Pictures to

"Multiflows and disjoint paths of minimum total cost"
For example, \( \varphi(H) = 1 \) if \( |EH| = 1 \). More generally, \( \varphi(H) = 1 \) for any complete bipartite graph \( H \), by the multi-terminal version of the min-cost max-flow problem [12]. On the other hand, it is easy to show that \( \varphi(H) \geq 2 \) for all other graphs \( H \). The next result is less trivial: if \( H = K_T \) then (1.2) has a half-integer o.s. [19]; hence, \( \varphi(K_T) = 2 \) if \( |T| \geq 3 \). This fact was proved by considering the following slightly more general \textit{parametric problem} which combines both objectives figured in (1.2):

\[(1.5) \text{ given } p \in Q_+, \text{ maximize the linear objective function } \text{pval}(f) - a_f \text{ among all multitflows } f \text{ for } G, K_T, c.\]

Obviously, (1.5) becomes equivalent to (1.2) when \( p \) is large enough. The above-mentioned result is an immediate corollary from the following theorem.

**Theorem 1** [19]. \( \text{If } H = K_T \text{ then for any } p \in Q_+ \text{ problem (1.5) has a half-integer optimal solution } f. \)

As a consequence, we observe that \( \varphi(H) = 2 \) for any complete multi-partite graph \( H \) with \( k \geq 3 \) parts (i.e., \( VH \) admits a partition \( \{T_1, \ldots, T_k\} \) such that \( \{s, t\} \in EH \) if and only if \( s \in T_i \) and \( t \in T_j \) for \( i \neq j \)). For we can add to \( G \) new nodes \( t_1, \ldots, t_k \) and edges \( t_is \) \( (s \in T_i) \) with the same rather large capacities and costs; then any o.s. for the resulting network with the complete graph on \( \{t_1, \ldots, t_k\} \) as commodity graph yields an o.s. for the original network. On the other hand, the following is true.

**Theorem 2** [20]. \( \text{If } H \text{ is not complete multi-partite then } \varphi(H) = \infty. \)

This theorem is reduced to examination of few instances of \( H \) because of the following simple fact.

**Statement 1.1.** \( \text{If } H' \text{ is an induced subgraph of } H \text{ then } \varphi(H') \leq \varphi(H). \)

**Proof.** Given a network \( N' = (G', c', a', \alpha') \), add to \( G' \) the elements \( s \in VH - VH' \) as isolated nodes and denote the resulting network by \( N \). Then \( N \) and \( N' \) have the same sets of optimal solutions, whence the result follows. \( \bullet \)

There are exactly three minimal, under taking induced subgraphs, graphs that are not complete multi-partite, namely, \( H_1, H_2, H_3 \) drawn in Fig. 1. Hence, by Statement 1.1, it suffices to show that \( \varphi(H_i) = \infty, i = 1, 2, 3 \). We explain why the fractionality for these \( H_i \)'s is unbounded in Section 3.

![Fig. 1](image)

2. The program dual of (1.5) can be written as

\[(1.6) \begin{align*}
\text{minimize } & c\gamma \\
\text{subject to } & \end{align*}\]
\[ \gamma \in \mathbb{Q}^E_{+} \text{ and } \text{dist}_{\gamma}(s,t) \geq p \text{ for all } s,t \in T, \ s \neq t, \]

where for \( \ell : EG \to \mathbb{Q}^E_{+} \), \( \text{dist}_\ell(u,v) \) denotes the \( \ell \)-distance between nodes \( u \) and \( v \), i.e., the minimum \( \ell \)-length \( \ell(P) \) of a path \( P \) in \( G \) that connects \( u \) and \( v \).

**Example 4.** Let \( G \) be as in Fig. 2a, \( T = \{s_1, \ldots, s_6\}, c = \mathbb{I} \) and \( a = \mathbb{I} \). There is an only optimal \( T \)-multiflow, namely, that takes value 1/2 on the six paths shown in Fig. 2b, and zero on the other \( T \)-paths. Suppose \( p = 7 \). Then an optimal \( \gamma \) to (1.6) is zero on the edge \( uv \) and 2.5 on the other edges.

![Fig. 2](image)

The original proof of Theorem 1 given in [19] was constructive and provided by a pseudo-polynomial algorithm. Being within frameworks of the primal-dual linear programming method, this algorithm is based on a parametric approach, like that used in the classic algorithm of Ford and Fulkerson [12] for the min-cost max-flow problem, but now in a more complicated context. In fact, it finds optimal primal and dual solutions simultaneously for all \( p \in \mathbb{Q}^E_{+} \). More precisely, it constructs, step by step, a sequence \( 0 = p_0 \leq p_1 < p_2 < \ldots < p_M \) of rationals, a sequence \( f_0, f_1, \ldots, f_M \) of half-integer \( T \)-multiflows and a sequence \( \gamma_0, \gamma_1, \ldots, \gamma_M, \gamma_{M+1} \) of functions on \( EG \) such that: (i) for \( i = 0, \ldots, M - 1 \) and \( 0 \leq \varepsilon \leq 1 \), \( f_i \) and \( (1 - \varepsilon)\gamma_i + \varepsilon \gamma_{i+1} \) are o.s. to (1.5) and (1.6) with \( p = (1 - \varepsilon)p_i + \varepsilon p_{i+1} \), respectively; and (ii) for \( 0 \leq \varepsilon < \infty \), \( f_M \) and \( \gamma_M + \varepsilon \gamma_{M+1} \) are o.s. to those programs with \( p = p_M + \varepsilon \). In particular, \( f_M \) is a maximum \( T \)-multiflow.

The key idea in [19] is that, at each iteration, the new optimal \( f \) and \( \gamma \) can be obtained by solving the usual maximum flow problem in a certain "skew-symmetric" digraph, called a double covering over \( G \). A shorter, though non-algorithmic, proof of Theorem 1 is described in [21]; it is also based on double covering techniques. We outline this proof in Section 2.

Two more results were obtained in [21]. It was shown that the dual program (1.6) has a half-integer o.s. whenever \( p \) is an integer. Also a strongly polynomial algorithm to find a half-integer o.s. to (1.2) with \( H = K_T \) was developed there. However, this algorithm is not "purely combinatorial" as it uses the ellipsoid method.

Recently Goldberg and the author [13] designed two polynomial algorithms for finding a half-integer o.s. to (1.2) with \( H = K_T \). Both algorithms are combinatorial and they handle within the original graph \( G \) itself rather than double coverings. One of these applies scaling on capaciticies, while the other scaling on costs (cf. [11,6] and [33,2] for the min-cost max-flow problem).
The construction of $f$ yields a natural mapping of $\sigma$ into $\mathcal{A} \cap \sigma \mathcal{D}$. If $\mathcal{I} \subseteq \sigma$, then $\mathcal{D} \cap f(\mathcal{I}) = \emptyset$.

For all distinct $s, t \in 3$.

Let $(a, \mathcal{L}, \mathcal{F}, \mathcal{G})$ and $(a', \mathcal{L}', \mathcal{F}', \mathcal{G}')$ be adjacent two arcs of $\mathcal{D}$, and let $a \in a'$. Then $a \subseteq a'$. If $a \in a'$, then $a \subseteq a'$. If $a \not\in a'$, then $a \not\in a'$. If $a = a'$, then $a = a'$.

For all distinct $s, t \in 3$, let $a \subseteq a'$.

Let $A \subseteq \mathcal{A}$ and $\mathcal{D} \in \mathcal{D}$.

If $a \in a'$, then $a \subseteq a'$. If $a \not\in a'$, then $a \not\in a'$. If $a = a'$, then $a = a'$.

Let $a \in a'$. Then $a = a'$.

For all distinct $s, t \in 3$, let $a \subseteq a'$. If $a \in a'$, then $a \subseteq a'$. If $a \not\in a'$, then $a \not\in a'$. If $a = a'$, then $a = a'$.

Let $a \in a'$. Then $a = a'$.
$G^p$. Third, find an integer flow $h$ in $\Gamma$ with the restrictions $h_b = c_b$ for $b \in A^+$ (such an $h$ must exist). Now an integer decomposition of $h$ determines the desired half-integer multiflow for $G$.

3. Unbounded fractionality

As mentioned in the Introduction, to prove Theorem 2 it suffices to show that $\varphi(H) = \infty$ for $H = H_1, H_2, H_3$ as in Fig 1. Following [20], we design “bad networks” $N = (G, H, c, a)$ for these $H$’s. Let $k$ be an odd positive integer. Take $k$ disjoint paths $(v_1^i, e_1^i, v_1^i, \ldots, e_{2k}^i, v_{2k}^i)$, $i = 1, \ldots, k$. Connect $v_{j-1}^i$ and $v_j^i$ by edge $u_j^i$ for all $i, j$ such that $i - j \equiv 1 \pmod{2}$. Add nodes $s, t, s', t', y, z, y', z'$ and edges

(i) $sy, tz, s'y', t'z'$;
(ii) $yv_1^i$ and $zv_{2k}^i$ for $i = 1, \ldots, k$;
(iii) $y'v_1^j$ for each odd $j$, and $z'v_{2k}^i$ for each even $j$,

obtaining graph $G$. Assign the capacity $k - 1$ to the edges $s'y', t'z'$, and 1 to the other edges of $G$. Assign the edge costs as follows:

$$
\begin{align*}
0 & \text{ for } tz \text{ and } e_{2j}^i, \quad i, j = 1, \ldots, k; \\
1 & \text{ for all edges } u_j^i \text{ and the remaining edges } e_j^i; \\
k & \text{ for } s'y', t'z' \text{ and the edges as in (ii) and (iii);} \\
2k & \text{ for } sy.
\end{align*}
$$

Fig. 4

(See Fig. 4 where $k = 3$ and the numbers on edges indicate non-zero costs.) We identify $s, t, s', t'$ with the corresponding nodes of the graph $H \in \{ H_1, H_2, H_3 \}$ in question; therefore $\{st, s't'\} \subseteq EH \subseteq \{st, s't', ss', st'\}$.

For $i = 1, \ldots, k$, let $P_i (L_i)$ be the simple path going through the nodes $s, y, v_1^i, \ldots, v_{2k}^i, z, t$ (respectively, $s', y', v_{2k-1}^i, v_{2k}^i, v_{2k-1}^i, \ldots, v_{2i-1}^k, v_{2i}^k, v_{2i-1}^k, v_{2i}^k, z', t'$). Assign