Combined tilings and separated set-systems

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Abstract. In 1998, Leclerc and Zelevinsky introduced the notion of *weakly separated* collections of subsets of the ordered n-element set $[n]$ (using this notion to give a combinatorial characterization for quasi-commuting minors of a quantum matrix). They conjectured the purity of certain natural domains $\mathcal{D} \subseteq 2^{[n]}$ (in particular, of the hypercube $2^{[n]}$ itself, and the hyper-simplex $\{X \subseteq [n] \colon |X| = m\}$ for m fixed), where $\mathcal{D}$ is called *pure* if all maximal weakly separated collections in $\mathcal{D}$ have the same cardinality. These conjectures have been answered affirmatively.

In this paper, generalizing those earlier results, we reveal wider classes of pure domains in $2^{[n]}$. This is obtained as a consequence of our study of a novel geometric–combinatorial model for weakly separated set-systems, so-called *combined (polygonal) tilings* on a zonogon, which yields a new insight in the area.

Keywords: weakly separated sets, strongly separated sets, quasi-commuting quantum minors, rhombus tiling, Grassmann necklace

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1 Introduction

For a positive integer n, the set $\{1, 2, \ldots, n\}$ with the usual order is denoted by $[n]$. For a set $X \subseteq [n]$ of elements $x_1 < x_2 < \cdots < x_k$, we use notation $(x_1, \ldots, x_k)$ for X, $\min(X)$ for x_1, and $\max(X)$ for x_k, where $\min(X) = \max(X) := 0$ if $X = \emptyset$.

We will deal with several binary relations on the set $2^{[n]}$ of all subsets of $[n]$. Namely, for distinct $A, B \subseteq [n]$, we write:

(1.1) (i) $A \prec B$ if $A = (a_1, \ldots, a_k)$, $B = (b_1, \ldots, b_m)$, $k \le m$, and $a_i \le b_i$ for $i = 1, \ldots, k$ (*termwise dominating*);

(ii) $A < B$ if $\max(A) < \min(B)$ (*global dominating*);

(iii) $A \lessdot B$ if $(A - B) < (B - A)$, where $A' - B'$ denotes $\{i' \colon A' \ni i' \notin B'\}$ (*global dominating after cancelations*);

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(iv) $A \rhd B$ if $A - B \neq \emptyset$, and $B - A$ can be expressed as a union of non-empty subsets B', B'' so that $B' < (A - B) < B''$ (*splitting*).

(Note that relations (i),(ii) are transitive, whereas (iii),(iv) are not in general.) Relations (iii) and (iv) give rise to two important notions introduced by Leclerc and Zelevinsky [5] (where these notions appear in characterizations of quasi-commuting flag minors of a generic q-matrix).

Definitions. Sets $A, B \subseteq [n]$ are called *strongly separated* (from each other) if $A \lessdot B$ or $B \lessdot A$ or $A = B$. Sets $A, B \subseteq [n]$ are called *weakly separated* if either they are strongly separated, or $A \rhd B$ and $|A| \geq |B|$, or $B \rhd A$ and $|B| \geq |A|$. Accordingly, a collection $\mathcal{F} \subseteq 2^{[n]}$ is called strongly (resp. weakly) separated if any two of its members are such. For brevity, we refer to strongly and weakly separated collections as *s-collections* and *w-collections*, respectively.

In what follows, we will distinguish one or another set-system $\mathcal{D} \subseteq 2^{[n]}$, referring to it as a ground collection, or a *domain*. We are interested in the situation when (strongly or weakly) separated collections in $\mathcal{D}$ possess the property of *max-size purity*, which means the following.

Definitions. Let us say that a domain $\mathcal{D}$ is *s-pure* if all (inclusion-wise) maximal s-collections in $\mathcal{D}$ have the same cardinality, which in this case is called the *s-rank* of $\mathcal{D}$ and denoted by $r^s(\mathcal{D})$. Similarly, we say that $\mathcal{D}$ is *w-pure* if all maximal w-collections in $\mathcal{D}$ have the same cardinality, called the *w-rank* of $\mathcal{D}$ and denoted by $r^w(\mathcal{D})$.

(In topology, the term "pure" is often applied to complexes in which all maximal cells have the same dimension. In our case we can interpret each s-collection (resp. w-collection) as a cell, forming an abstract simplicial complex with $\mathcal{D}$ regarded as the set of zero-dimensional cells. This justifies our terms "s-pure" and "w-pure".)

Leclerc and Zelevinsky [5] proved that the maximal domain (*hypercube*) $\mathcal{D} = 2^{[n]}$ is s-pure and conjectured that $2^{[n]}$ is w-pure as well (in which case there would be $r^w(2^{[n]}) = r^s(2^{[n]}) = \frac{n(n+1)}{2} + 1$). A sharper version of this conjecture deals with *ω-chamber sets* $X \subseteq [n]$ for a permutation ω on $[n]$, where X obeys the condition:

(1.2) if $i < j$, $\omega(i) < \omega(j)$, and $j \in X$, then $i \in X$.

They conjectured that the domain $\mathcal{D}(\omega)$ consisting of the ω-chamber sets is w-pure (in our terms), with the w-rank equal to $|\mathrm{Inv}(\omega)| + n + 1$. Here $\mathrm{Inv}(\omega)$ denotes the set of *inversions* of ω (i.e., pairs (i, j) in $[n]$ such that $i < j$ and $\omega(i) > \omega(j)$), and the number $|\mathrm{Inv}(\omega)|$ is called the *length* of ω. For the *longest* permutation ω_0 (where $\omega_0(i) = n - i + 1$), we have $\mathcal{D}(\omega_0) = 2^{[n]}$.

This conjecture was proved in [2]. The key part consisted in proving the w-purity for $2^{[n]}$; using this, the w-purity was shown for an arbitrary permutation ω, and more.

Theorem 1.1 ([2]) *The hypercube $2^{[n]}$ is w-pure. As a consequence, the following domains $\mathcal{D}$ are w-pure as well:*

(i) $\mathcal{D} = \mathcal{D}(\omega)$ *for any permutation ω on $[n]$;*

(ii) $\mathcal{D} = \mathcal{D}(\omega', \omega)$, *where* ω', ω *are two permutations on* $[n]$ *with* $\mathrm{Inv}(\omega') \subset \mathrm{Inv}(\omega)$, *and* $\mathcal{D}(\omega', \omega)$ *is formed by the* ω*-chamber sets* $X \subseteq [n]$ *satisfying the additional condition: if* $i < j$, $\omega'(i) > \omega'(j)$, *and* $i \in X$, *then* $j \in X$*; furthermore,* $r^w(\mathcal{D}(\omega, \omega')) = |Inv(\omega)| - |Inv(\omega')| + n + 1$*;*

(iii) $\mathcal{D} = \Delta_n^{m',m} := \{X \subseteq [n] \colon m' \le |X| \le m\}$ *for any* $m' \le m$*; furthermore,* $r^w(\Delta_n^{m',m}) = \binom{n+1}{2} - \binom{n-m+1}{2} - \binom{m'+1}{2} + 1$ *(this turns into* $m(n-m)+1$ *when* $m' = m$*).*

Note that (ii) generalizes (i) since $\mathcal{D}(\omega) = \mathcal{D}(\mathrm{id}, \omega)$, where id is the identical permutation ($\mathrm{id}(i) = i$). The domain $\Delta_n^{m',m}$ in (iii) generalizes the Boolean hyper-simplex, or *discrete Grassmannian*, $\Delta_n^m := \Delta_n^{m,m}$. The domains in cases (i) and (ii) turn out to be s-pure as well, and the w- and s-ranks are equal; see [2]. (Note that in general a domain $\mathcal{D}$ may be w-pure but not s-pure (e.g. for $\mathcal{D} = \Delta_5^2$), and vice versa; also when both w- and s-ranks exist, they may be different.) Using simple observations from [5], one can reduce case (iii) to $2^{[n]}$ as well. In its turn, the proof of w-purity for $2^{[n]}$ given in [2] is direct and essentially relies on a mini-theory of *generalized tilings* developed in [1].

Oh et al. [6] gave another proof for Δ_n^m using *plabic tilings* (and relying on a machinery of *plabic graphs* elaborated in [8]). Moreover, [6] established the w-purity for certain domains $\mathcal{D} \subseteq \Delta_n^m$ related to so-called *Grassmann necklaces*. They also explained that the w-purity of such necklace domains implies the w-purity for the ω-chamber domain $\mathcal{D}(\omega)$ with any permutation ω (however, it is not clear whether the w-purity of $\mathcal{D}(\omega', \omega)$ can be obtained directly from results in [6]).

The purpose of this paper is twofold: to extend the above mentioned w-purity results by demonstrating wider classes of domains whose w-purity follows from the w-purity of $2^{[n]}$, and to describe a novel geometric–combinatorial representation of maximal w-collections in $2^{[n]}$ (which is essentially used to carry out the first task, but also is interesting by its own right and can find other important applications).

More precisely, in the half-plane $\mathbb{R} \times \mathbb{R}_{\ge 0}$ we fix n generic vectors $\xi_1, \dots, \xi_n$ having equal norms and ordered as indicated around the origin. Under the linear projection $\mathbb{R}^{[n]} \to \mathbb{R}^2$ mapping ith orth to ξ_i, the image of the solid cube $conv(2^{[n]})$ is a *zonogon* Z (a central symmetric $2n$-gone), and we naturally identify each subset $X \subseteq [n]$ with the point $\sum(\xi_i \colon i \in X)$ of Z. A well-known fact is that the maximal s-collections in $2^{[n]}$ one-to-one correspond to the vertex sets of *rhombus tilings* on Z, i.e., planar subdivisions of Z into rhombi with the same side lengths (cf. [5] where the correspondence is described in dual terms of pseudo-line arrangements). In case of w-collections, the construction given in [1] is more sophisticated: each maximal w-collection one-to-one corresponds to a *generalized tiling* (briefly, *g-tiling*), which is viewed as a certain covering of Z by rhombi which may overlap. So a g-tiling need not be fully planar; also the maximal w-collection corresponding to the tiling is represented by a certain subset of its vertices, but not necessarily the whole vertex set.

The simpler model for w-collections elaborated in this paper deals with a sort of polygonal complexes on Z; they are fully planar and their vertex sets are exactly the maximal w-collections in $2^{[n]}$. We call them *combined tilings*, or *c-tilings* for short (as they consist of tiles of two types: “semi-rhombi” and “lenses”, justifying the term "combined"). A nice planar structure and appealing properties of c-tilings will enable us to reveal new classes of w-pure domains.

A particular class (extending that in [6] and a more general construction in [3]) is generated by a *cyclic pattern*. This means a sequence $\mathcal{S} = (S_1, S_2, \ldots, S_r = S_0)$ of different weakly separated subsets of $[n]$ such that, for each i, $|S_i - S_{i-1}| \le 1$ and $|S_{i-1} - S_i| \le 1$. We call $\mathcal{S}$ *simple* if $|S_{i-1}| \ne |S_i|$ holds for each i, and *generalized* otherwise. Considering the members of $\mathcal{S}$ as the corresponding points in the zonogon Z, we connect the pairs of consecutive points by straight-line segments, obtaining the piecewise linear curve $\zeta_\mathcal{S}$. We show that $\zeta_\mathcal{S}$ is non-self-intersecting when $\mathcal{S}$ is simple, and give necessary and sufficient conditions on $\mathcal{S}$ to provide that $\zeta_\mathcal{S}$ is non-self-intersecting in the generalized case. This allows us to represent the collection $\mathcal{D}_\mathcal{S}$ of all subsets of $[n]$ weakly separated from $\mathcal{S}$ as the union of two domains $\mathcal{D}^{\rm in}_\mathcal{S}$ and $\mathcal{D}^{\rm out}_\mathcal{S}$ whose members (points) are located inside and outside $\zeta_\mathcal{S}$, respectively. Domains of type $\mathcal{D}^{\rm in}_\mathcal{S}$ generalize the ones in Theorem 1.1 and the ones described in the necklace constructions of [6].

We show that both domains $\mathcal{D}^{\rm in}_\mathcal{S}$ and $\mathcal{D}^{\rm out}_\mathcal{S}$ are w-pure. Moreover, these domains form a *complementary pair*, in the sense that any $X \in \mathcal{D}^{\rm in}_\mathcal{S}$ and any $Y \in \mathcal{D}^{\rm out}_\mathcal{S}$ are weakly separated from each other (proving a conjecture on generalized necklaces in [3]). The latter property follows from the observation that for any two c-tilings T and T' on Z whose vertex sets contain the given cyclic pattern $\mathcal{S}$, one can extract and exchange their portions lying inside $\zeta_\mathcal{S}$, obtaining two correct c-tilings again.

The most general case of w-pure domains that we describe is obtained when the role of a pattern is played by a *planar graph* $\mathcal{H}$, embedded in a zonogon, whose vertices form a w-collection $\mathcal{S}$, and edges are given by pairs similar to those in cyclic patterns. Then, instead of two domains $\mathcal{D}^{\rm in}_\mathcal{S}, \mathcal{D}^{\rm out}_\mathcal{S}$ as above, we deal with a set of domains $\mathcal{D}^F_\mathcal{S}$, each being associated with a face F of $\mathcal{H}$. We prove that any two of them form a complementary pair. It follows that each $\mathcal{D}^F_\mathcal{S}$ is w-pure, as well as any union of them.

Note that for cyclic patterns consisting of strongly separated sets, we can consider analogous pairs of domains w.r.t. the strong separation relation. One shows that such domains are s-pure (which is relatively easy) and that the corresponding s-ranks and w-ranks are equal.

This paper is organized as follows. Section 2 considers simple cyclic patterns $\mathcal{S}$ and related domains $\mathcal{D}^{\rm in}_\mathcal{S}$ and $\mathcal{D}^{\rm out}_\mathcal{S}$ and state a w-purity result for them (Theorem 2.2). Section 3 starts with a brief review on rhombus tilings and then introduces combined tilings. The most attention in this section is drawn to transformations of c-tilings, so-called *raising* and *lowering flips*, which correspond to standard mutations for maximal w-collections. As a result, we establish a bijection between the c-tilings on the zonogon and the maximal w-collections in $2^{[n]}$. Section 4 further develops our "mini-theory" of c-tilings by describing operations of *contraction* and *expansion* (analogous to those elaborated for g-tilings in [1, Sec. 8] but looking more transparent for c-tilings). These operations transform c-tilings on the n-zonogon to ones on the $(n-1)$-zonogon, and conversely. This enables us to conduct needed proofs by induction on n, and using this and results from Sect. 3, we finish the proof of Theorem 2.2. The concluding Sect. 5 is devoted to special cases, illustrations and generalizations. Here we extend the w-purity result to generalized cyclic patterns (Theorem 5.3), and finish with planar graph patterns (Theorem 5.4).

Note that the preliminary version [4] of this paper gave one more combinatorial construction of w-pure domains (based on lattice paths in certain ladder diagrams), which is omitted here to make our description more compact.

Additional notations. An *interval* in $[n]$ is a set of the form $\{p, p+1, \dots, q\}$, and a *co-interval* is the complement of an interval to $[n]$. For $p \le q$, we denote by $[p..q]$ the interval $\{p, p+1, \dots, q\}$.

When sets $A, B \subseteq [n]$ are weakly (strongly) separated, we write $A \,\text{(weak)}\, B$ (resp. $A \,\text{(str)}\, B$). We use similar notation $A \,\text{(weak)}\, \mathcal{B}$ (resp. $A \,\text{(str)}\, \mathcal{B}$) when a set $A \subseteq [n]$ is weakly (resp. strongly) separated from all members of a collection $\mathcal{B} \subseteq 2^{[n]}$.

For a set $X \subset [n]$, distinct elements $i, \dots, j \in [n] - X$, and an element $k \in X$, we abbreviate $X \cup \{i\} \cup \ldots \cup \{j\}$ as $Xi \dots j$, and $X - \{k\}$ as $X - k$.

By a *path* in a directed graph, we mean a sequence $P = (v_0, e_1, v_1, \dots, e_k, v_k)$, where each e_i is an edge connecting vertices v_{i-1} and v_i. An edge e_i is called *forward* (*backward*) if it goes from v_{i-1} to v_i (resp. from v_i to v_{i-1}), and we write $e_i = (v_{i-1}, v_i)$ (resp. $e_i = (v_i, v_{i-1})$). The path is called *directed* if all its edges are forward. Sometimes we will use for P an abbreviated notation via vertices, writing $P = v_0 v_1 \dots v_k$.

In what follows, we will use the following simple fact for $A, B \subset [n]$:

(1.3) if $A \,\text{(weak)}\, B$ and $|A| \le |B|$, then relations $A \prec B$ and $A \lessdot B$ are equivalent.

Another useful property is as follows (a similar property is valid for the strong separation as well).

Proposition 1.2 *Let domains $\mathcal{D}, \mathcal{D}' \subset 2^{[n]}$ be such that $X \,\text{(weak)}\, Y$ for any $X \in \mathcal{D}$ and $Y \in \mathcal{D}'$. Let $\mathcal{D} \cup \mathcal{D}'$ be w-pure. Then each of $\mathcal{D}, \mathcal{D}'$ is w-pure.*

Indeed, let $A := \mathcal{D} \cap \mathcal{D}'$, fix a maximal w-collection X in $\mathcal{D}$, and take an arbitrary maximal w-collection Y in $\mathcal{D}'$. It is easy to see that $A = X \cap Y$ and that $X \cup Y$ is a maximal w-collection in $\mathcal{D} \cup \mathcal{D}'$. We have $|Y| = |X \cup Y| + |A| - |X|$, and now the w-purity of $\mathcal{D}'$ follows from that of $\mathcal{D} \cup \mathcal{D}'$. The argument for $\mathcal{D}$ is similar.

We say that domains $\mathcal{D}, \mathcal{D}'$ as in this proposition form a *complementary pair.*

2 Simple cyclic patterns and their geometric interpretation

Recall that by a *simple cyclic pattern* we mean a sequence $\mathcal{S}$ of subsets $S_1, S_2, \dots, S_r = S_0$ of $[n]$ such that $|S_{i-1} \triangle S_i| = 1$ for $i = 1, \dots, r$, where $A \triangle B$ denotes the symmetric difference $(A - B) \cup (B - A)$. We assume that $\mathcal{S}$ satisfies the following conditions:

(C1) All sets in $\mathcal{S}$ are different;

(C2) $\mathcal{S}$ is weakly separated.

(Condition (C1) can be slightly weakened, as we explain in Section 5.1.) We associate with $\mathcal{S}$ the collection

$$\mathcal{D}_{\mathcal{S}} := \{X \subseteq [n] : X \,\text{(weak)}\, \mathcal{S}\}.$$

We extract from $\mathcal{D}_\mathcal{S}$ two domains $\mathcal{D}^{\rm in}_\mathcal{S}$ and $\mathcal{D}^{\rm out}_\mathcal{S}$, using a geometric construction mentioned in Sect. 1.

More precisely, in the upper half-plane $\mathbb{R}\times\mathbb{R}_{\ge 0}$, we fix n vectors $\xi_1,\ldots,\xi_n$ which:

(2.1) (a) follow in this order clockwise around the origin $(0,0)$;

(b) have the *same euclidean norm*, say, $\|\xi_i\|=1$; and

(c) are $\mathbb{Z}$-independent (i.e., all integer combinations of these vectors are different).

We call these vectors *generators*. Then the set

$$Z=Z_n:=\{\lambda_1\xi_1+\cdots+\lambda_n\xi_n\colon \lambda_i\in\mathbb{R},\ 0\le\lambda_i\le 1,\ i=1,\ldots,n\}$$

is a $2n$-gon. Moreover, Z is a *zonogon*, as it is the sum of line segments $\{\lambda\xi_i\colon 0\le\lambda\le 1\}$, $i=1,\ldots,n$. Also it is the projection of the solid n-cube $conv(2^{[n]})$ into the plane, given by $\pi(x):=x_1\xi_1+\cdots+x_n\xi_n$, $x\in\mathbb{R}^{[n]}$. We identify a subset $X\subseteq[n]$ with the corresponding point $\sum(\xi_i\colon i\in X)$ in Z. Due to the $\mathbb{Z}$-independence, all such points in Z are different.

The *boundary* $bd(Z)$ of Z consists of two parts: The *left boundary* $lbd(Z)$ formed by the sequence of points (vertices) $z^\ell_i:=\xi_1+\cdots+\xi_i$ $(i=0,\ldots,n)$ connected by the line segments $z^\ell_{i-1}z^\ell_i$ congruent to ξ_i, and the *right boundary* $rbd(Z)$ formed by the sequence of points $z^r_i:=\xi_n+\xi_{n-1}+\cdots+\xi_{n-i+1}$ $(i=0,\ldots,n)$ connected by the line segments $z^r_{i-1}z^r_i$ congruent to ξ_{n-i+1}. Then $z^\ell_0=z^r_0=(0,0)$ is the bottom(most) vertex, and $z^\ell_n=z^r_n=\xi_1+\cdots+\xi_n$ is the top(most) vertex of Z.

Returning to $\mathcal{S}$ as above, we draw the closed piecewise linear curve $\zeta_\mathcal{S}$ by concatenating the line-segments connecting consecutive points S_{i-1} and S_i for $i=1,\ldots,r$. The following property is important.

Proposition 2.1 *For a simple cyclic pattern $\mathcal{S}$, the curve $\zeta_\mathcal{S}$ is non-self-intersecting, and therefore it divides the zonogon Z into two closed regions $R^{\rm in}_\mathcal{S}$ and $R^{\rm out}_\mathcal{S}$, where $R^{\rm in}_\mathcal{S}\cap R^{\rm out}_\mathcal{S}=\zeta_\mathcal{S}$, $R^{\rm in}_\mathcal{S}\cup R^{\rm out}_\mathcal{S}=Z$, and $bd(Z)\subseteq R^{\rm out}_\mathcal{S}$.*

(So $R^{\rm in}_\mathcal{S}$ is a disk and $R^{\rm out}_\mathcal{S}$ is viewed as a "ring".) This enables us to define the desired domains:

$$\mathcal{D}^{\rm in}_\mathcal{S}:=\mathcal{D}_\mathcal{S}\cap R^{\rm in}_\mathcal{S}\qquad\text{and}\qquad \mathcal{D}^{\rm out}_\mathcal{S}:=\mathcal{D}_\mathcal{S}\cap R^{\rm out}_\mathcal{S}. \tag{2.2}$$

Proposition 2.1 is proved in Sect. 4 relying on properties of combined tilings introduced in the next section. Moreover, using such tilings, we will prove the following w-purity result (where a complementary pair is defined in Sect. 1).

Theorem 2.2 *For a simple cyclic pattern $\mathcal{S}$, the domains $\mathcal{D}^{\rm in}_\mathcal{S}$ and $\mathcal{D}^{\rm out}_\mathcal{S}$ form a complementary pair. As a consequence, both $\mathcal{D}^{\rm in}_\mathcal{S}$ and $\mathcal{D}^{\rm out}_\mathcal{S}$ are w-pure.*

This theorem has a natural strong separation counterpart.

Theorem 2.3 *Let $\mathcal{S}$ be a cyclic pattern consisting of different pairwise strongly separated sets in $[n]$. Define $\widehat{\mathcal{D}}^{\rm in}_\mathcal{S}$, $\widehat{\mathcal{D}}^{\rm out}_\mathcal{S}$ as in (2.2) with $\mathcal{D}_\mathcal{S}$ replaced by $\widehat{\mathcal{D}}_\mathcal{S}:=\{X\subseteq[n]: X \circledS\mathcal{S}\}$ (str). Then X (str) Y for any $X\in\widehat{\mathcal{D}}^{\rm in}_\mathcal{S}$ and $Y\in\widehat{\mathcal{D}}^{\rm out}_\mathcal{S}$. Hence both $\widehat{\mathcal{D}}^{\rm in}_\mathcal{S}$ and $\widehat{\mathcal{D}}^{\rm out}_\mathcal{S}$ are s-pure. Also the s- and w-ranks of $\widehat{\mathcal{D}}^{\rm in}_\mathcal{S}$ are equal, and similarly for $\widehat{\mathcal{D}}^{\rm out}_\mathcal{S}$.*

3 Tilings

We know that maximal s-collections (w-collections) are representable via rhombus (resp. generalized) tilings on the zonogon $Z = Z_n$. In this section, we start with a short review on pure (rhombus) tilings and then introduce the notion of *combined tilings* on Z and show that the latter objects behave similarly to g-tilings: They represent maximal w-collections. At the same time, this new combinatorial model is viewed significantly simpler than the one of g-tilings, and we will take advantage from this fact to obtain a rather transparent proof of Theorem 2.2.

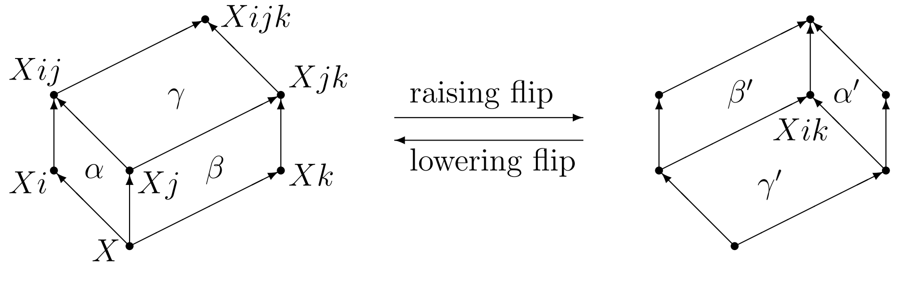
3.1 Rhombus tilings

By a *planar tiling* on the zonogon Z, we mean a collection T of convex polygons, called *tiles*, such that: (i) the union of tiles is Z, (ii) any two tiles either are disjoint or intersect by a common vertex or by a common side, and (iii) each boundary edge of Z belongs to exactly one tile. Then the set of vertices and the set of edges (sides) of tiles in T, ignoring multiplicities, form a planar graph, denoted by $G_T = (V_T, E_T)$. Usually the edges of G_T are equipped with directions. Speaking of vertices or edges of T, we mean those in G_T.

A *pure (rhombus) tiling* is a planar tiling T in which all tiles are rhombi; then each edge of T is congruent to some generator ξ_i, and we direct it accordingly (upward). More precisely, each tile τ is of the form $X + \{\lambda\xi_i + \lambda'\xi_j : 0 \le \lambda, \lambda' \le 1\}$ for some $i < j$ and some subset $X \subseteq [n] - \{i, j\}$ (regarded as a point in Z). We call τ a tile of *type* ij and denote it by $\tau(X; i, j)$. By a natural visualization of τ, its vertices X, Xi, Xj, Xij are called the *bottom, left, right, top* vertices of τ and denoted by $b(\tau)$, $\ell(\tau)$, $r(\tau)$, $t(\tau)$, respectively.

The vertex set V_T of G_T is also called the *spectrum* of T.

Pure tilings admit a sort of *mutations*, called strong raising and lowering flips, which transform one tiling T into another, and back. More precisely, suppose that T contains a hexagon H formed by three tiles $\alpha := \tau(X; i, j)$, $\beta := \tau(X; j, k)$, and $\gamma := \tau(Xj; i, k)$. The *strong raising flip* replaces α, β, γ by the other possible combination of three tiles in H, namely by $\alpha' := \tau(Xk; i, j)$, $\beta' := \tau(Xi; j, k)$, and $\gamma' := \tau(X; i, k)$. The *strong lowering flip* acts conversely: it replaces α', β', γ' by α, β, γ. See the picture.



In terms of sets, a strong flip is applicable to a set-system when for some X and $i < j < k$, it contains six sets $X, Xi, Xk, Xij, Xjk, Xijk$ and one more set $Y \in \{Xj, Xik\}$. The flip ("in the presence of six witnesses", in terminology of [5]) replaces Y by the other member of $\{Xj, Xik\}$; the replacement $Xj \rightsquigarrow Xik$ gives a raising flip, and $Xik \rightsquigarrow Xj$

a lowering flip. Leclerk and Zelevinsky described in [5] important properties of strongly separated set-systems. Among those we use the following.

(i) If $\mathcal{F} \subseteq 2^{[n]}$ is a maximal s-collection and if $\mathcal{F}'$ is obtained by a strong flip from $\mathcal{F}$, then $\mathcal{F}'$ is a maximal s-collection as well.

(ii) Let $\mathbf{S_n}$ be the set of maximal s-collections in $2^{[n]}$, and for $\mathcal{F}, \mathcal{F}' \in \mathbf{S_n}$, let us write $\mathcal{F} \prec_s \mathcal{F}'$ if $\mathcal{F}'$ is obtained from $\mathcal{F}$ by a series of strong raising flips. Then the poset $(\mathbf{S_n}, \prec_\mathbf{s})$ has a unique minimal element and a unique maximal element, which are the collection $\mathcal{I}_n$ of all intervals and the collection co-$\mathcal{I}_n$ of all co-intervals in $[n]$, respectively.

Moreover, they established a correspondence between maximal s-collections and so-called commutation classes of pseudo-line arrangements. In terms of pure tilings (which are dual to pseudo-line arrangements, in a sense), this is viewed as follows.

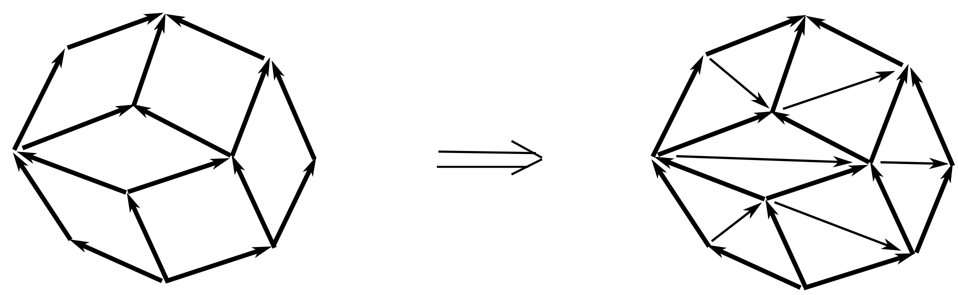
Theorem 3.1 [5] *For each pure (rhombus) tiling T, its spectrum V_T is a maximal s-collection (regarding a vertex as a subset of $[n]$). This correspondence gives a bijection between the set of pure (rhombus) tilings on Z_n and $\mathbf{S_n}$.*

3.2 Combined tilings

In addition to the generators ξ_i (satisfying (2.1)), we will deal with vectors $\epsilon_{ij} := \xi_j - \xi_i$, where $1 \le i < j \le n$. A *combined tiling*, abbreviated as a *c-tiling*, or simply a *combi*, is a sort of planar tilings (defined later) generalizing rhombus tilings in essence in which each edge e is congruent to either ξ_i for some i or ϵ_{ij} for some $i < j$; we say that e has *type* i in the former case and *type* ij in the latter case.

Remark 1. In fact, we are free to choose arbitrary basic vectors ξ_i (subject to (2.1)) without affecting our model (in the sense that corresponding structures and set-systems remain equivalent when the generators vary). Sometimes, to simplify visualization, it is convenient to think of vectors $\xi_i = (x_i, y_i)$ as "almost vertical" ones (i.e., to assume that $|x_i| = o(|y_i|)$). Then the vectors ϵ_{ij} become "almost horizontal". For this reason, we will liberally refer to edges congruent to $\xi_\bullet$ as *V-edges*, and those congruent to $\epsilon_\bullet$ as *H-edges*. Also we say that V-edges point *upward*, and H-edges point *to the right*.

The simplest case of combies arises from arbitrary rhombus tilings T by subdividing each rhombus τ of T into two isosceles triangles Δ and ∇, where the former (the "upper" triangle) uses the vertices $\ell(\tau), t(\tau), r(\tau)$ and the latter (the "lower" triangle) uses the vertices $\ell(\tau), b(\tau), r(\tau)$. Then the resulting combi has as V-edges all edges of T and has as H-edges the diagonals $(\ell(\tau), r(\tau))$ of rhombi τ of T. We refer to such a combi as a *semi-rhombus tiling.*



The above picture illustrates the transformation of a rhombus tiling into a combi; here V-edges (H-edges) are drawn by thick (resp. thin) lines.

In a general case, a combi is a planar tiling K formed by tiles of three sorts: Δ-tiles, ∇-tiles, and lenses. As before, the vertices of K (or G_K) represent subsets of $[n]$.

A Δ-*tile* is an isosceles triangle Δ with vertices A, B, C and edges $(B, A), (C, A), (B, C)$, where (B, C) is an H-edge, while (B, A) and (C, A) are V-edges (so (B, C) is the *base* side and A is the *top* vertex of Δ). We denote Δ as $\Delta(A|BC)$.

A ∇-*tile* is symmetric. It is an isosceles triangle ∇ with vertices A', B', C' and edges $(A', B'), (A', C'), (B', C')$, where (B', C') is an H-edge, while (A', B') and (A', C') are V-edges (so (B', C') is the base side and A' is the *bottom* vertex of ∇). We denote ∇ as $\nabla(A'|B'C')$. The picture illustrates Δ- and ∇-tiles.

We say that a Δ- or ∇-tile τ has type ij if its base edge has this type (and therefore the V-edges of τ have types i and j).

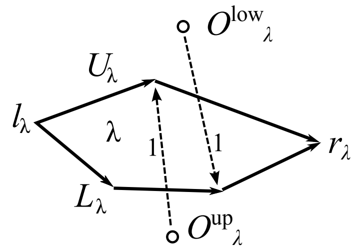
In a *lens* λ, the boundary is formed by two directed paths U_λ and L_λ with at least two edges in each. The paths U_λ and L_λ have the same beginning vertex ℓ_λ and the same end vertex r_λ (called the *left* and *right* vertices of λ, respectively), contain merely H-edges, and form the *upper* and *lower* boundaries of λ, respectively. More precisely,

(3.1) the path $U_\lambda = (\ell_\lambda = X_0, e_1, X_1, \ldots, e_q, X_q = r_\lambda)$ (where $q \geq 2$) is associated with a sequence $i_0 < i_1 < \cdots < i_q$ in $[n]$ so that for $p = 1, \ldots, q$, the edge e_p (going from the vertex X_{p-1} to the vertex X_i) is of type $i_{p-1}i_p$.

This implies that the vertices of U_λ are expressed as $X_p = Xi_p$ for one and the same set $X \subset [n]$ (equal to $X_p \cap X_{p'}$ for any $p \neq p'$). In other words, the point X_p in Z is obtained from X by adding the vector ξ_{i_p}, and all vertices of U_λ lie on the upper half of the circumference of radius 1 centered at X. We call X the *center* of U_λ and denote by O_λ^{up}. In its turn,

(3.2) the path $L_\lambda = (\ell_\lambda = X'_0, e'_1, X'_1, \ldots, e'_{q'}, X'_{q'} = r_\lambda)$ (with $q' \geq 2$) is associated with a sequence $j_0 > j_1 > \cdots > j_{q'}$ in $[n]$ so that for $p = 1, \ldots, q'$, the edge e'_p is of type $j_p j_{p-1}$.

Then the vertices of L_λ are expressed as $X'_p = Y - j_p$ for the same set $Y \subseteq [n]$ (equal to $X'_p \cup X'_{p'}$ for any $p \neq p'$). So the vertices of L_λ lie on the lower half of the circumference of radius 1 centered at Y, called the *center* of L_λ and denoted by O_λ^{low}. See the picture (where $q = 2$ and $q' = 3$):



Note that $\ell_\lambda = Xi_0 = Y - j_0$ and $r_\lambda = Xi_q = Y - j_{q'}$ imply $i_0 = j_{q'}$ and $i_q = j_0$. We say that the lens λ *has type* $i_0 i_q$. The intersection of the circles of radius 1 centered at O^{up}_λ and O^{low}_λ is called the *rounding* of λ and denoted by Ω_λ. Observe that all vertices of λ lie on the boundary of Ω_λ. It is not difficult to realize that none of the vertices of the combi K lies in the interior (i.e., strictly inside) of Ω_λ.

Clearly all vertices of a lens λ have the same size (regarding a vertex as a subset of $[n]$); we call it the *level* of λ. For an H-edge e, consider the pair of tiles in K containing e. Then either both of them are lenses, or one is a lens and the other is a triangle, or both are triangles (Δ- and ∇-tiles); in the last case, we regard e as a *degenerate lens.*

The union of lenses of level h (including degenerate lenses) forms a closed simply connected region meeting $lbd(Z)$ at the vertex $z^\ell_h = [h]$, and $rbd(Z)$ at the vertex $z^r_h = [(n-h+1)..n]$; we call it hth *girdle* and denote by Λ_h. A vertex v in Λ_h having both entering and leaving V-edges is called *critical*; it splits the girdle into two (left and right) closed sets, and the part of Λ_h between two consecutive critical vertices is either a single H-edge or a disk (being the union of some non-degenerate lenses). The region between the upper boundary of Λ_h and the lower boundary of Λ_{h+1} is filled up by triangles; we say that these triangles have level $h + \frac{1}{2}$.

Figure 1 illustrates a combi for $n = 4$ having one lens λ; here the bold (thin) arrows indicate V-edges (resp.H-edges); note that the girdle Λ_2 is the union of the lens λ and the edge $(24, 34)$.

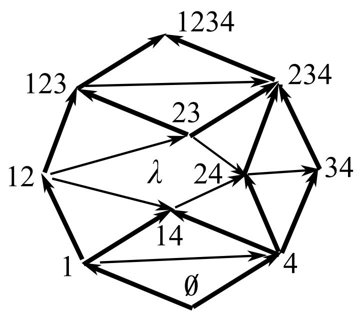


Figure 1: The combi K with $V_K = \{\emptyset, 1, 4, 12, 14, 23, 24, 34, 123, 234, 1234\}$

Remark 2. Fix a girdle Λ_h of K and transform each lens λ in it by drawing the line segment between ℓ_λ and r_λ, thus subdividing λ into two "semi-lenses", the upper and lower ones. Let us color each of the obtained upper semi-lenses in black and each lower semi-lens in white. Then the resulting planar tiling (within the region of Λ_h), in which the tiles are colored black and white and any two tiles sharing an edge have different colors, is viewed as a *plabic tiling.* Recall that plabic tilings are studied in [6] where it is shown, in particular, that the set of vertices of such a tiling represents a weakly separated collection in the discrete Grassmannian Δ_n^h for corresponding n, h. A description of plabic tilings and their properties in more details is beyond our paper.

3.3 Flips in c-tilings

Now our aim is to show that the spectrum (viz. vertex set) V_K of a combi K is a maximal w-collection and that any maximal w-collection is obtained in this way. To show this, we elaborate a technique of flips on combies (which, due to the planarity of a combi, looks simpler than a technique of this sort for g-tilings in [1]).

We will rely on the following two statements.

Proposition 3.2 *Suppose that a combi K contains two vertices of the form X and Xi (where $i \in [n] - X$). Then K has edge from X to Xi.*

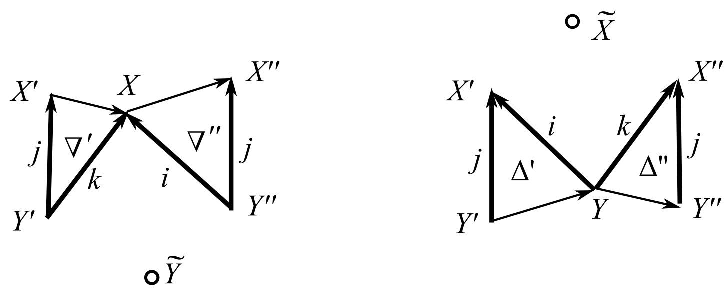
Proposition 3.3 *Suppose that for some $i < j'' \le j' < k$, a combi K contains an H-edge $e = (A, B)$ of type $j'k$ and an H-edge $e' = (B, C)$ of type ij''. Then $j' = j''$ and the edges e, e' belong to either the lower boundary of one lens of K, or two Δ-tiles with the same top vertex, namely $\Delta(D|AB)$ and $\Delta(D|BC)$, where $D = Ak = Bj' = Ci$. Symmetrically, if $e = (A, B)$ has type ij'' and $e' = (B, C)$ has type $j'k$, then $j' = j''$ and e, e' belong to either the upper boundary of one lens of K, or two ∇-tiles with the same bottom vertex, namely $\nabla(D'|AB)$ and $\nabla(D'|BC)$, where $D' = A - i = B - j' = C - k$.*

These propositions are proved in Sect. 4. Assuming their validity, we show the following.

Theorem 3.4 *For a combi K, the spectrum V_K is a maximal w-collection.*

Proof First of all, we introduce W- and M-configurations in K (their counterparts for g-tilings are described in [1, Sec. 4]).

A *W-configuration* is formed by two ∇-tiles $\nabla' = \nabla(Y'|X'X)$ and $\nabla'' = \nabla(Y''|XX'')$ (resembling letter W), as indicated in the left fragment of the picture.



Here for some $i < j < k$, the left edge (Y', X') of ∇' and the right edge (Y'', X'') of ∇'' have type j, the right edge (Y', X) of ∇' has type k, and the left edge (Y'', X) of ∇'' has type i. In other words, letting $\widetilde{Y} := Y' \cap Y''$ (which need not be a vertex of K), we have

$$Y' = \widetilde{Y}i, \quad Y'' = \widetilde{Y}k, \quad X' = \widetilde{Y}ij, \quad X = \widetilde{Y}ik, \quad X'' = \widetilde{Y}jk, \qquad (3.3)$$

and denote such a W-configuration as $W(\widetilde{Y};i,j,k)$.

In its turn, an M-configuration is formed by two Δ-tiles $\Delta'=\Delta(X'|Y'Y)$ and $\Delta''=\Delta(X''|YY'')$ in K (resembling letter M), as indicated in the right fragment of the above picture. Here for $i<j<k$, the V-edges $(Y',X'),(Y,X'),(Y,X''),(Y'',X'')$ have types j,i,k,j, respectively. For $\widetilde{Y}:=Y'\cap Y''$ (as before), the vertices Y',X',Y'',X'' are expressed as in (3.3), and

$$Y=\widetilde{Y}j. \tag{3.4}$$

We denote such an M-configuration as $M(\widetilde{Y};i,j,k)$.

When K has a W-configuration (M-configuration), we can make a *weak lowering* (resp. *raising*) *flip* to transform K into another combi K'. (This is somewhat similar to "strong" flips in pure tilings but now the flip is performed "in the presence of four (rather than six) witnesses", in terminology of Leclerc and Zelevinsky [5], namely the vertices X',X'',Y',Y'' as above.) Roughly speaking, the weak lowering flip applied to $W=W(\widetilde{Y};i,j,k)$ replaces the "middle" vertex $X=\widetilde{Y}ik$ by the new vertex Y as in (3.4), updating the tile structure in a neighborhood of X. In particular, W is replaced by the M-configuration $M(\widetilde{Y};i,j,k)$. Weak raising flips act conversely.

In what follows, speaking of a flip, we default mean a weak flip.

Next we describe the lowering flip for W in detail (using notation as above). Define $\widetilde{X}:=X'\cup X''$ $(=\widetilde{Y}ijk)$. Two cases are possible.

<u>*Case 1*</u>: $\widetilde{X}$ is a vertex of K. Note that $\widetilde{X}=X'k=Xj=X''i$. Therefore, by Proposition 3.2, K has the edges $e'=(X',\widetilde{X})$, $e=(X,\widetilde{X})$, and $e''=(X'',\widetilde{X})$. This implies that K contains the Δ-tiles $\rho':=\Delta(\widetilde{X}|X'X)$ and $\rho'':=\Delta(\widetilde{X}|XX'')$. We replace $\nabla',\nabla'',\rho',\rho''$ by the triangles

$$\Delta(X'|Y'Y),\quad \Delta(X''|YY''),\quad \Delta(\widetilde{X}|X'X''),\quad \nabla(Y|X'X''),$$

as illustrated in Fig. 2.

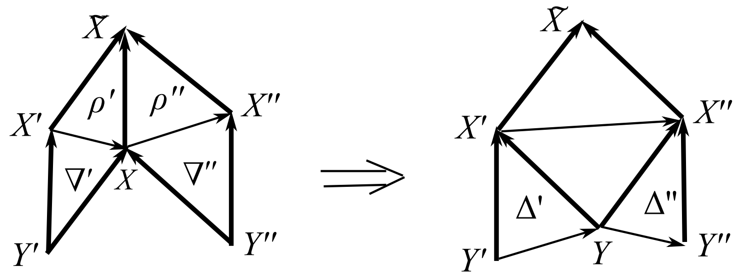


Figure 2: Lowering flip in Case 1

The tile structure of K in a neighborhood of X below the edges (Y',X) and (Y'',X) can be of three possibilities.

<u>*Subcase 1a*</u>: The V-edges (Y',X) and (Y'',X) belong to one and the same Δ-tile $\tau=\Delta(X|Y'Y'')$, and the base (Y',Y'') is shared by τ and a ∇-tile τ'. Since $Y'=\widetilde{Y}i$ and $Y''=\widetilde{Y}k$ (cf. (3.3)), τ' is of the form $\nabla(\widetilde{Y}|Y'Y'')$. We replace τ,τ' by the ∇-tiles

$$\nabla(\widetilde{Y}|Y'Y)\quad\text{and}\quad\nabla(\widetilde{Y}|YY''),$$

as illustrated in the left fragment of Fig. 3.

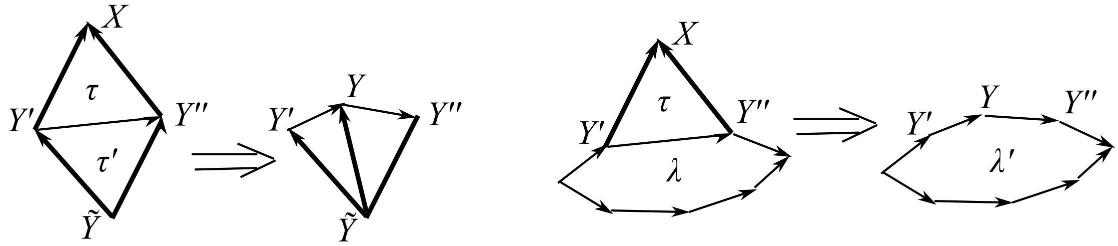


Figure 3: Transformations below X in Subcases 1a (left) and 1b (right)

(Therefore, taking together, the six triangles in the hexagon with the vertices $\widetilde{Y}, Y', X', \widetilde{X}, X'', Y''$ are replaced by another combination of six triangles so as to change the inner vertex X to Y. This matches lowering flips in pure tilings.)

<u>*Subcase 1b*</u>: (Y', X) and (Y'', X) belong to the same Δ-tile $\tau = \Delta(X|Y'Y'')$, but the edge $e = (Y', Y'')$ is shared by τ and a lens λ. Then e belongs to the upper boundary of λ and the center of U_λ is just $\widetilde{Y}$ (since $Y' \cap Y'' = \widetilde{Y}$). We replace the edge e by the path with two edges (Y', Y) and (Y, Y''), thus transforming λ into a larger lens λ' (which is correct since $L_{\lambda'} = L_\lambda$, $|U_{\lambda'}| = |U_\lambda| + 1$, and $Y = \widetilde{Y}j$). The transformation is illustrated in the right fragment of Fig. 3.

<u>*Subcase 1c*</u>: The edges (Y', X) and (Y'', X) belong to different Δ-tiles. Then the "angle" between these edges is filled by a sequence of two or more consecutive Δ-tiles $\Delta_1 = \Delta(X|Y_0Y_1)$, $\Delta_2 = (X|Y_1Y_2), \dots, \Delta_r = (X|Y_{r-1}Y_r)$, where $r \ge 2$, $Y_0 = Y'$, and $Y_r = Y''$. We replace these triangles by one lens λ with U_λ formed by the path $Y'YY''$ and with L_λ formed by the path $Y_0Y_1 \dots Y_r$ (using path notation via vertices). This transformation is illustrated in Fig. 4.

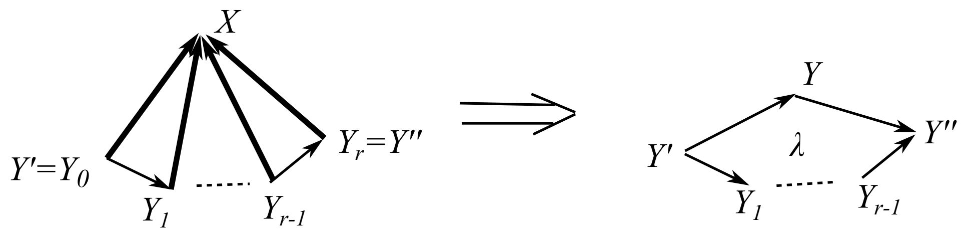


Figure 4: Transformation below X in Subcase 1c

<u>*Case 2*</u>: $\widetilde{X}$ is not a vertex of K. Then the H-edges $e' = (X', X)$ and $e'' = (X, X'')$ belong to the lower boundaries of some lenses λ' and λ'', respectively. Since e' has type jk and e'' has type ij with $i < j < k$, it follows from Proposition 3.3 that $\lambda' = \lambda'' =: \lambda$. Two cases are possible.

<u>*Subcase 2a*</u>: $|L_\lambda| \ge 3$ (i.e., L_λ contains e', e'' and at least one more edge). We replace e', e'' by one H-edge $\widetilde{e} = (X', X'')$, which has type ik. This reduces λ to lens $\widetilde{\lambda}$ with $U_{\widetilde{\lambda}} = U_\lambda$ and $L_{\widetilde{\lambda}} = (L_\lambda - \{e', e''\}) \cup \{\widetilde{e}\}$. (It is indeed a lens since $|L_{\widetilde{\lambda}}| = |L_\lambda| - 1 \ge 2$ and $X' \cup X'' = \widetilde{X}$, thus keeping the lower center: $O^{\rm low}_{\widetilde{\lambda}} = O^{\rm low}_\lambda$.) Also, acting as in

Case 1, we remove the ∇-tiles ∇', ∇'' and add the ∇-tile $\nabla(Y|X'X'')$ and the two Δ-tiles $\Delta(X'|Y'Y)$ and $\Delta(X''|YY'')$. The transformation is illustrated in the left fragment of Fig. 5 (where $|L_\lambda| = 4$ and $|U_\lambda| = 3$).

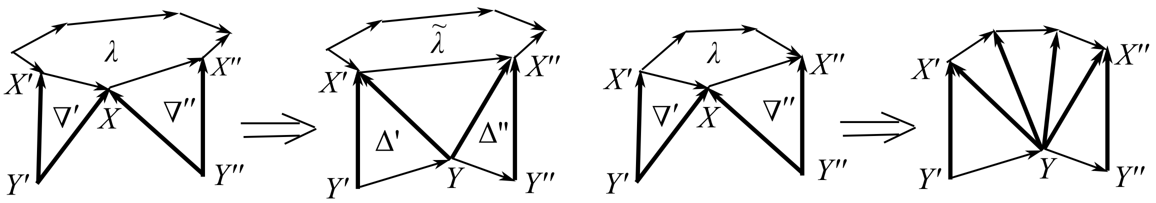


Figure 5: Lowering flip in Subcases 2a (left) and 2b (right)

<u>*Subcase 2b*</u>: $|L_\lambda| = 2$. Then $\ell_\lambda = X'$, $r_\lambda = X''$, and L_λ is formed by the edges e', e''. Let $X_0X_1 \ldots X_q$ be the upper boundary U_λ, where $X_0 = X'$ and $X_q = X''$. We replace λ by the sequence of ∇-tiles $\nabla(Y|X_0X_1), \ldots, \nabla(Y|X_{q-1}X_q)$ (taking into account that $Y = O_\lambda^{\rm up}$). See the right fragment of Fig. 5.

In addition, in both Subcases 2a,2b we should perform an appropriate update of the tile structure in a neighborhood of X below the edges (Y', X) and (Y'', X). This is done in the same way as described in Subcases 1a–1c.

A straightforward verification shows that in all cases we obtain a correct combi K'. As a result of the flip, the vertex $X = \widetilde{Y}ik$ is replaced by $Y = \widetilde{Y}j$, and the sum of the sizes of vertices decreases by 1. The arising Δ-tiles $\Delta(X'|Y'Y)$ and $\Delta(X''|YY'')$ give an M-configuration M in K', and making the corresponding raising flip with M, we return the initial K. (The description of raising flip is "symmetric" to that of lowering flip given above. In fact, a raising flip is equivalent to the corresponding lowering flip in the combi whose vertices are the complements $[n] - X$ of the vertices X of K.)

In what follows, we utilize some results from [1, 2, 5].

(A) Let $\mathcal{F} \subset 2^{[n]}$ be a maximal w-collection. Suppose that for some triple $i < j < k$ in $[n]$ and some set $A \subseteq [n] - \{i, j, k\}$, $\mathcal{F}$ contains the four sets Ai, Ak, Aij, Ajk and one set $B \in \{Aj, Aik\}$ (both Aj and Aik cannot be simultaneously in $\mathcal{F}$ as they are not weakly separated from each other). Then replacing in $\mathcal{F}$ the set B by the other set from $\{Aj, Aik\}$ (a *weak flip* on set-systems) makes a maximal w-collection as well [5].

(B) Let $\mathcal{P} = (\mathbf{W}_n, \prec_w)$ be the poset where $\mathbf{W}_n$ is the set of maximal w-collections for $[n]$ and we write $\mathcal{F} \prec_w \mathcal{F}'$ if $\mathcal{F}'$ can be obtained from $\mathcal{F}$ by a series of (weak) raising flips. Then $\mathcal{P}$ has a unique minimal element and a unique maximal element, which are the set $\mathcal{I}_n$ of intervals and the set co-$\mathcal{I}_n$ of co-intervals in $[n]$, respectively [1, 2].

Now we finish the proof of the theorem as follows. We associate with a combi K the parameter $\eta(K) := \sum(|X| \colon X \in V_K)$. Then a lowering (raising) flip applied to K decreases (resp. increases) η by 1. We assert that if K admits no lowering flip, then $V_K = \mathcal{I}_n$.

Indeed, suppose that K has a (non-degenerate) lens λ. Let λ be chosen so that L_λ is entirely contained in the lower boundary of the girdle Λ_h, where h is the level of λ (see Sect. 3.2); one easily shows that such a λ does exist. Take two consecutive edges e, e' in L_λ. Then e has type jk and e' has type ij for some $i < j < k$. Furthermore,

e, e' are the bases of some ∇-tiles ∇, ∇', respectively (lying in level $h - \frac{1}{2}$). But then ∇, ∇' form a W-configuration. So K admits a lowering flip; a contradiction.

Hence, K is a semi-rhombus tiling. Let T be its underlying rhombus tiling. We know (see Section 3.1) that T admits no strong lowering flip (concerning a hexagon) if and only if $V_T = \mathcal{I}_n$. Since $V_T = V_K$ and a (strong) lowering flip in T is translated as a (weak) lowering flip in K, we obtain the desired assertion.

It follows that starting with an arbitrary combi K and making a finite number of lowering flips, we can always reach the combi K_0 with $V_{K_0} = \mathcal{I}_n$ (taking into account that each application of lowering flips decreases η). Then K is obtained from K_0 by a series of raising flips. Each of such flips changes the spectrum of the current combi in the same way as described for flips concerning w-collections in (A). Thus, V_K is a maximal w-collection.

This completes the proof of the theorem. ∎

A converse property is valid as well.

Theorem 3.5 *For any maximal w-collection $\mathcal{F}$ in $2^{[n]}$, there exists a combi K such that $V_K = \mathcal{F}$. Moreover, such a K is unique.*

Proof To see the existence of K with $V_K = \mathcal{F}$, it suffices to show that flips in combies and in maximal w-collections are consistent (taking into account (A),(B) in the proof of Theorem 3.4). This is provided by the following:

(3.5) for a combi K, if V_K contains the sets Ai, Ak, Aij, Ajk, Aik for some $i < j < k$ and $A \subseteq [n] - \{i, j, k\}$, then K contains the ∇-tiles $\nabla = \nabla(Ai|Aij, Aik)$ and $\nabla' = \nabla(Ak|Aik, Ajk)$.

(So ∇, ∇' form a W-configuration and we can make a lowering flip in K to obtain a combi K' with $V_{K'} = (V_K - \{Aik\}) \cup \{Aj\}$. This matches the flip $Aik \rightsquigarrow Aj$ in the corresponding w-collection. The assertion on raising flips is symmetric.)

Indeed, Proposition 3.2 ensures the existence of V-edges $e_1 = (Ai, Aij)$, $e_2 = (Ai, Aik)$, $e_3 = (Ak, Aik)$, $e_4 = (Ak, Ajk)$ in K. Obviously, the angle between the edges e_1 and e_2 is covered by one or more ∇-tiles; let $\widetilde{\nabla} = \nabla(Ai|Aij', Aik)$ be the rightmost tile among them (namely, the one containing the edge e_2). Similarly, let $\widetilde{\nabla}' = \nabla(Ak|Aik, Aj''k)$ be the leftmost tile among the ∇-tiles lying between the edges e_3 and e_4. Clearly $j \le j' < k$ and $i < j'' \le j$. By Proposition 3.3, $j' = j = j''$. Therefore, $\widetilde{\nabla} = \nabla$ and $\widetilde{\nabla}' = \nabla'$, as required in (3.5).

Finally, K is determined by its spectrum V_K (yielding the required uniqueness). Indeed, using Proposition 3.2, we can uniquely restore the V-edges of K. This determines the set of Δ- and ∇-tiles of K. Now for $h = 1, \ldots, n-1$, consider the region (girdle) Λ_h in Z bounded from below by the directed path L_h formed by the base edges of ∇-tiles of level $h - \frac{1}{2}$, and bounded from above by the directed path U_h formed by the base edges of Δ-tiles of level $h + \frac{1}{2}$. Then Λ_h is the union of lenses of level h, and we have to show that these lenses are restored uniquely.

It suffices to show this for the disk D in Λ_h between two consecutive critical vertices u, v (see Sect. 3.2). The lower boundary L_D of D is the part of L_h beginning at u and ending at v; let $L_D = (u = X_0, e_1, X_1, \dots, e_r, X_r = v)$. Each edge $e_p = (X_{p-1}, X_p)$ belongs to the lower boundary of some lens λ with center $O_\lambda^{\rm low} = X_{p-1} \cup X_p$ (and we have $X_{p-1} = O_\lambda^{\rm low} - k$ and $X_p = O_\lambda^{\rm low} - j$ for some $j < k$). From Proposition 3.3 it follows that consecutive edges e_p, e_{p-1} belong to the same lens if and only if $X_{p-1} \cup X_p = X_p \cup X_{p+1}$. Also there exists a lens λ such that L_λ is entirely contained in L_D. Such a λ is characterized by the following conditions:

(i) L_λ is a maximal part $X_p X_{p+1} \dots X_q$ of L_D satisfying $q \ge p+2$ and $X_p \cup X_{p+1} = \dots = X_{q-1} \cup X_q$, and

(ii) there exists a vertex $Y \ne X_p, X_q$ in D such that $X_p \cap Y = Y \cap X_q = X_p \cap X_q$.

In this case, $\ell_\lambda = X_p$, $r_\lambda = X_q$, $L_\lambda = X_p X_{p+1} \dots X_q$, and U_λ is formed by X_p, X_q and all those vertices Y that satisfy (ii). (Conditions (i),(ii) are necessary for a lens λ with $L_\lambda \subseteq L_D$. To see the sufficiency, suppose that $L_\lambda \cap L_D = X_p X_{p+1} \dots X_q$ but $L_\lambda \not\subseteq L_D$, i.e., either $\ell_\lambda \ne X_p$ or $r_\lambda \ne X_q$ or both. Consider the rounding Ω_λ of λ (see Sect. 3.2). One can realize that a point Y as in (ii) is located in the *interior* of Ω_λ, which is impossible.)

Once a lens λ satisfying (i),(ii) is chosen, we "remove" it from D and repeat the procedure with the lower boundary of the updated (reduced) D, and so on. Upon termination of the process for all h, we obtain the list of lenses of K, and this list is constructed uniquely. ∎

4 Proofs

In this section, we prove the assertions stated but left unproved in Sects. 2 and 3, namely Propositions 2.1, 3.2, 3.3 and Theorem 2.2.

We will use a technique of contractions and expansions which transform combies on Z_n into ones on Z_{n-1}, and back. These are analogous to those developed for g-tilings in [1, Sec. 8] (see also [2, Sec. 6]), but are arranged simpler because of the planarity of objects we deal with.

4.1 n-contraction

The n-contraction operation is, in fact, well known and rather transparent when we deal with a rhombus tiling T on the zonogon $Z = Z_n$. In this case, we take the sequence Q of rhombi of type $*n$ (where $*$ means any number i different from n) in which the first rhombus contains the first edge (having type n) in $rbd(Z)$, the last rhombus contains the last edge in $\ell bd(Z)$, and each pair of consecutive rhombi shares an edge of type n. The operation consists in shrinking each rhombus $\tau = \tau(X; i, n)$ in Q into the only edge (X, Xi). Under this operation, the right boundary of Q is shifted by $-\xi_n$, getting merged with the left one, each vertex Y of T containing the element n (i.e., Y lies on the right from Q) turns into $Y - n$, and the resulting set T' of rhombi forms a correct rhombus tiling on the zonogon Z_{n-1} (generated by $\xi_1, \dots, \xi_{n-1}$). This T' is called the *n-contraction* of T.

In case of c-tilings, the n-contraction operation becomes less trivial since, besides triangles ("semi-rhombi"), it should involve lenses of type $*n$. Below we describe this in detail.

Consider a combi K on Z. Let $\mathcal{T}^n$ be the set of tiles of types $*n$ in K; it consists of the Δ-tiles $\Delta(A|BC)$ whose left edge (B,A) is of type n, the ∇-tiles $\nabla(A'|B'C')$ whose right edge (A',C') is of type n, and the lenses λ of type $*n$. Note that the first edge of L_λ has type jn, and the last edge of U_λ has type $j'n$ for some j,j'.

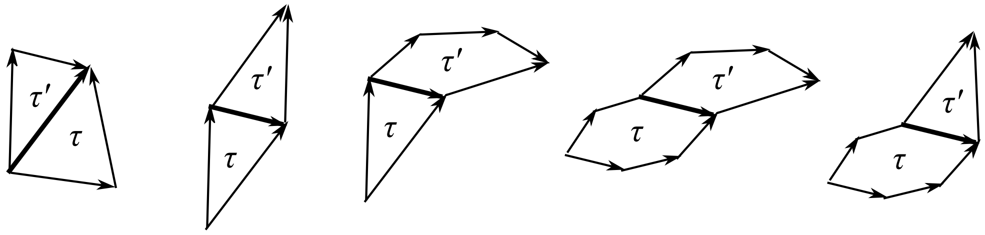
Let E^n denote the set of edges of type n or $*n$ in K.

By an *n-strip*, we mean a maximal sequence $Q=(\tau_0,\tau_1,\ldots,\tau_N)$ of tiles in $\mathcal{T}^n$ such that for any two consecutive $\tau=\tau_{p-1}$ and $\tau'=\tau_p$, their intersection $\tau\cap\tau'$ consists of an edge $e\in E^n$, and:

(i) if e has type n, then τ is a Δ-tile and τ' is a ∇-tile;

(ii) if e has type $*n$, then τ lies below e and τ' lies above e.

In case (ii), there are four possibilities: (a) τ is a ∇-tile, τ' is a Δ-tile, and e is their common base; (b) τ is a ∇-tile, τ' is a lens, and e is the base of τ and the first edge of $L_{\tau'}$; (c) τ,τ' are lenses, and e is the last edge of U_τ and the first edge of $L_{\tau'}$; and (d) τ is a lens, τ' is a Δ-tile, and e is the last edge of U_τ and the base of τ'. The possible cases of τ,τ' are illustrated in the picture, where their common edge e is drawn bold.



The following property is useful:

(4.1) there is only one n-strip Q as above; each tile of $\mathcal{T}^n$ occurs in Q exactly once; Q begins with the ∇-tile in K containing the first edge $e_1^r=(z_0^r,z_1^r)$ of $rbd(Z)$ and ends with the Δ-tile in K containing the last edge $e_n^\ell=(z_{n-1}^\ell,z_n^\ell)$ of $lbd(Z)$.

This follows from three facts: (i) Each tile in $\mathcal{T}^n$ contains exactly two edges from E^n; (ii) each edge in E^n belongs to exactly two tiles from $\mathcal{T}^n$, except for the edges e_1^r and e_n^ℓ, which belong to one tile each; and (iii) Q cannot be cyclic. To see validity of (iii), observe that if τ_{p-1},τ_p are triangles, then either the level of τ_p is greater than that of τ_{p-1} (when τ_{p-1} is a ∇-tile), or the level of the *base* of τ_p is greater than that of τ_{p-1} (when τ_{p-1} is a Δ-tile). As to traversing across lenses, we rely on the fact that the relation: A lens λ is "higher" than a lens λ' if L_λ and $U_{\lambda'}$ share an edge, induces a poset on the set of all lenses in K (which in turn appeals to evident topological reasonings using the facts that the lenses are convex and all H-edges are "directed to the right").

Note that for each tile $\tau\in\mathcal{T}^n$ and its edges $e,e'\in E^n$, $bd(\tau)-\{e,e'\}$ consists of two directed paths, *left* and *right* ones (of which one is a single vertex when τ is a triangle). The concatenation of the left (resp. right) paths along Q gives a directed path from $z_0^r=(0,0)$ to z_{n-1}^ℓ (resp. from z_1^r to $z_n^\ell=z_n^r$); we call this path the *left* (resp. *right*) *boundary* of the strip Q and denote as $P^{\rm left}$ (resp. $P^{\rm right}$).

Let Z^{left} (Z^{right}) be the region in Z bounded by P^{left} and the part of $lbd(Z)$ from $(0,0)$ to z^{ℓ}_{n-1} (resp. bounded by P^{right} and the part of $rbd(Z)$ from z^r_1 to z^r_n). Accordingly, the graph $(V_K, E_K - E^n)$ consists of two connected components, the *left subgraph* G^{left} and the *right subgraph* G^{right} (lying in Z^{left} and Z^{right}, respectively), and we denote by K^{left} and K^{right} the corresponding subtilings in $K - Q$. It is easy to see that $n \notin X$ (resp. $n \in X$) holds for each vertex X of G^{left} (resp. G^{right}).

Figure 6 illustrates the construction of $P^{\text{left}}, P^{\text{right}}, G^{\text{left}}, G^{\text{right}}$ for the combi K with $n = 4$ from Fig. 1; here the edges of types 4 and $*4$ in K are drawn by dotted lines.

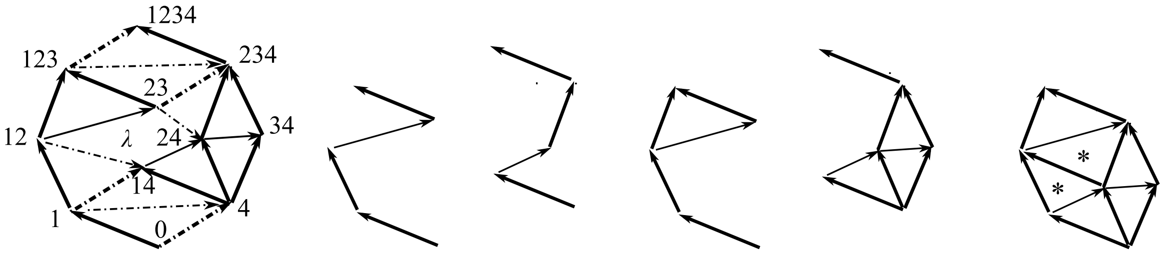


Figure 6: From left to right: K; P^{left}; P^{right}; G^{left}; G^{right}; K'

The *n-contraction operation* applied to K makes the following.

(Q1) The region Z^{left} preserves, while Z^{right} is shifted by the vector $-\xi_n$. As a result, the right boundary of the shifted Z^{right} becomes the right boundary of the zonogon Z_{n-1} (while the left boundary of Z^{left} coincides with $lbd(Z_{n-1})$). Accordingly, the subtiling K^{left} preserves, and K^{right} is shifted so that each vertex X becomes $X - n$.

(Q2) The edges in E^n and the triangles in $\mathcal{T}^n$ vanish.

(Q3) The lenses λ in $\mathcal{T}^n$ are transformed as follows. Let $U_\lambda = X_0X_1\dots X_p$ and $L_\lambda = Y_0Y_1\dots Y_q$ (using path notation via vertices); so $X_0 = Y_0 = \ell_\lambda$, $X_p = Y_q = r_\lambda$, and $(Y_0, Y_1), (X_{p-1}, X_p) \in E^n$. Under the above shift, each vertex Y_d with $d > 0$ becomes $Y'_d := Y_d - n$. Since $r_\lambda = O^{\text{up}}_\lambda + \xi_n$ and $Y'_q = Y_q - n$, the point Y'_q coincides with O^{up}_λ; hence Y'_q becomes the center for $X_0, \dots, X_{p-1}$. In its turn, the equality $\ell_\lambda = O^{\text{low}}_\lambda - \xi_n$ implies that X_0 becomes the center for $Y'_1, \dots, Y'_q$. We fill the space between the paths $X_0 \dots X_{p-1}$ and $Y'_1 \dots Y'_q$ with new Δ-tiles $\Delta(X_0|Y'_1Y'_2), \dots, \Delta(X_0|Y'_{q-1}Y'_q)$ and new ∇-tiles $\nabla(Y'_q|X_0X_1), \dots, \nabla(Y'_q|X_{p-2}X_{p-1})$.

The transformation of a lens with $p = 4$ and $q = 3$ is illustrated in Fig. 7.

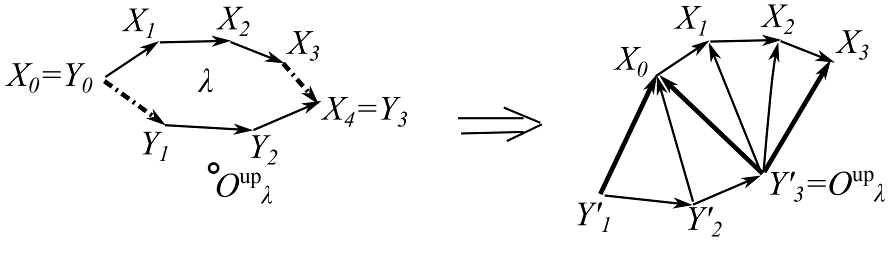


Figure 7: Contraction on a lens

We associate with the lens λ as above in K the zigzag path $Y'_1X_0Y'_qX_{p-1}$, denoted as P_λ, in the resulting object (it is drawn in bold in the right fragment of Fig. 7). One may say that the transformation described in (Q3) "replaces the lens λ by the zigzag path P_λ with two inscribed fillings" (consisting of Δ- and ∇-tiles), and we refer to such a transformation as a *lens reduction*. (As one more illustration, observe that the lens λ with $p = q = 2$ from the leftmost fragment of Fig. 6 turns, under the lens reduction, into two triangles marked by $*$ in the rightmost fragment of that figure.)

Let K' be the set of tiles occurring in $K^{\rm left}$ and in the shifted $K^{\rm right}$ plus those Δ- and ∇-tiles that arise instead of the lenses in $\mathcal{T}^n$ as described in (Q3). We call K' the *n-contraction* of K and denote it as K/n. Analyzing the above description, one can conclude with the following

Proposition 4.1 *K/n is a correct combi on the zonogon Z_{n-1}.*

Acting in a similar fashion w.r.t. the set $\mathcal{T}^1$ of tiles of types $1*$ in K, one can construct the corresponding *1-contraction* $K/1$ of K, which is a combi on the $(n-1)$-zonogon generated by the vectors $\xi_2, \ldots, \xi_n$. This corresponds to the n-contraction of the combi $\widehat{K}$ that is the mirror reflection of K w.r.t. the vertical axis. (More precisely, $\widehat{K}$ is obtained from K by changing each generator $\xi_i = (x_i, y_i)$ to $\widehat{\xi}_{n-i+1} = (-x_i, y_i)$.)

4.2 n-expansion

Now we describe a converse operation, called the *n-expansion one*. It is easy when we deal with a rhombus tiling T' on the zonogon $Z' = Z_{n-1}$. In this case, we take a *directed* path P in $G_{T'}$ going from the bottom $(0,0)$ to the top z_{n-1} of Z'. Such a P splits Z' and T' into the corresponding left and right parts. We shift the right region of Z' by the vector ξ_n (and shift the right subtiling of T' by adding the element n to its vertices) and then fill the space between P and its shifted copy with corresponding rhombi of type $*n$. This results in a corrected rhombus tiling T on the zonogon Z_n in which the new rhombi form its n-strip, and the n-contraction operation applied to T returns T'.

In case of c-tilings, the operation becomes somewhat more involved since now we should deal with a path P in Z' which is not necessarily directed.

More precisely, we consider a combi K' on Z' and a simple path $P = (A_0, e_1, A_1, \ldots, e_M, A_M)$ in the graph $G_{K'}$ such that:

(P1) P goes from the bottom to the top of Z', i.e., $A_0 = \emptyset$ and $A_M = [n-1]$, and all edges of P are V-edges;

(P2) for any two consecutive edges of P, at least one is a forward edge, i.e., $|A_{d-1}| > |A_d|$ implies $|A_d| < |A_{d+1}|$;

(P3) any zigzag subpath in P goes to the right; in other words, if $|A_{d-1}| = |A_{d+1}| \neq |A_d|$, then either $A_{d-1} = A_d\, i$ and $A_{d+1} = A_d\, j$ for some $i < j$, or $A_{d-1} = A_d - i'$ and $A_{d+1} = A_d - j'$ for some $i' > j'$.

Borrowing terminology in [1, Sec. 8], we call such a P a *legal* path for K'. It splits Z' into two closed simply connected regions R_1, R_2 (the *left* and *right* ones, respectively), where $R_1 \cup R_2 = Z'$, $R_1 \cap R_2 = P$, R_1 contains $lbd(Z')$, and R_2 contains $rbd(Z')$. Let G_i and K_i denote, respectively, the subgraph of $G_{K'}$ and the subtiling of K' contained in R_i, $i = 1, 2$.

We call a vertex A_d of P a *slope* if $|A_{d-1}| < |A_d| < |A_{d+1}|$, a *peak* if $|A_{d-1}| = |A_{d+1}| < |A_d|$, and a *pit* if $|A_{d-1}| = |A_{d+1}| > |A_d|$. (By (P2), the case $|A_{d-1}| > |A_d| > |A_{d+1}|$ is impossible.) When A_d is a peak, the angle between the edges e_d and e_{d+1} is covered by a sequence of Δ-tiles $\Delta(A_d|Y_{r-1}Y_r)$, $r = 1, \dots, q$, ordered from left to right, where $Y_0 = A_{d-1}$ and $Y_q = A_{d+1}$. We call this sequence the *(lower) filling* at A_d w.r.t. P. Similarly, when A_d is a pit, the angle between e_d and e_{d+1} is covered by a sequence of ∇-tiles $\nabla(A_d|X_{r-1}X_r)$, $r = 1, \dots, p$, called the *(upper) filling* at A_d w.r.t. P.

The *n-expansion operation* applied to K' and P constructs a combi K on Z_n as follows.

(E1) K inherits all tiles of K_1 except for those in the fillings of pits of P. For each tile τ of K_2 not contained in the filling of any peak of P, K receives the shifted tile $\tau + \xi_n$. Accordingly, the vertex set of K consists of the vertices of G_1 except for the pits of P, and the vertices of the form Xn for all vertices X of G_2 except for the peaks of P. In particular, each slope X of P (and only these vertices of K') creates two vertices in K, namely X and Xn.

(E2) Each slope A_d creates two additional tiles in K: the ∇-tile $\nabla(A_d|A_{d+1}, A_d\, n)$ of type in and the Δ-tile $\Delta(A_d\, n|A_d, A_{d-1}\, n)$ of type jn, where i, j are the types of the edges (A_d, A_{d+1}) and (A_{d-1}, A_d), respectively. See the left fragment of Fig. 8.

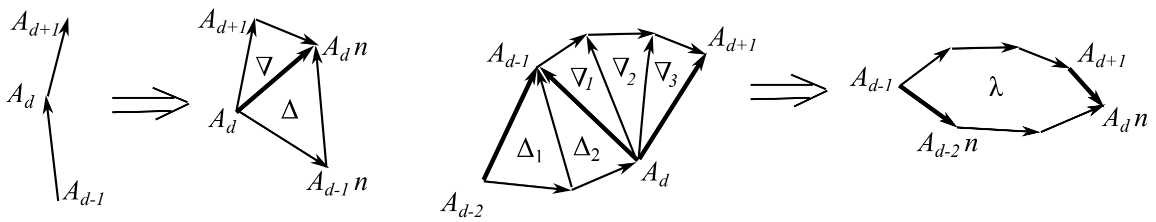


Figure 8: n-expansion in a neighborhood of a slope (left) and a zigzag (right)

In addition, the first and last vertices of P create two extra tilings: $\nabla(A_0 = \emptyset|A_1, A_0\, n = z_1^r)$ and $\Delta(A_M\, n = [n]|A_M, A_{M-1}\, n)$.

(E3) Each backward edge $e_d = (A_d, A_{d-1})$ of P (equivalently, each peak-pit pair A_{d-1}, A_d) creates a new lens λ in K by the following rule. Let $\Delta(A_{d-1}|Y_{r-1}Y_r)$, $r = 1, \dots, q$, be the filling at A_{d-1} w.r.t. P, and let $\nabla(A_d|X_{r-1}X_r)$, $r = 1, \dots, p$, be the filling at A_d (so $Y_0 = A_{d-2}$, $Y_q = A_d$, $X_0 = A_{d-1}$, and $X_p = A_{d+1}$). Then λ is such that: $\ell_\lambda = A_{d-1}$, $r_\lambda = A_d\, n$, the upper boundary U_λ has the vertex sequence $X_0, X_1, \dots, X_p, A_d\, n$, and the lower boundary L_λ the sequence $A_{d-1}, Y_0\, n, Y_1\, n, \dots, Y_q\, n$. See the right fragment of Fig. 8 (where $p = 3$ and $q = 2$). We refer to the transformation in (E3) as a *lens creation*; this is converse to a lens reduction described in (Q3) of Sect. 4.1.

One can check that K obtained in this way is indeed a correct combi on Z_n. It is called the *n-expansion* of K' w.r.t. the legal path P. The corresponding sequence Q of tiles of types $*n$ in K is just formed by the Δ- and ∇-tiles induced by the slopes of P (together with A_0, A_M) as described in (E2), and by the lenses induced by the backward edges (or the three-edge zigzags) of P as described in (E3).

A straightforward examination shows that the n-contraction operation applied to

K returns the initial K' (and the legal path P in it is obtained by a natural deformation of the left boundary of Q); details are left to the reader. As a result, we conclude with the following.

Theorem 4.2 *The correspondence $(K', P) \mapsto K$, where K' is a c-tiling on Z_{n-1}, P is a legal path for K', and K is the n-expansion of K' w.r.t. P, gives a bijection between the set of such pairs (K', P) and the set of c-tilings on Z_n. Under this correspondence, K' is the n-contraction K/n of K.*

Note that when we consider in $G_{K'}$ a path P' defined similarly to P with the only difference (in (P3)) that any zigzag subpath in P' goes *to the left*, then, duly modifying the expansion operation described in (E1)–(E3), we obtain what is called the *1-expansion* of K'; this gives an analog of Theorem 4.2 concerning type 1. (Again, to clarify the construction we can make the mirror reflection w.r.t. the vertical axis.)

4.3 Proofs

We first prove Propositions 3.2 and 3.3, thus completing the proof of Theorem 3.4.

Proof of Proposition 3.2. Let X and Xi be vertices of a combi K on the zonogon Z_n. We use induction on n and, assuming w.l.o.g. that $i \ne n$, consider the n-contraction $K' = K/n$ of K (in case $i = n$, we should consider the 1-contraction $K/1$ and argue in a similar way). Also we use terminology, notation and constructions from the previous subsections. Two cases are possible.

(i) Let $n \notin X$. Then X and Xi belong to the left subgraph $G^{\rm left}$ of G_K (w.r.t. the n-strip Q). Hence X, Xi are vertices of K', and by induction K' has edge $e = (X, Xi)$ (which is a V-edge). Let P be the legal path in K' such that the n-expansion of K' w.r.t. P gives K (i.e., P is the "image" of the strip Q in K'). Then e is a V-edge of the left subgraph G_1 of $G_{K'}$ (w.r.t. P). The only situation when e might be destroyed under the n-expansion operation is that e belongs to a ∇-tile in the filling at some pit A_d of P, implying $X = A_d$. But this is impossible since A_d induces only one vertex $A_d n$ in K.

(ii) Let $n \in X$. Then X, Xi belong to $G^{\rm right}$. So K' has vertices $X' = X - n$ and $X'' = Xi - n$, and by induction K' has V-edge $e = (X', X'')$. For the corresponding legal path P in K', the edge e could not generate the edge $(X'n, X''n) = (X, Xi)$ in K only if e belongs to a Δ-tile in the filling at some peak A_d of P, implying $X'' = A_d$. But A_d induces only one vertex A_d in K (instead of the required vertex $A_d n = Xi$); a contradiction.

Thus, in all cases, (X, Xi) is an edge of K, and we are done. ∎

Proof of Proposition 3.3. Let $e = (A, B)$, $e' = (B, C)$, and $i < j'' \le j' < k$ be as in the hypotheses of the first statement in this proposition. Consider two cases.

<u>*Case 1*</u>. Suppose that the vertex B has at least one outgoing V-edge. Let $\widetilde{e} = (B, D)$ and $\widetilde{e}' = (B, D')$ be the leftmost and rightmost edges among such V-edges, respectively. Then there exist Δ-tiles of the form $\Delta := \Delta(D|A'B)$ and $\Delta' := \Delta(D'|BC')$ (i.e., lying on the left of $\widetilde{e}$ and on the right of $\widetilde{e}'$, respectively). Let the V-edges

$(A', D), (B, D), (B, D'), (C', D')$ be of types $\widetilde{k}, \widetilde{j}', \widetilde{j}'', \widetilde{i}$, respectively. By the choice of $\widetilde{e}$ and $\widetilde{e}'$, we have $\widetilde{j}' \le \widetilde{j}''$.

Next, the base (A', B) of Δ has type $\widetilde{j}'\widetilde{k}$ and, by the planarity of K, should lie above the edge (A, B) (admitting the equality $(A', B) = (A, B)$). This implies $j' \le \widetilde{j}'$ (and $k \le \widetilde{k}$), by comparing Δ with the abstract Δ-tile with the base (A, B), which, obviously, has type $j'k$. For a similar reason, since the base (B, C') of Δ' lies above (B, C), we have $\widetilde{j}'' \le j''$. Therefore, $\widetilde{j}'' \le j'' \le j' \le \widetilde{j}' \le \widetilde{j}''$. This gives equality throughout, implying that $\Delta = \Delta(D|AB)$ and $\Delta' = \Delta(D|BC)$, as required.

<u>*Case 2*</u>. Let B have no outgoing V-edges. Then B is an intermediate vertex in the lower boundary of some lens λ. Let $\widetilde{e} = (A', B)$ and $\widetilde{e}' = (B, C')$ be the edges of L_λ entering and leaving B, respectively. Then $\widetilde{e}, \widetilde{e}'$ have types $\widetilde{j}\widetilde{k}$ and $\widetilde{i}\widetilde{j}$ for some $\widetilde{i} < \widetilde{j} < \widetilde{k}$. Since the edge $\widetilde{e}$ lies above e, and $\widetilde{e}'$ lies above e' (admitting equalities), we have $j' \le \widetilde{j} \le j''$. This together with $j'' \le j'$ gives $e = \widetilde{e}$ and $e' = \widetilde{e}'$, and the result follows.

The second assertion in the proposition is symmetric. ∎

This completes the proof of Theorem 3.4.

Now we return to a simple cyclic pattern $\mathcal{S} = (S_1, \dots, S_r = S_0)$, i.e., such that $|S_{i-1} \triangle S_i| = 1$ for all i, and the sets S_i are different and weakly separated from each other (conditions (C1),(C2) in Sect. 2). By Theorem 3.5, $\mathcal{S}$ is included in the spectrum (vertex set) V_K of some combi K on Z_n. Proposition 3.2 implies that each pair of consecutive vertices S_{i-1}, S_i is connected in G_K by a V-edge directed from the smaller set to the bigger one. This gives the corresponding cycle in G_K; we identify it with the curve $\zeta_\mathcal{S}$ in Z_n (defined in Sect. 2). The planarity of G_K implies that $\zeta_\mathcal{S}$ is non-self-intersecting, as required in Proposition 2.1.

Next we finish the proof of Theorem 2.2, as follows. Consider arbitrary $X \in \mathcal{D}_\mathcal{S}^{\rm in}$ and $Y \in \mathcal{D}_\mathcal{S}^{\rm out}$. Since $\mathcal{S} \cup \{X\}$ is weakly separated, it is included in the spectrum V_K of some combi K, by Theorem 3.5. Let $K^{\rm in}$ be the subtiling of K formed by the tiles occurring in $R_\mathcal{S}^{\rm in}$. Then X is a vertex of $K^{\rm in}$. In its turn, $\mathcal{S} \cup \{Y\} \subseteq V_{K'}$ for some combi K', and Y is a vertex of the subtiling $K'^{\rm out}$ of K' formed by the tiles occurring in $R_\mathcal{S}^{\rm out}$ (or Y is a vertex in $bd(Z_n)$). Since $R_\mathcal{S}^{\rm in} \cap R_\mathcal{S}^{\rm out} = \zeta_\mathcal{S}$, the union $K^{\rm in} \cup K'^{\rm out}$ gives a correct combi $\widetilde{K}$ on Z_n containing both X and Y. Since $V_{\widetilde{K}}$ is a w-collection (by Theorem 3.4), we have $X \text{ (weak) } Y$, as required. Therefore, $\mathcal{D}_\mathcal{S}^{\rm in}, \mathcal{D}_\mathcal{S}^{\rm out}$ form a complementary pair (their union $\mathcal{D}$ is w-pure). This implies the w-purity of $\mathcal{D}_\mathcal{S}^{\rm in}$ and $\mathcal{D}_\mathcal{S}^{\rm out}$, by Proposition 1.2. ∎

The strong separation counterpart of this theorem, namely Theorem 2.3, is obtained in a similar way. Given a cyclic pattern $\mathcal{S}$ consisting of different strongly separated sets, the result immediately follows from Theorem 3.1 and the fact that any rhombus tiling on $R_\mathcal{S}^{\rm in}$ and any rhombus tiling on $R_\mathcal{S}^{\rm out}$ can be combined to form a rhombus tiling on Z_n. The equality $r^s(\mathcal{D}) = r^w(\mathcal{D})$ is obvious, where $\mathcal{D} = \widehat{\mathcal{D}}_\mathcal{S}^{\rm in}$ or $\widehat{\mathcal{D}}_\mathcal{S}^{\rm out}$.

Remark 3. Using combies, one can essentially simplify the proof of the fundamental property that $2^{[n]}$ is w-pure, which is given in [2] by use of generalized tilings. On this way, we first weaken the statement of Theorem 3.4 as follows: for a combi K, the spectrum V_K is a *largest* w-collection (i.e., it has the maximal possible size $n(n+1)/2+1$). This is proved just as in Sect. 3.3 but without appealing to results in [2]. Second, as is explained in [2, Sec. 4], the assertion that any maximal w-collection is largest is reduced to the following: *the partial order* $\prec^*$ *on a largest w-collection* $\mathcal{F}$*, given by* $A \prec^* B \Leftrightarrow (A \lessdot B \;\&\; |A| \le |B|)$*, forms a lattice*, where $\lessdot$ is defined in (1.1)(iii). The core of the whole proof consists in showing that the partial order $(\mathcal{F}, \prec^*)$ coincides with the natural partial order on the vertices of the combi K associated with $\mathcal{F}$ (where a vertex u is less than v if there is a directed path from u to v in G_K); cf. Theorem 6.1 in [2]. This is proved by induction on n, using a technique of n-contractions and n-expansions on combies (which is simpler than an analogous technique for g-tilings). We omit details here.

5 Special cases and generalizations

In this section, we discuss possible ways to extend the w-purity result to more general patterns. Along the way we demonstrate some representative special cases.

5.1 Semi-simple cyclic patterns

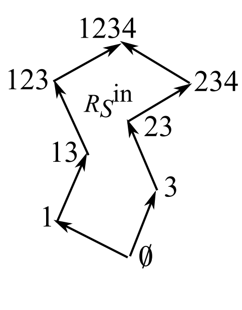
We can slightly generalize Theorem 2.2 by weakening condition (C1) in Section 2. Now we admit, with a due care, cyclic patterns $\mathcal{S}$ having some repeated elements $C_i = C_j$. More precisely, in the corresponding closed curve $\zeta_{\mathcal{S}}$ in $Z = Z_n$ we allow only touchings but not crossings, which is equivalent to saying that under a "very small" deformation the curve becomes non-self-intersecting. We refer to $\mathcal{S}$ satisfying this requirement and condition (C2) as *semi-simple*. The definitions of regions $R^{\rm in}_{\mathcal{S}}, R^{\rm out}_{\mathcal{S}}$ and domains $\mathcal{D}^{\rm in}_{\mathcal{S}}, \mathcal{D}^{\rm out}_{\mathcal{S}}$ are modified in a natural way. The desired generalization reads as follows:

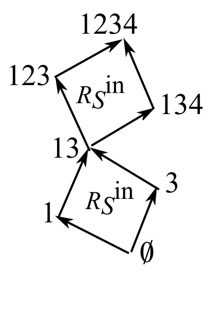
($*$) *For a semi-simple cyclic pattern* $\mathcal{S}$*, the domains* $\mathcal{D}^{\rm in}_{\mathcal{S}}$ *and* $\mathcal{D}^{\rm out}_{\mathcal{S}}$ *form a complementary pair; as a consequence, both* $\mathcal{D}^{\rm in}_{\mathcal{S}}$ *and* $\mathcal{D}^{\rm out}_{\mathcal{S}}$ *are w-pure.*

This is shown in a similar way as Theorem 2.2 and we omit a proof here.

One can see that domains of types $\mathcal{D}(\omega)$ and $\mathcal{D}(\omega', \omega)$ exposed in Theorem 1.1(i),(ii) are representable as $\mathcal{D}^{\rm in}_{\mathcal{S}}$ for special simple or semi-simple cyclic patterns $\mathcal{S}$.

In particular, for the domain $\mathcal{D} = \mathcal{D}(\omega', \omega)$ with permutations $\omega', \omega : [n] \to [n]$ such that ${\rm Inv}(\omega') \subset {\rm Inv}(\omega)$, the cyclic pattern $\mathcal{S}$ involves $\emptyset$, $[n]$, and the sets $\omega'^{-1}([i])$ and $\omega^{-1}([i])$ for $i = 1, \ldots, n-1$; such an $\mathcal{S}$ is simple if $\omega'^{-1}([i]) \ne \omega^{-1}([i])$ for all i, and semi-simple otherwise. The picture below illustrates $\mathcal{S}$ for $\mathcal{D}(\omega', \omega)$ with $n = 4$ in the case $\omega' = 1324$ and $\omega = 3241$ (left), and in the case $\omega' = 1324$ and $\omega = 3142$ (right).





5.2 Generalized cyclic patterns

Next we discuss another, more important way to extend the obtained w-purity results, namely, we consider a *generalized* cyclic pattern $\mathcal{S} = (S_1, \ldots, S_r = S_0)$ in $2^{[n]}$ (defined in Sect. 1). As before, $\mathcal{S}$ obeys (C1) and (C2), and now we assume that

(5.1) for each $p = 1, \ldots, r$, either $|S_{p-1} \triangle S_p| = 1$, or $|S_{p-1} \triangle S_p| = 2$ and $|S_{p-1}| = |S_p|$.

In the former (latter) case, we say that $\{S_{p-1}, S_p\}$ is a *1-distance pair* (resp. *2-distance pair*) in $\mathcal{S}$. In the latter case, $S_{p-1} - S_p$ and $S_p - S_{p-1}$ consist of some singletons i and j, respectively, and setting $X := S_{p-1} \cap S_p$ and $Y := S_{p-1} \cup S_p$, we have

$$S_{p-1} = Xi = Y - j \quad \text{and} \quad S_p = Xj = Y - i.$$

We rely on the following assertion. (A property of a somewhat similar flavor for plabic graphs is established in [6].)

Proposition 5.1 *Suppose that a combi K has two vertices of the form $A = Xi$ and $B = Xj$. Then at least one of the following takes place:*

(i) *K contains the vertex X (and therefore the edges (X, A) and (X, B));*

(ii) *K contains the vertex Xij (and therefore the edges (A, Xij) and (B, Xij));*

(iii) *both A, B belong to the lower boundary of some lens in K;*

(iv) *both A, B belong to the upper boundary of some lens in K.*

Proof Let for definiteness $i < j$. Like the proof of Proposition 3.2, we use induction on n and consider the n-contraction K' of K. Then K is the n-expansion of K' w.r.t. a legal path P in $G_{K'}$. We first assume that $j < n$ and consider two possible cases.

<u>*Case 1*</u>. Let $n \notin X$. Then A, B are vertices of K' contained in the left subgraph G_1 of $G_{K'}$ (w.r.t. P). By induction either (a) both A, B belong to one boundary (L_λ or U_λ) of some lens λ in K', or (b) K' contains $X = A \cap B$, or (c) K' contains $Xij = A \cup B$. In case (a), λ continues to be a lens in K, yielding (iii) or (iv) in the proposition. In case (b), if X is not a pit of P, then X is a vertex of K, as required in (i). And if X is a pit of P, then A, B belong to ∇-tiles in the filling at X w.r.t. P, and therefore A, B become vertices of U_λ for the lens λ arising in K in place of the three-edge zigzag in P containing X; this yields (iv). In case (c), Xij cannot be a pit of P (otherwise A, B would be not in G_1). Hence Xij is a vertex of K, as required in (ii).

<u>*Case 2*</u>. Let $n \in X$. Then $A' := A - n$ and $B' := B - n$ are vertices of K' contained in the right subgraph G_2 of $G_{K'}$. By induction either (a′) both A', B' belong to one of L_λ, U_λ for some lens λ of K', or (b′) K' contains $A' \cap B'$, or (c′) K' contains $A' \cup B'$. Cases (a′) and (b′) are easy, as well us (c′) unless $A' \cup B'$ is a peak of P. And if $A' \cup B'$ is a peak of P, then A', B' belong to Δ-tiles in the filling at $A' \cup B'$ w.r.t. P, implying that $A = A'n$ and $B = B'n$ are vertices of L_λ for the corresponding lens λ of K.

Now assume that $j = n$. Then $n \notin X$ and $n \notin A$. Hence A is a vertex of the subgraph G_1, and $B \cap [n-1] = X$ is a vertex of G_2 of K'. Also $A = Xi$, and by Proposition 3.2, K' has the V-edge (X, A) of type i. Consider two cases.

Case 3a. X is a slope or a peak of P. Then X is a vertex of K, yielding (i) in the proposition.

Case 3b. X is a pit of P. Then A belongs to a ∇-tile in the filling at X w.r.t. P, whence A is a vertex of U_λ for the corresponding lens λ of K. The right vertex r_λ of λ is just of the form $Xn = B$, and we obtain (iv) in the proposition. ∎

Like simple cyclic patterns in Sect. 2, the sets S_p are identified with the corresponding points in the zonogon $Z = Z_n$, and we connect each pair S_{p-1}, S_p by line segment e_p, obtaining the closed piecewise linear curve $\zeta_{\mathcal{S}}$ in Z. We direct each e_p so as to be congruent to the corresponding generator ξ_i or vector ϵ_{ij}.

A reasonable question arises: When $\zeta_{\mathcal{S}}$ is non-self-intersecting? Proposition 5.1 enables us to find necessary and sufficient conditions in terms of "forbidden quadruples" in $\mathcal{S}$. These conditions are as follows (cf. Proposition 5.2).

(C3) S contains no quadruple $S_{p-1}, S_p, S_{q-1}, S_q$ such that either $\{S_{p-1}, S_p\} = \{Xi, Xk\}$ and $\{S_{q-1}, S_q\} = \{Xj, X\ell\}$, or $\{S_{p-1}, S_p\} = \{X-i, X-k\}$ and $\{S_{q-1}, S_q\} = \{X-j, X-\ell\}$, where $i < j < k < \ell$.

(C4) S contains no quadruple $S_{p-1}, S_p, S_{q-1}, S_q$ such that either $\{S_{p-1}, S_p\} = \{Xi, Xk\}$ and $\{S_{q-1}, S_q\} = \{X, Xj\}$, or $\{S_{p-1}, S_p\} = \{X-i, X-k\}$ and $\{S_{q-1}, S_q\} = \{X, X-j\}$, where $i < j < k$.

To prove the assertion below and for further purposes, we need to refine some definitions. Fix a combi K and consider a vertex A in it. By the *(full) upper filling* at A we mean the sequence $\nabla(A|X_0X_1), \dots, \nabla(A|X_{q-1}X_q)$ of all ∇-tiles having the bottom A and ordered from left to right (i.e., $X_0X_1 \dots X_q$ is a directed path in G_K). The union of these tiles is called the *upper sector* at A and denoted by $\Sigma_A^{\rm up}$, and the path $X_0X_1 \dots X_q$ is called the *upper boundary* of $\Sigma_A^{\rm up}$ and denoted by U_A. Symmetrically, the *(full) lower filling* at A is the sequence $\Delta(A|Y_0Y_1), \dots, \Delta(A|Y_{q'-1}Y_{q'})$ of all Δ-tiles having the top A and ordered from left to right, the *lower sector* $\Sigma_A^{\rm low}$ at A is the union of these tiles, and the *lower boundary* L_A of $\Sigma_A^{\rm low}$ is the directed path $Y_0Y_1 \dots Y_{q'}$. Note that one of these fillings or both may be empty.

Proposition 5.2 *For a generalized cyclic pattern $\mathcal{S}$, the curve $\zeta_{\mathcal{S}}$ is non-self-intersecting if and only if $\mathcal{S}$ satisfies (C1)–(C4).*

Proof It is easy to see that if $\mathcal{S}$ has a quadruple as in (C3) or (C4), then the corresponding segments (edges) e_p and e_q are crossing (have a common interior point), and therefore $\zeta_{\mathcal{S}}$ is self-intersecting.

Conversely, suppose that $\zeta_{\mathcal{S}}$ is self-intersecting. To show the existence of a pair as in (C3) or (C4), fix a combi K with V_K including $\mathcal{S}$. Since all sets in $\mathcal{S}$ are different, $\zeta_{\mathcal{S}}$ has two crossing segments e_p and e_q. The edges of K are non-crossing (since K is planar); therefore, at least one of e_p, e_q is not an edge of K. Let for definiteness e_p be such, i.e., $\{S_{p-1}, S_p\}$ is a 2-distance pair.

By Proposition 5.1, at least one of the following takes place: (i) $X := S_{p-1} \cap S_p \in V_K$; (ii) $Y := S_{p-1} \cup S_p \in V_K$; (iii) both S_{p-1}, S_p belong to the same boundary, either U_λ or L_λ, for some lens λ of K. In case (iii), the only possibility for e_q to cross e_p is when

$\{S_{q-1}, S_q\}$ is a 2-distance pair occurring in the same boundary (either U_λ or L_λ) of λ where S_{p-1}, S_p are contained; moreover, the elements of these two pairs should be intermixing in this boundary. This gives a quadruple as in (C3).

In case (i), both vertices S_{p-1}, S_p (being of the form Xi, Xk for some i, k) lie in the boundary U_X of the upper sector $\Sigma_X^{\rm up}$ at X. Then e_q can cross e_p only in two cases: (a) the pair $\{S_{q-1}, S_q\}$ lies in U_X as well and, moreover, its elements and those of $\{S_{p-1}, S_p\}$ are intermixing in U_X (yielding a quadruple as in (C3)); and (b) one of S_{q-1}, S_q is just X while the other belongs to U_X and, moreover, the latter lies between S_{p-1} and S_p (yielding a quadruple as in (C4)). The case (ii) is symmetric to (i) and we argue in a similar way. ∎

To extend Theorem 2.2 to a generalized cyclic pattern $\mathcal{S} = (S_1, \ldots, S_r)$, we will consider one or another combi K with $\mathcal{S} \subseteq V_K$. Unlike the case of simple cyclic patterns, it now becomes less trivial to split K into two subtilings $K^{\rm in}$ and $K^{\rm out}$ (lying in the regions $R_{\mathcal{S}}^{\rm in}$ and $R_{\mathcal{S}}^{\rm out}$, respectively). A trouble is that some 2-segment $e_p = [S_{p-1}, S_p]$ may cut some tile τ of K (i.e., e_p and τ have an interior point in common), in contrast to 1-segments, which correspond to V-edges and therefore cannot cut any tile of K. Hereinafter we refer to the line segment $[S_{p-1}, S_p]$ connecting points S_{p-1}, S_p in the zonogon as a *1-segment* (resp. *2-segment*) if these points form a 1-distance (resp. 2-distance) pair.

We overcome this trouble by use of the *splitting method* described below. It works somewhat differently for lenses and for triangles. We use an important fact that can be deduced from Proposition 5.1: For a 2-segment $[S_{p-1}, S_p]$ cutting a tile τ of K, if τ is a lens, then both vertices S_{p-1}, S_p belong to the boundary either U_τ or L_τ; if τ is a Δ-tile $\Delta(A|BC)$, then S_{p-1}, S_p belong to the lower boundary of the sector $\Sigma_A^{\rm low}$; and if τ is a ∇-tile $\nabla(A|BC)$, then S_{p-1}, S_p belong to the upper boundary of $\Sigma_A^{\rm up}$.

I. First we consider a lens λ of K such that $\zeta_{\mathcal{S}}$ cuts λ (otherwise there is no problem with λ at all). The curve $\zeta_{\mathcal{S}}$ may go across λ several times; let $e_{p(1)}, \ldots, e_{p(d)}$ be the 2-segments cutting λ. These segments are pairwise non-crossing (by (C3)) and subdivide λ into $d+1$ polygons $D_1, \ldots, D_{d+1}$; so each $e_{p(i)}$ is $D_j \cap D_{j'}$ for some j, j' (and one of $D_j, D_{j'}$ lies in $R_{\mathcal{S}}^{\rm in}$, and the other in $R_{\mathcal{S}}^{\rm out}$).

If $\{S_{p-1}, S_p\} = \{\ell_\lambda, r_\lambda\}$, we say that $e_{p(i)}$ is the *central segment*. Otherwise, both ends of $e_{p(i)}$ belong to either U_λ or L_λ; in the former (latter) case, $e_{p(i)}$ is called an *upper* (resp. *lower*) *segment*, and we associate to it the path $P_{p(i)}$ in U_λ (resp. L_λ) connecting $S_{p(i)-1}$ and $S_{p(i)}$. In view of (C3), such paths form a *nested family*, i.e., for any $i \ne i'$, either the interiors of $P_{p(i)}$ and $P_{p(i')}$ are disjoint, or $P_{p(i)} \subset P_{p(i')}$, or $P_{p(i)} \supset P_{p(i')}$.

Accordingly, polygons D_j can be of three sorts. When D_j has all vertices in U_λ, its lower boundary is formed by exactly one (upper or central) segment (while its upper boundary is formed by some edges of $P_{p(i)}$ and segments $e_{p(i')}$); we call such a D_j an *upper semi-lens*. Symmetrically, when D_j has all vertices in L_λ, its upper boundary is formed by exactly one (lower or central) segment, and we call D_j a *lower semi-lens*. Besides, when the central segment does not exist, there appears one more polygon D_j; it is viewed as an (abstract) lens λ' with $\ell_{\lambda'} = \ell_\lambda$ and $r_{\lambda'} = r_\lambda$, and we call it (when exists) a *secondary* lens.

A possible splitting of a lens λ is illustrated in Fig. 9, where $d = 4$ and the four cutting segments are indicated by dotted lines.

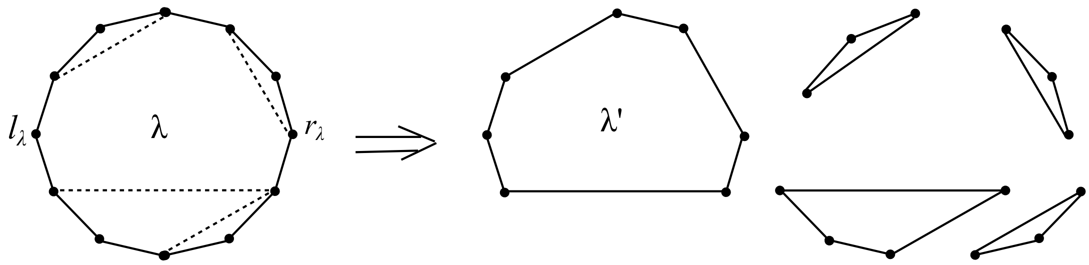


Figure 9: Splitting a lens

II. Next suppose that ζ_S cuts some Δ-tile $\Delta(A|BC)$ of K. Then the lower sector $\Sigma_A^{\rm low}$ at A is cut by some 2-segments in ζ_S; let $e_{p(1)}, \ldots, e_{p(d)}$ be these 2-segments. (Besides, $\Sigma_A^{\rm low}$ can be cut by some 1-segments.) For each segment $e_{p(i)}$, its ends $S_{p(i)-1}$ and $S_{p(i)}$ belong to the lower boundary L_A of the sector, and we denote by $P_{p(i)}$ the directed path in L_A connecting these vertices. Such paths form a nested family, by (C3).

The sector $\Sigma_A^{\rm low}$ is subdivided by the above 2-segments into $d + 1$ polygons $D_1, \ldots, D_{d+1}$. Among these, d polygons are lower semi-lenses, each being associated with the segment $e_{p(i)}$ forming its upper boundary. The remaining polygon contains the vertex A and is viewed as a lower sector Σ' whose lower boundary $L(\Sigma')$ is formed by the segments $e_{p(i)}$ with $P_{p(i)}$ maximal and the edges of L_A between these segments. We fill Σ' with the corresponding Δ-tiles (so each edge (B, C) of $L(\Sigma')$ generates one Δ-tile, namely $(A|BC)$). We call them *secondary* Δ-tiles.

In view of (C4), each 1-segment e_i cutting $\Sigma_A^{\rm low}$ connects some vertex of $L(\Sigma')$ and the top A, and therefore e_i coincides with the common V-edge of some two neighboring Δ-tiles in the filling of Σ'.

A possible splitting of a sector by three 2-segments and one 1-segment (drawn by dotted lines) is illustrated in Fig. 10.

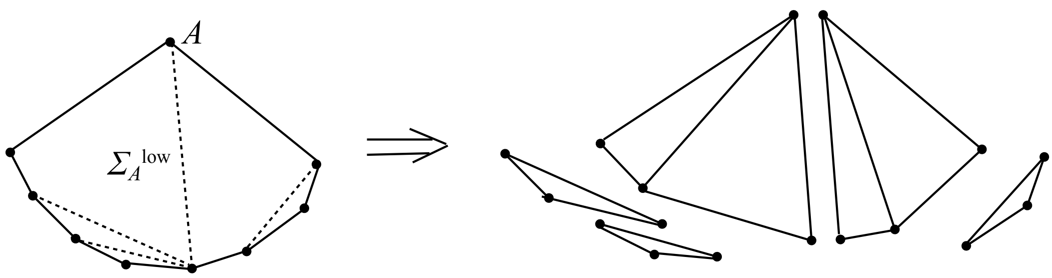


Figure 10: Splitting a sector

When ζ_S cuts some ∇-tile $\nabla(A|BC)$, we consider the upper sector $\Sigma_A^{\rm up}$ and make a splitting in a similar way (which subdivides $\Sigma_A^{\rm up}$ into corresponding upper semi-lenses and secondary ∇-tiles).

Let $\widehat{K}$ be the resulting set of tiles upon termination of the splitting process for K w.r.t. ζ_S (it consists of the tiles of K not cut by ζ_S and the appeared semi-lenses and secondary tiles). Then the desired $K^{\rm in}$ ($K^{\rm out}$) is defined to be the set of tiles of $\widehat K$ lying in $R^{\rm in}_{\mathcal S}$ (resp. $R^{\rm out}_{\mathcal S}$). We refer to $\widehat K$, $K^{\rm in}$, $K^{\rm out}$ as *quasi-combies* on Z, $R^{\rm in}_{\mathcal S}$, $R^{\rm out}_{\mathcal S}$, respectively, *agreeable with* $\mathcal S$.

Now we are ready to generalize Theorem 2.2. As before, the domain $\mathcal D^{\rm in}_{\mathcal S}$ ($\mathcal D^{\rm out}_{\mathcal S}$) consists of the sets (points) $X\subseteq[n]$ such that X (weak) $\mathcal S$ and X lies in $R^{\rm in}_{\mathcal S}$ (resp. $R^{\rm out}_{\mathcal S}$).

Theorem 5.3 *Let $\mathcal S$ be a generalized cyclic pattern satisfying (C1)–(C4). Then the domains $\mathcal D^{\rm in}_{\mathcal S}$ and $\mathcal D^{\rm out}_{\mathcal S}$ form a complementary pair. As a consequence, both $\mathcal D^{\rm in}_{\mathcal S}$ and $\mathcal D^{\rm out}_{\mathcal S}$ are w-pure.*

Proof Given arbitrary $X\in\mathcal D^{\rm in}_{\mathcal S}$ and $Y\in\mathcal D^{\rm out}_{\mathcal S}$, take a combi K with V_K including $\mathcal S\cup\{X\}$ and a combi K' with $V_{K'}$ including $\mathcal S\cup\{Y\}$. Split K into the corresponding quasi-combies $K^{\rm in}$ and $K^{\rm out}$ on $R^{\rm in}_{\mathcal S}$ and $R^{\rm out}_{\mathcal S}$, respectively, and similarly, split K' into quasi-combies $K'^{\rm in}$ and $K'^{\rm out}$. Then X is a vertex in $K^{\rm in}$, and Y is a vertex in $K'^{\rm out}$. Their union $\widetilde K:=K^{\rm in}\cup K'^{\rm out}$ is a quasi-combi on Z. We transform $\widetilde K$, step by step, in order to obtain a correct combi with the same vertex set.

More precisely, in a current $\widetilde K$ choose a semi-lens λ. If λ is a lower semi-lens with $\ell_\lambda=X$ and $r_\lambda=Y$, then its upper boundary is formed by the unique edge $e=(X,Y)$, and this edge belongs to another tile τ in $\widetilde K$. Three cases are possible: (a) τ is a Δ-tile (and e is its base); (b) τ is a lens or a lower semi-lens (and e belongs to its lower boundary L_τ); and (c) τ is an upper semi-lens (and e forms its lower boundary). We remove the edge e, combining λ and τ into one polygon ρ.

In case (a), ρ looks like an upper sector, and we fill it with the corresponding Δ-tiles. In case (b), ρ is again a lens or a lower semi-lens (like τ). In case (c), ρ is a lens. The new $\widetilde K$ is a quasi-combi on Z, and the number of semi-lenses becomes smaller.

If λ is an upper semi-lens, it is treated symmetrically.

We repeat the procedure for the current $\widetilde K$, and so on, until we get rid of all semi-lenses. Then the eventual $\widetilde K$ is a correct combi containing both X,Y, and the result follows. ∎

5.3 Planar graph patterns

We can further extend the w-purity result by considering an arbitrary graph $\mathcal H=(\mathcal S,\mathcal E)$ with the following properties:

(H1) the vertex set $\mathcal S$ is a w-collection in $2^{[n]}$;

(H2) each edge $e\in\mathcal E$ is formed by a 1- or 2-distance pair in $\mathcal S$;

(H3) the edges of $\mathcal H$ obey (C3) and (C4), in the sense that there are no quadruple of vertices of $\mathcal H$ that can be labeled as S_{p-1},S_p,S_{q-1},S_q so that both $\{S_{p-1},S_p\}$ and $\{S_{q-1},S_q\}$ are edges of $\mathcal H$ and they behave as indicated in (C3) or (C4).

Representing the vertices of $\mathcal H$ as corresponding points in $Z=Z_n$, and the edges as line segments, we observe from (H3) and the proof of Proposition 5.2 that the graph $\mathcal H$ is planar (has a planar layout in Z). Let $\mathcal F$ be the set of its (closed 2-dimensional) faces. (W.l.o.g., we may assume that $\mathcal H$ includes the entire boundary of Z, since $bd(Z)$

is weakly separated from any subset of $[n]$.) For a face $F \in \mathcal{F}$, the set of elements of $\mathcal{D}_{\mathcal{S}}$ contained in F is denoted by $\mathcal{D}_{\mathcal{S}}(F)$.

Theorem 5.4 *Let $\mathcal{H}$ be a graph satisfying (H1)–(H3). Then for any two different faces F, F' of $\mathcal{H}$, the domains $\mathcal{D}_{\mathcal{S}}(F)$ and $\mathcal{D}_{\mathcal{S}}(F')$ form a complementary pair. As a consequence, $\mathcal{D}_{\mathcal{S}}(F)$ is w-pure for each face F, and similarly for any union of faces in $\mathcal{H}$.*

(When $\mathcal{H}$ is a simple cycle, this turns into Theorem 5.3. Also this generalizes the result on semi-simple cyclic patterns in Sect. 5.1.)

Proof If sets (points) X, Y lie in F, F', respectively, then they are separated by the curve corresponding to some simple cycle $\mathcal{C}$ in $\mathcal{H}$. This $\mathcal{C}$ is, in fact, a generalized cyclic pattern obeying conditions (C1)–(C4). Also one of X, Y lies in the region $R_{\mathcal{C}}^{\rm in}$, and the other in $R_{\mathcal{C}}^{\rm out}$. Now the result follows from Theorem 5.3. ▮

One can reformulate this theorem as follows: For each face F of $\mathcal{H}$, take an arbitrary maximal w-collection $\mathcal{X}_F$ in $\mathcal{D}_{\mathcal{S}}(F)$; then for any set $\mathcal{F}'$ of faces of $\mathcal{H}$, $\cup(\mathcal{X}_F\colon F \in \mathcal{F})$ is a maximal w-collection in $\cup(\mathcal{D}_{\mathcal{S}}(F)\colon F \in \mathcal{F}')$.

Theorem 5.4 was originally stated and proved in [4]. An alternative proof can be given based on nice properties of mutations of w-collections in a discrete Grassmannian, established in a subsequent work of Oh and Speyer [7].

We conclude this paper with a special case of generalized cyclic patterns, namely a *Grassmann necklace* of [6]. This is a sequence $\mathcal{N} = (S_1, S_2, \ldots, S_n = S_0)$ of sets in Δ_n^m such that $S_{i+1} - S_i = \{i\}$ for each i. One can check that $\mathcal{N}$ is a w-collection satisfying (C3). As is shown in [6], the domain $\mathcal{D}_{\mathcal{N}}^{\rm in}$ is w-pure. A sharper result in [3] says that the domains $\mathcal{D}_{\mathcal{N}}^{\rm in}$ and $\mathcal{D}_{\mathcal{N}}^{\rm out} \cap \Delta_n^m$ form a complementary pair within the hyper-simplex Δ_n^m. This is a special case of Theorem 5.4. Indeed, take as $\mathcal{H}$ the union of the (natural) cycle $\mathcal{C}$ on $\mathcal{N}$ and the cycle $\mathcal{C}_0$ on the "maximal" necklace (formed by the intervals and co-intervals if size m in $[n]$). Then $\mathcal{H}$ has the face surrounded by $\mathcal{C}$ (giving the domain $\mathcal{D}_{\mathcal{N}}^{\rm in}$) and the face (or the union of several faces) "lying between" $\mathcal{C}$ and $\mathcal{C}_0$ (giving the domain $\mathcal{D}_{\mathcal{N}}^{\rm out} \cap \Delta_n^m$).

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