Pictures to

"Multiflows and disjoint paths of minimum total cost"

For example, $\varphi(H) = 1$ if $|EH| = 1$. More generally, $\varphi(H) = 1$ for any complete bipartite graph H, by the multi-terminal version of the min-cost max-flow problem [12]. On the other hand, it is easy to show that $\varphi(H) \geq 2$ for all other graphs H. The next result is less trivial: if $H = K_T$ then (1.2) has a *half-integer* o.s. [19]; hence, $\varphi(K_T) = 2$ if $|T| \geq 3$. This fact was proved by considering the following slightly more general *parameteric problem* which combines both objectives figured in (1.2):

(1.5) *given* $p \in Q_+$, *maximize the linear objective function* $\text{pval}(f) - a_f$ *among all multiflows* f *for* G, K_T, c.

Obviously, (1.5) becomes equivalent to (1.2) when p is large enough. The above-mentioned result is an immediate corollary from the following theorem.

Theorem 1 [19]. *If $H = K_T$ then for any $p \in Q_+$ problem (1.5) has a half-integer optimal solution f.*

As a consequence, we observe that $\varphi(H) = 2$ for any *complete multi-partite* graph H with $k \geq 3$ parts (i.e., VH admits a partition $\{T_1, \dots, T_k\}$ such that $\{s,t\} \in EH$ if and only if $s \in T_i$ and $t \in T_j$ for $i \neq j$). For we can add to G new nodes $t_1, \dots, t_k$ and edges $t_i s$ ($s \in T_i$) with the same rather large capacities and costs; then any o.s. for the resulting network with the complete graph on $\{t_1, \dots, t_k\}$ as commodity graph yields an o.s. for the original network. On the other hand, the following is true.

Theorem 2 [20]. *If H is not complete multi-partite then $\varphi(H) = \infty$.*

This theorem is reduced to examination of few instances of H because of the following simple fact.

Statement 1.1. *If H' is an induced subgraph of H then $\varphi(H') \leq \varphi(H)$.*

Proof. Given a network $N' = (G', H', c', a')$, add to G' the elements $s \in VH - VH'$ as isolated nodes and denote the resulting network by N. Then N and N' have the same sets of optimal solutions, whence the result follows. •

There are exactly three minimal, under taking induced subgraphs, graphs that are not complete multi-partite, namely, H_1, H_2, H_3 drawn in Fig. 1. Hence, by Statement 1.1, it suffices to show that $\varphi(H_i) = \infty$, $i = 1, 2, 3$. We explain why the fractionality for these H_i's is unbounded in Section 3.

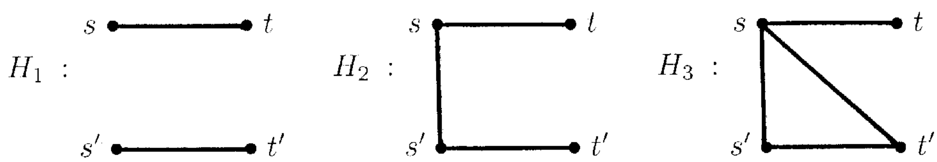

Fig. 1

2. The program dual of (1.5) can be written as

(1.6) minimize $c\gamma$ subject to

$$\gamma \in \mathbb{Q}_+^{EG} \text{ and } \mathrm{dist}_{a+\gamma}(s,t) \geq p \text{ for all } s,t \in T,\ s \neq t,$$

where for $\ell : EG \to \mathbb{Q}_+$, $\mathrm{dist}_\ell(u,v)$ denotes the *ℓ-distance* between nodes u and v, i.e., the minimum ℓ-length $\ell(P)$ of a path P in G that connects u and v.

Example 4. Let G be as in Fig. 2a, $T = \{s_1, \ldots, s_6\}$, $c = \mathbb{1}$ and $a = \mathbb{1}$. There is an only optimal T-multiflow, namely, that takes value 1/2 on the six paths shown in Fig. 2b, and zero on the other T-paths. Suppose $p = 7$. Then an optimal γ to (1.6) is zero on the edge uv and 2.5 on the other edges.

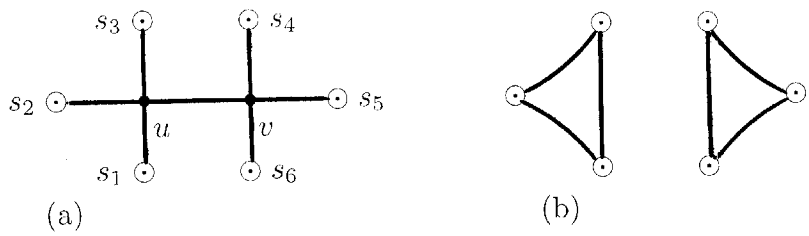

Fig. 2

The original proof of Theorem 1 given in [19] was constructive and provided by a pseudo-polynomial algorithm. Being within frameworks of the primal-dual linear programming method, this algorithm is based on a parametric approach, like that used in the classic algorithm of Ford and Fulkerson [12] for the min-cost max-flow problem, but now in a more complicated context. In fact, it finds optimal primal and dual solutions simultaneously for all $p \in \mathbb{Q}_+$. More precisely, it constructs, step by step, a sequence $0 = p_0 \leq p_1 < p_2 < \ldots < p_M$ of rationals, a sequence $f_0, f_1, \ldots, f_M$ of half-integer T-multiflows and a sequence $\gamma_0, \gamma_1, \ldots, \gamma_M, \gamma_{M+1}$ of functions on EG such that: (i) for $i = 0, \ldots, M-1$ and $0 \leq \varepsilon \leq 1$, f_i and $(1-\varepsilon)\gamma_i + \varepsilon\gamma_{i+1}$ are o.s. to (1.5) and (1.6) with $p = (1-\varepsilon)p_i + \varepsilon p_{i+1}$, respectively; and (ii) for $0 \leq \varepsilon < \infty$, f_M and $\gamma_M + \varepsilon\gamma_{M+1}$ are o.s. to these programs with $p = p_M + \varepsilon$. In particular, f_M is a maximum T-multiflow.

The key idea in [19] is that, at each iteration, the new optimal f and γ can be obtained by solving the usual maximum flow problem in a certain "skew-symmetric" digraph, called a *double covering* over G. A shorter, though non-algorithmic, proof of Theorem 1 is described in [21]; it is also based on double covering techniques. We outline this proof in Section 2.

Two more results were obtained in [21]. It was shown that the dual program (1.6) has a half-integer o.s. whenever p is an integer. Also a strongly polynomial algorithm to find a half-integer o.s. to (1.2) with $H = K_T$ was developed there. However, this algorithm is not "purely combinatorial" as it uses the ellipsoid method.

Recently Goldberg and the author [13] designed two polynomial algorithms for finding a half-integer o.s. to (1.2) with $H = K_T$. Both algorithms are combinatorial and they handle within the original graph G itself rather than double coverings. One of these applies scaling on capacities, while the other scaling on costs (cf. [11,6] and [33,2] for the min-cost max-flow problem).

(2.3) Let $e = uv \in EG$ and $u, v \in VG^p$. Then $e \in EG^p$ if and only if, up to permutation of u and v, either (i) $u \in V_s$, $v \in V_s \cup V^\bullet$ and $\pi(v) - \pi(u) = \ell_e$ for some $s \in T$, or (ii) $u \in V_s$, $v \in V_t$ and $\pi(u) + \pi(v) + \ell_e = p$ for some distinct $s, t \in T$.

(2.4) Let $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ be a path in G^p connecting distinct terminals $s = v_0$ and $t = v_k$. Then $P \in \mathcal{P}^p$ if and only if there is $0 \le i < k$ such that $v_0, \dots, v_i \in V_s$; $v_{i+2}, \dots, v_k \in V_t$; $\pi(v_0) < \dots < \pi(v_i)$; $\pi(v_{i+2}) > \dots > \pi(v_k)$; and either $v_{i+1} \in V^\bullet$, or $v_{i+1} \in V_t$ and $\pi(v_{i+1}) > \pi(v_{i+2})$.

Property (2.3) enables us to construct a digraph $\Gamma = (V\Gamma, A\Gamma)$, the *double covering* over G^p, as follows. Split each $v \in VG^p$ into $2|T(v)|$ nodes v_s^1 and v_s^2 ($s \in T(v)$). If $T(v) = \{s\}$, we also denote v_s^i as v^i. The arcs of Γ are assigned as follows:

(i) each $e = uv \in EG^p$ with $u \in V_s$, $v \in V_s \cup V^\bullet$ and $\pi(u) < \pi(v)$ generates two arcs (u_s^1, v_s^1) and (v_s^2, u_s^2);

(ii) each $e = uv \in EG^p$ with $u \in V_s$ and $v \in V_t$ ($s \ne t$) generates two arcs (u_s^1, v_t^2) and (v_t^1, u_s^2);

(iii) each $v \in V^\bullet$ generates arcs (v_s^1, v_t^2) for all distinct $s, t \in T(v)$;

see Fig. 3 where $T = \{s, t, q\}$, $p = 4$, the numbers on edges indicate values of ℓ, and the arcs of Γ are directed up. Arcs in (i) and (ii) have capacities c_e, and arcs in (iii) have capacity ∞. We use the same notation c for the capacities in Γ and think of $T^1 = \{s^1 : s \in T\}$ and $T^2 = \{s^2 : s \in T\}$ as the sets of *sources* and *sinks* of Γ, respectively. Define $\sigma(v_s^i) = v_s^{3-i}$. This gives a *skew symmetry* of Γ because for each $b = (u_s^i, v_t^j) \in A\Gamma$, (v_t^{3-j}, u_s^{3-i}) is also an arc of Γ, denoted as $\sigma(b)$. We extend σ to the dipaths of Γ in a natural way.

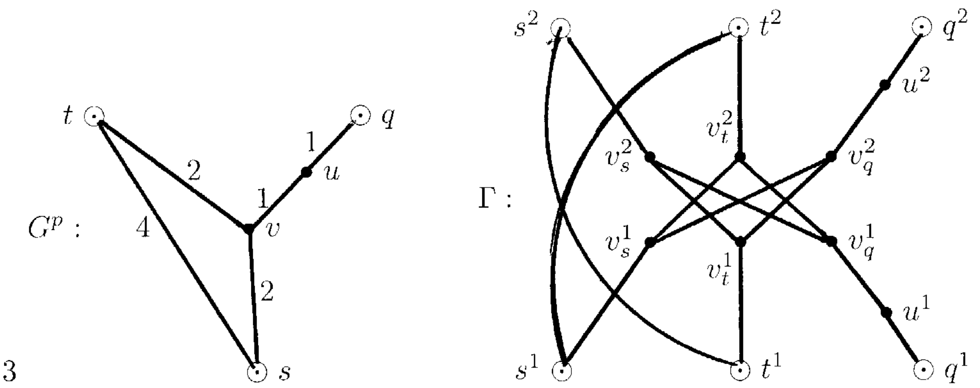

Fig. 3

The construction of Γ yields a natural mapping ω of $V\Gamma \cup A\Gamma$ to $VG^p \cup EG^p$; it brings a node v_s^i to v, an arc (y_s^i, z_t^j) as in (i) or (ii) to the edge yz, and an arc (v_s^1, v_t^2) as in (iii) to the node v. We extend ω in a natural way to a mapping of the dipaths of Γ into paths of G^p. From (2.4) one can derive the following key property:

(2.5) (i) for a dipath P in Γ, P and $\sigma(P)$ are disjoint, and $\omega(\sigma(P))$ is reverse to $\omega(P)$;

G^p. Third, find an integer flow h in Γ with the restrictions $h_b = c_b$ for $b \in A^+$ (such an h must exist). Now an integer decomposition of h determines the desired half-integer multiflow for G.

3. Unbounded fractionality

As mentioned in the Introduction, to prove Theorem 2 it suffices to show that $\varphi(H) = \infty$ for $H = H_1, H_2, H_3$ as in Fig 1. Following [20], we design "bad networks" $N = (G, H, c, a)$ for these H's. Let k be an odd positive integer. Take k disjoint paths $(v_1^i, e_2^i, v_2^i, \dots, e_{2k}^i, v_{2k}^i)$, $i = 1, \dots, k$. Connect v_j^i and v_j^{i+1} by edge u_j^i for all i, j such that $i - j \equiv 1 \pmod 2$. Add nodes $s, t, s', t', y, z, y', z'$ and edges

(i) $sy, tz, s'y', t'z'$;

(ii) yv_1^i and zv_{2k}^i for $i = 1, \dots, k$;

(iii) $y'v_j^1$ for each odd j, and $z'v_j^k$ for each even j,

obtaining graph G. Assign the capacity $k - 1$ to the edges $s'y', t'z'$, and 1 to the other edges of G. Assign the edge costs as follows:

0 for tz and e_{2j}^i, $i, j = 1, \dots, k$;

1 for all edges u_j^i and the remaining edges e_j^i;

k for $s'y', t'z'$ and the edges as in (ii) and (iii);

$2k$ for sy.

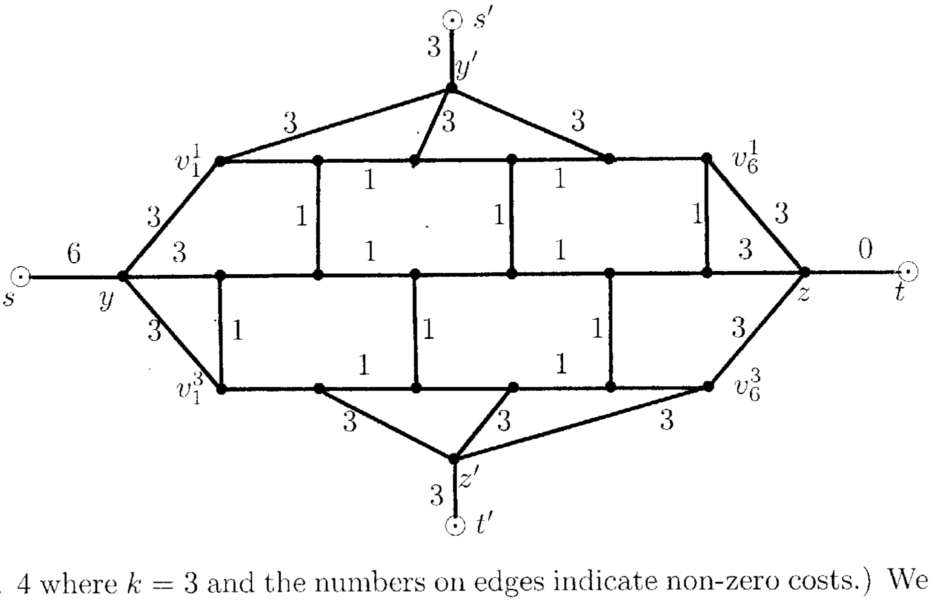

Fig. 4

(See Fig. 4 where $k = 3$ and the numbers on edges indicate non-zero costs.) We identify s, t, s', t' with the corresponding nodes of the graph $H \in \{H_1, H_2, H_3\}$ in question; therefore $\{st, s't'\} \subseteq EH \subseteq \{st, s't', ss', st'\}$.

For $i = 1, \dots, k$, let P_i (L_i) be the simple path going through the nodes $s, y, v_1^i, \dots, v_{2k}^i, z, t$ (respectively, $s', y', v_{2i-1}^1, v_{2i}^1, v_{2i}^2, v_{2i-1}^2, \dots, v_{2i-1}^{k-1}, v_{2i-1}^k, v_{2i}^k, z', t'$). Assign