On the structure of the system of minimum edge cuts of a graph[1]

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1. We consider a class of undirected connected finite graphs. A graph G from this class has the vertex set $V = V(G)$ and the edge set $U = U(G)$, and each edge is endowed with a *positive* real weight $c(u)$.

A *cut* in a graph is meant to be a partition $R = (V_1, V_2)$ of the vertices into two subsets V_1 and V_2, i.e. $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$. An edge $u \in U$ having one endvertex in V_1 and the other in V_2 is called an *edge of the cut* R. The set of these edges is denoted by $U(R)$. The *weight* of the cut R is the value $c(R) := \sum_{u \in U(R)} c(u)$. We assume by definition that if V_1 or V_2 is empty, then $c(V_1, V_2) := \infty$. A cut having the minimum possible weight, denoted by $\widehat{c} = \widehat{c}(G)$, is called a *minimum cut*, or an *m-cut*; the set of m-cuts is denoted by $\mathbf{M}(G)$.

In this paper, we associate to an arbitrary graph G a certain "structural graph" $\Gamma = \Gamma(G)$ yielding information about all minimum cuts and their interrelations in G.

For any graph G, its associated "structural graph" Γ possesses the property

I: any two distinct simple cycles of Γ have at most one vertex in common.

This is equivalent to the property

II: any edge of Γ belongs to at most one simple cycle

(see Fig. 1b). The graphs of this sort generalize *trees* and so-called *cactuses*[2]. We call a graph satisfying I (or, equivalently, II) a *plant*.

Let us call a cut *proper* if the removal of its edges makes exactly two connected subgraphs. It is easy to see that a minimum cut is always proper (since c is positive). The set of proper cuts of G is denoted by $\mathbf{P}(G)$. It is not difficult to realize that the set of proper cuts of a plant consists of: (a) *one-edge cuts*, each corresponding to a *non-cyclic* edge (an edges contained in no cycle); and (b) *two-edge cuts*, each corresponding to a pair of *cyclic* edges contained in the same cycle.

The weights of edges of the "structural graph" Γ are defined as follows:

1) if $u \in U(\Gamma)$ is a non-cyclic edge, then $c(u) := \widehat{c}$;

2) if $u \in U(\Gamma)$ is a cyclic edge, then $c(u) := \widehat{c}/2$.

A plant with the edge weights defined in this way is called a *$\widehat{c}$-plant*.

By the above observations, the "structural graph" Γ has the following properties: (a) $\widehat{c}(\Gamma) = \widehat{c}(G) = \widehat{c}$; and (b) each proper cut in Γ is a minimum cut, and vice versa, i.e. $\mathbf{M}(\Gamma) = \mathbf{P}(\Gamma)$.

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[1]Translated by A.V. Karzanov (preserving the original style and notation as much as possible) from: Е.А. Диниц, А.В. Карзанов, М.В. Ломоносов, О структуре системы минимальных реберных разрезов графа, В кн.: *Исследования по Дискретной Оптимизации* (ред. А.А. Фридман), Наука, Москва, 1976, с. 290–306. (E.A. Dinitz, A.V. Karzanov, and M.V. Lomonosov, O strukture sistemy minimal'nykh rebernykh razrezov grafa, In.: *Issledovaniya po Diskretnoĭ Optimizatsii* (A.A. Fridman, ed.), Nauka, Moscow, 1976, pp. 290–306, in Russian.) For pictures, see the Russian original.

[2]Here by a "cactus" one means a graph in which each edge belongs to *exactly* one simple cycle.

Let $G' = (V', U')$ and $G = (V, U)$ be two graphs, and $\rho : V' \to V$ a map. Then each cut $R = (V_1, V_2)$ of G determines the cut $\rho^*(R) := (\rho^{-1}(V_1), \rho^{-1}(V_2))$ of G'.

Main result. *For any graph G with edge weights c, there exists a $\widehat{c}$-plant Γ and a map $\varphi : V(G) \to V(\Gamma)$ such that:*

(a) *for $v_1, v_2 \in V(\Gamma)$, $\varphi(v_1) = \varphi(v_2)$ holds if and only if the vertices v_1 and v_2 are separated by none of the m-cuts in G;*

(b) *the map φ^* brings the set of proper (=minimum) cuts of Γ onto the set of minimum cuts of G.* (See Fig. 1.)

In fact, φ^* establishes an almost one-to-one correspondence between $\mathbf{M}(G)$ and $\mathbf{P}(\Gamma)$. At the same time, the formation of the "structural graph" and its interrelation to the initial graph are rather simple. This makes the graph Γ as a good model in studying the minimum cuts in G. It should be noted that there exists an algorithm of constructing the "structural graph" whose complexity (in standard operations) is of order close to np, where n is the number of vertices and p is the number of edges of the initial graph (this algorithm is beyond this paper).

It will be shown later that the main result implies the following two properties.

Theorem on circumference disposition. *The vertices of G can be represented as points on a circumference so that each minimum cut of G corresponds to a section of the circumference into two arcs.*

Theorem on the number of minimum cuts. *The number of minimum cuts of G does not exceed $n(n-1)/2$, where $n := |V(G)|$. This bound is attained by the cycle with n vertices whose edges have equal weights.*[3]

2. Let $R = (V_1, V_2)$ and $R' = (V_1', V_2')$ be distinct cuts in a graph. Two cases of their mutual disposition are possible: 1) all sets $V_1 \cap V_1'$, $V_1 \cap V_2'$, $V_2 \cap V_1'$, $V_2 \cap V_2'$ are nonempty; and 2) some of these sets is empty.

The cuts R and R' are called *transversal* in the former case (see Fig. 2a), and *parallel* in the latter case (Fig 2b).[4]

We call an m-cut of G a *p-cut* if it is parallel to any other m-cut; otherwise we call it a *t-cut*. The set of p-cuts of G is denoted by $\mathbf{M}_p(G)$, and the set of t-cuts is denoted by $\mathbf{M}_t(G)$.

In this section we study interrelations of p-cuts and show the following.

Proposition on the structure of the system of p-cuts. *For any graph G, there exists a "structural tree" Δ (with all edges of weight $\widehat{c}$) along with a map $\psi : V(G) \to V(\Delta)$ such that:*

(a) *$\psi(v_1) = \psi(v_2)$ if and only if v_1 and v_2 are separated by none of p-cuts;*

(b) *the map ψ^* gives a one-to-one correspondence between the proper (=minimal, =one-edge) cuts of the tree Δ and the p-cuts of G.*[5]

[3]Using the proof of this theorem given later, one can show that this bound is attained by only those graphs which result in cycles with the edges of the same weight after combining each tuple of parallel edges into one edge.

[4]Transversal and parallel cuts are often called *crossing* and *laminar*, respectively.

[5]From the further description the reader can realize that the fact that $V(G)$ is the set of vertices of a graph is not important; an important thing is that we deal with a finite system of "parallel partitions" of a set V.

Let $R = (V_1, V_2)$ and $R' = (V'_1, V'_2)$ be p-cuts and let $V_1 \cap V'_2 = \emptyset$. We say that a cut $R'' = (V''_1, V''_2)$ *lies between* R and R' (or *separates* R and R') if $V_1 \subset V''_1$ and $V'_2 \subset V''_2$ (or $V_1 \subset V''_2$ and $V'_2 \subset V''_1$). We say that a vertex $v \in V$ *lies between* R and R' if $v \in V'_1 \cap V_2$.

For a set $X \subset V$, define $\overline{X} := V - X$. Consider an arbitrary p-cut $R = (X, \overline{X})$ and all p-cuts separating X (then they do not separate $\overline{X}$). A p-cut R' from this set is called *neighboring* to R if there exists no p-cut lying between R and R'. Denote the set of p-cuts separating X and neighboring to R by $S(X)$. Let $R' = (X', \overline{X}') \in S(X)$ and let R separates X'. Then it is easy to check that any p-cut $R'' \in S(X) \cup \{R\}$ different from R' is neighboring to R'. Therefore, each (maximal) collection of pairwise neighboring p-cuts is constructed canonically and is determined by a p-cut and one of the two sets $X, \overline{X}$ forming it. We call such a collection a *p-bundle* (see Fig. 3).

Let us say that a vertex $v \in V$ *is attributed to a p-bundle* S if v lies between any two cuts in S (when S consists of a single cut $R = (X, \overline{X})$ and is determined by R and X, v is attributed to S if $v \in X$). The set of such vertices for S is denoted by $V(S)$. It is possible that $V(S) = \emptyset$, in which case we say that the p-bundle S is *degenerate.* One can see that every vertex $v \in V$ is attributed to exactly one p-bundle.

Two distinct bundles S' and S'' are called *neighboring* if they share a cut. Such a cut is unique; we denote it by $R(S', S'')$. Every p-cut R belongs to (or "separates") two bundles S' and S''; they "lie on different sides" from R, and R is just of the form $R(S', S'')$.

Let $\{v_1, v_2\}$ be an edge of G. The set of p-cuts separating v_1 and v_2 (if any) can be ranged as a sequence $R_1, R_2, \dots, R_k$, where $R_i = (X_i, \overline{X}_i)$, $v_1 \in X_i$, and $X_1 \subset X_2 \subset \dots \subset X_k$. One easily shows that the pairs of consecutive cuts in this sequence are neighboring, i.e. belong to the same p-bundle: for $i = 2, \dots, k$, the cuts R_{i-1}, R_i belong to the bundle $S(X_i) \cup \{R_i\} = S(\overline{X}_{i-1}) \cup \{R_{i-1}\}$, denoted as $S_i(v_1, v_2)$. Also denote the p-bundle $S(v_1)$ by $S_1(v_1, v_2)$, and $S(v_2)$ by $S_{k+1}(v_1, v_2)$.

In order to obtain the desired "structural graph" Δ for the system of p-cuts of G, we transform the graph G as follows.

1) For each p-bundle S, merge all vertices attributed to S into one vertex, denoted as x_S; if S is degenerate, we add the "dummy" vertex x_S (see Fig. 4, fragments 1,2).

2) Accordingly, each edge $\{v_1, v_2\} \in U$ is transformed into the "edge" $\{x_{S(v_1)}, x_{S(v_2)}\}$; the latter is replaced by the corresponding sequence of edges $\{x_{S_i(v_1,v_2)}, x_{S_{i+1}(v_1,v_2)}\}$, $i = 1, \dots, k$, when $S(v_1) \neq S(v_2)$, and is deleted otherwise (see Fig. 4, fragments 2,3).

3) Merge each set of parallel edges into one edge whose weight is equal to the sum of weights of merged edges (see Fig. 4, fragments 3,4).

Proposition. (a) *The constructed graph* Δ *is a tree in which all edges have the weight* $\widehat{c}(\Delta) = \widehat{c} = \widehat{c}(G)$.

(b) *The vertices* x_S *of* Δ *one-to-one correspond to the p-bundles* S *of the graph* G*, and the edges* $\{x_S, x_{S'}\}$ *to the pairs of neighboring p-bundles* S *and* S'*, or to the p-cuts* $R = R(S, S')$ *of* G.

Proof. Each proper cut in the tree consists of a single edge, and conversely, each edge determines a proper cut. The weight of each of these cuts is equal to $\widehat{c}(G)$. Therefore, $\mathbf{P}(\Delta) = \mathbf{M}(\Delta) = \mathbf{M}_p(\Delta)$.

There is a natural map $\psi : V(G) \to V(\Delta)$ which brings a vertex $v \in V(G)$ attributed to a bundle S to the vertex $x_S \in V(\Delta)$. The induced map ψ^* brings the one-edge cut determined by an edge $\{x_{S'}, x_{S''}\}$ to the cut $R(S', S'')$. It is easy to check that ψ^* is bijective.

This implies the proposition. ▮

3. Now we include into consideration the t-cuts of G. Let R be a t-cut. Any p-cut in G is parallel to R (by the definition of p-cuts). Consider the system of mutually parallel cuts $M_p \cup \{R\}$. Select in it the cuts neighboring to R (from both sides). One can check that these p-cuts form a p-bundle S. Obviously, R separates no two cuts of a bundle different from S. We say that the cut R is *internal* for S.

The main role in our study of t-cuts plays the following lemma. Let R' and R'' be transversal minimum-weigh cuts of a graph, and let these cuts partition the vertices of the graph into the (nonempty) subsets A_1, A_2, A_3, A_4; let for definiteness $R' = (A_1 \cup A_2,\ A_3 \cup A_4)$ and $R'' = (A_1 \cup A_4,\ A_2 \cup A_3)$ (see Fig. 5a). Denote by $c(A_i, A_j)$ the sum of weights of the edges connecting vertices in A_i and in A_j. Also we denote a cut $(X, \overline{X})$ by R_X.

Lemma on a quadrangle. (a) $c(A_1, A_3) = c(A_2, A_4) = 0$*;* $c(A_1, A_2) = c(A_2, A_3) = c(A_3, A_4) = c(A_4, A_1) = \widehat{c}/2$*;*

(b) *The "corner" cuts* R_{A_i}*,* $i = 1, 2, 3, 4$*, are minimum.* (See Fig. 5b.)

Proof. We have:

$$c(R_{A_i}) \geq \widehat{c} \quad \text{implies} \quad 2\widehat{c} \leq \frac{1}{2}\sum_i c(R_{A_i}) = \sum_{i<j} c(A_i, A_j); \quad \text{and}$$

$$2\widehat{c} = c(R') + c(R'') = c(A_1, A_3) + c(A_2, A_4) + \sum_{i<j} c(A_i, A_j).$$

Comparing these, we obtain $c(A_1, A_3) + c(A_2, A_4) \leq 0$. This implies two facts, in view of the nonnegativity of c. Firstly, $c(A_1, A_3) = c(A_2, A_4) = 0$. Secondly, $\frac{1}{2}\sum_i R(A_i) = 2\widehat{c}$, whence $c(R_{A_i}) = \widehat{c}$ for each i, i.e. the cuts R_{A_i} are minimum.

Let $c(A_1, A_2) = d$. Then from proved above it follows that $c(A_4, A_1) = c(A_2, A_3) = \widehat{c} - d$. Similarly, $c(A_3, A_4) = d$. Therefore, $2d = c(R'') = \widehat{c}$, implying $d = \widehat{c} - d = \widehat{c}/2$. ▮

This lemma implies the following

Theorem on crossing minimum cuts. *If* $R' = (X, \overline{X})$ *and* $R'' = (Y, \overline{Y})$ *are minimum cuts of a graph and if* $X \cap Y \neq \emptyset \neq \overline{Y} \cap \overline{Y}$*, then* $R := (X \cap Y, \overline{X} \cup \overline{Y})$ *is a minimum cut as well.*[6]

Given a collection L of cuts, we refer to an (inclusion-wise) maximal subset of vertices separated by none of the cuts in L as an *L-class* (obviously the L-classes are pairwise disjoint and their union is the whole vertex set).

Proposition. *The set of* $\mathbf{M}_p$*-classes coincides with the set of* $\mathbf{M}$*-classes.*

Proof. It suffices to show that if vertices v_1 and v_2 are separated by some m-cut, then they are separated by some p-cut as well. Take a minimal subset X of vertices

[6]Below we prove in essence that if a set V and a system of its partitions satisfy the assertion in this theorem, then the related "structural graph" for this system is a plant.

such that $v_1 \in X$, $v_2 \in \overline{X}$, and $(X, \overline{X})$ is an m-cut. We assert that $(X, \overline{X})$ is a p-cut. Indeed, suppose that this cut is transversal to some m-cut $(Y, \overline{Y})$, and let for definiteness $v_1 \in Y$. By the theorem on crossing minimum cuts, $(X \cap Y, \overline{X} \cup \overline{Y})$ is an m-cut. Since it separates the vertices v_1 and v_2, and $X \cap Y$ is strictly included in X, we obtain a contradiction with the minimality of X. ∎

4. In this section we study a local structure of the system of t-cuts in G. Namely, we consider the t-cuts which are internal for some p-bundle S. As a model in our study, we introduce the graph G_S obtained from G as follows. Each subset of vertices of G separated from S by a cut $R \in S$ is shrunk into one vertex x_R, followed by merging possible parallel edge and deleting loops (see Fig. 6). (The graph G_S is viewed, to some extent, as the star of the vertex x_S in the tree Δ: each pendant vertex $x_{S'}$ of this star, related to a p-bundle S' neighboring to S and separated from S by the cut $R(S, S')$, corresponds to the vertex $x_{R(S,S')}$ in G_S.)

The "images" in G_S of p-cuts in S are called *extreme*; they separate one vertex (of the form x_R) from the other vertices in G_S. These and only these are (minimum) p-cuts in the graph G_S.

If two t-cuts internal for S are transversal, then their images in G_S are transversal as well. The images of internal t-cuts for S and only these are (minimum) t-cuts in the graph G_S.

A cycle whose all edges have the weight $\widehat{c}/2$ is called a $\widehat{c}$-cycle.

Theorem on cycle. *If a p-bundle S has an interior t-cut, then the graph G_S is a $\widehat{c}$-cycle on the vertices x_R, $R \in S$.*

Proof. Suppose we are given some partition of the vertex set of G_S. We refer to each set in this partition as a *class*. The *factor-graph* of G_S induced by this partition is the graph whose vertices are the classes; two classes A and B are connected by an edge in the factor-graph if and only if they are connected by at least one edge in G_S, and the sum of weights of such edges is regarded as the weight $c(A, B)$ of the edge $\{A, B\}$.

We are interested in those partitions of $V(G_S)$ whose induced factor-graphs are $\widehat{c}$-cycles. When there are several partitions of this sort, we will construct a new partition, with a greater number of elements than each of those, and prove that its factor-graph is again a $\widehat{c}$-cycle. Eventually we will come to a partition consisting of single vertices of G_S, which will imply that G_S is a $\widehat{c}$-cycle. Each vertex v of the latter is separated from the other vertices by a minimum cut; this is a p-cut, and therefore, v is of the form x_R, implying the theorem.

We start with taking an arbitrary pair of interior t-cuts in G_S. By the lemma on quadrangle, the factor-graph induced by the partition (into four sets) of $V(G_S)$ by these cuts is a $\widehat{c}$-cycle on four vertices.

Suppose we have a partition P whose factor-graph is a $\widehat{c}$-cycle Q with m vertices. Consider a cut in G_S separating one set in P, say, A, from the other ones; then R_A is a minimum cut. Suppose that A contains two or more vertices. Since any p-cut in G_S separates a single vertex of G_S, R_A is a t-cut. Then R_A is transversal to another t-cut R in G_S, and the latter separates A into two subsets A_1 and A_2 (see Fig. 7). Applying the lemma on quadrangle to R and R_A, we have $c(A_1, A_2) = \widehat{c}/2$. Since $c(R) = \widehat{c}$, it follows that R separates at most two classes in P. Therefore, at least one of the two classes neighboring to A in the cycle Q is not separated by R. Let B be such a class, and let for definiteness R separates B from A_2 (otherwise take A_1). By the lemma on quadrangle (applied to R, R_A), $c(B, A_2) = 0$; this together with $c(B, A) = \widehat{c}/2$ gives $c(B, A_1) = \widehat{c}/2$. Let C be the other class in Q neighboring to A. Since A is connected by edges only with vertices in B and C, and taking into account the equalities $c(A_2, \overline{A}_2) = \widehat{c}$, $c(A_2, A_1) = \widehat{c}/2$ and $c(A_2, B) = 0$, one can conclude that $c(A_2, C) = \widehat{c}/2$.

Thus, replacing A by two sets A_1, A_2, we obtain a partition of $V(G)$ whose factor-graph is a $\widehat{c}$-cycle with $m + 1$ vertices, and the theorem follows. ▮

Remark. The t-cuts in G_S are exactly the cuts separating the cycle G_S into two connected parts, each of which consisting of at least two vertices.

5. The "structural graph" $\Gamma = \Gamma(G)$ is constructed from the "structural tree" $\Delta = \Delta(G)$, as follows. Label the vertices x_S of Δ such that the p-bundle S has an interior t-cut. For each labeled vertex x_S and each of its incident edges $\{x_S, x_{S'}\}$, "insert" a new vertex $x_{S,S'}$ on this edge "in a vicinity" of the vertex x_S. As a result, some edges turn into pairs of edges in series or even into triples of edges in series (with two intermediate vertices of the form $x_{S,S'}$ and $x_{S',S}$). Let Δ' be the obtained tree (see Fig. 8, fragments 1,2).

Remove from Δ' the stars of all labeled vertices x_S. The pendant vertices $x_{S,S'}$ of the star of x_S are not removed and we connect them by edges of weight $\widehat{c}/2$ in the same order as the corresponding vertices $x_{S'}$ are connected in the cycle G_S. The resulting graph Γ' is, obviously, a $\widehat{c}$-plant.