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ON MULTICOMMODITY FLOW PROBLEMS WITH INTEGER-VALUED OPTIMAL SOLUTIONS

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We give sufficient conditions for the existence of an integer-valued optimal solution in the problem of a maximal multicommodity flow in an undirected network, and we suggest an algorithm for finding such a solution with a polynomial number of operations. This strengthens a number of known results on integer-valued and half-integer-valued multicommodity flows.

1. Let $G = (V, E)$ and $H = (T, U)$ be finite undirected graphs with $T \subseteq V$; the edge of a graph with end vertices x and y will be denoted by xy. An *st-chain* in G, where $s, t \in V$ and $s \neq t$, is understood to be a set $L \subseteq E$ of edges such that $L = \{x_i x_{i+1} : i = 0, 1, \dots, k\}$ for certain distinct vertices $s = x_0, x_1, \dots, x_k = t$. The problem of a maximal multicommodity flow in an undirected network admits the following formulation (*the problem* $\mathcal{P}(G, c, H)$): for a given function $c\colon E \to \mathbf{R}_+$ (for given edge capacities) find a function $f\colon \mathcal{L} \to \mathbf{R}_+$ satisfying the load condition

$$(*) \qquad \varsigma^f(e) \stackrel{\text{def}}{=} \sum (f(L) : e \in L \in \mathcal{L}) \leq c(e) \quad \forall e \in E$$

and maximizing the quantity $1 \cdot f = \sum(f(L) : L \in \mathcal{L})$. Here $\mathcal{L} = \mathcal{L}(G, H)$ is the set of all st-chains in G for $st \in U$, and $\mathbf{R}_+$ is the set of nonnegative real numbers. A function f satisfying $(*)$ is called an *admissible* multiflow (multicommodity flow) in the network (G, c) with flow scheme H; the maximum of $1 \cdot f$ over all admissible f is denoted by $v(G, c, H)$.

The function c is said to be *intrinsically even* if it is integer-valued and $\sum(c(xy) : y \in V - \{x\})$ is even for all $x \in V - T$. We say that H is *solvable* in $\frac{1}{k}\mathbf{Z}_+$ (*solvable in* $\frac{1}{k}\mathbf{Z}_+$ *under the condition of intrinsic evenness*) if for any graph $G = (V, E)$ with $V \supseteq T$ and any function $c\colon E \to \mathbf{Z}_+$ (any intrinsically even function c) the problem $\mathcal{P}(G, kc, H)$ has an integer-valued optimal solution f, where $\mathbf{Z}_+$ is the set of nonnegative integers, and k is some positive integer.

It is known that H is solvable in $\mathbf{Z}_+$ when $|U| = 1$ [1], and is solvable in $\mathbf{Z}_+$ under the condition of intrinsic evenness when $|U| = 2$ [2] or when H is a complete graph [3]. In [4] a large class of flow schemes solvable in $\frac{1}{2}\mathbf{Z}_+$ is given. Namely, let $\mathcal{A} = \mathcal{A}(H)$ denote the set of all anticliques (i.e., maximal (with respect to inclusion) independent sets of vertices) in H. The set $\mathcal{A}$ is said to be *bipartite* if there exists a partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of it in which each $\mathcal{A}_i$ consists of pairwise disjoint anticliques. For example: a) if $T = \{s, t, p, q\}$ and $U = \{st, pq\}$, then $\mathcal{A} = \{\{s, p\}, \{x, q\}, \{t, p\}, \{t, q\}\}$; b) if H is a complete graph, then $\mathcal{A} = \{\{s\} : s \in T\}$. In both cases $\mathcal{A}$ is bipartite. It is shown in [4] (see [5] for details) that if $\mathcal{A}(H)$ is bipartite, then H is solvable in $\frac{1}{2}\mathbf{Z}_+$, and an algorithm is proposed for finding a half-integer-valued optimal solution with number of operations bounded by a polynomial in $|V|$, multiplied by $c(E)$ (here and below, $g(S')$ denotes $\sum(g(s) : s \in S')$ for $g\colon S \to \mathbf{R}$ and a finite subset $S' \subseteq S$). In this note we improve these results.

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THEOREM 1. *If the set $\mathcal{A}(H)$ is bipartite and the function c is intrinsically even, then the problem $\mathcal{P}(G, c, H)$ has an integer-valued optimal solution.*

We give a scheme for proving Theorem 1 and describe an algorithm (for partite $\mathcal{A}$), with number of operations bounded by a polynomial in $|V|$, that finds a real optimal solution for $c\colon E \to \mathbf{R}_+$ and an integer-valued optimal solution when c is intrinsically even. We remark that Theorem 1 and an effective construction in [4] and [5] give us

THEOREM 2. *If each vertex in H belongs to at most two anticliques, then H is solvable in $\frac{1}{2}\mathbf{Z}_+$ under the condition of intrinsic evenness.*

2. Scheme of proof. Without loss of generality it will be assumed that the graph G is complete. Suppose that $c\colon E \to \mathbf{R}_+, \{\mathcal{A}_1, \mathcal{A}_2\}$ is a corresponding partition of $\mathcal{A}$, and $f\colon \mathcal{L} \to \mathbf{R}_+$ is an optimal solution (OS) of the problem $\mathcal{P}(c)$ (for $\mathcal{P}(G, c, H)$ and $v(G, c, H)$, the abbreviated notation $\mathcal{P}(c)$ and $v(c)$ is used in what follows). Let $\mathcal{L}^+(f) = \{L \in \mathcal{L}\colon f(L) > 0\}$.

For $X \subseteq V$ let $\partial X = \partial^G X$ denote the set of edges in G with one end in X and the other in $V - X$ (a *cut* of the graph G). A family of subsets $\mathcal{X} = \{X_A \subset V\colon A \in \mathcal{A}\}$ (allowing empty subsets) will be called *semiregular* if $X_A \cap T \subseteq A\ \forall A \in \mathcal{A}$ and each element of T belongs to exactly one X_A, and it will be called *regular* if, in addition, the sets in $\mathcal{X}$ are disjoint. Let $c(\mathcal{X}) = \frac{1}{2}\sum(c(\partial X_A)\colon A \in \mathcal{A})$. Obviously, $c(\mathcal{X}) \geq v(c)$ for any semiregular $\mathcal{X}$.

LEMMA. *$c(\mathcal{X}) = v(c)$ for some regular $\mathcal{X}$.*

This lemma actually follows from the algorithm in [4]. Another proof of it, direct and simpler, consists in the following. Let $l\colon E \to \mathbf{R}_+$ be an optimal solution of the problem dual to $\mathcal{P}(c)$ in the linear programming sense, i.e., $l(L) \geq 1\ \forall L \in \mathcal{L}$ and $c \cdot l = v(c)$. Denote by μ the metric in G generated by l, i.e., $\mu(xy) = \min\{l(L)\colon L \text{ an } xy\text{-chain in } G\}$ for $x, y \in V$ with $x \neq y$, and $\mu(xy) = 0$ for $x = y$. For $x, t \in T$ (allowing $s = t$) we write $s \not\sim t$ if there are no elements $A, B \in \mathcal{A}$ with $A \neq B$ such that $s, t \in A \cap B$. For $A \in \mathcal{A}$ and $x \in V$ let

$$r_A(x) = \min\{\mu(sx)\colon s \in A\}, \quad d_A(x) = \min\{\mu(sx) + \mu(tx)\colon s, t \in A, s \not\sim t\}.$$

We define the required family $\mathcal{X} = \{X_A\colon A \in \mathcal{A}\}$ as

$$X_A = \{x \in V\colon d_A(x) < \tfrac{1}{2}\}, \qquad A \in \mathcal{A}_1,$$

$$X_A = \{x \in V\colon d_A(x) \leq \tfrac{1}{2}\} \cup \{x \in V\colon r_A(x) = 0,\ d_B(x) \geq \tfrac{1}{2}\ \forall B \in \mathcal{A} - \{A\}\}, \quad A \in \mathcal{A}_2.$$

It can be proved that:

a) $\mathcal{X}$ is a regular family,

b) $\varsigma^f(e) = c(e)$ for any $A \in \mathcal{A}$ and $e \in \partial X_A$, and

c) $|\partial X_A \cap L| \leq 1$ for any $A \in \mathcal{A}$ and $L \in \mathcal{L}^+(f)$, which easily implies that $c(\mathcal{X}) = 1 \cdot f = v(c)$.

Suppose, further, that c is an intrinsically even function. It is not hard to see that $c(\mathcal{X})$ is an integer for any semiregular $\mathcal{X}$, and thus

COROLLARY. *The quantity $v(c)$ is an integer.*

We say that a family $\mathcal{X}$ is *c-minimal* if $c(\mathcal{X}) = v(c)$; let $\mathcal{M}(c)$ be the set of all c-minimal regular families. The rest of the proof is by induction. Assume that for fixed G and H the theorem is true for all intrinsically even c' such that either $|\mathcal{M}(c')| > |\mathcal{M}(c)|$, or $|\mathcal{M}(c')| = |\mathcal{M}(c)|$ and $c'(E) < c(E)$. The theorem is obvious for $c = 0$ (note that in this case every regular family is c-minimal, i.e., $|\mathcal{M}(c)|$ is the largest possible). A triple of vertices $\tau = xyz$ in which $y \neq x, z$ will be called a *fork*. Define $\theta_\tau : E \to \mathbf{R}_+$ and $\beta_\tau, \alpha_\tau \in \mathbf{R}_+$ as follows:

a) $\theta_\tau(xy) = \theta_\tau(yz) = 1$, $\theta_\tau(xz) = -1$, $\theta_\tau(e) = 0$ $(e \in E - \{xy, yz, xz\})$ for $x \neq z$, and $\theta_\tau(xy) = 2$, $\theta_\tau(e) = 0$ $(e \in E - \{xy\})$ for $x = z$;

b) $\beta_\tau = \min\{c(xy), c(yz)\}$ for $x \neq z$, and $\beta_\tau = \frac{1}{2}c(xy)$ for $x = z$; and

c) $\alpha_\tau = \max\{a\colon a \leq \beta_\tau,\ v(c - a\theta_\tau) = v(c)\}$.

From the lemma it is not hard to get that

1) (i) If $0 < \alpha_\tau < \beta_\tau$, then $\mathcal{M}(c) \subset \mathcal{M}(c - \alpha_\tau\theta_\tau)$.

(ii) There exists a γ, $\alpha_\tau \leq \gamma \leq \beta_\tau$, such that $v(c - a\theta_\tau) = v(c) - (a - \alpha_\tau)$ for $\alpha_\tau \leq a \leq \gamma$, and $v(c - a\theta_\tau) = v(c) - (\gamma - \alpha_\tau) - 2(a - \gamma)$ for $\gamma \leq a \leq \beta_\tau$.

A fork $\tau = xyz$ will be said to be *essential* (with respect to f) if $x \neq z$ and there is a chain $L \in \mathcal{L}^+(f)$ containing xy and yz; obviously, $\alpha_\tau \geq f(L) > 0$. If $|L| = 1$ for all $L \in \mathcal{L}^+(f)$, then the multiflow f is clearly integer-valued; therefore we can assume that the set of essential forks is nonempty. Our goal is to prove that there is a fork $\tau = xyz$ such that $\alpha_\tau \geq 1$. Then the proof of Theorem 1 is concluded as follows. Let $c' = c - \theta_\tau$. It is clear that $c'(E) \leq c(E) - 1$, the function c' is intrinsically even, and $\mathcal{M}(c) \subseteq \mathcal{M}(c')$, which implies by induction that the problem $\mathcal{P}(c')$ has an integer-valued OS f'. The required integer-valued OS f^* for $\mathcal{P}(c)$ is not determined as follows:

a) $f^* = f'$ if either $x = z$, or $x \neq z$ and $\varsigma^{f'}(xz) \leq c(xz)$;

b) $f^*(L) = f'(L) - 1$, $f^*(L') = f'(L') + 1$, $f^*(L'') = f'(L'')$ $(L'' \in \mathcal{L} - \{L, L'\})$ if $x \neq z$ and $\varsigma'(xz) = c(xz) + 1$ $(= c'(xz))$, where L is some chain in $\mathcal{L}^+(f')$ containing the edge xz, and L' is a chain in $\mathcal{L}$ contained in $(L = \{sz\}) \cup \{xy, yz\}$.

Assume that $\alpha_\tau < 1$ for each essential fork τ. The following is an easy consequence of the lemma.

2) If $\tau = xyz$ is an essential form and $c' = c - \frac{1}{2}\theta_\tau$, then

(i) $\alpha_\tau = \frac{1}{2}$, and

(ii) there is a regular family $\mathcal{X} = \{X_A\colon A \in \mathcal{A}\}$ such that $c(\mathcal{X}) - 1 = v(c) = c'(\mathcal{X})$, and $x, z \in X_A \subseteq V - \{y\}$ and $y \in X_B \subseteq V - \{x, z\}$ for some $A, B \in \mathcal{A}$.

From 2)(i) and 1)(i) we get that $\mathcal{M}(c) \subset \mathcal{M}(c'')$ for the intrinsically even function $c'' = 2c'$, and therefore the problem $\mathcal{P}(c'')$ has an integer-valued OS by induction. Consequently, the problem $\mathcal{P}(c)$ has a *half-integer-valued* OS; we use the previous notation f for it. Assume, moreover, that $\varsigma^f(E)$ is minimal over all half-integer-valued OS's of $\mathcal{P}(c)$.

3) Let $x_1yx_2, x_2yx_3, \ldots, x_kyx_1$, $k \geq 3$, be a sequence of essential forks, where all the vertices $x_1, \ldots, x_k$ are distinct. Then $\alpha_\tau \geq 1$ for $\tau = x_1yx_3$.

4) Suppose that for every $y \in V$ the sequence of essential forks indicated in 3) does not exist. Then $|L| = 1$ for all $L \in \mathcal{L}^+(f)$.

3) and 4) are the key assertions; they conclude the proof of the theorem. In them we use the fact that $f(L) = \frac{1}{2}$ for all $L \in \mathcal{L}^+(f)$ with $|L| > 1$, along with a consequence of 2)(ii): if $xyz, \mathcal{X}, A$, and B are the objects indicated in 2), and $L \in \mathcal{L}^+(f)$ contains xy and yz, then

a) $\varsigma^f(e) = c(e)$ for any $C \in \mathcal{A}$ and $e \in \partial X_C$,

b) $|L' \cap \partial X_C| \leq 1$ $\forall L' \in \mathcal{L}^+(f) - \{L\}$, $C \in \mathcal{A}$, and

c) $|L \cap \partial X_A| = 3$, $|L \cap \partial X_D| \leq 1$ $\forall D \in \mathcal{A} - \{A, B\}$.

3. The algorithm. This is based on the same idea as in the problem of Theorem 1 for composing a network by "separation" of forks. It uses the procedure described in §4 for finding the number $v(c)$. We first consider the case $c\colon E \to \mathbf{R}_+$. First of all, we determine the number $v = v(c)$. In the *main* step of the algorithm we examine successively the vertices in G, and for each $y \in V$ we examine sucessively the forks xyz. For fork $\tau = xyz$ under consideration and the current function c we find α_τ as follows (using 1)(ii)). We let $a := \beta_\tau$ and, if $a > 0$, we compute $v' = v(c - a\theta_\tau)$. If $h' = v - v' > 0$, then we let $a := a - \frac{1}{2}h'$ and compute $v'' = v(c - a\theta_\tau)$. If again $h'' = v - v'' > 0$, then we set $a := a - h''$. The a obtained is the required α_τ. Let $c := c - \alpha_\tau\theta_\tau$ and proceed to the next fork. For the final function $\tilde{c}$ we have that $\tilde{c}(e) = 0$ for all $e \in E - U$, i.e., the function $\tilde{f}$ defined as

$$\tilde{f}(\{st\}) = \tilde{c}(st), \quad st \in U, \quad \tilde{f}(L) = 0, \quad L \in \mathcal{L} - u,$$

is an OS of the problem $\mathcal{P}(\tilde{c})$. At the *concluding* step of the algorithm, an OS of the original problem $\mathcal{P}(c)$ is constructed in an obvious way from $\tilde{f}$ and the sequence of numbers α_τ.

The only difference in the algorithm in the case where c is intrinsically even and an integer-valued OS must be constructed for $\mathcal{P}(c)$ is that the current function c is transformed every time like $c := c - [\alpha_\tau]\theta_\tau$, where $[b]$ is the integer part of a number b.

4. Construction of a c-minimal family. For each $A \in \mathcal{A}$ take a copy G_A of the graph G; x_A denotes the copy in G_A of a vertex $x \in V$. We determine the edge capacities in G_A: $d(x_Ay_A) = 2c(xy)$ if $x, y \in A \cap B$ for some $B \in \mathcal{A} - \{A\}$, and $d(x_Ay_A) = c(xy)$ otherwise. We glue together these graphs, identifying the vertices s_A and s_B, as well as the edges s_At_A and s_Bt_B for each $A, B \in \mathcal{A}$ and $s, t \in A \cap B$. Finally, we form the graph $\Gamma = (\mathcal{V}, \mathcal{E})$ by adding the vertices s^0 and t^0 to the graph obtained along with the following edges of infinite capacity: 1) $s^0s_A, A \in \mathcal{A}_1, s \in \tilde{A}$; 2) s^0t_A, $A \in \mathcal{A}_2$, $t \in T - A$; 3) t^0t_A, $A \in \mathcal{A}_1$, $t \in T - A$; 4) t^0s_A, $A \in \mathcal{A}_2$, $s \in \tilde{A}$, where $\tilde{A} = A - \bigcup(B\colon B \in \mathcal{A} - \{A\})$.

Let $\mathcal{Y}$ be the set of all $Y \subset \mathcal{V}$ such that $s^0 \in Y \subseteq \mathcal{V} - \{t^0\}$ and the cut $\partial^\Gamma Y$ does not contain edges of infinite capacity. It can be verified that the mapping φ associating with each $Y \in \mathcal{Y}$ the family $\{X_A\colon A \in \mathcal{A}\}$, where $X_A = \{x \in V\colon x_A \in Y\}$ for $A \in \mathcal{A}_1$ and $X_A = \{x \in V\colon x_A \in \mathcal{V} - Y\}$ for $A \in \mathcal{A}_2$, is a bijection between $\mathcal{Y}$ and the set of semiregular families, and, moreover, $d(\partial^\Gamma Y) = 2c(\varphi(Y))$. Consequently, the construction of a c-minimal family and the determination of the quantity $v(c)$ reduce to the construction of a minimal cut for Γ and d "separating" s^0 from t^0.

We remark that $|\mathcal{V}| \leq nt + 2$ because $\mathcal{A}$ is bipartite, where $n = |V|$ and $t = |T|$. Thus, the algorithm in §3 has the estimate $O(n^3\sigma(tn))$ for the number of operations, where $\sigma(q)$ is an estimate for the number of operations in the procedure used to construct a maximal flow and a minimal cut in a network with q vertices. There is a modification of the algorithm in which only $O(n)$ forks are considered for each $y \in V$, and it has the estimate $O(n^2\sigma(n))$.

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