

## A GENERALIZED MFMC-PROPERTY AND MULTICOMMODITY CUT PROBLEMS

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### ABSTRACT

Let  $K_V$  be the complete undirected graph with the vertex-set  $V$  whose edges have nonnegative weights (lengths)  $l$ ,  $T$  be a distinguished subset of its vertices and  $\mathcal{C}$  be a family of cuts in  $K_V$ . In the work mainly the maximum fractional packing problem for  $\mathcal{C}$  is considered and the question is studied: if  $T$  and  $\mathcal{C}$  have the property that, for any  $l$ , the maximum total value of packings is completely determined by distances (with respect to  $l$ ) between elements of  $T$ . It is known that the answer is affirmative when

(i)  $T = \{s, s', t, t'\}$  and  $\mathcal{C}$  consists of all cuts containing both edges  $ss'$  and  $tt'$  (such a  $\mathcal{C}$  is said to be a two-commodity cut family)

(ii)  $|T|$  is even and  $\mathcal{C}$  is the set of all cuts induced by subsets  $X \subset V$  such that  $X \cap T$  is odd (so-called  $T$ -cut family). As main result, a large class (complete under some additional assumptions) of  $T$  and  $\mathcal{C}$  having this property is described. To that end we introduce a concept of a generalized MFMC-property for abstract families of subsets of a set and

establish some interconnection between multicommodity cut problems and multicommodity flow ones. Also other results about cut packing problems are presented.

## 1. INTRODUCTION

For a finite set  $Y$ , let  $K_Y$  denote the complete undirected graph with the vertex-set  $Y$  and  $E(Y)$  denote the set of edges of  $K_Y$ ;  $xy$  (or  $yx$ ) denotes the edge with ends  $x$  and  $y$ . For  $X \subseteq Y$ ,  $\partial X = \partial^Y X$  is the set of edges in  $K_Y$  with one end in  $X$  and the other in  $Y - X$ .  $E \subseteq E(Y)$  is called a *cut* if  $E = \partial X$  for some *proper* subset  $X$  of  $Y$  (i.e.  $\phi \neq X \neq Y$ ). Obviously,  $X \neq X'$  and  $\partial X = \partial X'$  imply  $X' = Y - X$ .

We consider a quadruple  $(V, l, T, S)$  consisting of a (basic) finite set  $V$ , a function  $l: E(V) \rightarrow \mathbf{R}_+$  ( $\mathbf{R}_+$  is the set of nonnegative reals), a distinguished subset  $T \subseteq V$  and a nonempty collection  $S \subseteq 2^T$  of proper subsets of  $T$ ; we refer to  $l, T$  and  $S$  as an *edge length* function, a set of *terminals* and a *scheme*, respectively. Let  $\mathcal{X}(V, S)$  denote the family of sets  $X \subset V$  such that  $X \cap T \in S$ , and  $\mathcal{C}(V, S)$  the set  $\{\partial^V X: X \in \mathcal{X}(V, S)\}$  of cuts (note that  $\mathcal{C}(V, S)$  may contain repeated members, clearly this is so if and only if there are two complementary members  $A$  and  $T - A$  in  $S$ ). A function  $\alpha: \mathcal{X}(V, S) \rightarrow \mathbf{R}_+$  is called *l-admissible* if, for any  $e \in E(V)$ , the value

$$\lambda^\alpha(e) := \sum \{\alpha(X): X \in \mathcal{X}(V, S), e \in \partial X\}$$

does not exceed  $l(e)$  (in other words, the function  $f$  on  $\mathcal{C}(V, S)$  defined by  $f(\partial X) = \alpha(X)$  ( $X \in \mathcal{X}(V, S)$ ) is a packing of  $\mathcal{C}(V, S)$  into  $E(V)$  weighted by  $l$ ). Also *l-admissible*  $\alpha$  will be referred to as a *multicommodity cut*, or a *multicut* for the sake of brevity.

In this work we study mainly the maximum packing problem for  $\mathcal{C}(V, S)$  and  $l$ : find *l-admissible*  $\alpha: \mathcal{X}(V, S) \rightarrow \mathbf{R}_+$  whose total value  $1 \cdot \alpha$  is maximum; this maximum value is denoted by  $p(V, S, l)$ . Such a problem will be shortly called a *multicut max- $\sum$  problem* and denoted by  $\sum(V, S, l)$ .

Applying the linear programming (l.p.) duality theorem to  $\sum(V, S, l)$  we have

$$(1.1) \quad p(V, S, l) = \min \{l \cdot w: w \in \mathbf{R}_+^{E(V)}, w(\partial X) \geq 1 \quad \forall X \in \mathcal{X}(V, S)\}.$$

(For functions  $f, g \in \mathbf{R}^Y$ ,  $f \cdot g$  denotes  $\sum (f(y)g(y): y \in Y)$  and if  $Y' \subseteq Y$  then  $f(Y')$  denotes  $\sum (f(y): y \in Y')$ .)

A number of families of cuts is known for which there is a minimax relation of a more special kind than (1.1). We need some known notions. Let  $\mathcal{F} \neq \phi$  be a collection of nonempty subsets of a set  $E$  (we admit that  $F \subset F'$  for some  $F, F' \in \mathcal{F}$  and that  $\mathcal{F}$  contains repeated members). The *blocker*  $b(\mathcal{F})$  of  $\mathcal{F}$  is the set of minimal subsets of  $E$  meeting each member of  $\mathcal{F}$ .  $\mathcal{F}$  is said to have the (weak) *MFMC-property* (cf. [1]) if, for any  $c \in \mathbf{R}_+^E$ ,

$$\max (1 \cdot f) = \min \{c(B): B \in b(\mathcal{F})\},$$

where the maximum is taken over all  $f: \mathcal{F} \rightarrow \mathbf{R}_+$  satisfying the packing condition  $\sum (f(F): e \in F \in \mathcal{F}) \leq c(e)$  for all  $e \in E$ .

According to the definition a cut family  $\mathcal{C}(V, S)$  has the MFMC-property if the equality

$$(1.2) \quad p(V, S, l) = \min \{l(B): B \in b(\mathcal{C}(V, S))\}$$

holds for any  $l \in \mathbf{R}_+^{E(V)}$ . It can be shown (see Theorem 7.1 and Corollary 7.2) that if  $\mathcal{C}(V, S)$  has the MFMC-property then any member  $B$  of  $b(\mathcal{C}(V, S))$  is the union of some disjoint  $T$ -terminus chains in  $K_Y$  and, moreover, for any  $l \in \mathbf{R}_+^{E(V)}$ , (1.2) may be rewritten in the following more nice form:

$$(1.3) \quad p(V, S, l) = \min \{ \sum (\mu_i(u): u \in U) \},$$

where the minimum is taken over all  $U \subseteq E(T)$  such that  $U \cap \partial^T A \neq \phi$  for each  $A \in S$ .

[Some terminology and notation throughout the paper: a *chain* in  $K_Y$  is a set  $L \subseteq E(Y)$  of *edges* such that a subgraph in  $K_Y$  induced by  $L$  is connected and it has the vertices of valency 2 except two vertices  $x$  and  $y$  which are of valency 1; these  $x$  and  $y$  are called the ends of  $L$  and the edge  $xy$  is denoted by  $eL$ ; a *circuit* is a similar nonempty subset such that each vertex in the induced (connected) subgraph is of valency 2;

a chain  $L$  in  $K_V$  is called  $T$ -terminus (for a distinguished  $T$ ) if  $eL \in E(T)$ ; for  $l \in \mathbf{R}_+^{E(V)}$ ,  $\mu_l \in \mathbf{R}_+^{E(V)}$  denotes the distance function induced by  $l$ , i.e.  $\mu_l(xy)$  is the minimum of  $l(L)$  over all chains  $L$  in  $K_V$  with  $eL = xy$ .]

Here are some known examples of cut families having the MFMC-property (Examples 2, 3 will occur in Section 5, 6).

**Example 1.**  $T = \{s, t\}$  and  $S = \{\{s\}\}$ , i.e.  $\mathcal{C}(V, S)$  consists of the cuts in  $K_V$  "separating"  $s$  and  $t$ . The maximum packing problem for  $\mathcal{C}(V, S)$  with such  $S$  is usually called one-commodity cut problem. R o b a c k e r [2] proved that  $p(V, S, l) = \mu_l(st)$ .

**Example 2.**  $T = \{s, s', t, t'\}$  and  $S = \{\{s, t\}, \{s, t'\}\}$ , i.e.  $\mathcal{C}(V, S)$  consists of the cuts in  $K_V$  each "separating"  $s$  and  $s'$  as well as  $t$  and  $t'$ . The corresponding maximum packing problem is known as two-commodity cut problem. It follows from results of L e h m a n [3] and H u [4] that  $p(V, S, l) = \min \{\mu_l(ss'), \mu_l(tt')\}$  (see also [5], [6]).

**Example 3.**  $|T|$  is a positive even integer and  $S$  consists of the odd-size subsets of  $T$  ( $\mathcal{C}(V, S)$  for such  $T$  and  $S$  is usually called the family of  $T$ -cuts, or *odd-terminus cuts*). E d m o n d s and J o h n s o n [7] (see also [8]) proved that  $p(V, S, l)$  is equal to the minimum of  $l(J)$  over all  $T$ -joins  $J$  in  $K_V$  (a  $T$ -join is a minimal subset  $J \subseteq E(V)$  such that the odd valency vertex-set of the subgraph in  $K_V$  induced by  $J$  is exactly  $T$ ). It is easy to show that Edmonds–Johnson's relation can be expressed also in the form (1.3), where  $U$  ranges over all perfect matchings in  $K_T$ .

Unfortunately, the list of cut families  $\mathcal{C}(V, S)$  having the MFMC-property is rather short (the complete collection of such families for  $|V| \geq |T| + 2$  will be given (without proof) in Section 7). In this paper, we are interested in the cut families for which a special minimax relation, weaker than (1.3), hold. Namely, we study such triples  $V, T, S$  for which, for any  $l \in \mathbf{R}_+^{E(V)}$ , the value  $p(V, S, l)$  is determined by certain non-negative linear combinations of distances between terminals. More precisely, let  $\beta: E(T) \rightarrow \mathbf{R}_+$  be a function satisfying

$$\beta(\partial^T A) (= \sum (\beta(u): u \in \partial^T A)) \geq 1 \quad \text{for all } A \in S,$$

and let  $L^u$  ( $u \in E(T)$ ) be a shortest chain in  $K_V$  (i.e.  $l(L) = \mu_l(eL)$ ) with  $eL = u$ . Then for arbitrary  $l$ -admissible  $\alpha: \mathcal{X}(V, S) \rightarrow \mathbf{R}_+$  we have

$$\begin{aligned} 1 \cdot \alpha &= \sum (\alpha(X): X \in \mathcal{X}(V, S)) \leq \\ &\leq \sum (\alpha(X) \beta(\partial^T (X \cap T)): X \in \mathcal{X}(V, S)) = \\ &= \sum_{u \in E(T)} \beta(u) \sum (\alpha(X): X \in \mathcal{X}(V, S), u \in \partial^V X) \leq \\ &\leq \sum_{u \in E(T)} \beta(u) \sum (\lambda^\alpha(e): e \in L^u) \leq \sum_{u \in E(T)} \beta(u) l(L^u), \end{aligned}$$

whence

$$p(V, S, l) \leq \sum (\beta(u) \mu_l(u): u \in E(T)).$$

**Definition.** We say a scheme  $S \subset 2^T$  belongs to the DSC-class with respect to the multicut max- $\sum$  problems (shortly, to the  $\sum$ -DSC-class) if, for any  $V \supseteq T$  and  $l \in \mathbf{R}_+^{E(V)}$ , the following equality is true:

$$(1.4) \quad \begin{aligned} p(V, S, l) &= \\ &= \min \left\{ \sum_{u \in E(T)} \beta(u) \mu_l(u): \beta \in \mathbf{R}_+^{E(T)}, \beta(\partial^T A) \geq 1 \quad \forall A \in S \right\}. \end{aligned}$$

(The term "DSC-class" abbreviates that of "the class of schemes for which the output of any corresponding packing problem is *determined* by the lengths of *shortest chains* connecting terminals".)

Clearly, a scheme  $S \subset 2^T$  belongs to the  $\sum$ -DSC-class if  $\mathcal{C}(V, S)$  has the MFMC-property for each  $V \supseteq T$  since the relation (1.3) (subject to  $U \cap \partial^T A \neq \emptyset$  for all  $A \in S$ ) is a special case of (1.4) with 0, 1-functions  $\beta$ .

**Example 4.**  $S$  is the set of 1-element subsets of  $T$ . For any  $V \supseteq T$  and  $l$ , the equality (1.4) is known to hold for some  $\beta$  taking values 0, 1 and  $\frac{1}{2}$ ; thus, such  $S$  belongs to the  $\sum$ -DSC-class. At the same time  $\mathcal{C}(V, S)$  does not have the MFMC-property if  $3 \leq |T| \neq 4$ .

Our aim is to characterize the  $\Sigma$ -DSC-class. We need some definitions. Two subsets  $X, Y \subseteq W$  are said to be *crossing* if none of  $X \cap Y, X - Y, Y - X$  and  $W - (X \cup Y)$  is empty and to be *laminar* otherwise. A collection  $\mathcal{X} \subset 2^W$  is called *i-crossing* (for an integer  $i \geq 2$ ) if there are at least  $i$  members of  $\mathcal{X}$  each two of which are crossing. For example, the collection  $\{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{3, 4, 5\}, \{5\}\}$  of subsets of the five elements set is 2- and 3-crossing but not 4-crossing. A collection is called *laminar* if it is not 2-crossing.

**Definition.** A scheme  $S \subset 2^T$  is called *3-complete* if, for any three pairwise crossing members  $A_1, A_2, A_3 \in S$  (if any), there exist non-negative reals  $\gamma(A_i)$  ( $i = 1, 2, 3$ ), and a function  $\epsilon: S \rightarrow \mathbf{R}_+$  such that

$$(1.5) \quad \begin{aligned} \gamma(A_1) + \gamma(A_2) + \gamma(A_3) &\leq \sum(\epsilon(A): A \in S), \\ \sum(\gamma(A_i)\rho_{A_i}: i = 1, 2, 3) &> \sum(\epsilon(A)\rho_A: A \in S), \end{aligned}$$

where, for  $B \subseteq T$ ,  $\rho_B$  denotes the characteristic vector of the subset  $\partial^T B$  in  $\mathbf{R}^{E(T)}$ , and the vector inequality  $>$  in (1.5) means that the left hand side value is no less than the right hand side value for any component  $u \in E(T)$  and it is strictly more for at least one component  $u$ .

For example, where  $T = \{0, \dots, 5\}$ , the scheme  $S$  consisting of six subsets  $A_i = \{i, i + 1, i + 2\}$  ( $i = 1, 2, 3$ ),  $A_4 = \{1\}$ ,  $A_5 = \{3\}$ ,  $A_6 = \{5\}$  is 3-complete because for the triple of pairwise crossing members  $\{A_1, A_2, A_3\}$  (it is the unique triple in  $S$  with such a property) (1.5) holds with  $\gamma(A_i) = 1$ ,  $\epsilon(A_i) = 0$  ( $i = 1, 2, 3$ ) and  $\epsilon(A_j) = 1$  ( $j = 4, 5, 6$ ) (the corresponding component inequality is strict for the edge  $u = 02$ ). Also the schemes in Examples 1, 2 are obviously 3-complete. One can verify that the schemes in Example 3 are also 3-complete (see Section 5).

#### Theorem A.

- (i) A scheme belongs to the  $\Sigma$ -DSC-class if it is 3-complete.
- (ii) Let a scheme  $S \subset 2^T$  be not 3-complete. Then  $S$  does not belong to the  $\Sigma$ -DSC-class. Moreover, for any  $V \supset T$  with  $|V| \geq |T| + 2$  there exists  $l \in \mathbf{R}_+^{E(V)}$  such that (1.4) is not true.

This theorem is central in the paper. It will be proved in Section 4. Note that there is a good characterization of the  $\Sigma$ -DSC-class since a polynomial (in  $|T|, |S|$ ) algorithm can be produce which decides if a scheme  $S$  is 3-complete. Indeed, we can generate all triples in  $S$  and select the triples of pairwise crossing members. Further each triple  $\{A_1, A_2, A_3\}$  of pairwise crossing members can be examined in a constant time whether  $\gamma$  and  $\epsilon$  exist for it satisfying (1.5). For one can identify (in a new vertex) each maximal subset of terminals having the property that  $st \notin \partial^T A_i$  for  $i = 1, 2, 3$  and any two elements  $s$  and  $t$  of the subset, and next solve a corresponding linear program for a graph with at most  $2^3 = 8$  vertices.

Now we introduce one more type of cut packing problems. Such problems will arise in the proof of Theorem A, but they are interesting also by themselves. For arbitrary  $\alpha: \mathcal{X}(V, S) \rightarrow \mathbf{R}_+$  and  $A \in S$  define the *partial value* of  $\alpha$  to be

$$\zeta^\alpha(A) \triangleq \sum(\alpha(X): X \subset V, X \cap T = A).$$

*Multicut existence problem*  $EX(V, S, l, d)$ : given  $V, T, S, l$  (defined as above) and a function  $d: S \rightarrow \mathbf{R}_+$  (as demands on partial values), find  $l$ -admissible  $\alpha: \mathcal{X}(V, S) \rightarrow \mathbf{R}_+$  satisfying  $\zeta^\alpha(A) \geq d(A)$  for all  $A \in S$  (or establish that such  $\alpha$  does not exist).

If  $\alpha$  is  $l$ -admissible and  $L$  is a shortest chain joining terminals  $s$  and  $t$ , then

$$\begin{aligned} \mu_l(st) &= l(L) \geq \sum(\lambda^\alpha(e): e \in L) = \\ &= \sum(\alpha(X)|\partial X \cap L|: X \in \mathcal{X}(V, S)) \geq \\ &\geq \sum(\alpha(X): X \in \mathcal{X}(V, S), st \in \partial^V X) = \\ &= \sum(\zeta^\alpha(A): A \in S, st \in \partial^T A). \end{aligned}$$

And so, in order to have a solution of a problem  $EX(V, S, l, d)$  it is necessary (but, in general, not sufficient) that the inequality

$$(1.6) \quad \mu_l(u) \geq \sum(d(A): A \in S, u \in \partial^T A)$$

should hold for all  $u \in E(T)$ .

**Definition.** We say a scheme  $S \subset 2^T$  belongs to the DSC-class with respect to the multicut existence problems (shortly, to the EX-DSC-class) if, for any  $V \supseteq T$ ,  $l \in \mathbf{R}_+^{E(V)}$  and  $d \in \mathbf{R}_+^S$ ,  $EX(V, S, l, d)$  has a solution whenever the inequality (1.6) holds for each  $u \in E(T)$ .

In Section 3 we prove the following theorem which gives a complete description of the EX-DSC-class.

**Theorem B.**

- (i) A scheme belongs to the EX-DSC-class if it is not 3-crossing.
- (ii) Let a scheme  $S \subset 2^T$  be 3-crossing,  $V \supset T$  and  $|V| \geq |T| + 2$ .

Then there exist  $l \in \mathbf{R}_+^{E(V)}$  and  $d \in \mathbf{R}_+^S$  such that (1.6) holds for each  $u \in E(T)$  but  $EX(V, S, l, d)$  has no solution.

(Note that there is a stronger, half-integer, version of this theorem, it will be mentioned in Section 5.) Theorem B is one of statements on which the proof of Theorem A is based. The proof of Theorem B uses in turn some relationship between multicut existence problems and one class of multicommodity flow problems, it allow us to apply one known result about multicommodity flows. Such a relationship can be shown in abstract terms. To that end, in Section 2, we consider an arbitrary family of subsets of a set with a fixed partition into subfamilies (such a family is named us to be a *compound* one), introduce concepts of the generalized  $\Sigma$ -MFMC- and EX-MFMC-properties for compound families and establish some facts about families having such properties. It will follow from the definitions for our special case that  $S \subset 2^T$  belongs to the  $\Sigma$ -DSC- (resp., EX-DSC-) class if and only if the cut family  $\mathcal{C}(V, S)$  (divided into subfamilies  $\mathcal{C}^V(A) = \{\partial^V X : X \subset V, X \cap T = A\} \quad (A \in S)$ ) has the generalized  $\Sigma$ -MFMC- (resp., EX-MFMC-) property for every  $V \supseteq T$ .

In Section 5 we study a special subclass of 3-complete schemes, so-called 2-complete ones, and show that multicut  $\max\text{-}\Sigma$  problems with such schemes have optimum solutions with some nice features. As an application of this, we give a simple proof of one of Seymour's theorems on multicommodity cuts announced in [9] and proved in [10]. Also here we show that unbounded least "fractionality" of optimum solutions in

multicut  $\max\text{-}\Sigma$  problems with integer-valued lengths and laminar schemes is possible. Finally, in Section 6 the concepts of the generalized MFMC-properties are illustrated with multicommodity flows and some known results are surveyed.

2. COMPOUND FAMILIES AND THE GENERALIZED MFMC-PROPERTIES

Let  $\mathcal{F}$  be a family of nonempty subsets of a finite set  $E$ .  $\mathcal{F}$  is called *compound* and denoted by  $(\mathcal{F}_i : i \in I)$  if some partition of  $\mathcal{F}$  into nonempty subfamilies  $\mathcal{F}_i \quad (i \in I)$  is fixed. For  $c \in \mathbf{R}_+^E$ , a function  $f : \mathcal{F} \rightarrow \mathbf{R}_+$  is  $c$ -admissible if

$$(2.1) \quad \sum(f(F) : e \in F \in \mathcal{F}) \leq c(e) \quad \text{for all } e \in E.$$

We consider the following two packing problems.

- A. *Max- $\Sigma$  problems*  $\Sigma(\mathcal{F}, c)$ : given  $c \in \mathbf{R}_+^E$ , find  $c$ -admissible  $f$  on  $\mathcal{F}$  with  $1 \cdot f$  maximum (this maximum is denoted by  $p(\mathcal{F}, c)$ ).
- B. *Existence problem*  $EX(\mathcal{F}, c, d)$ : given  $c \in \mathbf{R}_+^E$  and  $d \in \mathbf{R}_+^I$ , find  $c$ -admissible  $f$  on  $F$  such that

$$(2.2) \quad \sum(f(F) : F \in \mathcal{F}_i) \geq d(i) \quad \text{for all } i \in I$$

(or establish that such  $f$  does not exist).

Applying the l.p. duality theorem to  $\Sigma(\mathcal{F}, c)$ ,

$$(2.3) \quad p(\mathcal{F}, c) = \min \{c \cdot w : w \in \mathbf{R}_+^E, w(F) \geq 1 \quad \text{for all } F \in \mathcal{F}\}.$$

For a function  $w$  on  $E$  and  $i \in I$ , let  $w^i$  denote the value  $\min \{w(F) : F \in \mathcal{F}_i\}$ . The following gives a criterion of solvability of existence problems.

**Proposition 2.1.**  $EX(\mathcal{F}, c, d)$  has a solution if and only if the inequality

$$(2.4) \quad c \cdot w \geq \sum(d(i)w^i : i \in I)$$

holds for any  $w \in \mathbf{R}_+^E$ .

**Proof.** Solvability of  $EX(\mathcal{F}, c, d)$  means that the system of the linear inequalities (2.1)–(2.2) has a feasible solution. Farkas' lemma applying to this system implies that  $EX(\mathcal{F}, c, d)$  has a solution if and only if the inequality

$$c \cdot w - \sum(d(i)v(i): i \in I) \geq 0$$

holds for any  $w \in \mathbf{R}_+^E$  and  $v \in \mathbf{R}_+^I$  satisfying

$$w(F) - v(i) \geq 0 \text{ for all } i \in I \text{ and } F \in \mathcal{F}_i,$$

whence the result easily follows. ■

Now we define the collection

$$\hat{b}(\mathcal{F}) = \bigcup (b(\mathcal{F}_i): i \in I)$$

(where  $b(\mathcal{F}_i)$  is the blocker of  $\mathcal{F}_i$ ) which is called the *imperfect blocker* of the compound family  $(\mathcal{F}_i: i \in I)$ . For  $B \subseteq E$ , we say that  $B$  meets  $\mathcal{F}_i$  if  $B$  meets each member of  $\mathcal{F}_i$ .

Let  $\theta(E') = \theta^E(E')$  denote the characteristic vector in  $\mathbf{R}^E$  of a subset  $E' \subseteq E$ . Consider a function  $\delta: \hat{b}(\mathcal{F}) \rightarrow \mathbf{R}_+$  satisfying

$$(2.5) \quad \sum(\delta(B): B \in \hat{b}(\mathcal{F}), B \text{ meets } \mathcal{F}_i) \geq 1 \text{ for all } i \in I.$$

For  $w = \sum(\delta(B)\theta(B): B \in \hat{b}(\mathcal{F}))$  and any  $F \in \mathcal{F}_i$ ,  $i \in I$ , we have

$$\begin{aligned} w(F) &= \sum(\delta(B)\theta(B)\theta(F): B \in \hat{b}(\mathcal{F})) \geq \\ &\geq \sum(\delta(B): B \in \hat{b}(\mathcal{F}), B \text{ meets } \mathcal{F}_i) \geq 1 \end{aligned}$$

and hence (2.3) implies

$$p(\mathcal{F}, c) \leq \sum(\delta(B)c(B): B \in \hat{b}(\mathcal{F})).$$

**Definition.** We say a compound family  $\mathcal{F}$  has the generalized MFMC-property with respect to the  $\max\text{-}\sum$  problems (shortly, the *gen.  $\sum$ -MFMC-property*) if the equality

$$(2.6) \quad p(\mathcal{F}, c) = \min \{ \sum(\delta(B)c(B): B \in \hat{b}(\mathcal{F})) \}$$

holds for any  $c \in \mathbf{R}_+^E$ , where the minimum is taken over all  $\delta: \hat{b}(\mathcal{F}) \rightarrow \mathbf{R}_+$  satisfying (2.5). (Obviously, the notion of the gen.  $\sum$ -MFMC-prop-

erty is equivalent to that of the MFMC-property in the case  $|I| = 1$ , i.e. when an ordinary (noncompound) family is considered.)

Now consider an existence problem  $EX(\mathcal{F}, c, d)$ . It follows from Proposition 2.1 that it is necessary (but generally not sufficient) for solvability of this problem that the inequality

$$(2.7) \quad c(B) \geq \sum(d(i): i \in I, B \text{ meets } \mathcal{F}_i)$$

holds for all  $B \in \hat{b}(\mathcal{F})$ , because, for  $B \in \hat{b}(\mathcal{F})$  and  $w = \theta(B)$ , we have  $c \cdot w = c(B)$  and  $w^i = \min \{ |B \cap F|: F \in \mathcal{F}_i \} \geq 1$  for all  $i \in I$  such that  $B$  meets  $\mathcal{F}_i$ .

**Definition.** We say a compound family  $\mathcal{F}$  has the generalized MFMC-property with respect to the existence problems (shortly, the *gen. EX-MFMC-property*) if, for any  $c \in \mathbf{R}_+^E$  and  $d \in \mathbf{R}_+^I$ ,  $EX(\mathcal{F}, c, d)$  has a solution whenever (2.7) holds for all  $B \in \hat{b}(\mathcal{F})$ .

(It is easy to show that in the case  $|I| = I$  the notions of the gen. EX-MFMC- and  $\sum$ -MFMC-properties are equivalent and, therefore, both of them are equivalent to the MFMC-property.)

Now we introduce some packing problem for the imperfect blockers. Given  $w \in \mathbf{R}_+^E$ , let  $\kappa: \hat{b}(\mathcal{F}) \rightarrow \mathbf{R}_+$  be a  $w$ -admissible function, i.e.

$$(2.8) \quad \sum(\kappa(B)\theta(B): B \in \hat{b}(\mathcal{F})) \leq w.$$

Then, for any  $F \in \mathcal{F}$ , we get

$$\begin{aligned} w(F) &\geq \theta(F) \sum(\kappa(B)\theta(B): B \in \hat{b}(\mathcal{F})) \geq \\ &\geq \sum(\kappa(B): B \in \hat{b}(\mathcal{F}), B \cap F \neq \emptyset), \end{aligned}$$

therefore, for any  $i \in I$ ,

$$(2.9) \quad w^i (= \min \{ w(F): F \in \mathcal{F}_i \}) \geq \sum(\kappa(B): B \in \hat{b}(\mathcal{F}), B \text{ meets } \mathcal{F}_i).$$

We say that a  $w$ -admissible  $\kappa$  locks  $\mathcal{F}_i$  if

$$(2.10) \quad w^i = \sum(\kappa(B): B \in \hat{b}(\mathcal{F}), B \text{ meets } \mathcal{F}_i).$$

*Locking problem* LOCK  $(\mathcal{F}, w)$ : given  $w \in \mathbf{R}_+^E$ , find a  $w$ -admissible  $\kappa: \hat{b}(\mathcal{F}) \rightarrow \mathbf{R}_+$  which locks  $\mathcal{F}_i$  for all  $i \in I$  (or establish that such a

$\kappa$  does not exist).  $\mathcal{F}$  is called *lockable* if  $\text{LOCK}(\mathcal{F}, w)$  has a solution (i.e. such a  $\kappa$  exists) for any  $w \in \mathbf{R}_+^E$ .

**Lemma 2.2.** *The following statements are equivalent:*

- (i)  $\mathcal{F}$  has the gen. EX-MFMC-property;
- (ii)  $\mathcal{F}$  is lockable.

**Proof.** (i)  $\rightarrow$  (ii). In view of Proposition 2.1, the fact that  $\mathcal{F}$  has the gen. EX-MFMC-property means that, for any fixed  $w \in \mathbf{R}_+^E$ , (2.4) is true for every  $c \in \mathbf{R}_+^E$  and  $d \in \mathbf{R}_+^I$  satisfying (2.7) for all  $B \in \hat{b}(\mathcal{F})$ . Applying Farkas' lemma to the implication  $\forall c \geq 0, d \geq 0 (\forall B(2.7) \rightarrow (2.4))$  we get that there is a  $\kappa: \hat{b}(\mathcal{F}) \rightarrow \mathbf{R}_+$  satisfying (2.8) and

$$(2.11) \quad -\sum(\kappa(B): B \in \hat{b}(\mathcal{F}), B \text{ meets } \mathcal{F}_i) \leq -w^i \text{ for all } i \in I.$$

But the  $w$ -admissibility of  $\kappa$  implies (2.9) for any  $i \in I$ , and (2.10) holds because of (2.9) and (2.11). Considering all  $w \in \mathbf{R}_+^E$  we conclude that  $\mathcal{F}$  is lockable. (ii)  $\rightarrow$  (i) is proved by conversion of the above arguments. ■

Lemma 2.2 will be used in the proof of Theorem B. The following lemma will be needed for proving Theorem A.

**Lemma 2.3.** *Let  $\mathcal{F}$  be a compound family and  $c \in \mathbf{R}_+^E$ . The following statements are equivalent:*

- (i) the equality (2.6) is true;
- (ii) the inequality

$$(2.12) \quad c \cdot w \geq \sum(d(i): i \in I)$$

is valid for any  $d \in \mathbf{R}_+^I$  and  $w \in \mathbf{R}_+^E$  such that  $d$  satisfies (2.7) for all  $B \in \hat{b}(\mathcal{F})$  and  $w$  does

$$(2.13) \quad w(F) \geq 1 \text{ for all } F \in \mathcal{F}.$$

**Proof.** First of all, we observe that validity of (2.6) is equivalent to feasibility (by  $f: \mathcal{F} \rightarrow \mathbf{R}_+$  and  $\delta: \hat{b}(\mathcal{F}) \rightarrow \mathbf{R}_+$ ) of the system of linear inequalities  $\{(2.1), (2.5)\}$  together with the inequality

$$(2.14) \quad -\sum(f(F): F \in \mathcal{F}) + \sum(c(B)\delta(B): B \in \hat{b}(\mathcal{F})) \leq 0.$$

By Farkas' lemma, the system  $\{(2.1), (2.5), (2.14)\}$  has a feasible solution if and only if the inequality

$$(2.15) \quad c \cdot \tilde{w} - \sum(\tilde{d}(i): i \in I) \geq 0$$

holds for any  $\tilde{w} \in \mathbf{R}_+^E, \tilde{d} \in \mathbf{R}_+^I$  and  $\psi \in \mathbf{R}_+$  satisfying

$$(2.16) \quad \tilde{w}(F) - \psi \geq 0 \text{ for all } F \in \mathcal{F},$$

$$(2.17) \quad -\sum(\tilde{d}(i): i \in I, B \text{ meets } \mathcal{F}_i) + c(B)\psi \geq 0 \text{ for all } B \in \hat{b}(\mathcal{F})$$

( $\tilde{w}, \tilde{d}$  and  $\psi$  are variables dual to (2.1), (2.5) and (2.14), respectively). But the implication  $\{(2.16), (2.17)\} \rightarrow (2.15)$  is equivalent to  $\{(2.13), (2.7)\} \rightarrow (2.12)$ . For, partly, if  $\psi = 0$  then (2.17) implies  $\tilde{d} = 0$  (as  $\tilde{d} \geq 0$ , each  $F \in \mathcal{F}$  is nonempty and each  $\mathcal{F}_i$  is nonempty), whence (2.15) trivially follows, partly, for arbitrary  $\psi > 0$ , the mapping  $(\tilde{w}, \tilde{d}, \psi) \rightarrow (w, d)$  defined by  $w = \tilde{w}/\psi$  and  $d = \tilde{d}/\psi$  turns the former implication into the latter one. ■

**Remark 2.4.** It follows from Lemmas 2.1 and 2.3 that if  $\mathcal{F}$  has the gen. EX-MFMC-property then it has the gen.  $\Sigma$ -MFMC-property as well (because (2.4) and (2.13) together imply (2.12)).

Now we return to cut packing problem. A cut family  $\mathcal{C}(V, S)$  has a natural representation as a compound one, namely  $(\mathcal{C}^V(A): A \in S)$ , where  $\mathcal{C}^V(A)$  denotes  $\{\partial^V X: X \subset V, X \cap T = A\}$  for  $A \in S$ . Obviously, the blocker of  $\mathcal{C}^V(A)$  consists of all the minimal chains in  $K_V$  with one end in  $A$  and the other one in  $T - A$ . Thus  $\hat{b}(\mathcal{C}(V, S)) = \mathcal{L}^S$ , where  $\mathcal{L}^S$  is the set of minimal  $T$ -terminus chains  $L$  such that  $eL \in \partial^T A$  for some  $A \in S$  (such a chain  $L$  is called an  $S$ -chain). According to the definition,  $\mathcal{C}(V, S)$  has the gen.  $\Sigma$ -MFMC-property if, for any  $l \in \mathbf{R}_+^{E(V)}$ ,

$$(2.18) \quad p(V, S, l) = \min \{\sum(\delta(L)l(L): L \in \mathcal{L})\},$$

where  $\mathcal{L} = \mathcal{L}^S$  and the minimum is taken over all  $\delta: \mathcal{L} \rightarrow \mathbf{R}_+$  satisfying

$$(2.19) \quad \sum(\delta(L): L \in \mathcal{L}, eL \in \partial^T A) \geq 1 \text{ for all } A \in S.$$

Let  $\mathcal{L}(V, T)$  denote the set of  $T$ -terminus chains in  $K_V$ . Note that if  $eL \in \partial^T A$  for some  $T$ -terminus  $L$  and  $A \in S$ , then there is an  $S$ -chain  $L' \subseteq L$  for which also  $eL' \in \partial^T A$ . It easily follows from this that if (2.18) (subject to (2.19)) holds for  $\mathcal{L} = \mathcal{L}(V, T)$ , then the same is true for  $\mathcal{L} = \mathcal{L}^S$ , and vice versa. Next, let  $\mathcal{L} = \mathcal{L}(V, T)$ , and assume that (2.18) (subject to (2.19)) holds and  $\delta^*$  attains the minimum in (2.18). Then, for  $L \in \mathcal{L}(V, T)$ ,  $\delta^*(L) > 0$  implies that  $L$  is a shortest chain (since if  $k(L') < k(L)$  and  $eL' = eL$  for some chain  $L'$ , the the function  $\delta'$  defined by  $\delta'(L) = 0$ ,  $\delta'(L') = \delta^*(L) + \delta^*(L')$  and  $\delta'(L'') = \delta^*(L'')$  for the remaining  $L''$ 's in  $\mathcal{L}(V, T)$  would satisfy (2.19), which would lead to a contradiction to the minimality of  $\delta^*$ ). Put  $\beta^*(u) = \sum(\delta^*(L): L \in \mathcal{L}(V, T), eL = u)$  for each  $u \in E(T)$ . Then  $\beta^*$  attains the equality in (1.4). Conversely, validity of (1.4) easily implies that of (2.18) (subject to (2.19)) for  $\mathcal{L} = \mathcal{L}(V, S)$ . Thus, a scheme  $S \subset 2^T$  belongs to the  $\sum$ -DSC-class if and only if  $\mathcal{L}(V, S)$  has the gen.  $\sum$ -MFMC-property for all  $V \supseteq T$ .

Next, for  $\mathcal{F} = \mathcal{C}(V, S)$ ,  $c = l \in \mathbf{R}_+^{E(V)}$  and  $d \in \mathbf{R}_+^S$ , the inequality (2.7) is specified as

$$(2.20) \quad k(L) \geq \sum(d(A): A \in S, eL \in \partial^T A),$$

where  $L$  is an  $S$ -chain in  $K_V$ . Easy arguments show that if, given  $l$  and  $d$ , the inequality (2.20) holds for any  $S$ -chain  $L$ , then it does for any  $T$ -terminus chain  $L$ , whence validity of (2.20) for all  $L \in \mathcal{L}^S$  is equivalent to that of (1.6) for all  $u \in E(T)$ . Thus, a scheme  $S \subset 2^T$  belongs to the EX-DSC-class if and only if  $\mathcal{C}(V, S)$  has the gen. EX-MFMC-property for all  $V \supseteq T$ .

For  $w \in \mathbf{R}_+^{E(V)}$  and  $A \in S$ , let  $w^A$  denote the value  $\min\{w(E'): E' \in \mathcal{C}^V(A)\}$ . According to the above definition, a  $w$ -admissible function  $\varphi: \mathcal{L}^S \rightarrow \mathbf{R}_+$  locks  $\mathcal{C}^V(A)$  if

$$(2.21) \quad w^A = \sum(\varphi(L): L \in \mathcal{L}, eL \in \partial^T A),$$

where  $\mathcal{L} = \mathcal{L}^S$ . As above, we extend  $\mathcal{L}$  to  $\mathcal{L}(V, T)$ , and let  $\varphi: \mathcal{L} \rightarrow \mathbf{R}_+$  (where  $\mathcal{L} = \mathcal{L}(V, T)$ ) be a  $w$ -admissible functions satisfying (2.21). Then the function  $\varphi'$  on  $\mathcal{L}^S$  defined by

$$\varphi'(L') = \sum(\varphi(L): L \in \mathcal{L}(V, T), L' \subseteq L), \quad L' \in \mathcal{L}^S,$$

is also  $w$ -admissible and

$$\begin{aligned} \sum(\varphi'(L'): L \in \mathcal{L}^S, eL' \in \partial^T A) &\geq \\ &\geq \sum(\varphi(L): L \in \mathcal{L}(V, T), eL \in \partial^T A), \end{aligned}$$

whence it follows that  $\varphi'$  locks  $\mathcal{C}^V(A)$ . Therefore, we may consider the following slightly different (but equivalent, in essence) form of the locking problem for  $\mathcal{C}(V, S)$ : given  $w \in \mathbf{R}_+^{E(V)}$ , find a  $w$ -admissible  $\varphi: \mathcal{L}(V, T) \rightarrow \mathbf{R}_+$  satisfying (2.21) for all  $A \in S$  (or establish that such  $\varphi$  does not exist). This form is denoted by  $\text{LOCK}^f(V, S, w)$  and called a *multiflow locking problem*.

The problem  $\text{LOCK}^f(V, S, w)$  will appear in the proof of Theorem B. Note that such a problem has also other applications, in particular, it is used for solving a number of multicommodity flow problems (see [11], [12]).

### 3. PROOF OF THEOREM B

A proof of Theorem B can be immediately obtained from Lemma 2.2 and the following theorem.

#### Theorem 3.1.

- (i) *Let a scheme  $S \subset 2^T$  be not 3-crossing. Then, for any  $V \supseteq T$ , the cut family  $\mathcal{C}(V, S)$  is lockable.*
- (ii) *Let a scheme  $S \subset 2^T$  be 3-crossing,  $V \supset T$  and  $|V| \geq |T| + 2$ . Then  $\mathcal{C}(V, S)$  is not lockable.*

Theorem 3.1 is due to M. V. Lomonosov and the author (see [13], [12]). However, in order to make the paper self-contained I give another proof of Theorem B. Namely, a direct proof of the part (ii) of Theorem B and a new proof of the part (i) Theorem 3.1 are given here (the part (i) of Theorem B follows from the part (i) of Theorem 3.1 and Lemma 2.2).

**Proof of the part (ii) of Theorem B.** Let  $S \subset 2^T$  be 3-crossing,  $V \supset T$  and  $|V| \geq |T| + 2$ . Choose an arbitrary triple of pairwise crossing



members in  $S$ , say  $S^* = \{A_1, A_2, A_3\}$ . Put  $d(A_i) = 1$  ( $i = 1, 2, 3$ ) and  $d(A) = 0$  for all  $A \in S - S^*$ . Now our purpose is to determine a function  $l$  on  $E(V)$  such that the inequality (1.6) holds for any  $u \in E(T)$  but there exists a function  $w \in \mathbf{R}_+^{E(V)}$  such that

$$l \cdot w < \sum (d(A)w^A : A \in S)$$

(then by Proposition 2.1,  $\text{EX}(V, S, l, d)$  has no solution).

First of all we introduce two sorts of triples of pairwise crossing sets:

1.  $S_1 = \{\{0, i\} : i = 1, 2, 3\} \subset 2^{T_1}$ , where  $T_1 = \{0, 1, 2, 3\}$ ,
2.  $S_2 = \{\{i, i + 1, i + 2\} : i = 1, 2, 3\} \subset 2^{T_2}$ , where  $T_2 = \{0, \dots, 5\}$ .

(see Fig. 3.1, where the corresponding families of cuts are shown).

**Proposition 3.2.** *Let  $T'$  be a minimal subset of  $T$  having the property that the subsets  $A'_i = A_i \cap T'$  ( $i = 1, 2, 3$ ), are still pairwise crossing with respect to  $T'$ . Then the triple  $S' = \{A'_1, A'_2, A'_3\} \subset 2^{T'}$  is equivalent to either  $S_1$  or  $S_2$ , that is, there is a one-to-one mapping  $\chi: T' \rightarrow T_1$  (resp.,  $\rightarrow T_2$ ) such that,  $\chi(A'_i)$  is either  $\{0, i\}$  or  $T_1 - \{0, i\}$  (resp., is either  $\{i, i + 1, i + 2\}$  or  $T_2 - \{i, i + 1, i + 2\}$ ) for  $i = 1, 2, 3$ .*

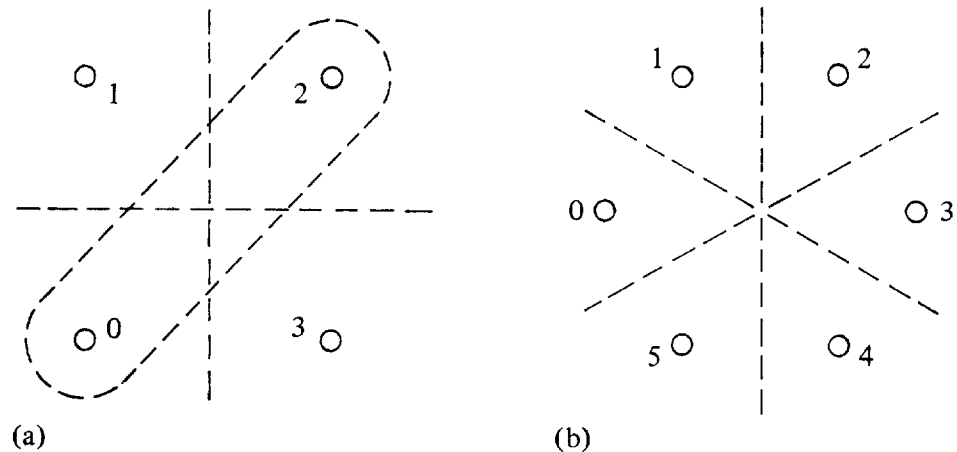


Figure 3.1

This statement occurred in [12]. A sketch of that proof is the following. Obviously,  $|A_1^* \cap A_2^* \cap A_3^*| \leq 1$  for arbitrary  $A_i^* \in \{A'_i, T' - A'_i\}$  ( $i = 1, 2, 3$ ). Without loss of generality we may assume  $A'_1 \cap A'_2 \cap A'_3 \neq \emptyset$ . The following two cases are possible:

1. for  $j = 1, 2, 3$ ,  $A'_j$  does not lie in  $A'_p \cup A'_q$ , where  $\{p, q\} = \{1, 2, 3\} - \{j\}$ ,
2.  $A'_j \subseteq A'_p \cup A'_q$  for some  $j \in \{1, 2, 3\}$  and  $\{p, q\} = \{1, 2, 3\} - \{j\}$ .

Using the minimality of  $T'$  one shows that  $S'$  is equivalent to  $S_1$  in the first case and  $S'$  is equivalent to  $S_2$  in the second case. ■

Now, let  $T'$  and  $S'$  be defined for above  $T$  and  $S^*$  as in Proposition 3.2.

*Case 1.*  $|T'| = 4$  (and  $S'$  is equivalent to  $S_1$ ).

We may assume that  $T'$  is identified with  $T_1$  by use of the map  $\chi$ . Take an arbitrary element  $x$  in  $V - T$ , and let  $G_1 = (V_1, E_1)$  be the subgraph of the graph  $K_V$  drawn in Fig. 3.2a. Put  $l(e) = 1$  for all  $e \in E_1$  and  $l(e) \geq 2$  for all  $e \in E(V) - E_1$ .

*Case 2.*  $|T'| = 6$  (and  $S'$  is equivalent to  $S_2$ ).

Similarly we assume that  $T'$  is identified with  $T_2$ . Take arbitrary two elements  $x$  and  $y$  in  $V - T$  (which exist because  $|V| \geq |T| + 2$ ), and let  $G_2 = (V_2, E_2)$  be the subgraph of  $K_V$  drawn in Fig. 3.2b. Put  $l(e) = 1$  for all  $e \in E_2$  and  $l(e) \geq 3$  for all  $e \in E(V) - E_2$ .

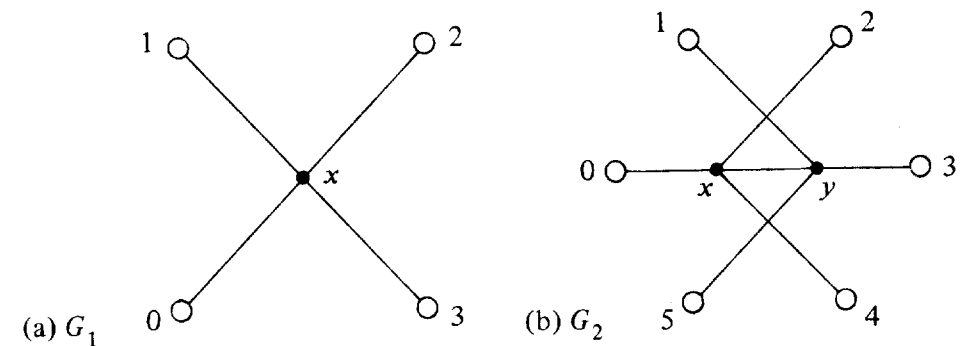


Figure 3.2

One can check that in both cases the inequality (1.6) holds for all  $u \in E(T)$ . However, in both cases the problem  $EX(V, S, l, d)$  has no solution. Indeed, consider the function  $w$  on  $E(V)$  defined to be 1 on  $E_1$  (respectively, on  $E_2$ ) and 0 on the remaining edges in  $E(V)$ . It is easy to check that  $w^{A_i} = 2$  in Case 1 and  $w^{A_i} = 3$  in Case 2 ( $i = 1, 2, 3$ ). If Case 1 takes place we have  $l \cdot w = 4 < 6 = \sum(d(A)w^A : A \in S)$  and in Case 2 does we have  $l \cdot w = 7 < 9 = \sum(d(A)w^A : A \in S)$ . ■

**Proof of the part (i) of Theorem 3.1.** We say that a nonnegative integer-valued function  $w$  on  $E(V)$  is *inner cut even* if the value  $w(\partial^V\{x\})$  is even for all  $x \in V - T$ .

**Lemma 3.3.** *Let  $S \subset 2^T$  be not 3-crossing.  $V \supseteq T$ , and let  $w$  be inner cut even function on  $E(V)$ . Then there exists a  $w$ -admissible  $\varphi: \mathcal{L}(V, T) \rightarrow \mathbf{Z}_+$  locking all  $A \in S$  ( $\mathbf{Z}_+$  is the set of nonnegative integers).*

(Clearly this lemma is a strengthening of (i) in Theorem 3.1. It is due to Lomonosov and the author [13], [12]; their proof is based on an algorithm whose running time is bounded by  $w(E(V))$  times a polynomial in  $|V|$  and  $|S|$ . In [12] also another algorithm is developed which works with arbitrary nonnegative real-valued  $w$  and uses  $O(|T|^5|V|^5)$  operations.) Here I present another and simpler proof of this lemma.

**Proof.** We proceed by induction on  $\eta(w) = \sum(w(\partial\{x\}) : x \in V - T)$ . Assume that  $\eta(w) = 0$ , and put  $\varphi(L) = w(u)$  for all  $L = \{u\}$ . ( $u \in E(T)$ ) and  $\varphi(L) = 0$  for the remaining  $L$ 's in  $\mathcal{L}(V, T)$ . Then  $w^A = \sum(w(u) : u \in \partial^T A)$  and (2.21) is obviously true for all  $A \in S$ . Now we assume that  $\eta(w) > 0$ , and let  $x \in V - T$  be such a vertex that  $w(\partial\{x\}) > 0$ . Define the set  $V(x)$  to be  $\{y : w(xy) > 0\}$ . For  $X \subset V$ , let  $X^*$  denote  $X$  if  $x \notin X$  and it do  $V - X$  if  $x \in X$ .

Assume that  $|V(x)| = 1$ , and let  $V(x) = \{y\}$ . Put  $w'(xy) = 0$  and  $w'(e) = w(e)$  ( $e \in E(V) - \{xy\}$ ). Since  $w(xy) (= w(\partial\{x\}))$  is even,  $w'$  is inner cut even. We show that  $w'^A = w^A$  for all  $A \in S$  (then, by induction, there is a  $w'$ -admissible  $\varphi: \mathcal{L}(V, T) \rightarrow \mathbf{Z}_+$  locking all  $A \in S$  (concerning  $w'$ ), and hence  $\varphi$  is  $w$ -admissible and it locks all  $A \in S$  for  $w$ ). Let  $A \in S$ ,  $X \subset V$ ,  $X \cap T = A$  and  $w'(\partial X) = w'^A$ . If  $xy \notin \partial X$ ,

then  $w'(\partial X) = w(\partial X) \geq w^A$ . Suppose  $xy \in \partial X$ , and let  $Y = X^* \cup \{x\}$ . Then  $Y \cap T = X^* \cap T$  and  $w'(\partial X) = w'(\partial Y) = w(\partial Y) \geq w^A$ , therefore  $w'^A = w^A$ .

Thus, we may assume that  $|V(x)| \geq 2$ . For two distinct  $y, z \in V(x)$ , we say that  $w'$  is obtained from  $w$  by *righting* on  $y, z$  if  $w'(xy) = w(xy) - 1$ ,  $w'(xz) = w(xz) - 1$ ,  $w'(yz) = w(yz) + 1$  and  $w'(e) = w(e)$  for the remaining  $e$ 's in  $E(V)$ . Obviously,  $\eta(w') < \eta(w)$  and  $w'$  is inner cut even when  $w$  does so. We shall show that there are  $y, z \in V(x)$  such that the function  $w'$  obtained from  $w$  by righting on  $y, z$  satisfies  $w'^A = w^A$  for all  $A \in S$ . Then, by induction, there exists a  $w'$ -admissible  $\varphi': \mathcal{L}(V, T) \rightarrow \mathbf{Z}_+$  which locks all  $A \in S$  (for  $w'$ ). If  $w'(yz) > \sum(\varphi'(L) : L \in \mathcal{L}(V, T), yz \in L)$ , then  $\varphi'$  is also  $w$ -admissible and hence  $\varphi'$  locks all  $A \in S$  for  $w$ . Otherwise, we take an arbitrary  $L \in \mathcal{L}(V, T)$  such that  $yz \in L$  and yield the chain  $L'$  for which  $eL' = eL$  and  $L' \subseteq (L - \{yz\}) \cup \{xy, xz\}$ . Then the function  $\varphi$  defined by  $\varphi(L) = \varphi'(L) - 1$ ,  $\varphi(L') = \varphi'(L') + 1$  and  $\varphi(L'') = \varphi'(L'')$  ( $L'' \in \mathcal{L}(V, T) - \{L, L'\}$ ) is obviously  $w$ -admissible and it locks all  $A \in S$  for  $w$ .

We need three simple claims. We say that  $X$  *separates*  $Y$  and  $Z$  if either  $Y \subseteq X$ ,  $Z \cap X = \emptyset$  or  $Z \subseteq X$ ,  $Y \cap X = \emptyset$ .

(1) Let  $w$  be inner cut even, and  $w'$  be obtained from  $w$  by righting on  $y, z$  ( $y, z \in V(x)$ ). Next, let  $w'^A < w^A$  for some  $A \in S$ , and  $X \subset V$  be such that  $X \cap T = A$  and  $w'(\partial X) = w'^A$ . Then  $X$  separates  $\{x\}$  and  $\{y, z\}$ , and  $w(\partial X) = w^A$ .

Indeed, let  $E_1 = \partial^V X$ .  $w'(E_1) = w'^A < w^A \leq w(E_1)$  easily implies that  $xy, xz \in E_1$  and  $yz \notin E_1$ ; and so  $X$  separates  $\{x\}$  and  $\{y, z\}$ , and  $w'(E_1) = w(E_1) - 2$ , whence  $w'^A \geq w^A - 2$ . Let  $Y \subset V$  be such that  $Y \cap T = A$  and  $w(E_2) = w^A$ , where  $E_2 = \partial Y$ . Since  $X \cap T = Y \cap T$  and  $w$  is inner cut even, then  $w(E_1) - w(E_2)$  is even. Hence  $w'^A = w^A - 2$  and  $w(E_1) = w^A$ .

$X$  is called  $V(x)$ -*maximal* if  $X \in \mathcal{X}(V, S)$ ,  $w(\partial X) = w^{X \cap T}$  and the set  $X^* \cap V(x)$  is maximal under these conditions ( $X^*$  is defined as above).

(2) Let  $X_1$  and  $X_2$  be two  $V(x)$ -maximal sets such that  $X_1^* \cap V(x) \neq X_2^* \cap V(x)$  and  $X_1^* \cap X_2^* \cap V(x) \neq \emptyset$ , and let  $A_i = X_i \cap T$  ( $i = 1, 2$ ). Then  $A_1$  and  $A_2$  (considered as subsets of  $T$ ) are crossing.

Suppose, for a contradiction, that  $A_1$  and  $A_2$  are laminar. Let  $A'_i = X_i^* \cap T$  ( $i = 1, 2$ ). Assume that  $A'_1 \cap A'_2 = \emptyset$ , and let  $Y_1 = X_1^* - X_2^*$  and  $Y_2 = X_2^* - X_1^*$ .  $A'_1 \cap A'_2 = \emptyset$  implies  $Y_i \cap T = A'_i$ , and we have  $w(\partial X_i^*) = w^{A_i} \leq w(\partial Y_i)$  ( $i = 1, 2$ ). Hence, for any  $e \in E(V)$  with  $w(e) > 0$ , the obvious submodular inequality

$$|\{e\} \cap \partial X_1^*| + |\{e\} \cap \partial X_2^*| \geq |\{e\} \cap \partial Y_1| + |\{e\} \cap \partial Y_2|$$

holds as equality. Thus,  $w(zv) = 0$  for any  $z \in X_1^* \cap X_2^*$  and  $v \in V - (X_1^* \cup X_2^*)$  which contradicts to  $X_1^* \cap X_2^* \cap V(x) \neq \emptyset$  and  $x \in V - (X_1^* \cup X_2^*)$ . A similar contradiction is produced when  $A'_1 \cup A'_2 = T$ . It remains to consider the case  $\{A'_1 \cap A'_2, A'_1 \cup A'_2\} = \{A'_1, A'_2\}$ . Let, for definiteness,  $A'_1 \cup A'_2 = A'_1$ , and let  $Y = X_1^* \cup X_2^*$ . Then, by arguments as above,  $w(\partial Y) = w^{A'_1}$ . But the set  $Y \cap V(x)$  strongly includes  $X_i^* \cap V(x)$  ( $i = 1, 2$ ), contrary to the  $V(x)$ -maximality of  $X_i$ .

(3) Let  $X \subset V$ ,  $X \cap T = A$  and  $w(\partial X) = w^A$ . Then  $w(E') \leq w(E'')$ , where  $E' = \{xy: y \in X^* \cap V(x)\}$  and  $E'' = \{xy: y \in V(x) - X^*\}$ .

Indeed, let  $Y = X^* \cup \{x\}$ . Then  $Y \cap T = X^* \cap T$ , whence  $w(\partial X^*) = w^A \leq w(\partial Y)$ , where  $A = X \cap T$ . We have  $\partial X^* - \partial Y = E'$  and  $\partial Y - \partial X^* = E''$ , and the result follows.

Now we continue proving the lemma. Suppose, for a contradiction, that each two  $y, z \in V(x)$  have the property that  $w^{A'} < w^A$  for some  $A \in S$  and the function  $w'$  obtained from  $w$  by righting on  $y, z$ . Then, by (1), for each two  $y, z \in V(x)$ , there are  $A \in S$  and  $X \subset V$  such that  $X \cap T = A$ ,  $w(\partial X) = w^A$  and  $X$  separates  $\{x\}$  and  $\{y, z\}$ . Choose a collection  $Q$  of  $V(x)$ -maximal sets such that  $X \cap V(x) \neq Y \cap V(x)$  for any distinct  $X, Y \in Q$  and, for any  $V(x)$ -maximal  $Z$ , there is an  $X \in Q$  for which  $X \cap V(x) = Z \cap V(x)$ . Assuming that  $|Q| \leq 2$  we get that there is an  $X \in Q$  such that  $\sum (w(xy): y \in X^* \cap V(x)) > \frac{1}{2} \sum (w(xy): y \in V(x))$ , contrary to (3). Thus,  $|Q| \geq 3$ . Let  $X_1, X_2$  be elements

of  $Q$  such that  $X_1^* \cap X_2^* \neq \emptyset$ , and let  $y \in (X_1^* - X_2^*) \cap V(x)$  and  $z \in (X_2^* - X_1^*) \cap V(x)$ . Take an  $X_3 \in Q$  such that  $X_3$  separates  $\{x\}$  and  $\{y, z\}$ . Finally, let  $A_i = X_i \cap T$  ( $i = 1, 2, 3$ ). By (2), any two members of  $\{A_1, A_2, A_3\}$  are crossing. This contradiction completes the proof of the lemma.

#### 4. PROOF OF THEOREM A

**Proof of the part (i).** Consider an arbitrary 3-complete scheme  $S \subset 2^T$ , a set  $V \supseteq T$  and an edge length function  $l \in \mathbf{R}_+^{E(V)}$ . We have to prove that

$$(4.1) \quad p(V, S, l) = \sum (\beta^*(u) \mu_l(u): u \in E(T))$$

for some  $\beta^*: E(T) \rightarrow \mathbf{R}_+$  satisfying

$$(4.2) \quad \beta^*(\partial^T A) \geq 1 \quad \text{for all } A \in S.$$

Define  $\mu \in \mathbf{R}_+^{E(T)}$  by  $\mu(u) = \mu_l(u)$  ( $u \in E(T)$ ), and consider the "reduced" multicut max- $\sum$  problem  $\sum(T, S, \mu)$ . By linear programming duality there is a  $\beta^*: E(T) \rightarrow \mathbf{R}_+$  which satisfies (4.2) and

$$(4.3) \quad p(T, S, \mu) = \sum (\beta^*(u) \mu_l(u): u \in E(T)).$$

It suffices to prove that  $p(V, S, l) \geq p(T, S, \mu)$  (then (4.3) implies (4.1) since  $p(V, S, l) \leq \sum (\beta^*(u) \mu_l(u): u \in E(T))$  always holds subject to (4.2)). Let  $\tilde{\alpha}: S \rightarrow \mathbf{R}_+$  be an optimum solution of  $\sum(T, S, \mu)$ , i.e.

$$(4.4) \quad \lambda^{\tilde{\alpha}}(u) (= \sum (\tilde{\alpha}(A): A \in S, u \in \partial^T A)) \leq \mu(u) \quad \text{for all } u \in E(T)$$

and  $1 \cdot \tilde{\alpha} = p(T, S, \mu)$ . Assume, in addition, that  $\tilde{\alpha}$  is chosen so that the value  $\psi(\tilde{\alpha}) = \sum (\lambda^{\tilde{\alpha}}(u): u \in E(T))$  is minimum.

**Claim 4.1.** *Let  $S^* = \{A_1, A_2, A_3\}$  be a triple of pairwise crossing members in  $S$ . Then  $\tilde{\alpha}(A_i) = 0$  for at least one  $i \in \{1, 2, 3\}$ .*

Indeed, let  $a = \min \{\tilde{\alpha}(A_i): i = 1, 2, 3\}$ , and suppose, for a contradiction, that  $a > 0$ . Since  $S$  is 3-complete, there is  $\gamma: S^* \rightarrow \mathbf{R}_+$  and  $\epsilon: S \rightarrow \mathbf{R}_+$  satisfying (1.5). Put  $\tilde{\alpha}'(A_i) = \tilde{\alpha}(A_i) - b\gamma(A_i) + b\epsilon(A_i)$

( $i = 1, 2, 3$ ) and  $\tilde{\alpha}'(A) = \tilde{\alpha}(A) + b\epsilon(A)$  ( $A \in S - S^*$ ), where  $b = \frac{a}{\max\{\gamma(A_i): i = 1, 2, 3\}}$  ( $0 < b < \infty$  since  $a > 0$  and  $\gamma(A_i) > 0$  for at least one  $i$ , because of (1.5)). Obviously,  $\tilde{\alpha}' \geq 0$ . From (1.5) we get  $1 \cdot \tilde{\alpha}' \geq 1 \cdot \tilde{\alpha}$ ,  $\lambda^{\tilde{\alpha}'} \leq \lambda^{\tilde{\alpha}}$  and  $\lambda^{\tilde{\alpha}'}(u) < \lambda^{\tilde{\alpha}}(u)$  for some  $u \in E(T)$ , contradicting the choice of  $\tilde{\alpha}$ . ■

Now let  $S^+ = \{A \in S: \tilde{\alpha}(A) > 0\}$ . By Claim 4.1,  $S^+$  is not 3-crossing. Define the demand function  $d: S^+ \rightarrow \mathbf{R}_+$  by  $d(A) = \tilde{\alpha}(A)$  ( $A \in S^+$ ), and consider the multicut existence problem  $\text{EX}(V, S^+, l, d)$ . Since  $S^+$  is not 3-crossing and the inequalities in (4.4) hold (compare with (1.6)), then, by Theorem B, this problem has a solution  $\alpha': \mathcal{X}(V, S^+) \rightarrow \mathbf{R}_+$ . Let the function  $\alpha$  be the extension of  $\alpha'$  with zero on  $\mathcal{X}(V, S - S^+)$ . Then, obviously,  $1 \cdot \alpha = 1 \cdot \tilde{\alpha}$ , hence  $p(V, S, l) \geq 1 \cdot \alpha = p(T, S, \mu)$ , as required.

**Proof of the part (ii).** This proof is more complicated. We are based here on Lemma 2.3. Let  $S \subset 2^T$  be not 3-complete,  $V \supset T$  and  $|V| \geq |T| + 2$ . According to Lemma 2.3 we have to prove that there are  $l \in \mathbf{R}_+^{E(V)}$ ,  $d \in \mathbf{R}_+^S$  and  $w \in \mathbf{R}_+^{E(V)}$  such that

$$(4.5) \quad w(\partial^V X) \geq 1 \quad \text{for all } X \in \mathcal{X}(V, S),$$

$$(4.6) \quad \mu_1(u) \geq \sum(d(A): A \in S, u \in \partial^T A) \quad \text{for all } u \in E(T)$$

but

$$(4.7) \quad l \cdot w < (d(A): A \in S)$$

((4.6) is equivalent to  $\{(2.7) \forall B \in \hat{b}(\mathcal{F})\}$  by  $\mathcal{F} = \mathcal{C}(V, S)$  and  $l = c$ , as it was explained in Section 2.)

Let  $S^* = \{A_1, A_2, A_3\}$  be a triple of pairwise crossing members in  $S$  such that there are no  $\gamma: S^* \rightarrow \mathbf{R}_+$  and  $\epsilon: S \rightarrow \mathbf{R}_+$  satisfying (1.5). Let us choose a minimal subset  $T' \subseteq T$  such that  $A'_i = A_i \cap T'$  ( $i = 1, 2, 3$ ), are still pairwise crossing (with respect to  $T'$ ). By Proposition 3.2, we may assume that either

$$(1) \quad T' = T_1 = \{0, 1, 2, 3\} \quad \text{and} \quad S' = S_1 = \{\{0, i\}: i = 1, 2, 3\}$$

or

$$(2) \quad T' = T_2 = \{0, \dots, 5\} \quad \text{and} \quad S' = S_2 = \{\{i, i+1, i+2\}: i = 1, 2, 3\}, \quad \text{where} \quad S' = \{A'_1, A'_2, A'_3\}.$$

First of all, we define a suitable function  $l$  as follows.

Case 1.  $S' = S_1$ .

Choose an arbitrary vertex  $x \in V - T$ , and let  $G_1 = (V_1, E_1)$  be the graph with the vertex-set  $\{0, 1, 2, 3, x\}$  as in Fig. 3.2a. Put

$$l(e) = \begin{cases} 1, & e \in E_1 \\ \rho_{A_1}(e) + \rho_{A_2}(e) + \rho_{A_3}(e), & e \in E(T), \\ \text{a large positive number,} & e \in E(V) - (E_1 \cup E(T)), \end{cases}$$

where  $\rho_A$  denote the characteristic vector of the subset  $\partial^T A$  in  $\mathbf{R}^{E(T)}$ .

Case 2.  $S' = S_2$ .

Choose arbitrary  $x, y \in V - T$  (existing because of  $|V| \geq |T| + 2$ ), and let  $G_2 = (V_2, E_2)$  be the graph with vertex-set  $\{0, \dots, 5, x, y\}$  as in Fig. 3.2b.  $l$  is defined similarly to the previous case (with  $G_2$  instead of  $G_1$ ).

Next,  $d$  is defined by  $d(A_i) = 1$  ( $i = 1, 2, 3$ ) and  $d(A) = 0$  ( $A \in S - S^*$ ). It is not difficult to verify that in both cases, (4.6) holds with given  $l$  and  $d$ . Now our aim is to define a function  $w \in \mathbf{R}_+^{E(V)}$  that (4.5) and (4.7) should be true (with given  $l$  and  $d$ ). At the beginning, we define the function  $w'$  as follows:

$$w'(e) = \begin{cases} \frac{1}{2}, & e \in E_1 \\ 0, & e \in E(V) - E_1 \end{cases} \quad \text{in Case 1;} \\ w'(e) = \begin{cases} \frac{1}{3}, & e \in E_2 \\ 0, & e \in E(V) - E_2 \end{cases} \quad \text{in Case 2.}$$

One can check that

$$(4.8) \quad w'(\partial X) \geq 1 \quad \text{for all } X \in \mathcal{X}(V, S^*)$$

in both cases. Furthermore, in Case 1, we have

$$l \cdot w' = 4 \cdot 1 \cdot \frac{1}{2} < 3 = \sum(d(A): A \in S)$$

and in Case 2, we have

$$l \cdot w' = 7 \cdot 1 \cdot \frac{1}{3} < 3 = \sum(d(A): A \in S).$$

The only reason why it is not fit, in general, to put  $w = w'$  is that the inequality  $w'(\partial X) < 1$  is possible for some  $X \in \mathcal{X}(V, S - S^*)$ . However, the following lemma is valid (its proof is the central point in our process and it will be given later).

**Lemma 4.2.** *There exists a  $\beta \in \mathbf{R}_+^{E(T)}$  such that*

$$(4.9) \quad \begin{aligned} \beta(\partial^T A_i) &= 1 \quad (i = 1, 2, 3), \\ \beta(\partial^T A) &> 1 \quad \text{for all } A \in S - S^*. \end{aligned}$$

Assuming that this lemma is valid we define suitable  $w$  as follows. Put  $w''(e) = \beta(e)$  ( $e \in E(T)$ ) and  $w''(e) = 0$  ( $e \in E(V) - E(T)$ ). Let  $\kappa = \min\{\beta(\partial^T A) - 1: A \in S - S^*\}$  and  $\xi = \frac{1}{2} \min\{1, \kappa\}$ , then  $0 < \xi \leq \frac{1}{2}$ . Now put  $w = \xi w' + (1 - \xi)w''$ . We observe that  $w''(\partial X) = \beta(\partial^T A)$  for any  $X \subset V$  and  $A = X \cap T$ , and hence, by (4.8) and (4.9),  $w(\partial X) \geq 1$  for all  $X \in \mathcal{X}(V, S^*)$ . If  $X \in \mathcal{X}(V, S - S^*)$  and  $A = X \cap T$ , then

$$w(\partial X) \geq (1 - \xi)\beta(\partial^T A) \geq (1 - \xi)(1 + \kappa) \geq 1.$$

Thus, (4.5) holds for given  $w$ . Next, by (4.9) and the definition of  $l$ , we have

$$l \cdot w'' = \sum(\rho_{A_i}' \beta: i = 1, 2, 3) = 3,$$

and now, since  $l \cdot w' < 3$  and  $\sum(d(A): A \in S) = 3$ , we obtain

$$l \cdot w = \xi l \cdot w' + (1 - \xi)l \cdot w'' < 3 = \sum(d(A): A \in S),$$

i.e. (4.7) is true.

**Proof of Lemma 4.2.** We prove that the linear program  $\mathcal{P}: \max \delta$  subject to

$$\begin{aligned} \beta(\partial^T A_i) &= 1 \quad (i = 1, 2, 3), & \quad | \tilde{\gamma} \\ -\beta(\partial^T A) + \delta &\leq -1 \quad (A \in S - S^*), & \quad | \epsilon \\ \beta &\in \mathbf{R}_+^{E(T)}, \quad \delta \in \mathbf{R}_+ \end{aligned}$$

has an optimum solution  $(\beta, \delta)$  with either  $\delta > 0$  or  $\delta = \infty$ . The program  $\mathcal{P}^*$  dual to  $\mathcal{P}$  is:

$$\begin{aligned} q(\tilde{\gamma}, \epsilon) &= \sum(\tilde{\gamma}(A_i): i = 1, 2, 3) - \sum(\epsilon(A): A \in S - S^*) \rightarrow \min \\ \sum(\tilde{\gamma}(A_i)\rho_{A_i}: i = 1, 2, 3) - \sum(\epsilon(A)\rho_A: A \in S - S^*) &\geq 0 \\ \sum(\epsilon(A): A \in S - S^*) &\geq 1 \\ \gamma(A_i) &\leq 0 \quad (i = 1, 2, 3), \quad \epsilon(A) \geq 0 \quad (A \in S - S^*). \end{aligned}$$

Suppose that  $\mathcal{P}^*$  has no feasible solution. Then one can see that, for any  $A \in S - S^*$ , there is an edge  $u \in \partial^T A$  such that  $u \notin \partial^T A_i$  ( $i = 1, 2, 3$ ). Define  $\beta$  as follows. Let  $T' \subseteq T$  and  $S' = \{A'_1, A'_2, A'_3\}$  be defined as in Proposition 3.2. If  $T' = \{0, 1, 2, 3\}$ , put  $\beta(0, i) = \frac{1}{2}$  ( $i = 1, 2, 3$ ), and if  $T' = \{0, \dots, 5\}$ , put  $\beta(0, 3) = 1$ . Next, put  $\beta(u) = 2$  for all  $u \in E(T)$  such that  $u \notin \partial^T A_i$  ( $i = 1, 2, 3$ ), and  $\beta(u) = 0$  for the remaining  $u$ 's in  $E(T)$ . Then, as it is easy to see,  $\beta(\partial^T A_i) = 1$  ( $i = 1, 2, 3$ ) and  $\beta(\partial^T A) \geq 2$  ( $A \in S - S^*$ ), hence  $(\beta, 1)$  is a feasible solution of  $\mathcal{P}$ .

Next, suppose that  $\mathcal{P}^*$  has an unbounded solution, and let  $(\tilde{\gamma}, \epsilon)$  be a feasible solution of  $\mathcal{P}^*$  such that  $q(\tilde{\gamma}, \epsilon) < 0$ . Put  $\gamma(A_1) = \max\{\tilde{\gamma}(A_1), 0\} - q(\tilde{\gamma}, \epsilon)$ ,  $\gamma(A_i) = \max\{\tilde{\gamma}(A_i), 0\}$  ( $i = 2, 3$ ) and  $\epsilon(A_i) = \max\{-\tilde{\gamma}(A_i), 0\}$  ( $i = 1, 2, 3$ ). Then  $\gamma \geq 0$ ,  $\epsilon \geq 0$  and (1.5) holds for these  $\gamma$  and  $\epsilon$ , contradicting the choice of  $S^*$ .

Now suppose that  $\mathcal{P}^*$  has an optimum solution  $(\tilde{\gamma}, \epsilon)$  with  $q(\tilde{\gamma}, \epsilon) = 0$ . Note that feasibility of  $(\tilde{\gamma}, \epsilon)$  implies that  $\epsilon(A) > 0$  for some  $A \in S - S^*$ , whence  $\tilde{\gamma}(A_i) > 0$  for some  $A_i \in S^*$ . But (1.5) holds for no  $(\gamma, \epsilon)$ , and now a contradiction to the supposition is immediately obtained from the following lemma (thus, it remains to consider only the situation

when  $\mathscr{P}^*$  has an optimum solution  $(\tilde{\gamma}, \epsilon)$  with  $q(\tilde{\gamma}, \epsilon) > 0$ ; then, by l.p. duality, there is a feasible solution  $(\beta, \delta)$  of  $\mathscr{P}$  with  $\delta > 0$ , as required).

**Lemma 4.3.** *Let  $\phi \neq I \subseteq \{1, 2, 3\}$ ,  $\tilde{S}^* = \{A_i; i \in I\}$  and  $\gamma(A_i) > 0$  for  $i \in I$ . Further, let  $S(T)$  be a collection of proper subsets of  $T$  such that  $\tilde{S}^* \subseteq S(T)$  and, for any proper subset  $A$  of  $T$ , exactly one of  $A$  and  $T - A$  is in  $S(T)$ . Then there exists no function  $\epsilon: S(T) - \tilde{S}^* \rightarrow \mathbf{R}_+$  such that both of the following equalities are true:*

$$(4.10) \quad \sum(\gamma(A_i)\rho_{A_i}; i \in I) = \sum(\epsilon(A)\rho_A; A \in S(T) - \tilde{S}^*),$$

$$(4.11) \quad \sum(\gamma(A_i); i \in I) = \sum(\epsilon(A); A \in S(T) - \tilde{S}^*).$$

**Proof.** We need some known easy assertions about metrics. A function  $\mu: E(T') \rightarrow \mathbf{R}_+$  is said to be a *metric on  $T'$*  if  $\mu(xy) + \mu(yz) \geq \mu(xz)$  for any distinct  $x, y, z \in T'$ . A chain  $L$  in  $K_{T'}$  is called a *geodetic* of  $\mu$  if  $\mu(eL) = \mu(L)$  ( $= \sum(\mu(e); e \in L)$ ). Let  $\mathcal{F}(\mu)$  denote the set of geodetics of  $\mu$ .

(1) If  $\mu, \mu', \mu'' \in E(T')$  are metrics and  $\mu = \mu' + \mu''$ , then  $\mathcal{F}(\mu) = \mathcal{F}(\mu') \cap \mathcal{F}(\mu'')$ .

(2) Let  $S' \subset 2^{T'}$  and  $\nu \in \mathbf{R}_+^{S'}$ . Then  $\mu = \sum(\nu(A)\rho'_A; A \in S')$  is a metric, and a chain  $L \subseteq E(T')$  is a geodetic of  $\mu$  if and only if  $|L \cap \partial^{T'}A| \leq 1$  for each  $A \in S'$  such that  $\nu(A) > 0$  ( $\rho'_A$  is the characteristic vector of  $\partial^{T'}A$  in  $\mathbf{R}^{T'}$ ).

(3) Let  $\mu$  be a metric on  $T'$ , and let  $\mu(st) = 0$  for some  $st \in E(T')$ . Then  $\mu(sp) = \mu(tp)$  for any  $p \in T - \{s, t\}$ .

(4) (a corollary of (2) and (3)). Let  $\mu$  be a metric on  $T'$ , and let  $\{T_1, T_2, \dots, T_m\}$  be the partition of  $T'$  into maximal subsets  $T_i$  such that  $\mu(u) = 0$  for any  $u \in E(T_i)$ . Define  $\mu^P(ij) = \mu(st)$  for  $ij \in E(P)$ ,  $s \in T_i, t \in T_j$ , where  $P = \{1, \dots, m\}$ . Then

(a)  $\mu^P$  is a metric on  $P$ ,

(b) if  $\mu = \sum(\nu(A)\rho'_A; A \in S')$ , where  $S' \subset 2^{T'}$ ,  $\nu \in \mathbf{R}_+^{S'}$  and  $\nu(A) > 0$  for all  $A \in S'$ , then, for any  $A \in S'$ ,  $A = \cup(T_i; i \in A^P)$  for

some  $A^P \subset P$ , and  $\mu^P = \sum(\nu(A)\rho''_{A^P}; A \in S')$ , where  $\rho''_{A^P}$  denotes the characteristic vector of  $\partial^P A^P$  in  $\mathbf{R}^P$ .

Now we start proving the lemma. Suppose, for a contradiction, that there is  $\epsilon$  satisfying (together with given  $\gamma$ ) (4.10) and (4.11). Let  $\mu = \sum(\gamma(A_i)\rho_{A_i}; i \in I)$ . By (2),  $\mu$  is a metric. According to (4), without loss of generality we may assume that  $\mu(u) > 0$  for all  $u \in E(T)$  (as otherwise, we could consider  $P$  and  $\mu^P$  instead of  $T$  and  $\mu$ , where  $P$  is the partition for  $\mu$  as in (4)). Thus, we have  $2 \leq |T| \leq 2^{|I|} \leq 8$ .

Case 1.  $|I| = 1$ .

Then  $|T| = 2$ ,  $\tilde{S}^* = S(T)$ , and (4.10) is impossible.

Case 2.  $|I| = 2$ .

Let  $\tilde{S}^* = \{A_1, A_3\}$ , say. Since  $A_1$  and  $A_3$  are crossing, we have  $|T| = 4$ . Let, for definiteness,  $T = \{0, 1, 2, 3\}$ ,  $A_1 = \{0, 1\}$ ,  $A_3 = \{0, 3\}$  and  $S(T) - \tilde{S}^* = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0, 2\}\}$ . It is easy to see that:

(i) each chain  $L_j = \{j(j+1), (j+1)(j+2)\}$  (indices are taken by modulo 4) is a geodetic of  $\mu$  (see Fig. 4.1),

(ii) for each  $A \in S(T) - \tilde{S}^*$  there is some geodetic  $L_j$  such that  $|L_j \cap \partial^T A| = 2$ . Thus, by (1) and (2),  $\epsilon = 0$ , a contradiction.

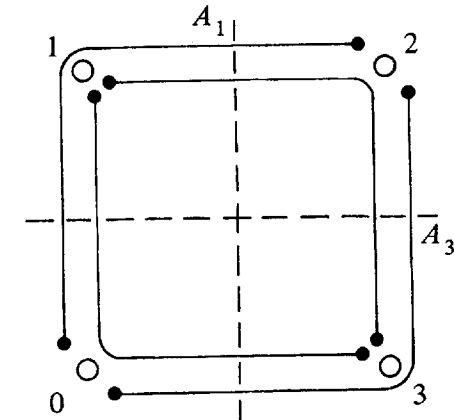


Figure 4.1

Case 3.  $|I| = 3$ .

Then  $\tilde{S}^* = S^*$  and  $4 \leq |T| \leq 8$ . Let  $T' \subseteq T$  and  $S' = \{A'_i = A_i \cap T' : i = 1, 2, 3\}$  be defined as in Proposition 3.2. Let  $\rho'_B$  be the characteristic vector of subset  $\partial^{T'}B$  in  $\mathbb{R}^{T'}$  for  $B \subset T'$ .

**Claim 4.4.** *The inequalities*

$$(4.12) \quad \sum(\psi(A'_i)\rho'_{A'_i} : i = 1, 2, 3) = \sum(\xi(A')\rho_{A'} : A' \in S(T')),$$

$$(4.13) \quad \sum(\psi(A'_i) : i = 1, 2, 3) = \sum(\xi(A') : A' \in S(T'))$$

imply

$$(4.14) \quad \xi(A') = 0 \quad \text{for all } A' \in S(T') - S',$$

$$(4.15) \quad \xi(A'_i) = \psi(A'_i) \quad \text{for } i = 1, 2, 3.$$

**Proof.** Assume that  $|T'| = 4$ , and let  $T' = \{0, 1, 2, 3\}$  and  $A'_i = \{0, i\}$  ( $i = 1, 2, 3$ ). Put  $A^j = \{j\}$  ( $j = 0, 1, 2, 3$ ). The vector equality (4.12) may be written as

$$(4.16) \quad \begin{aligned} \sum(\psi(A'_i) - \xi(A'_i) : i \in \{1, 2, 3\}, u \in \partial^{T'}A'_i) = \\ = \sum(\xi(A^j) : j \in \{0, 1, 2, 3\}, u \in \partial^{T'}A^j) \quad \text{for all } u \in E(T'). \end{aligned}$$

Summing up all the six equalities in (4.16) we get

$$4 \sum(\psi(A'_i) - \xi(A'_i) : i = 1, 2, 3) = 3 \sum(\xi(A^j) : j = 0, 1, 2, 3),$$

whence, taking into account (4.13), we obtain (4.14). Now (4.16) easily implies (4.15).

Now assume that  $|T'| = 6$ , and let  $T' = \{0, \dots, 5\}$ ,  $A'_i = \{i, i+1, i+2\}$  ( $i = 1, 2, 3$ ). One can see that:

(a) each chain  $L_j = \{j(j+1), (j+1)(j+2), (j+2)(j+3)\}$  ( $j = 0, \dots, 5$ ) (indices are taken by modulo 6) is a geodetic of  $\mu$  (see Fig. 4.2),

(b) for any  $A' \in S(T') - S'$  there is a geodetic  $L_j$  such that  $|L_j \cap \partial^{T'}A'| \geq 2$ . Thus, by (b), (1) and (2), (4.14) is valid. Now,

considering the equality (4.12) for the edge  $(i-1)i$  ( $i = 1, 2, 3$ ) we immediately obtain  $\psi(A'_i) = \xi(A'_i)$ , as required. ■

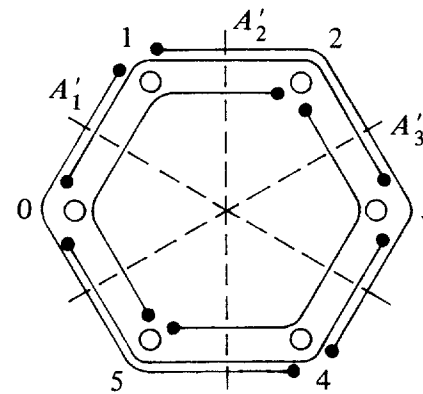


Figure 4.2

Thus, if  $T' = T$ , the result immediately follows from Claim 4.4. Now assume that  $T' \subset T$ . In view of Claim 4.4, in order to finish the proof it is enough to show that, if  $A \in S(T)$  is such that  $A \notin S^*$  and  $A \cap T' = A'_i$  (or  $A \cap T' = T' - A'_i$ ) for some  $i$ , then there is a geodetic  $L \subset E(T)$  of  $\mu$  for which  $|L \cap \partial^{T'}A| \geq 2$  (then, by (1) and (2), it must be  $\epsilon(A) = 0$ ). Let, for definiteness,  $A \cap T' = A'_1$  (and  $A \neq A_1$ ), and choose a vertex  $v \in T - T'$  such that either  $v \in A$  and  $v \notin A_1$  or  $v \notin A$  and  $v \in A_1$ . Take  $A^* \in \{A, T - A\}$  and  $A_i^* \in \{A_i, T - A_i\}$  ( $i = 1, 2, 3$ ) so that  $A^* \cap T' = A_1^* \cap T'$ ,  $v \notin A^*$  and  $v \in A_i^*$  ( $i = 1, 2, 3$ ). Since  $\mu(xy) > 0$  for any  $xy \in E(T)$ , then  $|A_1^* \cap A_2^* \cap A_3^*| \leq 1$ , and so  $A_1^* \cap A_2^* \cap A_3^* = \{v\}$ . Hence  $A_1^{*'} \cap A_2^{*'} \cap A_3^{*'} = \emptyset$ , where  $A_i^{*'} = A_i^* \cap T'$ . Choose  $p \in A_1^{*'} \cap A_2^{*'}$  and  $q \in A_1^{*'} \cap A_3^{*'}$  (see Fig. 4.3) (every  $A_i^{*'} \cap A_j^{*'}$  is nonempty because  $A_i^{*'}$  and  $A_j^{*'}$  are crossing). Now since  $p \notin A_3^{*'}$  and  $q \notin A_2^{*'}$ , we have  $\mu(pv) = \gamma(A_3)$ ,  $\mu(vq) = \gamma(A_2)$  and  $\mu(pq) = \gamma(A_2) + \gamma(A_3)$ , therefore the chain  $L = \{pv, vq\}$  is a geodetic of  $\mu$ . But  $p, q \in A^*$  and  $v \notin A^*$ , hence  $|L \cap \partial^{T'}A| = 2$ .

This completes the proof of Lemmas 4.3, 4.2 and Theorem 1. ■

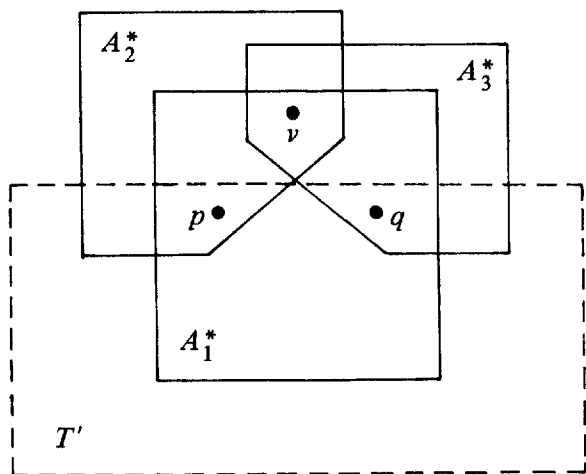


Figure 4.3

## 5. 2-COMPLETE SCHEMES

Consider a multicut  $\max\text{-}\Sigma$  problem  $\Sigma(V, S, l)$ , where  $S \subset 2^T$ , and let  $\mu = \mu_l|_{E(T)}$ . It was explained in Section 4 that the minimax relation (1.4) holds for such a problem if and only if

$$(5.1) \quad p(V, S, l) = p(T, S, \mu).$$

Thus, given  $S \subset 2^T$ , (5.1) is true for any  $V \supseteq T$  and  $l \in \mathbf{R}_+^{E(V)}$  if and only if  $S$  belongs to the  $\Sigma$ -DSC-class which, by Theorem A, is equivalent to that  $S$  is 3-complete. Moreover, the idea behind the proof of the part (i) of Theorem A prompts the following approach for solving  $\Sigma(V, S, l)$  with 3-complete  $S$ . First we must solve the reduced multicut  $\max\text{-}\Sigma$  problem  $\Sigma(T, S, \mu)$  (which may be named a *pre-problem* of  $\Sigma(V, S, l)$ ). Next, we reform the found optimum solution  $\gamma \in \mathbf{R}_+^S$  of  $\Sigma(T, S, \mu)$  into such an optimum solution  $\gamma'$  that the family  $S^+(\gamma') = \{A \in S: \gamma'(A) > 0\}$  is not 3-crossing. Finally, we solve the multicut existence problem  $\text{EX}(V, S^+(\gamma'), l, d)$ , where  $d(A) = \gamma'(A)$  ( $A \in S^+(\gamma')$ ), then the found solution  $\alpha$  is an optimum solution of  $\Sigma(V, S, l)$ .

There is a greedy algorithm for solving any existence problem  $\text{EX}(V, S, l, d)$  with real-valued  $l$  and  $d$  and non-3-crossing  $S$  [18]. The complexity of the algorithm is  $O(|T|^3|V|^3)$  operations. As a direct

corollary of the algorithm, the following half-integrality theorem is stated. We say that a function  $l$  on  $E(V)$  is *cyclically even* if it is nonnegative integer-valued and the value  $l(C) = \sum(l(e): e \in C)$  is even for any circuit  $C$  in  $K_V$ .

**Theorem 5.1.** *Let  $S \subset 2^T$  be not 3-crossing,  $V \supseteq T$ ,  $l \in \mathbf{Z}_+^{E(V)}$  and  $d \in \mathbf{Z}_+^S$ , and let (1.6) hold for all  $u \in E(T)$ . Further, let the function  $l'$ , defined by  $l'(e) = \sum(d(A): A \in S, e \in \partial^T A)$  ( $e \in E(T)$ ) and  $l'(e) = l(e)$  ( $e \in E(V) - E(T)$ ), be cyclically even. Then  $\text{EX}(V, S, l, d)$  has an integer solution.*

Return to the  $\max\text{-}\Sigma$  problem  $\Sigma(V, S, l)$  with 3-complete  $S$ , and suppose that we succeed in finding such an optimum solution  $\gamma$  of the pre-problem  $\Sigma(T, S, \mu)$  that  $S^+(\gamma)$  is not 3-crossing and both  $m\gamma$  and  $m\gamma$  are integer-valued, for some integer  $m$ . Then, by Theorem 5.1, there exists an optimum solution  $\alpha$  of  $\Sigma(V, S, l)$  such that  $2m\alpha$  is integer-valued.

Now we introduce one special class of 3-complete schemes. A scheme  $S \subset 2^T$  is called *2-complete* if, for any two crossing  $A_1, A_2 \in S$  (if any), there are  $B_1, B_2 \in S$  such that  $B_1^* = A_1^* - A_2^*$  and  $B_2^* = A_2^* - A_1^*$  for some  $A_i^* \in \{A_i, T - A_i\}$  and  $B_i^* \in \{B_i, T - B_i\}$  ( $i = 1, 2$ ). It is easy to see that such a relation can be expressed as the vector inequality

$$(5.2) \quad \rho_{A_1} + \rho_{A_2} > \rho_{B_1} + \rho_{B_2}.$$

Obviously, any laminar scheme (i.e. consisting of pairwise laminar subsets) is 2-complete. It is easy to show that, for any  $T$ , the scheme of all the odd-size subsets of  $T$  (Ex. 3 in Introduction) is 2-complete. Here are other examples of 2-complete schemes  $S$  (their examination is left to the reader).

**Example 5.**  $S$  consist of all proper subsets of  $T$ .

**Example 6.**  $S = \{A \subset T: 1 \leq |A| \leq k\}$  for arbitrary  $k < |T|$ .

**Example 7.**  $S = \{A \subset T: |\partial^T A \cap U| = 1\}$  for arbitrary  $U \subseteq E(T)$ .

**Example 8.**  $S = \{A \subset T: a(\partial^T A) \text{ odd}\}$  for arbitrary integer-valued function  $a$  on  $E(T)$ .



