ is an $(m, n)$ SE-graph (where $m := |R|$ and $n := |C|$). An example is drawn in Fig. 1.

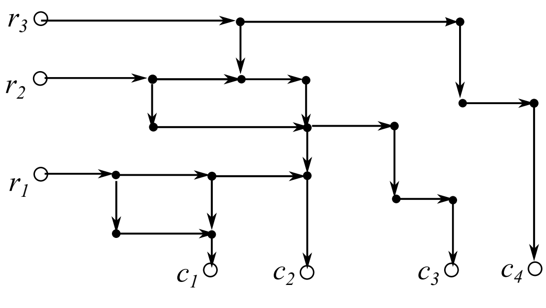



Figure 1: An SE-graph with $m = 3$ and $n = 4$

*Remark 1.* A representative special case is formed by the SE-graphs equivalent to *Cauchon graphs* introduced in [2] (which are associated with Cauchon diagrams [1]). In this case, $R = \{(0, i) \colon i \in [m]\}$, $C = \{(j, 0) \colon j \in [n]\}$, and $W \subseteq [m] \times [n]$. When $W = [m] \times [n]$, we refer to such a graph as the *extended* $(m, n)$*-grid* and denote by $\Gamma_{m,n}$.

Each inner vertex $v \in W$ of an SE-graph $G$ is regarded as a *generator*. We assign the weight $w(e)$ to each edge $e = (u, v) \in E$ in a way similar to that for Cauchon graphs in [2], namely:

(W1) $w(e) := v$ if $e$ is an H-edge with $u \in R$;

(W2) $w(e) := u^{-1}v$ if $e$ is an H-edge and $u, v \in W$;

(W3) $w(e) := 1$ if $e$ is a V-edge.

This gives rise to defining the weight $w(P)$ of a directed path $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ to be the ordered (from left to right) product, namely:

$$w(P) := w(e_1)w(e_2)\cdots w(e_k). \tag{2.1}$$

The generators $W$ are assumed to be subject to (quasi)commutation laws, which match those for Cauchon graphs in [2]. More precisely, for distinct $u, v \in W$,

(G1) if there is a directed *horizontal* path from $u$ to $v$ in $G$, then $uv = qvu$;

(G2) if there is a directed *vertical* path from $u$ to $v$ in $G$, then $vu = quv$;

(G3) otherwise $uv = vu$.

**Quantum matrices.** It is convenient for us to visualize matrices in the Cartesian form: for an $m \times n$ matrix $A = (a_{ij})$, the row indices $i = 1, \dots, m$ are assumed to increase upwards, and the column indices $j = 1, \dots, n$ from left to right.

Fix a field $\mathbb{K}$ and an element $q \in \mathbb{K}^*$, and consider the $m \times n$ matrix of indeterminates $x_{ij}$. The *quantized coordinate ring* $\mathcal{R} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ is the $\mathbb{K}$-algebra generated by the $x_{ij}$ satisfying (1.1). Such an $\mathcal{R}$ is shortly called the algebra of $m \times n$ *quantum matrices*.

**Quantum minors.** For an $m \times n$ matrix $A = (a_{ij})$, we denote by $A(I|J)$ the submatrix of $A$ whose rows and columns are indexed by $I \subseteq [m]$ and $J \subseteq [n]$, respectively. Let $|I| = |J| =: k$, and let $I$ consist of $i_1 < \dots < i_k$, and $J$ consist of $j_1 < \dots < j_k$. Then the *q-determinant* of $A(I|J)$, or the *q-minor* of $A$ for $(I|J)$, is defined as

$$[I|J]_{A,q} := \sum\nolimits_{\sigma \in S_k} (-q)^{\ell(\sigma)} \prod\nolimits_{d=1}^{k} a_{i_d j_{\sigma(d)}}, \tag{2.2}$$

where the product under $\prod$ is ordered by increasing $d$, and $\ell(\sigma)$ denotes the *length* (number of inversions) of a permutation $\sigma$. The terms $A$ and/or $q$ in $[I|J]_{A,q}$ may be omitted when they are clear from the context.

**Path matrix.** An important construction in [2] associates with a Cauchon graph $G$ a certain matrix, called the path matrix of $G$. This is extended to an arbitrary $(m, n)$ SE-graph $G = (V, E)$, namely: the *path matrix* $\mathrm{Path} = \mathrm{Path}_G$ associated with $G$ is the $m \times n$ matrix whose entries are defined by

$$\mathrm{Path}(i|j) := \sum\nolimits_{P \in \Phi_G(i|j)} w(P), \qquad (i, j) \in [m] \times [n], \tag{2.3}$$

where $\Phi_G(i|j)$ is the set of directed paths from $r_i$ to $c_j$ in $G$. In particular, $\mathrm{Path}(i|j) = 0$ if $\Phi_G(i|j) = \emptyset$. Thus, the entries of $\mathrm{Path}_G$ belong to the $\mathbb{K}$-algebra $\mathcal{L}_G$ of Laurent polynomials generated by the set $W$ of inner vertices of $G$ subject to (G1)–(G3).

**Flows.** Let $\mathcal{E}^{m,n}$ be the set of pairs $(I|J)$ with $I \subseteq [m]$, $J \subseteq [n]$ and $|I| = |J|$. Borrowing terminology from [5], for $(I|J) \in \mathcal{E}^{m,n}$, a set $\phi$ of pairwise disjoint directed paths from the source set $R_I := \{r_i : i \in I\}$ to the sink set $C_J := \{c_j : j \in J\}$ in $G$ is called an $(I|J)$-*flow*.

The set of $(I|J)$-flows $\phi$ in $G$ is denoted by $\Phi(I|J) = \Phi_G(I|J)$. We assume that the paths forming $\phi$ are ordered by increasing the source indices: if $I$ consists of $i(1) < i(2) < \cdots < i(k)$ and $J$ consists of $j(1) < j(2) < \cdots < j(k)$, then $\ell$-th path $P_\ell$ in $\phi$ begins at $r_{i(\ell)}$, and therefore, $P_\ell$ ends at $c_{j(\ell)}$ (which follows from the planarity of $G$, the ordering of sources and sinks in the boundary of $G$ and the fact that the paths in $\phi$ are disjoint). We write $\phi = (P_1, P_2, \ldots, P_k)$ and (similar to path systems in [2]) define the weight of $\phi$ to be the ordered product

$$w(\phi) := w(P_1)w(P_2)\cdots w(P_k). \tag{2.4}$$

Generalizing a $q$-analog of Lindström's Lemma shown for Cauchon graphs in [2], one can express minors of path matrices via flows as follows.

**Theorem 2.1 ([4])** *Let $G$ be an $(m, n)$ SE-graph. Then for the path matrix* Path $=$ Path$_G$ *and for any $(I|J) \in \mathcal{E}^{m,n}$, there holds*

$$[I|J]_{\mathrm{Path},q} = \sum\nolimits_{\phi \in \Phi(I|J)} w(\phi). \tag{2.5}$$

An important fact is that the (quasi)commutation relations for the entries of Path$_G$ are similar to those for the canonical generators $x_{ij}$ of the quantum algebra $\mathcal{R}$ in (1.1).

**Proposition 2.2** *For an SE-graph $G$, the entries of its path matrix* Path$_G$ *satisfy Manin's relations.*

(A proof, omitted here, can be given as an easy application of our flow-matching method.) This implies that the map $x_{ij} \mapsto \mathrm{Path}_G(i|j)$ determines a homomorphism of $\mathcal{R}$ to the subalgebra of $\mathcal{L}_G$ generated by the entries of Path$_G$, i.e., Path$_G$ is a $q$-matrix for any SE-graph $G$. A sharper property holds for the graph associated with the $m \times n$ Cauchon diagram without black cells. Namely, Corollary 3.2.5 in [3] relying on Cauchon's Algorithm [1] gives the following property (in our terms).

**Theorem 2.3** *Let $G$ be the extended grid $\Gamma_{m,n}$ (defined in Remark 1). Then* Path$_G$ *is a generic $q$-matrix, i.e., $x_{ij} \mapsto \mathrm{Path}_G(i|j)$ gives an injective map of $\mathcal{R}$ to $\mathcal{L}_G$.*

Due to this, the universal quadratic relations that we establish for minors of path matrices of SE-graphs turn out to be automatically valid for the algebra $\mathcal{R}$ of quantum matrices, and vice versa.

# 3 Double flows, matchings, and exchange operations

Quadratic identities of our interest involve products of the form $[I|J][I'|J']$, where $(I|J)$, $(I'|J') \in \mathcal{E}^{m,n}$. This leads us to a proper study of ordered pairs of flows $\phi \in \Phi(I|J)$

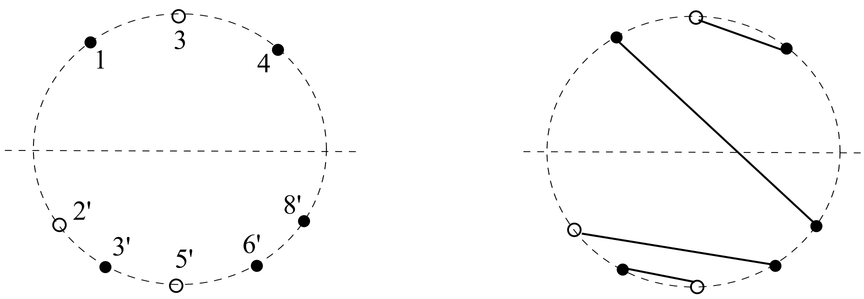
and $\phi' \in \Phi(I'|J')$. We need some definitions and conventions, borrowing terminology from [5].

Given $I, J, I', J', \phi, \phi'$ as above, we call the pair $(\phi, \phi')$ a *double flow* in $G$. Let

$$I^\circ := I - I', \quad J^\circ := J - J', \quad I^\bullet := I' - I, \quad J^\bullet := J' - J, \qquad (3.1)$$
$$Y^{\rm r} := I^\circ \cup I^\bullet \quad \text{and} \quad Y^{\rm c} := J^\circ \cup J^\bullet.$$

Then $|I| = |J|$ and $|I'| = |J'|$ imply that $|I^\circ| - |I^\bullet| = |J^\circ| - |J^\bullet|$ and that $|Y^{\rm r}| + |Y^{\rm c}|$ is even. As before, we refer to the quadruple $(I|J, I'|J')$ as a *cortege*, and call $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ the *refinement* of $(I|J, I'|J')$, or a *refined cortege*.

We interpret $I^\circ$ and $I^\bullet$ as the sets of *white* and *black* elements of $Y^{\rm r}$, respectively, and similarly for $J^\circ, J^\bullet, Y^{\rm c}$, and visualize these objects by use of a *circular diagram* $D$ in which the elements of $Y^{\rm r}$ (resp. $Y^{\rm c}$) are disposed in the increasing order from left to right in the upper (resp. lower) half of a circumference $O$. For example, if $I^\circ = \{3\}$, $I^\bullet = \{1, 4\}$, $J^\circ = \{2', 5'\}$ and $J^\bullet = \{3', 6', 8'\}$, then the diagram is viewed as in the left fragment of the picture below. (Here, to avoid a mess, we denote the elements of $Y^{\rm c}$ with primes.)




**Matchings.** A partition $M$ of $Y^{\rm r} \sqcup Y^{\rm c}$ into 2-element sets is called a *perfect matching* on $Y^{\rm r} \sqcup Y^{\rm c}$ (where $\sqcup$ stands for the disjoint union). We say that $\pi \in M$ is: an *R-couple* if $\pi \subseteq Y^{\rm r}$, a *C-couple* if $\pi \subseteq Y^{\rm c}$, and an *RC-couple* if $|\pi \cap Y^{\rm r}| = |\pi \cap Y^{\rm c}| = 1$ (as though $\pi$ "connects" two sources, two sinks, and one source and one sink, respectively). A perfect matching $M$ is called a *feasible* matching for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ (and for $(I|J, I'|J')$) if:

(FM1) for each $\pi = \{i, j\} \in M$, the elements $i, j$ have different colors if $\pi$ is an $R$- or $C$-couple, and have the same color if $\pi$ is an $RC$-couple; and

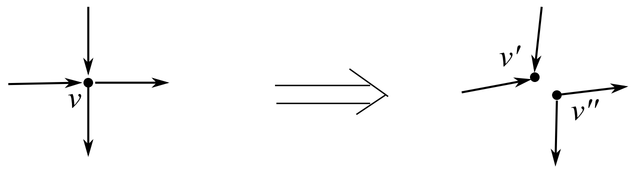
(FM2) $M$ is *planar*, in the sense that the chords connecting the couples in the circumference $O$ are pairwise non-intersecting.

The set of feasible matchings for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ is denoted by $\mathcal{M}_{I^\circ, I^\bullet, J^\circ, J^\bullet}$, or $\mathcal{M}(I|J, I'|J')$. One can show that this set is nonempty whenever $Y^{\rm r} \sqcup Y^{\rm c} \neq \emptyset$. The right fragment of the above picture illustrates an instance of feasible matchings.

Next we return to a double flow $(\phi, \phi')$ as above, and our aim is to associate to it a feasible matching for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$. Let $V_\phi$ and $E_\phi$ denote the sets of vertices and edges of $G$ occurring in $\phi$, respectively; and similarly for $\phi'$. Consider the subgraph $\langle U \rangle$ of $G$ induced by the set of edges

$$U := E_\phi \triangle E_{\phi'},$$

(where $A\triangle B$ denotes the symmetric difference $(A-B)\cup(B-A)$ of sets $A,B$). Then a vertex $v$ of $\langle U\rangle$ has degree 1 if $v\in R_{I^\circ}\cup R_{I^\bullet}\cup C_{J^\circ}\cup C_{J^\bullet}$, and degree 2 or 4 otherwise. We modify $\langle U\rangle$ by splitting each vertex $v$ of degree 4 in $\langle U\rangle$ into two vertices $v',v''$ disposed in a small neighborhood of $v$ so that the edges entering (resp. leaving) $v$ become entering $v'$ (resp. leaving $v''$):




The resulting graph, denoted as $\langle U\rangle'$, is planar and has vertices of degree only 1 and 2. Therefore, $\langle U\rangle'$ consists of pairwise disjoint (non-directed) simple paths $P'_1,\ldots,P'_k$ and, possibly, simple cycles $Q'_1,\ldots,Q'_d$. The corresponding images of $P'_1,\ldots,P'_k$ (resp. $Q'_1,\ldots,Q'_d$) give paths $P_1,\ldots,P_k$ (resp. cycles $Q_1,\ldots,Q_d$) in $\langle U\rangle$. When $\langle U\rangle$ has vertices of degree 4, some of the latter paths and cycles may be self-intersecting and may "touch", but not "cross", each other. It is not difficult to see the following

**Lemma 3.1** (i) *$k=(|I^\circ|+|I^\bullet|+|J^\circ|+|J^\bullet|)/2$;*

(ii) *the set of endvertices of $P_1,\ldots,P_k$ is $R_{I^\circ\cup I^\bullet}\cup C_{J^\circ\cup J^\bullet}$; moreover, each $P_i$ connects either $R_{I^\circ}$ and $R_{I^\bullet}$, or $C_{J^\circ}$ and $C_{J^\bullet}$, or $R_{I^\circ}$ and $C_{J^\circ}$, or $R_{I^\bullet}$ and $C_{J^\bullet}$;*

(iii) *in each path $P_i$, the edges of $\phi$ and the edges of $\phi'$ have different directions (say, the former edges are all forward, and the latter ones are all backward).*

Thus, each $P_i$ is representable as a concatenation $P_i^{(1)}\circ P_i^{(2)}\circ\ldots\circ P_i^{(\ell)}$ of forwardly and backwardly directed paths which are alternately contained in $\phi$ and $\phi'$, called the *segments* of $P_i$. We say that $P_i$ is an *exchange path*. The endvertices of $P_i$ determine a pair of elements of $Y^{\rm r}\sqcup Y^{\rm c}$, denoted by $\pi_i$. Then $M:=\{\pi_1,\ldots,\pi_k\}$ is a perfect matching on $Y^{\rm r}\sqcup Y^{\rm c}$. Moreover, it is feasible, since (FM1) follows from Lemma 3.1(ii), and (FM2) from the fact that $P'_1,\ldots,P'_k$ are disjoint simple paths in $\langle U\rangle'$. We denote $M$ as $M(\phi,\phi')$, and for $\pi\in M$, denote the exchange path $P_i$ corresponding to $\pi$ by $P(\pi)$.

**Corollary 3.2** $M(\phi,\phi')\in\mathcal{M}_{I^\circ,I^\bullet,J^\circ,J^\bullet}$.

**Flow exchange operation.** It rearranges a given double flow $(\phi,\phi')$ for $(I|J,\,I'|J')$ into another double flow $(\psi,\psi')$ for some $(\widetilde I|\widetilde J,\,\widetilde I'|\widetilde J')$, as follows. Fix a submatching $\Pi\subseteq M(\phi,\phi')$, and combine the exchange paths concerning $\Pi$, forming the set of edges

$$\mathcal{E}:=\cup(E_{P(\pi)}\colon \pi\in\Pi).$$

(where $E_{P(\pi)}$ is the edge set of $P(\pi)$). Using Lemma 3.1, one can show the following

**Lemma 3.3** *Let $V_\Pi:=\cup(\pi\in\Pi)$. Define*

$$\widetilde I:=I\triangle(V_\Pi\cap Y^{\rm r}),\ \ \widetilde I':=I'\triangle(V_\Pi\cap Y^{\rm r}),\ \ \widetilde J:=J\triangle(V_\Pi\cap Y^{\rm c}),\ \ \widetilde J':=J'\triangle(V_\Pi\cap Y^{\rm c}).$$

*Then the subgraph $\psi$ induced by $E_\phi \triangle \mathcal{E}$ gives a $(\widetilde{I}|\widetilde{J})$-flow, and the subgraph $\psi'$ induced by $E_{\phi'} \triangle \mathcal{E}$ gives a $(\widetilde{I'}|\widetilde{J'})$-flow in $G$. Furthermore, $E_\psi \cup E_{\psi'} = E_\phi \cup E_{\phi'}$, $E_\psi \triangle E_{\psi'} = E_\phi \triangle E_{\phi'}$ $(= U)$, and $M(\psi,\psi') = M(\phi,\phi')$.*

We call the transformation $(\phi,\phi') \overset{\Pi}{\longmapsto} (\psi,\psi')$ in this lemma the *flow exchange operation* for $(\phi,\phi')$ *using* $\Pi \subseteq M(\phi,\phi')$. Clearly the exchange operation applied to $(\psi,\psi')$ using the same $\Pi$ returns $(\phi,\phi')$.

So far our description has been close to that given for the commutative case in [5]. From now on we will essentially deal with the quantum version. The next theorem serves the main working tool in our arguments; its proof appealing to combinatorial techniques on paths and flows is given in [4].

**Theorem 3.4** *Let $\phi$ be an $(I|J)$-flow, and $\phi'$ an $(I'|J')$-flow in $G$. Let $(\psi,\psi')$ be the double flow obtained from $(\phi,\phi')$ by the flow exchange operation using a single couple $\pi = \{i,j\} \in M(\phi,\phi')$. Then:*

(i) *when $\pi$ is an $R$- or $C$-couple and $i<j$,*

$$w(\phi)w(\phi') = qw(\psi)w(\psi') \qquad \text{in case} \quad i \in I \cup J;$$
$$w(\phi)w(\phi') = q^{-1}w(\psi)w(\psi') \qquad \text{in case} \quad i \in I' \cup J';$$

(ii) *when $\pi$ is an $RC$-couple, $w(\phi)w(\phi') = w(\psi)w(\psi')$.*

An immediate consequence from this theorem is the following

**Corollary 3.5** *For an $(I|J)$-flow $\phi$ and an $(I'|J')$-flow $\phi'$, let $(\psi,\psi')$ be obtained from $(\phi,\phi')$ by the flow exchange operation using a set $\Pi \subseteq M(\phi,\phi')$. Then*

$$w(\phi)w(\phi') = q^{\zeta^\circ - \zeta^\bullet} w(\psi)w(\psi'), \tag{3.2}$$

*where $\zeta^\circ = \zeta^\circ(I|J,I'|J';\Pi)$ (resp. $\zeta^\bullet = \zeta^\bullet(I|J,I'|J';\Pi)$) is the amount of $R$- or $C$-couples $\pi = \{i,j\} \in \Pi$ such that $i<j$ and $i \in I \cup J$ (resp. $i \in I' \cup J'$).*

Indeed, the flow exchange operation using the whole $\Pi$ reduces to performing, step by step, the exchange operations using single couples $\pi \in \Pi$ (taking into account that for any current double flow $(\eta,\eta')$ occurring in the process, the sets $E_\eta \cup E_{\eta'}$ and $E_\eta \triangle E_{\eta'}$, as well as the matching $M(\eta,\eta')$, do not change; cf. Lemma 3.3). Then (3.2) follows from Theorem 3.4.

## 4 Quadratic identities and the $q$-balancedness

We deal with $(m,n)$ SE-graphs $G = (V,E;R,C)$ and consider (quantum) minors $[I|J] = [I|J]_{\mathrm{Path},q}$ of their path matrices $\mathrm{Path} = \mathrm{Path}_G$. In this section, based on Corollary 3.5 and developing a quantum version of the flow-matching method elaborated for the commutative case in [5], we establish sufficient conditions of a general form on quantized quadratic relations for minors to be valid independently of $G$ and some other data (mentioned in Remark 2 below), referring to them as "universal quadratic identities".

Relations of our interest are of the form

$$\sum\nolimits_{\mathcal{I}} q^{\alpha(I|J,I'|J')}[I|J][I'|J'] = \sum\nolimits_{\mathcal{K}} q^{\beta(K|L,K'|L')}[K|L][K'|L'], \qquad (4.1)$$

where $\alpha, \beta$ are integer-valued, $\mathcal{I}$ is a family of corteges $(I|J, I'|J') \in \mathcal{E}^{m,n} \times \mathcal{E}^{m,n}$ (with possible multiplicities), and similarly for $\mathcal{K}$. (Then (4.1) is equivalent to (1.2) with $\delta$ corresponding to $(\alpha, \beta)$.) We assume that $\mathcal{I}$ and $\mathcal{K}$ are *homogeneous*, in the sense that for any $(I|J, I'|J') \in \mathcal{I}$ and $(K|L, K'|L') \in \mathcal{K}$,

$$I \cup I' = K \cup K', \quad J \cup J' = L \cup L', \quad I \cap I' = K \cap K', \quad J \cap J' = L \cap L'. \qquad (4.2)$$

Moreover, we shall see that only the refinements $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ and $(K^\circ, K^\bullet, L^\circ, L^\bullet)$ are important, whereas the sets $I \cap I'$ and $J \cap J'$ are, in fact, indifferent.

To formulate our validity criterion, we need some definitions and notation.

• A tuple $(I|J, I'|J'; M)$, where $(I|J, I'|J') \in \mathcal{I}$ and $M \in \mathcal{M}_{I^\circ, I^\bullet, J^\circ, J^\bullet}$ (cf. (FM1)–(FM2)), is called a *configuration* for $\mathcal{I}$. The family of all configurations for $\mathcal{I}$ is denoted by $\mathbf{C}(\mathcal{I})$. Similarly, we define the family $\mathbf{C}(\mathcal{K})$ of configurations for $\mathcal{K}$.

• Define $\mathbf{M}(\mathcal{I})$ to be the family of all matchings $M$ (with possible multiplicities) occurring in the members of $\mathbf{C}(\mathcal{I})$. Define $\mathbf{M}(\mathcal{K})$ in a similar way.

• Families $\mathcal{I}$ and $\mathcal{K}$ are called *balanced* (borrowing terminology from [5]) if there exists a bijection $(I|J, I'|J'; M) \stackrel{\gamma}{\longmapsto} (K|K', L|L'; M')$ between $\mathbf{C}(\mathcal{I})$ and $\mathbf{C}(\mathcal{K})$ such that $M = M'$. In other words, $\mathcal{I}$ and $\mathcal{K}$ are balanced if $\mathbf{M}(\mathcal{I}) = \mathbf{M}(\mathcal{K})$.

• Families $\mathcal{I}$ and $\mathcal{K}$ along with functions $\alpha : \mathcal{I} \to \mathbb{Z}$ and $\beta : \mathcal{K} \to \mathbb{Z}$ are called *q-balanced* if there exists a bijection $\gamma$ as above such that, for each $(I|J, I'|J'; M) \in \mathbf{C}(\mathcal{I})$ and for $(K|K', L|L'; M) = \gamma(I|J, I'|J'; M)$, there holds

$$\beta(K|K', L|L') - \alpha(I|J, I'|J') = \zeta^\circ - \zeta^\bullet. \qquad (4.3)$$

(In particular, $\mathcal{I}, \mathcal{K}$ are balanced.) Here $\zeta^\circ, \zeta^\bullet$ are defined according to Corollary 3.5. Namely, $\zeta^\circ = \zeta^\circ(I|J, I'|J'; \Pi)$ and $\zeta^\bullet = \zeta^\bullet(I|J, I'|J'; \Pi)$, where $\Pi$ is the set of couples $\pi \in M$ such that the colorings of $\pi$ in the refined corteges $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ and $(K^\circ, K^\bullet, L^\circ, L^\bullet)$ are different. (Then $\zeta^\circ$ ($\zeta^\bullet$) is the number of $R$- and $C$-couples $\{i, j\} \in \Pi$ with $i < j$ and $i \in I^\circ \cup J^\circ$ (resp. $i \in I^\bullet \cup J^\bullet$).) We say that $(K^\circ, K^\bullet, L^\circ, L^\bullet)$ is obtained from $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ by the *index exchange operation* using $\Pi$.

**Theorem 4.1** *Let $\mathcal{I}$ and $\mathcal{K}$ be homogeneous families on $\mathcal{E}^{m,n} \times \mathcal{E}^{m,n}$, and let $\alpha : \mathcal{I} \to \mathbb{Z}$ and $\beta : \mathcal{K} \to \mathbb{Z}$. Suppose that $\mathcal{I}, \mathcal{K}, \alpha, \beta$ are q-balanced. Then* (4.1) *is valid for any $(m, n)$ SE-graph $G = (V, E; R, C)$.*

**Proof** We fix $G$ and denote by $\mathcal{D}(I|J, I'|J')$ the set of double flows for $(I|J, I'|J') \in \mathcal{I} \cup \mathcal{K}$ in $G$. A summand concerning $(I|J, J'|J') \in \mathcal{I}$ in the L.H.S. of (4.1) can be expressed via double flows as follows, ignoring the factor of $q^{\alpha(\cdot)}$:

$$[I|J][I'|J'] = \Big(\sum\nolimits_{\phi\in\Phi_G(I|J)} w(\phi)\Big) \times \Big(\sum\nolimits_{\phi'\in\Phi_G(I'|J')} w(\phi')\Big)$$
$$= \sum\nolimits_{(\phi,\phi')\in\mathcal{D}(I|J,I'|J')} w(\phi)w(\phi')$$
$$= \sum\nolimits_{M\in\mathcal{M}_{I^\circ,I^\bullet,J^\circ,J^\bullet}} \sum\nolimits_{(\phi,\phi')\in\mathcal{D}(I|J,I'|J')\,:\,M(\phi,\phi')=M} w(\phi)w(\phi'). \quad (4.4)$$

The summand for $(K|L,K'|L')\in\mathcal{K}$ in the R.H.S. of (4.1) is expressed similarly.

Consider a configuration $S=(I|J,I'|J';M)\in\mathbf{C}(\mathcal{I})$ and suppose that $(\phi,\phi')$ is a double flow for $(I|J,I'|J')$ with $M(\phi,\phi')=M$ (if such a double flow in $G$ exists). Since $\mathcal{I},\mathcal{K},\alpha,\beta$ are $q$-balanced, $S$ is bijective to some configuration $S'=(K|L,K'|L';M)\in\mathbf{C}(\mathcal{K})$ satisfying (4.3). As explained earlier, the cortege $(K|L,K'|L')$ is obtained from $(I|J,I'|J')$ by the index exchange operation using some $\Pi\subseteq M$. Then the flow exchange operation applying to $(\phi,\phi')$ using this $\Pi$ results in a double flow $(\psi,\psi')$ for $(K|L,K'|L')$ which satisfies relation (3.2) in Corollary 3.5. Comparing (3.2) with (4.3), we observe that

$$q^{\alpha(I|J,I'|J')}w(\phi)w(\phi') = q^{\beta(K|K',L|L')}w(\psi)w(\psi').$$

Furthermore, such a map $(\phi,\phi')\mapsto(\psi,\psi')$ gives a bijection between all double flows concerning configurations in $\mathbf{C}(\mathcal{I})$ and those in $\mathbf{C}(\mathcal{K})$. Now the desired equality (4.1) follows by considering the last term in expression (4.4) and the corresponding term in the analogous expression concerning $\mathcal{K}$. ∎

As a consequence of Theorems 2.3 and 4.1, the following result is obtained.

**Corollary 4.2** *If $\mathcal{I},\mathcal{K},\alpha,\beta$ as above are $q$-balanced, then relation* (4.1) *is valid for the corresponding minors in the algebra $\mathcal{R}$ of quantum $m\times n$ matrices.*

*Remark 2.* When speaking of a *universal quadratic identity* of the form (4.1) with homogeneous $\mathcal{I}$ and $\mathcal{K}$, abbreviated as a *UQ identity*, we mean that it depends neither on the graph $G$ nor on the field $\mathbb{K}$ and element $q\in\mathbb{K}^*$, and that the index sets can be modified as follows. Given $(I|J,I'|J')\in\mathcal{I}$, let $A:=I\triangle I'$, $B:=J\triangle J'$, $S:=I\cap I'$ and $T:=J\cap J'$ (by the homogeneity, these sets do not depend on $(I|J,I'|J)\in\mathcal{I}\cup\mathcal{K}$). Take arbitrary $\widetilde{m}\ge|A|$ and $\widetilde{n}\ge|B|$ and replace $A,B,S,T$ by disjoint sets $\widetilde{A},\widetilde{S}\subseteq[\widetilde{m}]$ and disjoint sets $\widetilde{B},\widetilde{T}\subseteq[\widetilde{n}]$ with $|\widetilde{A}|=|A|$, $|\widetilde{B}|=|B|$, $|\widetilde{S}|-|\widetilde{T}|=|S|-|T|$. Let $\lambda:A\to\widetilde{A}$ and $\mu:B\to\widetilde{B}$ be the order preserving maps. Transform each $(I|J,I'|J')\in\mathcal{I}$ into $(\widetilde{I}|\widetilde{J},\widetilde{I'}|\widetilde{J'})$, where

$$\widetilde{I}:=\widetilde{S}\cup\lambda(I-S),\quad \widetilde{I'}:=\widetilde{S}\cup\lambda(I'-S),\quad \widetilde{J}:=\widetilde{T}\cup\mu(J-T),\quad \widetilde{J'}:=\widetilde{T}\cup\mu(J'-T),$$

forming a new family $\widetilde{\mathcal{I}}$ on $\mathcal{E}^{\widetilde{m},\widetilde{n}}\times\mathcal{E}^{\widetilde{m},\widetilde{n}}$. Transform $\mathcal{K}$ into $\widetilde{K}$ in a similar way. One can see that if $\mathcal{I},\mathcal{K},\alpha,\beta$ are $q$-balanced, then so are $\widetilde{\mathcal{I}},\widetilde{\mathcal{K}}$, keeping $\alpha,\beta$. Therefore, if (4.1) is valid for $\mathcal{I},\mathcal{K}$, then it is valid for $\widetilde{\mathcal{I}},\widetilde{\mathcal{K}}$ as well.

One can say that identity (4.1), where all summands have positive signs, is written in the canonical form. Sometimes, however, it is more convenient to consider equivalent identities having negative summands in one or both sides (e.g., of the form (1.2)). Also one can multiply all summands in the identity by the same degree of $q$.

We can suggest a rather simple algorithm which has as the input a corresponding quadruple $\mathcal{I}, \mathcal{K}, \alpha, \beta$ and recognizes the $q$-balanced for it. Therefore, in light of Theorems 4.1 and 4.3, the algorithm decides whether or not the given quadruple determines a UQ identity of the form (4.1).

**Algorithm.** Compute the set $\mathcal{M}_{I^\circ, I^\bullet, J^\circ, J^\bullet}$ of feasible matchings $M$ for each $(I|J, I'|J') \in \mathcal{I}$, and similarly for $\mathcal{K}$. For each instance $M$ occurring there, extract the family $\mathbf{C}_M(\mathcal{I})$ of all configurations concerning $M$ in $\mathbf{C}(\mathcal{I})$, and extract a similar family $\mathbf{C}_M(\mathcal{K})$ in $\mathbf{C}(\mathcal{K})$. If $|\mathbf{C}_M(\mathcal{I})| \ne |\mathbf{C}_M(\mathcal{K})|$ for at least one instance $M$, then $\mathcal{I}$ and $\mathcal{K}$ are not balanced at all. Otherwise for each $M$, we seek for a required bijection $\gamma_M : \mathbf{C}_M(\mathcal{I}) \to \mathbf{C}_M(\mathcal{K})$ by solving the maximum matching problem in the corresponding bipartite graph $H_M$. More precisely, the vertices of $H_M$ are the tuples $(I|J, I'|J'; M)$ and $(K|L, K'|L'; M)$ occurring in $\mathbf{C}_M(\mathcal{I})$ and $\mathbf{C}_M(\mathcal{K})$, and such tuples are connected by edge in $H_M$ if they obey (4.3). Find a maximum matching $N$ in $H_M$. If $|N| = |\mathbf{C}_M(\mathcal{I})|$, then $N$ determines the desired $\gamma_M$ in a natural way. Taking together, these $\gamma_M$ give a bijection between $\mathbf{C}(\mathcal{I})$ and $\mathbf{C}(\mathcal{K})$ as required, implying that $\mathcal{I}, \mathcal{K}, \alpha, \beta$ are $q$-balanced. And if $|N| < |\mathbf{C}_M(\mathcal{I})|$ for at least one instance $M$, then the algorithm declares the non-$q$-balancedness.

**Example.** Next we give a simple illustration to our method. Recall that sets $I, J \subseteq [n]$ are called *weakly separated* if, up to renaming $I$ and $J$, there holds $(*)$: $|I| \ge |J|$, and $J - I$ has a partition $\{J_1, J_2\}$ such that $J_1 < I - J < J_2$ (where we write $X < Y$ if $x < y$ for any $x \in X$ and $y \in Y$). Let $k := |I|$ and $\ell := |J|$. Leclerc and Zelevinsky [9] proved that: *two flag quantum minors* $[[k]|I]$ *and* $[[\ell]|J]$ quasicommute, *i.e., satisfy* $[[k]|I][[\ell]|J] = q^c[[\ell]|J][[k]|I]$ *for some* $c \in \mathbb{Z}$, *if and only if* $I$ *and* $J$ *are weakly separated. Moreover, subject to* $(*)$, $c$ *is equal to* $|J_2| - |J_1|$.

We can show "if" part as follows. Assume $(*)$ as above and let $A := [k] - [\ell]$. One can see that $\mathcal{M}([k]|I, [\ell]|J)$ has exactly one feasible matching $M$; namely, $J_1$ is coupled with the first $|J_1|$ elements of $I - J$, $J_2$ is coupled with the last $|J_2|$ elements of $I - J$ (forming all $C$-couples), and the rest of $I - J$ is coupled with $A$ (forming all $RC$-couples). Observe that the index exchange operation applied to the cortege $([k]|I, [\ell]|J)$ using the whole $M$ swaps $([k]|I)$ and $([\ell]|J)$ (as it changes the colors of all elements of $I - J, J - I, A$. Also the $C$-couples of $M$ consist of $|J_1|$ couples $\{i, j\}$ with $i < j$ and $i \in J_1$, and $|J_2|$ couples $\{i, j\}$ with $i < j$ and $j \in J_2$. This gives $\zeta^\circ = |J_2|$ and $\zeta^\bullet = |J_1|$. Hence the (one-element) families $\mathcal{I} = \{([k]|I, [\ell]|J)\}$ and $\mathcal{K} = \{([\ell]|J, [k]|I)\}$ along with $\alpha(([k]|I, [\ell]|J)) = 0$ and $\beta(([\ell]|J, [k]|I)) = |J_2| - |J_1|$ are $q$-balanced. Now the result (with $c = |J_2| - |J_1|$) follows from Theorem 4.1.

Finally, we formulate (without a proof) a converse assertion to Theorem 4.1, saying that the $q$-balancedness condition is necessary as well. This gives a complete characterization for the UQ identities on quantized minors.

**Theorem 4.3** *Let* $\mathbb{K}$ *be a field of characteristic zero and let* $q \in \mathbb{K}^*$ *be transcendental over* $\mathbb{Q}$. *Suppose that* $\mathcal{I}, \mathcal{K}, \alpha, \beta$ *(as in Section 4) are not* $q$*-balanced. Then there exists* *(and can be explicitly constructed) an SE-graph G for which relation* (4.1) *is violated.*

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