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MAXIMUM MATCHING OF GIVEN WEIGHT IN COMPLETE AND COMPLETE BIPARTITE GRAPHS

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We consider the existence of maximum matching of a given weight in weighted complete and complete bipartite graphs. The corresponding problem is known to be NP-complete for integer nonnegative weights of the graph edges. A polynomial algorithm is proposed for its solution for the case of edges with weights 0 and 1.

Statement of the Problem and Results

Let G = (V, E) be a finite nondirected graph with an integer weight function $a: E \to \mathbf{Z}$ defined on its edges. A matching in G is a subset $M \subseteq E$ of pairwise nonadjacent edges (i.e., edges without common ends). A matching M is called bottleneck if it is maximal by inclusion, maximum if it has the maximum possible cardinality |M|, and perfect if |M| = |V|/2. Matching problems have a wide range of applications, and for many of them effective (polynomial) algorithms exists. In particular, this category includes the problem of maximum matching, the problem of matching M of the maximum weight $a(M) = \Sigma\,(a\,(e) : e \in M)$, the problem of perfect matching of minimum weight (see, e.g., [1]).

The situation is different with regard to matchings of strictly specified weight. Let P(G, a, k) denote the following problem: for a given integer k, decide if there exists in G a bottleneck matching M of weight a(M) equal to k. This problem is intractable already in the following particular cases.

1. G is a bipartite graph and a ≡ 1. This problem transforms to the NP-complete problem of minimum bottleneck matching [2, p. 239].

2. G is a complete bipartite graph and $a: E \to \mathbf{Z}^+$ (this is equivalent to a variant of assignment problem, which we may call the exact assignment problem: decide if a m × n matrix includes a subset of min{m, n} independent elements, i.e., at most one in each column and each row, which sum to a given number k). As noted in [3], this problem generalizes the NP-complete numerical partitioning problem [2, p. 66]. Note that while there is a pseudopolynomial algorithm for the latter problem, no such algorithm has been found so far for the former. A probabilistic pseudopolynomial algorithm was proposed in [3] which, for given G, a, k, and ε, 0 < ε < 1 generates a correct decision if the sought matching does not exist and errs with probability <ε if it exists; this algorithm runs in time $O\,(AQ\,(|V|)(\log_2 A)\varepsilon^{-1})$, where $A = \max\{|a\,(e)| : e \in E\}$ and Q(n) is some polynomial (in fact, the solution in [3] is for the problem of arbitrary matching of a given weight in a bipartite graph, but the algorithm is easily modified to our case).

3. G is a complete graph and $a: E \to \mathbf{Z}^+$. This problem is NP-complete, since it also transforms to the numerical partitioning problem.

The graph G = (V, E) is called bipartite if there exists a partition {I, J} of its vertices such that each edge in G has one end in I and the other in J. If I and J are nonempty and each pair of vertices $x \in I, y \in J$ is joined by an edge, G is called a complete bipartite graph; we will denote it by (I:J, E). A graph is called complete if any two distinct ~~edges~~ vertices are joined by an edge.

In this article, we develop a decidability criterion for the problem P(G, a, k) and propose a polynomial algorithm for its solution when a takes the value 0 and 1 and G is either complete or complete bipartite.

Let us first investigate the case of a complete bipartite graph G = (I:J, E). It suffices to consider a balanced graph, i.e., a graph with |I| = |J| = n: if, for instance,

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$|I| > |J|$, then we can pass to an equivalent problem by adding $|I| - |J|$ new vertices and joining them by edges of weight 0 with all the other vertices in I. In this graph, the sets of bottleneck, maximum, and perfect matchings obviously coincide, and all consist of matchings of cardinality n. Identify the subgraphs $H^0 = (V, E^0)$ and $H^1 = (V, E^1)$, where $E^0 = \{e \in E: a(e) = 0\}$ and $E^1 = E \setminus E^0$. Let k^0 and k^1 be the cardinality of the maximum matching in the graphs H^0 and H^1, respectively. The problem obviously has no solution (i.e., the sought matching does not exist) if $k > k^1$ or $n - k > k^0$. Other insoluble cases are less trivial. By $m(\Gamma)$ we denote the number of components (connected subgraphs maximal by inclusion) of the graph Γ.

Example 1. Let $m(H^1) \geqslant 3$ [respectively, $m(H^0) \geqslant 3$] and each component in H^1 (in H^0) is a balanced complete bipartite graph. Then the problem is soluble for all k from $n - k^0$ to $k^1 = n$, with the exception of $k = n - 1$ (respectively, for all k from 0 to k^1 with the exception of $k = 1$).

Example 2. Let $m(H^1) = 2$ and the components in H^1 are complete bipartite graphs $(I_i: J_i, E_i)$, $i = 1, 2$ (the same is obviously also true for H^0). Then $k^1 = n - \| |I_1| - |J_1| \|$, $k^0 = n - \| |I_1| - |J_2| \|$, and for $n - k^0 \leqslant k \leqslant k^1$ the problem is soluble if and only if $k + n + |I_1| + |J_1|$ is even.

The verification of these examples, and the following Examples 3 and 4, is left to the reader [the matrices (a_{ij}) corresponding to Examples 1, 2 are shown in Fig. 1].

THEOREM 1. Consider a balanced complete bipartite graph $G = (I:J, E)$, $|I| = n$, with the function $a: E \to \{0, 1\}$, and let the subgraphs H^0 and H^1 be different from those of Examples 1 and 2. Then G has a perfect matching of weight k for any integer k satisfying $n - k^0 \leqslant k \leqslant k^1$.

Let us now consider a complete graph $G = (V, E)$, with the subgraphs H^0, H^1 and the numbers k^0, k^1 defined as above. Without loss of generality, we may take $|V|$ to be even and equal to 2n. As above, the problem has no solution for $k > k^1$ or $n - k > k^0$.

Example 3. Let $m(H^1) = m \geqslant 2$ [respectively, $m(H^0) = m \geqslant 2$]; each component in H^1 (in H^0) is either a complete graph with an even number of vertices or a balanced complete bipartite graph, and if $m = 2$, then the set of the latter is nonempty. Then the problem has a solution for all k from $n - k^0$ to n with the exception of $k = n - 1$ [respectively, for all k from 0 to k^1 with the exception of $k = 1$].

Example 4. Let $m(H^1) = 2$ [respectively, $m(H^0) = 2$] and the components in H^1 (in H^0) are complete graphs (V_i, E_i), $i = 1, 2$. Let $n_i = |V_i|$. Then $k^1 = \lfloor n_1/2 \rfloor + \lfloor n_2/2 \rfloor$, $k^0 = \min\{n_1, n_2\}$ (respectively, $k^1 = \min\{n_1, n_2\}$, $k^0 = \lfloor n_1/2 \rfloor + \lfloor n_2/2 \rfloor$) and for $n - k^0 \leqslant k \leqslant k^1$ the problem has a solution if and only if $k + n + n_1$ (respectively, $k + n_1$) is even; $\lfloor c \rfloor$ is the integer ~~whole~~ part of c.

THEOREM 2. For a complete graph $G = (V, E)$ with an even number of vertices 2n and a function $a: E \to \{0, 1\}$, let the subgraphs H^0 and H^1 be different from those in Examples 3 and 4. Then G has a perfect matching of weight k for any integer k satisfying $n - k^0 \leqslant k \leqslant k^1$.

Theorems 1 and 2 will be proved later, and the proofs will directly lead to effective algorithms which either find the sought matching or establish that one of the above insoluble cases applies [the algorithm can identify the subgraphs H^1 and H^0 with those in Examples 1-4 in time $O(n^2)$]. The running time of these algorithms is comparable with that of the maximum matching algorithms for a bipartite graph and an arbitrary graph, respectively, i.e., the problem can be solved in time $O(n^{2.5})$ for a complete bipartite G and in time $O(n^{2.5} \log n)$ for a complete G.

In what follows we will require some additional definitions and notation. An edge with the ends x and y will be denoted by xy. A chain (a cycle) in G is a nonempty subset of edges $L \subseteq E$ such that $L = \{x_i x_{i+1} : i = 0, \ldots, r - 1\}$, where $x_0, \ldots, x_r$ are distinct vertices (with the obvious exception $x_0 = x_r$); we say that the chain L joins the vertices x_0 and x_r. The chain L is called alternating with respect to the matching M (or M-alternating) if in each pair of adjacent edges in L one edge belongs to M. The set of all perfect matchings in G will be denoted by $\mathcal{M} = \mathcal{M}(G)$. For $M \in \mathcal{M}$ and a M-alternating cycle C, we denote by q(M, C) the quantity $a(C \cap M) - a(C \setminus M)$; clearly, $M \Delta C$ is a perfect matching of weight $a(M) - q(M, C)$ [$A \Delta B$ is the symmetric difference $A \setminus B) \cup (B \setminus A)$]. For an arbitrary graph $\Gamma = (W, U)$ and nonintersecting subsets $X, Y \subseteq W$, we denote by $U(X:Y)$ the set of edges in Γ with one end in X and the other in Y; the bipartite subgraph $(X \cup Y, U(X:Y))$ generated by X and Y will be denoted by $\Gamma(X:Y)$. By $\Gamma(X) = (X, U(X))$ we denote the subgraph in Γ generated by X, i.e., $U(X) = \{xy \in U : x, y \in X\}$.

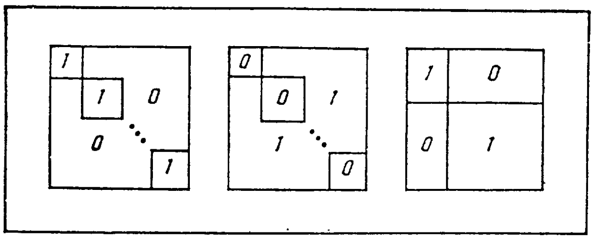

Fig. 1

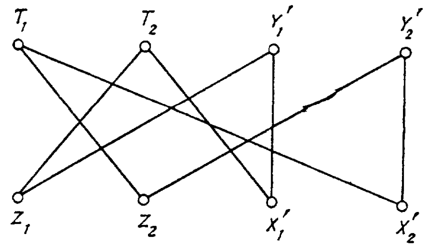

Fig. 2

<u>The Case of Complete Bipartite Graph</u>

Let G = (I:J, E) be a balanced complete bipartite graph, |I| = n, and a given k satisfies $n - k^0 \leqslant k \leqslant k^1$. If k = k^1 (respectively, k = n − k^0), the sought matching is obtained from the maximum matching in H^1 (in H^0) by arbitrarily extending it to a perfect matching in G, and we may therefore assume that $n - k^0 < k < k^1$.

<u>LEMMA 1.</u> There exists $M \in \mathcal{M}$ of weight k or k + 1.

Note that the lemma remains true if G is an arbitrary graph and $\mathcal{M}$ a set of all its bottleneck matchings.

<u>Proof.</u> We may assume that $k \leqslant k^1 - 2$. Take some $M_0 \in \mathcal{M}$ with $a(M_0) < k$ and choose any $M' \in \mathcal{M}$, such that $a(M') \geqslant k + 2$. To prove the lemma, it suffices to show that there exists $M'' \in \mathcal{M}$ such that $0 \leqslant a(M') - a(M'') \leqslant 2$ and $|M'' \cap M_0| > |M' \cap M_0|$. The set $M' \triangle M_0$ consists of pairwise nonintersecting cycles (alternating with respect to M' and M_0); among these cycles choose C with q(M', C) > 0, which exists since a(M') > $a(M_0)$. It is easy to see that C contains adjacent edges xy and yz such that $xy \in M' \cap E^1$ and $yz \in M_0 \cap E^0$. Let zt be an edge in C other than yz. Since G is a complete bipartite graph, then $tx \in E$; let C' = {xy, yz, zt, tx}. Clearly, C' is a M'-alternating cycle, and $0 \leqslant q(M', C') \leqslant 2$. For M" = M' Δ C' we have $(M' \cap M_0) \cup \{yz\} \subseteq M'' \cap M_0$, and a(M") = a(M') − q(M', C'). Q.E.D. □

Note that this proof suggests a $O(n^2)$ procedure to find $M \in \mathcal{M}$ with a(M) = k or k + 1, given $M_0, M_1 \in \mathcal{M}$ with $a(M_0) = n - k^0$ and $a(M_1) = k^1$.

Let $M \in \mathcal{M}$ be a matching of weight k + 1. Our aim is to find an alternating cycle C with q(M, C) = 1 (if we manage to do this, then M Δ C is the sought matching of weight k). Let $M^0 = M \cap E^0$, $M^1 = M \cap E^1$ and denote by X and Y the sets of vertices from I and J, respectively, which are incident to edges in M^1.

1. From the start, we can exclude from analysis the case when q(M, C) = 1 for any alternating cycle C with |C| = 4. In other words, we assume that the following conditions hold:

C1) if $xy, zt \in M^1$, $x, z \in I$, then either $xt, zy \in E^1$ or $xt, zy \in E^0$;

C2) if $xy \in M^1$, $zt \in M^0$, $x,z \in I$, then at least one of the edges xt, zy belongs to E^1.

2. Let us investigate the structure of the graph H^1(X, Y). For $x \in X$, let $D(x) = \{y \in Y : xy \in E^0\}$ and let $Q = \{x \in X : D(x) \neq \varnothing\}$. For $x \in X$ determine the sets X^x and Y^x of all vertices z in X and Y, respectively, such that the graph H^1(X:Y) has a M^1-alternating chain which joins x and z and contains an edge from M^1 incident ~~on~~ to x; the vertex x itself ~~may be~~ is regarded as contained in X^x. If for some $x \in Q$ we have $Y^x \cap D(x) = \varnothing$, then taking for ~~any~~ $y \in Y^x \cap D(x)$ this alternating chain between x and y and adding to it the edge xy, we obtain an alternating cycle C with q(M, C) = 1 (since all the edges in C, except xy, belong to E^1). At the same time, we have

<u>LEMMA 2.</u> If $Y^x \cap D(x) = \varnothing$ for all $x \in Q$, then each component in H^1(X:Y) is a balanced complete bipartite graph.

<u>Proof.</u> Let $x \in X$. Applying standard arguments about alternating chains in a bipartite graph, we find that a) for $vu \in M^1$, $v \in X^x$ implies that $u \in Y^x$ and vice versa; b) $vu \in E^0$ for any $v \in Y^x$ and $u \in X \setminus X^x$; c) $|X^x| = |Y^x|$. From a), b) and condition C1) we conclude that $vu \in E^0$ for any $v \in X^x$ and $u \in Y \setminus Y^x$, i.e., $H^1(X^x : Y^x)$—is a component in H^1(X:Y). This implies that for any $x, x' \in X$ the graphs $H^1(X^x:Y^x)$ and $H^1(X^{x'}:Y^{x'})$ either coincide or are nonintersecting. If now for some $x \in Q$ the graph $H^1(X^x:Y^x)$ is not a complete bipartite graph, then clearly there exists $x' \in X^x$ such that $Y^x \cap D(x') \neq \varnothing$ a contradiction, since $Y^x = Y^{x'}$. Q. E. D. □

Remark. We can find the required alternating cycle or establish that $H^1(X:Y)$ has the structure as described in Lemma 2 in time $O(|X|^2)$. Indeed, the sets X^x and Y^x (and the corresponding alternating cycles for the vertices in these sets) may be constructed by standard labeling methods in time $O(|X^x|^2)$. If for the current vertex $x \in Q$, $H^1(X^x:Y^x)$ is not a complete bipartite graph, then, by the proof of the lemma, we can take as the next vertex any $x' \in X^x$ such that $Y^x \cap D(x') \neq \emptyset$ and thus construct the alternating cycle.

3. Suppose that the components of the graph $H^1(X:Y)$ are the balanced complete bipartite graphs $G(X_1:Y_1), \ldots, G(X_m:Y_m)$, where $X_i \subset I$, $i = 1, \ldots, m$. In what follows, we do not fix the matching M in each $G(X_i:Y_i)$, i.e., we assume that M may contain arbitrary pairwise nonadjacent edges from each $G(X_i:Y_i)$.

We will show that for m = 1, G contains an alternating cycle C with q(M, C) = 1. First note that $M^0 \neq \emptyset$ (otherwise we would have $k + 1 = |M^1| = n$ and $k^0 = |E^0| = 0$, whence $k < n - k^0$). Let S and T be the sets of all vertices z in I and J, respectively, such that either $z \in X$ or the subgraph H^0 contains a M^0-alternating ~~cycle~~ chain which joins z with some vertex in X. Clearly, $|S \setminus X| = |T \setminus Y|$ and $xy \notin E^0$ for any $x \in S$, $y \in J \setminus T$. Since H^0 contains a matching of cardinality $k^0 \geqslant n - k$, then by the Koenig–Ore theorem (see [4]) we should have $|S| - |T| \leqslant k$, whence $|S \cap X| - |T \cap Y| \leqslant k$. But ~~He~~ $|S \cap X| = |X| = k + 1$ and so $T \cap Y \neq \emptyset$. Thus, H^0 contains an alternating chain L which joins some $x \in X$ and $y \in Y$, and then $C = L \cup \{xy\}$ is the sought alternating cycle with q(M, C) = 1.

4. Let $m \geqslant 2$. If $M^0 = \emptyset$, then $k = n - 1$ and we obtain the insoluble case from Example 1 (for $m \geqslant 3$) or Example 2 (for m = 2). Let $M^0 = \emptyset$. For an arbitrary edge $zt \in M^0$, $z \in I$, and vertices $x \in X_i$, $y \in Y_i$, $u \in X_j$, $v \in Y_j$, where $i \neq j$, we have the obvious alternative: either the alternating cycle C = {zt, tx, xy, yu, uv, vz} satisfies q(M, C) = 1, or a(tx) = a(zv). Let the latter be true for all ~~these~~ such z, t, x, y, u, v. Using condition C2), we thus conclude that for each $zt \in M^0$, $z \in I$, we may have only ~~one~~ some of the following two situations:

A1) $m \geqslant 2$, $E(\{z\}:Y) \cup E(X:\{t\}) \subset E^1$;

A2) m = 2 and up to indexing we have

$$E(\{z\}:Y_1) \cup E(X_2:\{t\}) \subset E^1,$$
$$E(\{z\}:Y_2) \cup E(X_1:\{t\}) \subset E^0.$$

Let $m \geqslant 3$. Then by A1), for any edge $zt \in M^0$, $z \in I$, and six vertices $x_i \in X_i$, $y_i \in Y_i$, i = 1, 2, 3, we have an alternating cycle $C = \{zt, tx_1, x_1y_1, y_1x_2, x_2y_2, y_2x_3, x_3y_3, y_3z\}$, with q(M, C) = 1.

5. It remains to consider the case m = 2. Choose four vertices $x_i \in X_i$, $y_i \in Y_i$, i = 1, 2 and take ~~a~~ the perfect matching $M' = M \Delta C$, where $C = \{x_1y_1, y_1x_2, x_2y_2, y_2x_1\}$, of weight $a(M') = a(M) - 2 = k - 1$. Let us transform M' to a perfect matching of weight k, using the same reasoning as for M (while interchanging the sets E^1 and E^0). Then we will find the sought matching, or else establish insolubility of the problem (by ending up with a subgraph H^0 from Example 1 or 2), or finally establish that

a) $a(M') = |M' \cap E^1| > 0$;

b) the graph $H^0(Z:T)$, where $Z = (I \setminus X) \cup \{x_1, x_2\}$ and $T = (J \setminus Y) \cup \{y_1, y_2\}$, is made up of two components, $G(Z_1:T_1)$ and $G(Z_2:T_2)$, $Z_i \subset I$, each a balanced complete bipartite graph;

c) for any $i \in \{1, 2\}$, $x \in X_i \setminus \{x_i\}$, $y \in Y_i \setminus \{y_i\}$, we have one of the following two situations [analogous to A1) and A2)]:

B1) $E(\{x\}:T) \cup E(Z:\{y\}) \subset E^0$;

B2) up to indexing, we have

$$E(\{x\}:T_1) \cup E(Z_2:\{y\}) \subset E^0,$$
$$E(\{x\}:T_2) \cup E(Z_1:\{y\}) \subset E^1.$$

Let $X'_i = X_i \setminus \{x_i\}$ and $Y'_i = Y_i \setminus \{y_i\}$, i = 1, 2. Examining arbitrary $i, j \in \{1, 2\}$, $z \in Z_i$, $t \in T_i$ $x \in X'_j$, $y \in Y'_j$, we establish that only A2) may hold for the pair (z, t) and only B2) for the pair (x, y). Hence it follows that for any $i, j \in \{1, 2\}$ all the edges of one of the sets $E(Z_i:Y'_j)$ or $E(X'_j:T_i)$ entirely belong to E^0 while all the edges of the other belong to E^1. Note that since $M' \cap E^1 \neq \emptyset$, at least one of the sets X'_1 and X'_2 is nonempty. For definiteness, let $X'_1 \neq \emptyset$ and $E(Z_1:Y'_1) \subset E^1$. Then

$$E(X'_1:T_2) \cup E(X'_2:T_1) \cup E(Z_2:Y'_2) \subset E^1,$$

$$E(Z_1:Y_2') \cup E(X_2':T_2) \cup E(Z_2:Y_1') \cup E(X_1':T_1) \subset E^n.$$

Hence it follows directly that H^1 is the union of two nonintersecting complete bipartite graphs $G(Z_1 \cup X_1' : T_2 \cup Y_1')$ and $G(Z_2 \cup X_2' : T_1 \cup Y_2')$ (see Fig. 2, where each of the eight vertex subsets is represented by a single vertex). Moreover, $|Z_1|+|T_2|=|M' \cap E^0|=n-k+1$ and $|X_1'|=|Y_1'|$ imply that $k+n+|Z_1 \cup X_1' \cup T_2 \cup Y_1'|$ is an odd number. We have thus obtained the insoluble case of Example 2.

This completes the proof of Theorem 1. Q.E.D.

The proof clearly suggests an algorithm for the solution of our problem. The algorithm first finds the maximum matchings in the graphs H^0 and H^1, which requires time $O(n^{2.5})$ (see [5-7]). All the other procedures of the algorithm require total time $O(n^2)$.

The Case of Complete Graph

Let $G = (V, E)$ be a complete graph, $|V| = 2n$, and for a given k let $n-k^0 \leqslant k \leqslant k^1$. We may assume that $n - k^0 < k < k^1$. It is easy to see that the proof of Lemma 1 remains valid for the case of complete graph also. Thus, we may assume that there exist $M, M' \in \mathcal{M}$, such that $a(M) = k + 1$ and $a(M') = k - 1$. Since all the cycles in $M \cup M'$ are of even cardinality, we may choose a subset $I \subset V$, $|I|=n$ such that M and M' are contained in a complete bipartite graph $G_I = G(I:J)$, where $J = V \setminus I$. Let E_I, H_I^0, H_I^1, k_I^0, k_I^1 be, respectively, the set of edges in G_I, the subgraphs $(V, E^0 \cap E_I)$, $(V, E^1 \cap E_I)$, and the cardinality of the maximum matchings in these subgraphs. Then $n-k_I^0<k<k_I^1$ and we may try to find the sought matching already in G_I. If it is not found there, then by Theorem 1 we have one of the insoluble cases of Example 1 or 2.

Case 1. $k = n - 1$ and H_I^1 consists of $m \geqslant 2$ balanced complete bipartite graphs $G(I_i:J_i)$, $I_i \subset I$, $i = 1,\dots,m$ (to the general case of Example 1 we have added a particular case from Example 2). As before, we assume that M is not fixed inside the graphs $G(I_i:J_i)$.

LEMMA 3. Assume that G contains no matching of weight k. Then

a) for each $i \in \{1,\dots,m\}$ either $E(I_i) \cup E(J_i) \subset E^0$, or $E(I_i) \cup E(J_i) \subset E^1$;

b) for any two different $i, j \in \{1,\dots,m\}$ either $E(I_i:I_j) \cup E(J_i:J_j) \subset E^0$ or $E(I_i:I_j) \cup E(J_i:J_j) \subset E^1$;

c) if $xz \in E^1$ for some $x \in I_i$, $z \in I_j$, $i, j \in \{1,\dots,m\}$, $i \neq j$, then $E(J_i) \subset E^0$ and for any $l \in \{1,\dots,m\} \setminus \{i, j\}$ we have $E(J_i:J_l) \subset E^0$.

Proof. Propositions a), b) derive from the following: if for any four different vertices $x \in I_i$, $y \in J_i$, $z \in I_j$, $t \in J_j$ (i = j is allowed) we had $a(xz) \neq a(yt)$, then for a M-alternating cycle $C = \{xy, yt, tz, zx\}$ we would have $q(M, C) = 1$. To prove c), consider any four different vertices $y \in J_i$, $t \in J_j$, $u \in I_l$, $v \in J_l$, where $\ell \neq j$ ($\ell = i$ is allowed). If $yv \in E^1$, then for the alternating cycle $C = \{xy, yv, vu, ut, tz, zx\}$ we would have $q(M, C) = 1$. Q.E.D. □

Lemma 3 directly implies that in case of no solution the components of the graph H^1 are only complete graphs of the form $G(I_i \cup J_i)$ and balanced complete bipartite graphs of the form $G(I_i:J_i)$, $G(I_i \cup J_j:I_j \cup J_i)$. We thus obtain the insoluble case of Example 3 or 4 (if H_I^1 consists of two complete graphs or is a complete bipartite graph, we have particular cases of Example 4).

The symmetrical case with $k = 1$ is analyzed similarly.

Case 2. The graph H_I^1 consists of two complete bipartite graphs, $G(I_i:J_i)$, $I_i \subset I$, $i=1, 2$. and $k+n+|I_1|+|J_1|$ is odd. We may assume that $M \cap E^0 \neq \varnothing$ and $M' \cap E^1 \neq \varnothing$, since otherwise we would get $k=n-1$, $|I_i|=|J_i|$, $i = 1, 2$, or $k=1$, $|I_1|=|J_2|$, $|J_1|=|I_2|$, which has been considered above. Note that $k>n-k_I^0$ implies that $M \cap E(I_i:J_i) \neq \varnothing$, $i = 1, 2$. For definiteness, let $|I_1| \geqslant |J_1|$. Clearly, $M \cap E^0$ contains an edge xy such that $x \in I_1$, $y \in J_2$. We may assume that M contains the edges xy, zt, uv for arbitrarily chosen different vertices $z, x \in I_1$, $t \in J_1$, $u \in I_2$, $v \in I_2$, $y, v \in J_2$.

LEMMA 4. Assume that G contains no matching of weight k. Then one of the following two propositions holds:

a) $E(I_1:I_2) \cup E(J_1:J_2) \subset E^0$, $E(I_1) \cup E(I_2) \cup E(J_1) \cup E(J_2) \subset E^1$;

b) $E(I_1:I_2) \cup E(J_1:J_2) \subset E^1$, $E(I_1) \cup E(I_2) \cup E(J_1) \cup E(J_2) \subset E^0$.

Proof. By assumption, for some $z_0 \in I_1$, $t_0 \in I_2$ we have $z_0t_0 \in E^0$, which leads to the following chain of propositions:

1) $uv \in E^0$ for any $u \in J_1$, $v \in J_2$ [otherwise, for an alternating cycle $C = \{z_0u, uv, vt_0, t_0z_0\}$ we would have q(M, C) = 1]; similarly, $zt \in E^0$ for all $z \in I_1$, $t \in I_2$;

2) $zx \in E^1$ for any two different $z, x \in I_1$ [otherwise $C = \{zx, xy, yu, uz\}$, where $u \in J_1$, $y \in J_2$, is an alternating cycle with q(M, C) = 1]; similarly $yv \in E^1$ for any two different $y, v \in J_2$;

3) if $M \cap E^0$ contains an edge x'y' with $x' \in I_2, y' \in J_1$, then from the same considerations as in 2), we obtain that $E(I_2) \cup E(J_1) \subset E^1$; assume that no such edges x'y' are contained in $M \cap E^0$, then $|M \cap E(I_1:J_1)| = |J_1|$ and therefore, for any different $u, u' \in J_1, z, z' \in I_1$, the cycle $C = \{zu, uu', u'z', z'z\}$ is alternating; since q(M, C) ≠ 1, then $uu' \in E^1$; we similarly show that $E(I_2) \subset E^1$.

We have thus proved a). The case b) arises when $z_0t_0 \in E^1$ for some $z_0 \in I_1$, $t_0 \in I_2$ (we should repeat the argument for M', interchanging E^0 and E^1). Q.E.D.

Lemma 4 directly implies that if the problem is insoluble, then either H^1 consists of the complete graphs $G(I_i \cup J_i)$, i = 1, 2 or H^0 consists of the complete graphs $G(I_1 \cup J_2)$ and $G(I_2 \cup J_1)$. Finally, it is easily seen that both numbers $k + n + |I_1 \cup J_1|$ and $k + |I_1 \cup J_2|$ are odd. i.e., in both variants we obtain the insoluble cases of Example 4.

This completes the proof of Theorem 2. Q.E.D.

Our proof is easily transformed into an algorithm for the solution of the relevant problem. It starts by constructing the maximum matchings in the graphs H^0 and H^1 in time $O(n^{2.5} \times \log n)$ [8] (the simpler algorithms in [9, 10] run in time $O(n^3)$). The other procedures of the algorithm run in time $O(n^2)$.

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