On a Class of Maximum Multicommodity Flow Problems with Integer Optimal Solutions

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In [4] we announced and briefly outlined a new proof of a theorem, due to Karzanov and Lomonosov, on the existence of an integral optimal solution for a special case of the maximum undirected multicommodity flow problem. In this case the capacity function is inner Eulerian and the commodity graph H possesses the following property: the set of its anticliques has a partition into two subsets, each consisting of pairwise disjoint anticliques (in other words, H is the complement to the line graph of a bipartite graph). Based on the splitting-off techniques (rather than alternating path approaches), this proof looks considerably simpler than the original one and, moreover, it provides a strongly polynomial algorithm to solve the problem.

In this paper we give a detailed description of these results.

1. Introduction

By a *graph* we mean a finite undirected graph without loops and multiple edges; the edge with endpoints x and y is denoted by xy. By a *chain* (or an *xy-chain*) of a graph we mean its subgraph $L = (VL, EL)$ whose vertex-set and edge-set have the following form: $VL = \{x = x_0, x_1, \dots, x_k = y\}$, $EL = \{x_i x_{i+1} : i = 0, \dots, k-1\}$; the chain L is denoted by $x_0 x_1 \cdots x_k$.

We deal with the following objects: a graph $G = (V, E)$, *capacities* $c(e) \geqslant 0$ defined on its edges $e \in E$, and a graph $H = (T, U)$ having no isolated vertices and satisfying the relation $T \subseteq V$. The pair (G, c), the graph H, and its vertex set T are said to be the *network*, the (flow) *scheme*, and the set of *terminals*, respectively.

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Consider the well-known problem of finding a maximum multicommodity flow in a network (G, c) with a scheme H. For our purposes, it is convenient to formulate this problem as a chain packing problem. Given $x, y \in V$, denote by $\mathscr{L}(G, xy)$ the set of all xy-chains in G. Put $\mathscr{L} := \mathscr{L}(G, H) := \bigcup_{st\in U} \mathscr{L}(G, st)$. A function $f: \mathscr{L} \to \mathbb{R}_+$ (here $\mathbb{R}_+$ is the set of all nonnegative reals) is said to be a *multicommodity flow*, or, briefly, a *multiflow*; a multiflow f is said to be *c-admissible* if the following capacity restrictions are valid:

$$\zeta^f(e) := \sum_{L\in\mathscr{L},\, e\in EL} f(L) \leqslant c(e), \qquad e \in E. \tag{1}$$

PROBLEM $\mathscr{P} = \mathscr{P}(G, c, H)$. *Find a c-admissible multiflow f having the maximum total value*

$$1 \cdot f = \sum_{L\in\mathscr{L}} f(L).$$

Denote by $v = v(G, c, H)$ the maximum of $1 \cdot f$ over all c-admissible f.

An interesting problem in discrete optimization theory consists in the following: Given a class of linear programs with integer-valued constraint matrices and right-hand side vectors, decide whether each program of this class has an optimal solution with all denominators bounded by a fixed number k. In our case, we have the following situation. Let us classify the problems $\mathscr{P}(G, c, H)$ according to the type of the scheme $H = (T, U)$. We say that a scheme H (and the corresponding class $\mathscr{P}(H)$ of the problems $\mathscr{P}(G, c, H)$ with H fixed) is *solvable in* $\frac{1}{k}\mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of all nonnegative integers and $k \in \mathbb{Z}_+ \setminus \{0\}$, if for any graph $G = (V, E)$, $V \supseteq T$, and any nonnegative integer-valued function $c: E \to \mathbb{Z}_+$ the problem $\mathscr{P}(G, c, H)$ possesses an optimal solution f such that $kf(L)$ is an integer for all $L \in \mathscr{L}(G, H)$; in other words, if the problem $\mathscr{P}(G, kc, H)$ possesses an integer optimal solution. A scheme H is said to be of *bounded fractionality* if H is solvable in $\frac{1}{k}\mathbb{Z}_+$ for some natural k.

We need also the following notions. Let $X \subseteq V$; the set of all edges of G having one endpoint in X and the other one in $V \setminus X$ is denoted by $\partial X = \partial^G X$ and is called a *cut* of G (the cases $X = \varnothing$ and $X = V$ are allowed). A function c is said to be *inner Eulerian* if it takes integer values and $c(\partial X)$ is even for any $X \subseteq V \setminus T$ (for arbitrary $g: E \to \mathbb{R}$ and $E' \subseteq E$ we denote by $g(E')$ the sum $\sum_{e\in E'} g(e)$). A scheme H is said to be *solvable in* $\frac{1}{k}\mathbb{Z}_+$ *for the inner Eulerian case* if the problem $\mathscr{P}(G, kc, H)$ possesses an integer optimal solution for any graph $G = (V, E)$, $V \supseteq T$, and any inner Eulerian function c on E. Evidently, if H is solvable in $\frac{1}{k}\mathbb{Z}_+$ for the inner Eulerian case, then it is solvable in $\frac{1}{2k}\mathbb{Z}_+$.

By the classical Ford–Fulkerson theorem, H is solvable in $\mathbb{Z}_+ = \frac{1}{1}\mathbb{Z}_+$ for $|U| = 1$ (in this case one has the usual maximum flow problem for

an undirected network); this fact is generalized easily to complete bipartite graphs H. One can show that if H is not a complete bipartite graph, then it is not solvable in $\mathbb{Z}_+$. It is well known that the scheme with two edges is solvable in $\frac{1}{2}\mathbb{Z}_+$ (Hu's theorem on half-integer two-commodity flows [3]), as well as schemes that are complete bipartite graphs with an arbitrary number of vertices [7], [9], [1]. These results were sharpened in [10], [9], and [1]; it was proved there that the corresponding schemes are solvable in $\mathbb{Z}_+$ for the inner Eulerian case. In [2], the solvability in $\frac{1}{2}\mathbb{Z}_+$ was proved for certain schemes represented as the union of two complete bipartite graphs. Finally, in [6] large classes of schemes solvable in $\frac{1}{2}\mathbb{Z}_+$ and in $\frac{1}{4}\mathbb{Z}_+$ were provided, which generalize all the previously known schemes with these properties. These classes are defined in terms of the family $\mathscr{A} = \mathscr{A}(H)$ of all anticliques of the graph H (an *anticlique* of a graph is a maximal (with respect to inclusion) independent (i.e., generating the empty subgraph) set of its vertices).

DEFINITIONS. A family $\mathscr{A}$ of anticliques is said to be *bipartite* if it has a partition $\{\mathscr{A}_1, \mathscr{A}_2\}$ such that each $\mathscr{A}_i$ consists of pairwise disjoint anticliques. A family $\mathscr{A}$ is said to be 3-*noncrossing* if it does not contain three pairwise intersecting anticliques. A family is said to be *perfect* if for any three distinct anticliques A, B, C such that $A \cap B \neq \varnothing$, $B \cap C \neq \varnothing$, $C \cap A \neq \varnothing$, one has $A \cap B = B \cap C = C \cap A$.

Each of the above three classes contains the previous one as a proper subset. Here are several instances of these classes:

a) If H contains only two edges, or H is a complete graph, then $\mathscr{A}$ is a bipartite family.

b) If H is the cycle on 5 vertices, then $\mathscr{A}$ is 3-noncrossing, but not bipartite.

c) If H consists of a triangle and an edge not adjacent to the triangle, then $\mathscr{A}$ is perfect, but not 3-noncrossing.

d) If H consists of three pairwise nonadjacent edges, then $\mathscr{A}$ is nonperfect.

According to [6], the following assertions are true:

(1) If $\mathscr{A}$ is bipartite, then H is solvable in $\frac{1}{2}\mathbb{Z}_+$.

(2) If $\mathscr{A}$ is 3-noncrossing, then H is solvable in $\frac{1}{4}\mathbb{Z}_+$.

The proof of the first statement is implied by the existence of a pseudopolynomial algorithm solving the problem $\mathscr{P}(G, c, H)$ for integer c and bipartite $\mathscr{A}(H)$ (the running time for the algorithm is bounded by $c(E)$ times a polynomial in $|V|$ and $|T|$). The second statement follows from a reduction of the problem $\mathscr{P}(G, c, H)$ with a 3-noncrossing $\mathscr{A}(H)$ to the bipartite case; this reduction assigns an optimal solution of the first problem to each optimal solution of the second one; moreover, the fractionality for the former is twice as big as that for the latter. For details of proofs see [4] and [8].

The question of what is the class of H's such that H is solvable in $\frac{1}{k}\mathbb{Z}_+$ for some natural k (depending only on H) is still open, and a conjecture is that this is exactly the set of H's with perfect $\mathscr{A}(H)$ (a strong version of the conjecture suggests that whenever such a k exists for a given H, k can be taken to be 4).[1]

The original proof of the first of the above statements, outlined in a sketched form in [6], provides, in essence, the following stronger result, as was pointed out in [8].

THEOREM 1. *If $\mathscr{A}(H)$ is bipartite and c is inner Eulerian, then the problem $\mathscr{P}(G, c, H)$ has an integer optimal solution (in other words, each scheme H such that $\mathscr{A}(H)$ is bipartite is solvable in $\mathbb{Z}_+$ for the inner Eulerian case).*

This implies that in the inner Eulerian case, H is solvable in $\frac{1}{2}\mathbb{Z}_+$ if $\mathscr{A}(H)$ is 3-noncrossing.

It should be noted that the above-mentioned proof of Theorem 1 used rather intricate augmenting-path techniques. In the present paper, we develop a new proof of this theorem, which seems to be significantly simpler; moreover, this proof provides a strongly polynomial algorithm to solve the problem in question. These proof and algorithm were announced and briefly described in [4].

The proof presented here relies on the following two ideas. First, one can directly establish the existence of a dual optimal solution having a special combinatorial, form (in fact, the existence of such dual solutions was stated in [6], but there it appeared as a consequence of the proof of Theorem 1). Second, on the basis of this result, one can apply the splitting-off technique that reduces the initial network step by step to a trivial one, for which an integer optimal solution exists evidently.

The rest of the paper is organized as follows. The special duality theorem is established in §2, and the proof of Theorem 1 is completed in §3. Finally, in §4, we describe a strongly polynomial algorithm for the problem $\mathscr{P}(G, c, H)$ with a bipartite $\mathscr{A}(H)$. It is capable of working with arbitrary "real-valued" capacities c and has complexity $O(n^3\sigma(tn))$; here $n = |V|$, $t = |T|$, and $\sigma(n')$ is the running time required for finding a maximum flow in a network on n' vertices. Whenever c is inner Eulerian, the algorithm determines an integer-valued optimal solution.

2. Minimal proper families

In what follows we use the abbreviated notation $\mathscr{P}(c)$, $v(c)$, $\mathscr{A}$, $\mathscr{L}$ for $\mathscr{P}(G, c, H)$, $v(G, c, H)$, $\mathscr{A}(H)$, $\mathscr{L}(G, H)$, respectively. Throughout the paper, we assume that $\mathscr{A}$ is bipartite, and $\{\mathscr{A}_1, \mathscr{A}_2\}$ is its partition ($A \cap B = \varnothing$ for any distinct $A, B \in \mathscr{A}_i$, $i = 1, 2$).

([1]) *Added in translation.* Recently, in [11] it was proved that: (1) if $\mathscr{A}(H)$ is perfect, then the dual problem is solvable in $\frac{1}{4}\mathbb{Z}_+$, and (2) if $\mathscr{A}(H)$ is not perfect, then the primal problem is not solvable in $\frac{1}{k}\mathbb{Z}_+$ for any k.

A family $\mathscr{X} = \{X_A : A \in \mathscr{A}\}$ is said to be *semiproper* if

(1) $X_A \subset V$ and $X_A \cap T \subseteq A$ for any $A \in \mathscr{A}$ (the case $X_A = \varnothing$ is allowed), and

(2) each terminal $s \in T$ is contained exactly in one set $X_A \in \mathscr{X}$.

A semiproper family $\mathscr{X}$ consisting of pairwise disjoint sets is said to be *proper*. Let us define the capacity of a semiproper family $\mathscr{X}$ by $c(\mathscr{X}) := \frac{1}{2}\sum_{A\in\mathscr{A}} c(\partial X_A)$.

In this section our aim is to prove the following theorem establishing a minimax relation between multiflows and proper families.

THEOREM 2.1. *The value $v(c)$ equals the minimum of $c(\mathscr{X})$ over all semiproper families $\mathscr{X}$, and the minimum is attained at a proper family.*

The proof includes several auxiliary facts. Assume that $f: \mathscr{L} \to \mathbb{R}_+$ is an optimal multiflow, and let $\mathscr{L}^+(f)$ denote the set $\{L \in \mathscr{L}: f(L) > 0\}$. A subset $X \subset V$ (as well as the corresponding cut ∂X) is said to be *saturated* by f if $\zeta^f(e) = c(e)$ for any $e \in \partial X$; X is said to be *compatible with* $L \in \mathscr{L}$ if $|\partial X \cap EL| \leqslant 1$, and *compatible with* f if it is compatible with each chain in $\mathscr{L}^+(f)$. By the definition, for a semiproper $\mathscr{X}$ and an arbitrary $st \in U$ there exists a unique pair $X_A, X_B \in \mathscr{X}$ such that $s \in X_A \not\ni t$ and $s \notin X_B \ni t$. This fact implies easily the following statement, which is often used here and in subsequent sections.

CLAIM 1. *For any semiproper family $\mathscr{X}$ one has $c(\mathscr{X}) \geqslant v(c)$, and equality holds if and only if each set $X_A \in \mathscr{X}$ is saturated and compatible with f.*

A terminal $s \in T$ is said to be a 1-*terminal* (a 2-*terminal*) if it belongs to exactly one (respectively, two) anticlique. Given two terminals s and t (not necessarily distinct), we write $s \sim t$ if $s, t \in A \cap B$ for two intersecting anticliques A and B, and $s \not\sim t$ otherwise. In particular, $s \sim s$ if s is a 2-terminal, while $s \not\sim s$ if s is a 1-terminal. One easily obtains the following useful statement.

CLAIM 2. *Let $\mathscr{A}$ be 3-noncrossing, and let $A, B \in \mathscr{A}$.*

1. *If $s, t \in A$, $s \not\sim t$, $p \in T \setminus A$, then at least one of the edges sp and tp belongs to U.*
2. *If $st \in U$ and $s' \sim s$, then $s't \in U$.*
3. *If A and B are intersecting and either $s \in A \cap B$, $t \in T \setminus (A \cup B)$, or $s \in A \setminus B$, $t \in B \setminus A$, then $st \in U$.*

Assume that $l: E \to \mathbb{R}_+$ is an optimal solution for the problem $\mathscr{P}^*(c)$ dual (in sense of linear programming) to $\mathscr{P}(c)$, that is, $l(EL) \geqslant 1$ for all $L \in \mathscr{L}$ and $\sum_{e\in E} c(e)l(e) = v(c)$. For arbitrary $x, y \in V$ put

$$\mu(xy) := \min_{L\in\mathscr{L}(G, xy)} l(EL);$$

thus, μ is the metric on V induced by the edge lengths l (observe that $\mu(xy) = 0$ if $x = y$, and $\mu(xy) = \infty$ if x and y are from different connected components of G). Then $\mu(st) \geqslant 1$ for any $st \in U$, and the complementary slackness conditions of linear programming applied to $\mathscr{P}(c)$ and $\mathscr{P}^*(c)$ are specified as

$$e \in E, \ \zeta^f(e) < c(e) \Rightarrow l(e) = 0; \tag{2}$$

$$L \in \mathscr{L}^+(f) \Rightarrow l(EL) = 1 \ (\Rightarrow \mu(st) = 1), \tag{3}$$

where s and t are the endpoints of L.

Let $A \in \mathscr{A}$ and $x \in V$; put

$$r_A(x) := \min\{\mu(sx) : s \in A\},$$
$$d_A(x) := \min\{\mu(sx) + \mu(xt) : s, t \in A, \ s \not\sim t\}.$$

Next, for $A \in \mathscr{A}_1$ put

$$X_A^* := Y_A := \{x \in V : d_A(x) < 1/2\},$$

and for $A \in \mathscr{A}_2$ put

$$Y_A := \{x \in V : d_A(x) \leqslant 1/2\}, \qquad R_A := \{x \in V : r_A(x) = 0\},$$
$$W_A := R_A \cap \{x \in V : d_B(x) \geqslant 1/2 \ \forall B \in \mathscr{A} \setminus \{A\}\}, \qquad X_A^* := Y_A \cup W_A.$$

Finally, put $\mathscr{X}^* = \{X_A^* : A \in \mathscr{A}\}$.

We want to prove that $\mathscr{X}^*$ is a proper family all of whose sets are saturated and compatible with f; by Claim 1, this would yield $c(\mathscr{X}^*) = v(c)$, thus proving the theorem. The proof is divided into several claims. For any two terminals s and t (not necessarily distinct), put

$$\tau(st) = \begin{cases} 1 & \text{if } st \in U, \\ 0 & \text{otherwise.} \end{cases}$$

For any four terminals s, t, p, q (not necessarily distinct), put

$$\tau(s, t, p, q) = \max\{\tau(st) + \tau(pq), \ \tau(sp) + \tau(tq), \ \tau(sq) + \tau(tp)\}.$$

CLAIM 3. 1. *If $\tau(s, t, p, q) = 0$, then $s, t, p, q \in A$ for some $A \in \mathscr{A}$.*

2. *If $\tau(s, t, p, q) = 1$, then there is an $A \in \mathscr{A}$ containing exactly three of the terminals s, t, p, q.*

PROOF. 1. This follows from the fact that no edge of H can have both of its endpoints in $\{s, t, p, q\}$ (otherwise $\tau(s, t, p, q) \geqslant 1$).

2. Evidently, each anticlique contains at most three elements out of $\{s, t, p, q\}$. There are two possible cases, up to a permutation:

1) $\tau(st) = \tau(sp) = \tau(tp) = 0$, and
2) $\tau(st) = \tau(sp) = \tau(sq) = 0$.

In the first case, the triple $\{s, t, p\}$ belongs to an anticlique. In the second case, each pair out of $\{s, t\}$, $\{s, p\}$, $\{s, q\}$ belongs to an anticlique, and since $\mathscr{A}$ is 3-noncrossing, two of these anticliques coincide, thus completing the proof.

CLAIM 4. *For any $s, t, p, q \in T$ and $x \in V$, one has*

$$\mu(sx) + \mu(tx) + \mu(px) + \mu(qx) \geqslant \tau(s, t, p, q).$$

PROOF. Observe that for any $s', t' \in T$ one has $\mu(s't') \geqslant \tau(s't')$. To be definite, let $\tau(s, t, p, q) = \tau(st) + \tau(pq)$. Then the required inequality follows immediately from the triangle inequalities:

$$\mu(sx) + \mu(xt) \geqslant \mu(st), \qquad \mu(px) + \mu(xq) \geqslant \mu(pq).$$

CLAIM 5. *Suppose that $A, B \in \mathscr{A}$ and $x \in V$. Then:*
(1) $d_A(x) + d_B(x) \geqslant 2$ if $A \cap B = \varnothing$;
(2) $d_A(x) + d_B(x) \geqslant 1$ if A and B intersect.

PROOF. Assume that $s, t \in A$ and $p, q \in B$ satisfy the relations

$$s \not\sim t, \qquad p \not\sim q,$$
$$d_A(x) = \mu(sx) + \mu(xt), \qquad d_B(x) = \mu(px) + \mu(xq).$$

By Claim 4, one has $d_A(x) + d_B(x) \geqslant \tau(s, t, p, q)$. Let $A \cap B = \varnothing$. If $\tau(s, t, p, q) \leqslant 1$, then by Claim 3 there exists an anticlique A' containing at least three elements out of s, t, p, q, say, $s, t, p \in A'$. Then $A \neq A'$, and $s, t \in A \cap A'$, contrary to $s \not\sim t$. Now let A and B intersect. Then $s \not\sim t$ and $p \not\sim q$ imply that there does not exist any anticlique containing all the four elements s, t, p, q. By Claim 3, this yields $\tau(s, t, p, q) \geqslant 1$.

CLAIM 6. *Suppose that A and B are two distinct anticliques and $x \in V$. Then:*

(1) If $r_A(x) + d_B(x) < 1$, then A and B intersect and

$$p \in A, \ \mu(px) = r_A(x) \Rightarrow p \in A \cap B.$$

(2) If $A \cap B = \varnothing$ and $r_A(x) + r_B(x) < 1$, then there exists an anticlique C intersecting both A and B and such that $d_C(x) \leqslant r_A(x) + r_B(x)$.

PROOF. 1. Let $s, t \in B$, $s \not\sim t$, and $d_B(x) = \mu(sx) + \mu(xt)$. Since $\mu(sp) \leqslant r_A(x) + d_B(x) < 1$, one has $sp \notin U$. Similarly, $tp \notin U$. Assume that B' is an anticlique containing s, t, p. Since $s \not\sim t$ and $\mathscr{A}$ is 3-noncrossing, one obtains $B' = B$.

2. Let $s \in A$, $t \in B$, $r_A(x) = \mu(sx)$, $r_B(x) = \mu(tx)$. Then $\mu(st) < 1$; hence $st \notin U$. Let C be an anticlique containing s and t. Since $\mathscr{A}$ is 3-noncrossing, the anticlique C is defined uniquely; hence $s \not\sim t$ and $d_C(x) \leqslant \mu(sx) + \mu(xt)$.

Claim 5 yields $Y_A \cap Y_B = \varnothing$ for any two distinct $A, B \in \mathscr{A}$. Next, let $A \in \mathscr{A}_2$ and $x \in R_A$. The definition of W_A implies that for any $B' \in \mathscr{A}_1$ the relations $x \in W_A$ and $x \in Y_{B'}$ are never valid simultaneously, while, by Claim 6(1), $d_{B''}(x) \geqslant 1$ for any $B'' \in \mathscr{A}_2 \setminus \{A\}$. Hence, $W_A \cap Y_B = \varnothing$ for any $B \in \mathscr{A} \setminus \{A\}$. Next, if $r_B(x) = 0$ for some $B \in \mathscr{A}_2 \setminus \{A\}$, then by Claim 6(2) there exists an anticlique $C \in \mathscr{A}$ such that $d_C(x) = 0$, thus implying $x \notin W_A, W_B$. Therefore, $\mathscr{X}^*$ consists of pairwise disjoint sets.

Consider an arbitrary terminal s. If s is a 1-terminal and $s \in A$, then $s \not\sim s$ and $d_A(s) = \mu(ss) + \mu(ss) = 0$, hence $s \in Y_A$. Assume that s is a 2-terminal and $s \in A \in \mathscr{A}_2$. Then $r_A(s) = \mu(ss) = 0$, and $s \notin W_A$ would imply $d_B(s) < 1/2$ for some $B \in \mathscr{A} \setminus \{A\}$, whence $s \in Y_B$; furthermore, taking Claim 6(1) into account, one obtains $s \in A \cap B$. Therefore each terminal s is contained in exactly one set X^*_C, and $s \in C$. Hence $\mathscr{X}^*$ is a proper family.

It remains to prove that each set in $\mathscr{X}^*$ is saturated and compatible with f. Let us make use of relations (2) and (3) for f and l. The following statement is trivial.

CLAIM 7. *Suppose that* $x, y \in V$, $\mu(xy) = 0$, *and* $A \in \mathscr{A}$. *Then*:

(1) *If* $x \in Y_A$, *then* $y \in Y_A$.

(2) *If* $x \in W_A$, *then* $y \in W_A$.

Given $A \in \mathscr{A}$ and $xy \in \partial X^*_A$, one applies Claim 7 to obtain $\mu(xy) > 0$, and thus $l(xy) > 0$. Therefore, by (2), the edge xy is saturated by f. Hence X^*_A is saturated by f.

To prove that X^*_A is compatible with f, consider an arbitrary pq-chain $L \in \mathscr{L}^+(f)$ and suppose that there exists a vertex $x \in VL$ contained in X^*_A. To be definite, assume that $q \notin A$, and let L' and L'' be the parts of L from p to x and from x to q, respectively. By (3), one has

$$l(EL) = l(EL') + l(EL'') = \mu(pq) = 1.$$

Let us show that $VL' \subseteq X^*_A$.

1. Suppose that $x \in Y_A$. Choose $s, t \in A$, $s \not\sim t$, such that $d_A(x) = \mu(sx) + \mu(xt)$. Then $\mu(sx) + \mu(tx) + \mu(px) + \mu(qx) \leqslant 3/2$; hence, by Claims 4 and 3, $\tau(s, t, p, q) = 1$ and there exists an anticlique B containing s, t and $p' \in \{p, q\}$. Evidently, $B = A$ (otherwise $s \sim t$); hence $p' = p$. According to Claim 2(1), one has $\{sq, tq\} \cap U \neq \varnothing$. If $s' \sim p$ for some $s' \in \{s, t\}$, then $s'q \in U$ (by Claim 2(2)), and $t' \not\sim p$, where $\{s', t'\} = \{s, t\}$. Therefore, one can assume that $sq \in U$ and $t \not\sim p$. The relations $\mu(sx) + l(EL'') \geqslant \mu(sq) \geqslant 1$ and $l(EL) = 1$ yield $\mu(tx) + l(EL') \leqslant d_A(x)$. Hence, for each vertex $y \in VL'$ one has $\mu(ty) + \mu(yp) \leqslant d_A(x)$, and thus $y \in Y_A$.

2. Suppose now that $A \in \mathscr{A}_2$ and $x \in W_A$. Choose $s \in A$ such that $\mu(sx) = r_A(x) = 0$; then $\mu(sx) + l(EL') + l(EL'') = 1$. If $sq \in U$, the above equality together with $\mu(sq) \geqslant 1$ implies $\mu(sx) + l(EL') = 0$. Hence, for an arbitrary $y \in VL'$ one has $\mu(sy) = 0$; therefore, by Claim 7, $y \in W_A$. Suppose now that $sq \notin U$, and let B be an anticlique containing s and q; evidently, $B \neq A$ and $s \not\sim q$. Since $x \in W_A$, one has $d_B(x) \geqslant 1/2$. Hence $l(EL'') + \mu(sx) \geqslant 1/2$, and thus $\mu(sp) \leqslant \mu(sx) + l(EL') \leqslant 1/2$. Therefore, $sp \notin U$; since $\mathscr{A}$ is 3-noncrossing, this yields $p \in A$. Now $pq \in U$ and $sq \notin U$ show that $s \not\sim p$. Then an arbitrary $y \in VL'$ satisfies $d_A(y) \leqslant \mu(sy) + \mu(yp) \leqslant 1/2$, i.e., $y \in Y_A$.

The proof of Theorem 2.1 is completed.

COROLLARY 2.2. *If* c *is an inner Eulerian function then* $c(\mathscr{X})$ *is an integer for any semiproper family* $\mathscr{X}$ *(hence* $v(c)$ *is an integer as well).*

PROOF. Indeed, assume that $A \in \mathscr{A}$ and $\mathscr{X}'$ is the proper family consisting of sets $X'_A = X_A \cup (V \setminus T)$ and $X'_B = X_B \cap T$ ($B \in \mathscr{A} \setminus \{A\}$). Since c is inner Eulerian, one obtains $c(\partial X'_C) \equiv c(\partial X_C) \bmod 2$ for any $C \in \mathscr{A}$ (taking into account that $X'_C \cap T = X_C \cap T$); hence $c(\mathscr{X}') - c(\mathscr{X})$ is an integer. Since $\mathscr{X}'$ is a partition of the set V, each edge $e \in E$ occurs in an even number of cuts $\partial X'_C$, $C \in \mathscr{A}$; therefore $c(\mathscr{X}')$ is an integer.

3. Splitting a network

In this section we complete the proof of Theorem 1. It will be convenient to assume that G is a complete graph, that is, $xy \in E$ for any distinct $x, y \in V$ (if the initial graph G lacks an edge xy, add the edge and put $c(xy) = 0$). Let c be inner Eulerian.

A semiproper family $\mathscr{X}$ such that $c(\mathscr{X}) = v(c)$ is called *c-minimal*. Denote by $\mathscr{M}(c)$ the set of all c-minimal proper families. The proof of the theorem proceeds by induction. More precisely, suppose that the assertion of the theorem is true for some fixed G and H and for all inner Eulerian functions c' such that either $|\mathscr{M}(c')| > |\mathscr{M}(c)|$, or $|\mathscr{M}(c')| = |\mathscr{M}(c)|$ and $c'(E) < c(E)$. The assertion is obvious for $c = 0$ (observe that in this case each proper family is c-minimal, hence $|\mathscr{M}(c)|$ takes the maximal possible value).

Denote by f an optimal solution for the problem $\mathscr{P}(G, c, U)$, as above. A triple xyz of vertices such that $y \neq x, z$ (x and z may coincide) is said to be a *fork* for the function c if $c(xy) > 0$ and $c(yz) > 0$. Given a fork xyz, define a function $\theta = \theta_{xyz}$ on E by the following relations:

$$\theta(e) = \begin{cases} 2 & \text{for } e = xy, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{if } x = z;$$

$$\theta(e) = \begin{cases} 1 & \text{for } e = xy, yz, \\ -1 & \text{for } e = xz, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{if } x \neq z.$$

A fork xyz is said to be *essential* (with respect to f) if $x \neq z$ and the set $\mathscr{L}^+(f)$ contains a chain passing through both xy and yz.

STATEMENT 3.1. *Suppose that* xyz *is a fork, and* $c' = c - \theta_{xyz}$. *Then:*

(1) c' *is inner Eulerian.*

(2) $v(c) \geqslant v(c') \geqslant v(c) - 2$.

(3) *If* $v(c') = v(c)$, *then* $\mathscr{M}(c') \supseteq \mathscr{M}(c)$.

(4) *If* xyz *is essential and* $\mathscr{X} = \{X_A : A \in \mathscr{A}\}$ *is a proper family such that* $c'(\mathscr{X}) < v(c)$, *then* $c'(\mathscr{X}) = v(c) - 1$, $c(\mathscr{X}) = v(c) + 1$, *and there*

exist $X_A, X_B \in \mathscr{X}$ such that $y \notin X_A \ni x, z$ and $x, z \notin X_B \ni y$ (this fact implies in particular that if xyz is essential, then either $v(c') = v(c)$ or $v(c') = v(c) - 1$).

PROOF. The first assertion is obvious. Let $\mathscr{X}$ be an arbitrary proper family. By Corollary 2.2, both $c(\mathscr{X})$ and $c'(\mathscr{X})$ are integers. For any $X_C \in \mathscr{X}$ one has

$$c'(\partial X_C) = \begin{cases} c(\partial X_C) - 2 & \text{if } xy, yz \in \partial X_C, \\ c(\partial X_C) & \text{otherwise.} \end{cases}$$

Since the members of $\mathscr{X}$ are pairwise disjoint, $\mathscr{X}$ contains at most two sets X_C such that $xy, yz \in \partial X_C$; hence $c(\mathscr{X}) \geqslant c'(\mathscr{X}) \geqslant c(\mathscr{X}) - 2$. Thus, the second and the third assertions are proved. Now let xyz and $\mathscr{X}$ be defined as in the last assertion, and assume that $X_C \in \mathscr{X}$ is a set such that $xy, yz \in \partial X_C$. Then X_C is not compatible with f; therefore, by Claim 1 of §2, $\mathscr{X}$ fails to be c-minimal. Now the required assertion follows from $c(\mathscr{X}) - c'(\mathscr{X}) \leqslant 2$.

Denote by $K(f)$ the set of all essential forks for f. If $K(f) = \varnothing$ (that is, $|EL| = 1$ for all $L \in \mathscr{L}^+(f)$), then f is evidently an integer multiflow. Hence, one may assume that $K(f) \neq \varnothing$. A fork xyz is said to be *separable* if $v(c') = v(c)$ with $c' := c - \theta_{xyz}$. We shall prove that there exists at least one separable fork. Once this fact is proved, the proof of Theorem 1 is completed in the following way. Assume that xyz is a separable fork and $c' = c - \theta_{xyz}$. Since $c'(E) < c(E)$ and $\mathscr{M}(c') \supseteq \mathscr{M}(c)$ (by Statement 3.1(3)), the problem $\mathscr{P}(c')$ possesses an integer solution f' by induction. If $x = z$, or $x \neq z$ and $\zeta^{f'}(xz) \leqslant c(xz)$, then f' is c-admissible, whence it is an optimal solution for $\mathscr{P}(c)$. If $x \neq z$ and $\zeta^{f'}(xz) = c(xz) + 1 = c'(xz)$, put

$$f^*(L) = f'(L) - 1, \qquad f^*(L') = f'(L') + 1,$$
$$f^*(L'') = f'(L''), \qquad L'' \in \mathscr{L} \setminus \{L, L'\},$$

where L is a chain from $\mathscr{L}^+(f')$ passing through the edge xz and L' is a chain from $\mathscr{L}$ such that $EL' \subseteq (EL \setminus \{xz\}) \cup \{xy, yz\}$. Evidently, f^* is c-admissible and $1 \cdot f^* = 1 \cdot f' = v(c)$; therefore f^* is an integer optimal solution for $\mathscr{P}(c)$.

Let us now prove the existence of a separable fork. The proof breaks up into several claims. In what follows we may assume that none of the essential forks is separable. It is then necessary to prove that there exists a separable nonessential fork. Choose an essential fork xyz and put $c' := c - \theta_{xyz}$, $c'' := c' - \frac{1}{2}\theta_{xyz}$, $\tilde{c} := 2c''$. For any proper family $\mathscr{X}$ one has $c''(\mathscr{X}) = (c(\mathscr{X}) + c'(\mathscr{X}))/2$; therefore, by Statement 3.1(4), the following are true:

1) $v(c'') = v(c)$.
2) If $c(\mathscr{X}) = v(c)$, then $c''(\mathscr{X}) = v(c'')$.
3) If $c'(\mathscr{X}) < v(c)$, then $c(\mathscr{X}) > v(c)$ and $c''(\mathscr{X}) = v(c'')$.

Hence $\mathscr{M}(\tilde{c}) = \mathscr{M}(c'') \supset \mathscr{M}(c)$. Since $\tilde{c} = 2c - \theta_{xyz}$ is evidently an inner Eulerian function, the problem $\mathscr{P}(\tilde{c})$ possesses an integer optimal solution $\tilde{f}$ by induction. Furthermore, the fork xyz is separable for the function $2c$; hence, acting as above, one can reform the multiflow $\tilde{f}$ to obtain a $2c$-admissible integer multiflow $\tilde{f}^*$ such that $1 \cdot \tilde{f}^* = 1 \cdot \tilde{f}$. Therefore, the problem $\mathscr{P}(2c)$ possesses an integer optimal solution, and so $\mathscr{P}(c)$ possesses a half-integer optimal solution.

Thus, one may assume that f is *half-integral*, that is, it takes its values in $\frac{1}{2}\mathbb{Z}_+$. Assume as well that the multiflow f has the least possible value of $\zeta^f(E)$ among all the half-integer optimal solutions for $\mathscr{P}(c)$. Given an essential fork xyz, we say that the proper family $\mathscr{X}$ indicated in Statement 3.1(4) is *critical*, while X_A and X_B are said to be the *external* and the *internal* sets (with respect to xyz), respectively.

CLAIM 1. *Suppose that xyz is an essential fork, $\mathscr{X}$ is a critical proper family for xyz, $X_A, X_B \in \mathscr{X}$ are the external and the internal sets, respectively, and $L \in \mathscr{L}^+(f)$ is a chain passing through xy and yz. Then:*

(1) *Each set $X_C \in \mathscr{X}$ is saturated by f.*
(2) *Each set $X_C \in \mathscr{X} \setminus \{X_A, X_B\}$ is compatible with f.*
(3) *The sets X_A and X_B are compatible with each chain in $\mathscr{L}^+(f) \setminus \{L\}$.*
(4) $|EL \cap \partial X_A| = 3$.
(5) $f(L) = 1/2$.

PROOF. Put $c' := c - \frac{1}{2}\theta_{xyz}$ and define a multiflow f' by the following:

$$f'(L) := f(L) - 1/2, \qquad f'(L') = f(L') + 1/2,$$
$$f'(L'') := f(L''), \qquad L'' \in \mathscr{L} \setminus \{L, L'\},$$

where L' is the chain in $\mathscr{L}$ such that $EL' = (EL \setminus \{xy, yz\}) \cup \{xz\}$. Then $v(c) = v(c') = c'(\mathscr{X})$, and f' is an optimal solution for $\mathscr{P}(c')$. Now the first three assertions, as well as the last one, follow immediately from Claim 2 of §2 applied to c', f' and $\mathscr{X}$. Finally, $|EL' \cap \partial X_A| = 1$ (since $VL' \cap X_A \neq \varnothing$) implies $|EL \cap \partial X_A| = |EL' \cap \partial X_A| + 2 = 3$.

Let $\tilde{V}$ be the set of vertices $y \in V$ occurring in at least one essential fork xyz. Consider a vertex $y \in \tilde{V}$ and an essential fork xyz. Claim 1(3) shows that $\mathscr{L}^+(f)$ contains exactly one chain L passing through both edges xy and yz. Since $f(L) = \frac{1}{2}$ and the edge xy is saturated by f, at least one of the following is true:

1) There exists an essential fork $x'yx$, $x' \neq z$.
2) The edge xy belongs to a chain $L' \in \mathscr{L}^+(f)$ having an endpoint at y (in particular, y is a terminal).

This implies immediately that at least one of the following two cases holds:

(C1) G contains a vertex y and distinct vertices $x_1, x_2, \dots, x_k = x_0$, $k \geqslant 3$, such that $x_i y x_{i+1}$ is an essential fork for any $i = 1, \dots, k$.

(C2) $\tilde{V} \subseteq T$, and for some $s \in T$ and $x, z \in V$ there exist a chain $L \in \mathscr{L}^+(f)$ passing through the edges xs and sz and an st-chain $L' \in \mathscr{L}^+(f)$ passing through the edge sx.

Let us prove that case (C2) is in fact impossible. To do this, we need the following statement.

CLAIM 2. *Let $L \in \mathscr{L}^+(f)$ be a pq-chain and $T' := T \cap (VL \setminus \{p, q\}) \neq \varnothing$. Then $T' \subseteq A \cap B$ for some distinct $A, B \in \mathscr{A}$.*

PROOF. The minimality of $\zeta^f(E)$ implies that $p'q' \notin U$ for any pair $\{p', q'\} \subset VL \cap T$ distinct from $\{p, q\}$ (otherwise one would shorten the chain L). Choose $p' \in T'$, and assume that A is an anticlique containing p and p', while B is an anticlique containing p' and q. Let $q' \in T' \setminus \{p'\}$. The relation $p'q' \notin U$ together with Claim 2(3) of §2 implies $q' \in A \cup B$. Since $pq' \notin U$, the relation $q' \notin A$ would imply the existence of an anticlique $C \neq A, B$ containing p and q', contrary to the fact that $\mathscr{A}$ is 3-noncrossing. Hence $q' \in A$, and similarly, $q' \in B$.

Consider s, x, z, L, L' indicated in (C2). Let p and q be the endpoints of L, and let X_A be the external set of the critical proper family for xsz. Observe that $\tilde{V} \subseteq T$ yields $x, z \in T$. Since $|EL| \geqslant |EL \cap \partial X_A| = 3$, at least one of the terminals x and z, say x, is distinct from p and q. Claim 2 together with the fact that $x \in X_A \cap T \subseteq A$ yields $s \in A$. Let t be the endpoint of the chain L' distinct from s. Since X_A is compatible with L' (by Claim 1(3)) and $s \notin X_A$, one has $t \in X_A$. Hence both endpoints of L' belong to the anticlique A, a contradiction.

Let us now turn to case (C1); consider the vertices $y, x_1, \dots, x_k$ as in (C1). Let us prove that any fork of the form $x_i y x_{i+2}$ is separable. Assume that $\mathscr{X}^i = \{X_B^i : B \in \mathscr{A}\}$ is the critical proper family for $x_i y x_{i+1}$, $X_{A(i)}^i \in \mathscr{X}^i$ is the external set (with respect to $x_i y x_{i+1}$), T^i denotes $T \cap X_{A(i)}^i$, and $L^i \in \mathscr{L}^+(f)$ is a chain passing through the edges $x_i y$ and yx_{i+1}, $i = 1, \dots, k$. In what follows all the indices are regarded modulo k.

CLAIM 3. *For any $i = 1, \dots, k$, $A(i)$ intersects $A(i+1)$ (possibly, $A(i) = A(i+1)$).*

PROOF. Suppose that $A(i) \cap A(i+1) = \varnothing$. Then $T^i \cap T^{i+1} = \varnothing$; hence the families

$$\mathscr{Y} := (\mathscr{X}^i \setminus \{X\}) \cup \{X \setminus Y\}, \qquad \mathscr{Y}' := (\mathscr{X}^{i+1} \setminus \{Y\}) \cup \{Y \setminus X\},$$

are proper (here $X := X_{A(i)}^i$, $Y := X_{A(i+1)}^{i+1}$). Since the internal (with respect to $x_i y x_{i+1}$) set for $\mathscr{X}^i$ is not compatible with f and occurs in $\mathscr{Y}$, one has $c(\mathscr{Y}) \geqslant v(c) + 1$. Similarly, $c(\mathscr{Y}') \geqslant v(c) + 1$. Therefore,

$$c(\mathscr{Y}) + c(\mathscr{Y}') \geqslant c(\mathscr{X}^i) + c(\mathscr{X}^{i+1}).$$

On the other hand, $x_{i+1}y \in \partial X, \partial Y$ and $x_{i+1}y \notin \partial(X \setminus Y), \partial(Y \setminus X)$; hence the strict submodular inequality

$$c(\partial X) + c(\partial Y) > c(\partial(X \setminus Y)) + c(\partial(Y \setminus X))$$

is valid for X and Y. This yields

$$c(\mathscr{X}^i) + c(\mathscr{X}^{i+1}) > c(\mathscr{Y}) + c(\mathscr{Y}'),$$

a contradiction.

Denote by $L[zz']$ the part of a chain L from z to z', where $z, z' \in VL$. Let s_i denote the endpoint of the chain L^i such that x_i is contained in $L^i[s_i y]$, and let t_i be the other endpoint of L^i, $i = 1, \dots, k$. Claim 1(3,4) immediately implies the following proposition.

CLAIM 4. *The following assertions are true for any $j = 1, \dots, k$:*
(1) $t_{j-1}, s_{j+1} \in T^j$.
(2) *Exactly one of the terminals s_j and t_j belongs to T^j.*

CLAIM 5. *At least one of the terminals s_{i+1} and t_i belongs to $T^i \cap T^{i+1}$, $i = 1, \dots, k$.*

PROOF. Suppose the contrary. Claim 4(1) (for $j = i+1$ and $j = i$) yields $t_i \in T^{i+1}$ and $s_{i+1} \in T^i$. Then $t_i \notin T^i$ and $s_{i+1} \notin T^{i+1}$; hence, by Claim 4(2), $s_i \in T^i$ and $t_{i+1} \in T^{i+1}$. Therefore, $s_i, s_{i+1} \in A(i)$, $t_i, t_{i+1} \in A(i+1)$, thus implying $A(i) \neq A(i+1)$. By Claim 3, $A(i)$ intersects $A(i+1)$. Besides, Claim 2(3) of §2 yields $s_i t_{i+1}, s_{i+1} t_i \in U$. Form an $s_i t_{i+1}$-chain L and an $s_{i+1} t_i$-chain L' such that $EL \subseteq EL^i[s_i y] \cup EL^{i+1}[yt_{i+1}]$ and $EL' \subseteq EL^{i+1}[s_{i+1} x_{i+1}] \cup EL^i[x_{i+1} t_i]$. Define the multiflow f' by the relations

$$f'(L) := f(L) + 1/2, \qquad f'(L') := f(L') + 1/2, \qquad f'(L^i) := f'(L^{i+1}) = 0,.$$
$$f'(L'') = f(L'') \quad \text{for all the other chains} \quad L'' \in \mathscr{L}.$$

Evidently, $1 \cdot f' = 1 \cdot f = v(c)$, and $\zeta^{f'}(e) \leqslant \zeta^f(e)$ for any $e \in E$. Moreover, $\zeta^{f'}(x_{i+1}y) < \zeta^f(x_{i+1}y)$ (since the chains L and L' do not contain the edge $x_{i+1}y$), a contradiction to the minimality of $\zeta^f(E)$.

To be definite, let $s_1 \in T^1$. Then Claims 4 and 5 imply that $s_i \in T^{i-1} \cap T^i$ and $t_i \in T^{i+1} \setminus T^i$, $i = 1, \dots, k$ (see Figure 1). Taking Claim 3 into account, one obtains that the following facts are true for an arbitrary i:

(F1) $s_i \in A(i-1) \cap A(i)$, $t_i \in A(i+1) \setminus A(i)$.

(F2) $A(i)$ intersects (but does not coincide with) $A(i+1)$ (by Claim 3 and the relations $s_i \in T^i$, $t_i \in T^{i+1}$);

(F3) $A(i) \neq A(i+2)$ (since $s_{i+1} \in T_i$ and $t_{i+1} \in T^{i+2}$).

(F4) $A(i) \neq A(i+3)$ (otherwise $A(i)$, $A(i+1)$, $A(i+2)$ would be pairwise intersecting).

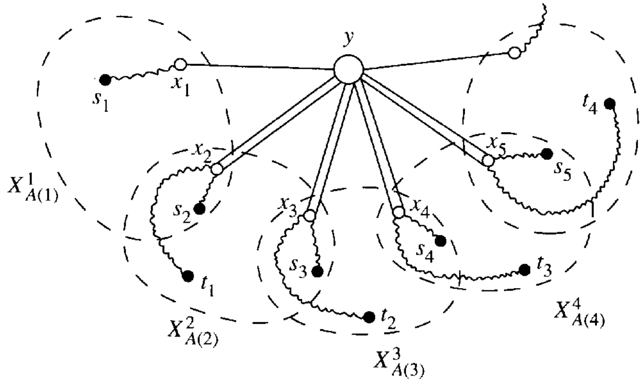

FIGURE 1

In particular, (F4) yields that $k \geqslant 4$. Besides, by (F2), (F3) and the fact that $\mathscr{A}$ is 3-noncrossing, one has

(F5) $A(i) \cap A(i+2) = \varnothing$.

Now we come to the final claim of our proof.

CLAIM 6. *The fork x_2yx_4 is separable.*

PROOF. By (F1) and (F5) one has $s_1 \notin A(2), A(3)$. Then, taking Claim 2(3) of §2 into account, one obtains $s_1s_3 \in U$; denote by P_1 an s_1s_3-chain such that $EP_1 \subseteq EL^1[s_1y] \cup EL^3[ys_3]$. Similarly, $s_2s_4 \in U$. There are two possible cases: 1) $t_1t_3 \in U$, and 2) $t_1t_3 \notin U$. In the first case, choose an s_2s_4-chain P_2 and a t_1t_3-chain P_3 such that

$$EP_2 \subseteq EL^2[s_2x_2] \cup \{x_2x_4\} \cup EL^4[x_4s_4],$$
$$EP_3 \subseteq EL^1[t_1x_2] \cup \{x_2x_4\} \cup EL^3[x_4t_3].$$

In the second case, let B be an anticlique containing t_1 and t_3. By (F1) and (F5), B is distinct from $A(1)$, $A(2)$, $A(3)$, $A(4)$. Then t_1 and t_3 are 2-terminals, $t_3 \notin A(1), A(2)$, $t_1 \notin A(3), A(4)$, and Claim 2(3) of §2 yields $s_2t_3 \in U$, $t_1s_4 \in U$. Choose an s_2t_3-chain P_2 and a t_1s_4-chain P_3 such that

$$EP_2 \subseteq EL^2[s_2x_2] \cup \{x_2x_4\} \cup EL^3[x_4t_3],$$
$$EP_3 \subseteq EL^1[t_1x_2] \cup \{x_2x_4\} \cup EL^4[x_4s_4].$$

In both cases, define a multiflow f' by the relations

$$f'(P_j) := f(P_j) + 1/2, \quad j = 1, 2, 3,$$
$$f'(L^m) := f(L^m) - 1/2 \ (= 0), \quad m = 1, 2, 3, 4,$$
$$f'(L) = f(L) \quad \text{for all the other chains } L \in \mathscr{L}.$$

Evidently, f' is c'-admissible with $c' := c - \theta_{x_2yx_4}$, and $v(c') \geqslant 1 \cdot f' = 1 \cdot f - 1/2 = v(c) - 1/2$. Now the fact that both $v(c)$ and $v(c')$ are integers implies $v(c') = v(c)$. Therefore, x_2yx_4 is a separable fork.

The proof of Theorem 1 is completed.

4. Algorithm

The algorithm is based on the same splitting-off idea as the proof of Theorem 1. In order to obtain an algorithm having strongly polynomial running time, we determine the maximum possible "weight" that can be assigned to a current fork so that splitting it off according to such a weight still preserves the maximum total value of a multiflow. To do this, one applies a subroutine that finds a c-minimal semiproper family $\mathscr{X}$ (for the current c).

It should be noted that the method of finding such a family presented in §2 is not efficient enough, since it involves the solution of the dual problem $\mathscr{P}^*(c)$. Let us show that finding a c-minimal family can be executed directly, via the reduction to an ordinary minimal cut problem for a certain extended network.

Let $c \colon E \to \mathbb{R}_+$. For each $A \in \mathscr{A}$, take a copy $G_A = (V_A, E_A)$ of the graph G and denote by x_A the copy of a vertex $x \in V$ in the graph G_A. Paste together graphs G_A, $A \in \mathscr{A}$, by identifying the following edges and vertices: for each pair of intersecting anticliques $A, B \in \mathscr{A}$, identify vertices x_A and x_B if $x \in A \cap B$, and identify edges x_Ay_A and x_By_B if $x, y \in A \cap B$ and $xy \in E$. Denote the resulting graph by $\mathscr{G}'$. Retain the same notation G_A for the corresponding subgraph of $\mathscr{G}'$. Let $\tilde{A}$ be the set of 1-terminals for an anticlique A, that is, $\tilde{A} = A - \bigcup_{B \in \mathscr{A} \setminus \{A\}} B$. Construct a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ by adding to $\mathscr{G}'$ the vertices s^0 (the *source*), t^0 (the *sink*) and the following edges:

$$\begin{array}{ll} s^0s_A & \text{for } s \in \tilde{A},\ A \in \mathscr{A}_1; \\ s^0t_A & \text{for } t \in T \setminus A,\ A \in \mathscr{A}_2; \\ t^0t_A & \text{for } t \in T \setminus A,\ A \in \mathscr{A}_1; \\ t^0s_A & \text{for } s \in \tilde{A},\ A \in \mathscr{A}_2 \end{array}$$

(see Figure 2 for an example of G, $\mathscr{A}$ and $\mathscr{G}$; here $\mathscr{A}_1 = \{A = \{s, t\},\ C = \{p, q, r\}\}$ and $\mathscr{A}_2 = \{B = \{t, p, q\}\}$). For $A \in \mathscr{A}$, denote by φ_A the natural inclusion map of G into $\mathscr{G}$ that takes G to the subgraph G_A. Define the edge capacities $d = d^c$ for the graph $\mathscr{G}$ in the following way. For $xy \in E$, $A \in \mathscr{A}$, $e = \varphi_A(xy) \in \mathscr{E}$, put

$$d(e) := \begin{cases} 2c(xy) & \text{if } x, y \in A \cap B \text{ for some } B \in \mathscr{A} \setminus \{A\}, \\ c(xy) & \text{otherwise}, \end{cases}$$

and put $d(e) := \infty$ for each edge $e \in \mathscr{E}$ incident to s^0 or t^0 (one can see that d is well-defined).

A cut ∂Z of the graph $\mathscr{G}$ is said to be an (s^0, t^0)-*cut* if $s^0 \in Z \subset \mathscr{V}$ and $t^0 \notin Z$. Let Q denote the set of all $Z \subset \mathscr{V}$ such that ∂Z is an (s^0, t^0)-cut containing no edges incident to s^0 or t^0. Given $Z \in Q$, define a family

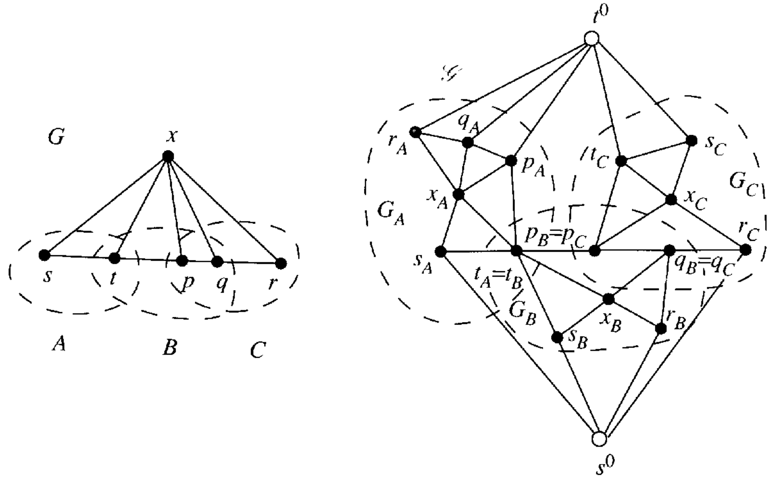

FIGURE 2

$\omega(Z) = \{X_A : A \in \mathscr{A}\}$ of subsets of V by the following rule:

$$X_A := \begin{cases} \varphi_A^{-1}(Z \cap V_A) & \text{for } A \in \mathscr{A}_1, \\ \varphi_A^{-1}(V_A \setminus Z) & \text{for } A \in \mathscr{A}_2. \end{cases}$$

The construction of the graph $\mathscr{G}$ immediately implies the following statement (details of the proof are left to the reader).

STATEMENT 4.1. *The mapping* ω *is a bijective correspondence between* Q *and the set of all semiproper families for* G *and* $\mathscr{A}$. *Moreover, for* $Z \in Q$ *and* $\mathscr{X} = \omega(Z)$ *one has* $d^c(\partial Z) = 2c(\mathscr{X})$.

Thus, to find a c-minimal semiproper family $\mathscr{X}$ and the value $v(c) = c(\mathscr{X})$ one has to construct an (s^0, t^0)-cut ∂Z in the graph $\mathscr{G}$ having the least possible capacity $d^c(\partial Z)$ (we have used the following fact: if $s^0 \in Z' \not\ni t$ and $Z' \notin Q$, then the cut $\partial Z'$ contains an edge having infinite capacity, hence it fails to be a minimum (s^0, t^0)-cut). Observe that one can transform a c-minimal semiproper family $\{X_A : A \in \mathscr{A}\}$ to a c-minimal proper family $\{X'_A : A \in \mathscr{A}\}$ by $X'_A = X_A \setminus (\bigcup_{B \in \mathscr{A} \setminus \{A\}} X_B)$ (this simple fact is left to the reader).

On the basis of on the above arguments, we now design an algorithm for solving the problem $\mathscr{P}(c)$. There are two cases to be distinguished:

1) c takes values in $\mathbb{R}_+$.
2) c is an inner Eulerian function.

In the first case, one has to find a "real-valued" optimal multiflow $f : \mathscr{L} \to \mathbb{R}_+$, and in the second case an integer-valued optimal multiflow $f : \mathscr{L} \to \mathbb{Z}_+$. As before, it is convenient to assume that G is a complete graph.

Let us start with the first case. Given a function $c' : E \to \mathbb{R}_+$ and a triple of vertices xyz ($y \neq x, z$), denote by $b(c', xyz)$ the maximal number $a \in \mathbb{R}_+$ such that $a \leqslant c'(xy)$, $a \leqslant c'(yz)$ and $v(c' - a\theta_{xyz}) = v(c)$. The algorithm consists of the main and the final stages. At the beginning, the number $v(c)$ is determined. At the main stage, the vertices of G are processed one by one in an arbitrary order, and for each $y \in V$ the triples xyz, $x, z \in V \setminus \{y\}$ (including $x = z$), are processed one by one in an arbitrary order. The processing of a current vertex y forms an *iteration* of the stage, while the processing of a current triple xyz forms a *step* of the iteration (therefore, the main stage consists of $|V|$ iterations, while each iteration consists of $|V|(|V|-1)/2$ steps). The processing of a triple xyz consists in finding the number $b = b(c, xyz)$ for the current function c and in changing the function: $c := c - b\theta_{xyz}$. Let $\tilde{c}$ be the current function c at the end of the main stage. Then an optimal solution f for the problem $\mathscr{P}(\tilde{c})$ is defined by $f(L_{st}) = \tilde{c}(st)$, $st \in U$, where L_{st} is the chain with $EL_{st} = \{st\}$ (f is assumed to be extended by zero to all the other chains in $\mathscr{L}$).

The number $b(c, xyz)$ can be computed in the following way. If $a_0 := \min\{c(xy), c(yz)\} = 0$, then, obviously, $b(c, xyz) := 0$; otherwise put $c_0 := c - a_0\theta_{xyz}$ and find $v(c_0)$ (to do this, according to Statement 4.1, one has to find the value q of the maximum flow from s^0 to t^0 in the network $(\mathscr{G}, d^{c_0})$; then $v(c_0) := q/2$). If $v(c_0) = v(c)$, then, clearly, $b(c, xyz) := a_0$. If $h_0 := v(c) - v(c_0) > 0$, put $a_1 := a_0 - h_0/2$, $c_1 := c - a_1\theta_{xyz}$ and find $v(c_1)$. The required number $b(c, xyz)$ is equal to $a_1 - h_1$, where $h_1 := v(c) - v(c_1)$.

Let us lay a foundation for the main stage. First of all, observe that during the stage the value of $v(c)$ (for current c) does not change. Let us prove that the numbers $b(c, xyz)$ are computed correctly, and that for any triple xyz one obtains $b(\tilde{c}, xyz) = 0$ for the final function $\tilde{c}$. For arbitrary $y \in V$, $x, z \in V \setminus \{y\}$, $a \in \mathbb{R}_+$ and a proper family $\mathscr{X}$ one has $c'(\mathscr{X}) = c(\mathscr{X}) - ak$, where $c' = c - a\theta_{xyz}$, while $k = k(\mathscr{X}, xyz)$ is the number of $X_A \in \mathscr{X}$ such that $xy, yz \in X_A$; so k can take only one of the following three values: 0, 1, 2. Hence

$$b(c, xyz) = \min\left\{c(xy),\ c(yz),\ \min_{\mathscr{X} \in \mathscr{K}} \Delta(\mathscr{X})\right\}, \tag{4}$$

where $\mathscr{K}$ is the set of all proper families, and

$$\Delta(\mathscr{X}) := \Delta(\mathscr{X}, xyz, c) := \begin{cases} \dfrac{c(\mathscr{X}) - v(c)}{k} & \text{for } k = k(\mathscr{X}, xyz) > 0, \\ \infty & \text{for } k = 0. \end{cases}$$

This fact easily implies that the numbers $b(c, xyz)$ calculated as above have correct values. Next, consider an arbitrary step where a triple xyz is processed. Let us prove the following fact: if relations $b(c, x'y'z') = 0$ are valid for all the previously processed triples $x'y'z'$ before executing the step, they remain valid after finishing it. Denote by c_1 and c_2 the function c at the beginning and at the end of the step, respectively, and let $x'y'z'$ be a previously processed triple. One may assume that both $c_2(x'y')$ and $c_2(y'z')$ are strictly positive. If $c_2(x'y') \leqslant c_1(x'y')$ and $c_2(y'z') \leqslant c_1(y'z')$, then the relation $b(c_2, x'y'z') = 0$ is implied by (4) (for $c = c_1$ and $c = c_2$), since evidently $\Delta(\mathscr{X}, x'y'z', c_2) \leqslant \Delta(\mathscr{X}, x'y'z', c_1)$ for all $\mathscr{X} \in \mathscr{K}$. Suppose now that $c_2(x'y') \neq 0 \neq c_2(y'z')$ and $c_2(x'y') > c_1(x'y')$ (the case $c_2(y'z') > c_1(y'z')$ is treated similarly). Then one has evidently $b(c_1, xyz) > 0$ and $x'y' = xz$, thus implying $y' \neq y$; hence y' is a previously processed vertex. To be definite, let $y' = z$. Since $yy'z'$ is a previously processed triple, one has $b(c_1, yy'z') = 0$, by the hypothesis. However, $c_1(yy') \geqslant b(c_1, xyz) > 0$ and $c_1(y'z') \geqslant c_2(y'z') > 0$; therefore there exists $\mathscr{X} \in \mathscr{K}$ such that $\Delta(\mathscr{X}, yy'z', c_1) = 0$. Choose $X_A \in \mathscr{X}$ satisfying $yy', y'z' \in \partial X_A$. The relations $c_1(\mathscr{X}) = v(c)$ and $b(c_1, xyz) > 0$ yield $k(\mathscr{X}, xyz) = 0$; hence, by $yz = yy' \in \partial X_A$, one obtains $xy \notin \partial X_A$ and $x'y' = xz \in \partial X_A$. Therefore, $k(\mathscr{X}, x'y'z') > 0$ and $\Delta(\mathscr{X}, x'y'z', c_2) = 0$ (since $c_2(\mathscr{X}) \leqslant c_1(\mathscr{X})$ and $c_1(\mathscr{X}) = v(c)$), whence $b(c_2, x'y'z') = 0$ as required. Observe that $b(c_2, xyz) = 0$ as well. Therefore, by induction, $b(\tilde{c}, xyz) = 0$ holds for the final function $\tilde{c}$ and arbitrary $y \in V$, $x, z \in V \setminus \{y\}$.

Assume now that f' is an optimal solution for the problem $\mathscr{P}(\tilde{c})$. If there were a chain $L \in \mathscr{L}^+(f')$ such that $|EL| \geqslant 2$, then L would contain two distinct edges xy and yz, thus implying $b(\tilde{c}, xyz) \geqslant f'(L) > 0$. Hence $|EL| = 1$ for all $L \in \mathscr{L}^+(f')$; therefore f' coincides with f.

At the final stage, f is used to determine in a natural way an optimal solution f^* for the initial problem $\mathscr{P}(c)$. Namely, let $c = c^0, c^1, \dots, c^N = \tilde{c}$ be the sequence of functions at the steps of the main stage. Starting from f, find successively multiflows $f = f^N, f^{N-1}, \dots, f^0 = f^*$, where f^i is an optimal solution for the problem $\mathscr{P}(c^i)$. How to find f^i via f^{i+1} is obvious.

Let $n = |V|$ and $t = |T|$. Since $|\mathscr{V}| \leqslant n|\mathscr{A}| + 2$ and $|\mathscr{A}| \leqslant t$, one has $|\mathscr{V}| \leqslant tn+2$. It is easy to see that the running time for the main stage, as well as for the entire algorithm, is $O(n^3\sigma(tn))$, where $\sigma(n')$ is the running time of the procedure for finding a maximum flow in a network with n' vertices (the final stage can be completed within $O(n^5)$ operations). In fact, one can organize the processing of triples at an iteration of the main stage in such a way that the procedure of finding a maximum flow in the network $(\mathscr{G}, d^c)$ is used only $O(n)$ times per iteration (we omit details here); this enables us to obtain an algorithm with running time $O(n^2\sigma(tn))$.

The algorithm for the case of an inner Eulerian function g differs from the one described above only in one point: we must put $c := c - \lfloor b(c, xyz) \rfloor \theta_{xyz}$ for each triple xyz at the corresponding processing step (here $\lfloor a \rfloor$ stands for the greatest integer not exceeding a). The verification of the algorithm follows mostly the same lines as above: one proves that at the end of the main stage $b(\tilde{c}, xyz) < 1$ for the final function $\tilde{c}$ and arbitrary $y \in V$, $x, z \in V \setminus \{y\}$ (the proof goes by induction, and is similar to that described

above; it relies on Statement 3.1(4) and relation (4) with $b(c, xyz)$ replaced by $\lfloor b(c, xyz) \rfloor$ and $\Delta(\mathscr{X})$ replaced by $\lfloor \Delta(\mathscr{X}) \rfloor$). This implies $|EL| = 1$ for all $L \in \mathscr{L}^+(f')$, provided f' is an optimal solution for the problem $\mathscr{P}(\tilde{c})$; hence $f' = f$.

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