Maximum- and Minimum-cost Multicommodity Flow Problems Having Unbounded Fractionality

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1. Introduction

Among the questions that are of interest in combinatorial optimization theory is the one dealing with fractionality of solutions for certain classes of linear programs. Let us give the formal setting of the problem. Assume that $\mathscr{P}$ is a set of programs of the form $Ax \leqslant b$ (*admissibility* problems), or $\max\{cx \mid Ax \leqslant b\}$ (*optimization* problems), with integer matrices A and integer vectors b and c; usually the set $\mathscr{P}$ (called a *class* of problems, or a *problem*) has a specific combinatorial meaning. By the *fractionality* $k(P)$ of an instance $P \in \mathscr{P}$ we mean the least natural number k such that for some admissible (respectively, optimal) solution x for P, the vector kx has only integer entries; if P has no solutions, we put $k(P) = 0$. The fractionality $k(\mathscr{P})$ for a class $\mathscr{P}$ is defined as $\max\{k(P) \mid P \in \mathscr{P}\}$; if $k(\mathscr{P}) = \infty$, we say that $\mathscr{P}$ has unbounded fractionality.

At present, no general theorems are known dealing with rather large classes $\mathscr{P}$ of fractionality $\geqslant 2$ (in contrast to well-known results on problems with totally unimodular constraint matrices, or on problems with submodular restrictions, which provide two representative examples of classes of fractionality 1). To find the fractionality, or even to establish that it is bounded, turns out to be a very difficult task for many combinatorial problems; one can mention, for instance, the Tutte–Seymour conjecture that the problem of finding an exact covering of the edges of an undirected graph by cycles has fractionality 2 (see [9]), which has withstood many efforts to solve it.

In the present paper, we discuss unbounded fractionality phenomena in multicommodity flow problems on undirected networks (the case of directed

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networks appears to be significantly simpler, as we explain below).

For our purposes, it is convenient to consider multicommodity flows as certain chain packings, rather than collections of flows (these two definitions are known to be equivalent [1]). By a *graph* we mean a finite undirected graph without loops and multiple edges; an edge with endpoints x and y is denoted by xy. By a *chain*, or an *xy-chain*, of a graph we mean a nonvoid subgraph $L=(VL, EL)$ of it with vertices $VL=\{x=x_0, x_1, \dots, x_k=y\}$, $x_i \neq x_j$, and edges $EL=\{x_ix_{i+1} \mid i=0, \dots, k-1\}$; the chain L is denoted by $x_0x_1\cdots x_k$.

The main objects of our studies are a graph $G=(V, E)$, a distinguished subset $T \subseteq V$ of vertices (called *terminals*), a nonnegative integer-valued function $c: E \to \mathbb{Z}_+$ (of edge *capacities*), and a graph $H=(T, U)$ without isolated vertices (H is called a *commodity graph*, or a *scheme*). Given $x, y \in V$, we denote by $\mathscr{L}(G, xy)$ the set of all xy-chains in G. Put

$$\mathscr{L} := \mathscr{L}(G, H) := \bigcup_{st \in U} \mathscr{L}(G, st).$$

A function $f: \mathscr{L} \to \mathbb{R}_+$ is said to be a *multicommodity flow*, or a *multiflow*. A multiflow f is *c-admissible* if the following capacity restrictions are valid:

$$\zeta^f(e) := \sum_{L \in \mathscr{L}, e \in EL} f(L) \leqslant c(e), \qquad e \in E.$$

Given a multiflow f, one defines its *value* between terminals s and t by $v(f, st) := \sum_{L \in \mathscr{L}(G, st)} f(L)$, and its *total value* by $v(f) := \mathbf{1} \cdot f = \sum_{L \in \mathscr{L}} f(L)$.

Below we formulate three main types of multiflow problems.

Demand problem (to be denoted hereafter by $\mathrm{D}(G, H, c, d)$): given a function $d: U \to \mathbb{Z}_+$ (flow *demands*), find a c-admissible multiflow f satisfying the condition

$$v(f, st) = d(st) \quad \text{for all } st \in U$$

(or prove that such a multiflow does not exist).

Maximum multiflow problem (to be denoted hereafter by $\mathrm{M}(G, H, c)$): find a maximum c-admissible multiflow, that is, a multiflow f with the greatest total value $v(f)$.

Minimum cost maximum multiflow problem (to be denoted hereafter by $\mathrm{C}(G, H, c, a)$): given a function $a: E \to \mathbb{Z}_+$ (of edge *costs*), find among the maximum multiflows the one (denote it by f) having the least cost

$$\sum_{L \in \mathscr{L}(G, H)} a(EL) f(L) = \sum_{e \in E} a(e) \zeta^f(e);$$

hereafter $g(S')$, $S' \subseteq S$, stands for $\sum_{e \in S'} g(e)$ for a function g on S.

Let us classify the problems according to their schemes, thus combining in one class all the problems of a given type having the same scheme H.

Namely, given a scheme $H=(T, U)$, denote by $\mathrm{D}(H)$ the set of all the demand problems $\mathrm{D}(G, H, c, d)$ for arbitrary graphs $G=(V, E)$, $V \supseteq T$, and arbitrary functions $c: E \to \mathbb{Z}_+$, $d: U \to \mathbb{Z}_+$; in a similar way one defines the sets $\mathrm{M}(H)$ and $\mathrm{C}(H)$. The fractionalities $k(\mathscr{P})$ for the classes $\mathscr{P}=\mathrm{D}(H)$, $\mathscr{P}=\mathrm{M}(H)$, $\mathscr{P}=\mathrm{C}(H)$ are denoted by $k_1(H)$, $k_2(H)$, $k_3(H)$, respectively.

The following simple proposition is true.

Claim 1. 1. *If a scheme H' is a subgraph of a scheme H, then $k_1(H') \leqslant k_1(H)$.*

2. *If a scheme H' is an induced subgraph of a scheme H (that is, H' is the subgraph induced by a vertex subset of H), then $k_i(H') \leqslant k_i(H)$, $i=2, 3$.*

Proof. Indeed, let $H'=(T', U') \subset H=(T, U)$; then the problem $\mathrm{D}(G'=(V', E'), H', c', d')$ is equivalent to the problem $\mathrm{D}(G, H, c, d)$ with $G=(V' \cup (T \setminus T'), E')$, $c(e) := c'(e)$ for $e \in E'$, $d(u) := d'(u)$ for $u \in U'$, $d(u) := 0$ for $u \in U \setminus U'$; this yields the first assertion. The second one is proved in a similar way.

The aim of this paper is to prove the following theorems.

Theorem 1. *If a scheme H contains three distinct pairwise intersecting anticliques A, B, C such that $A \cap B \neq A \cap C$, then $k_2(H)=\infty$.*

Theorem 2. *If a scheme H contains two distinct intersecting anticliques (i.e., H is not a complete multipartite graph), then $k_3(H)=\infty$.*

Recall that an *anticlique* of a graph is defined as a maximal (with respect to inclusion) independent (that is, inducing an empty graph) subset of its vertices.

We make a few comments. Denote by K_n the complete graph on n vertices, by Z_r the star with r edges (that is, the graph all of whose r edges possess a common vertex), and by $\Gamma_1+\Gamma_2+\cdots+\Gamma_m$ the union of pairwise disjoint graphs $\Gamma_1, \Gamma_2, \dots, \Gamma_m$; instead of $\Gamma+\cdots+\Gamma$ (m times) we write $m\Gamma$. Let $\mathscr{A}(H)$ stand for the set of all anticliques of H.

1) It is known that (a) $k_1(H)=1$, provided $H=Z_r$ [1]; (b) $k_1(H)=2$, provided H is the union of two stars and $H \neq Z_r$ (according to Dinits, this fact is an easy consequence of the half-integrality theorem for two-commodity flows [2]); and (c) $k_1(H)=2$, provided H is K_4, or H is the cycle on 5 vertices [7] (see also [10] and [8]).

Recently, the author proved that $k_1(H)=2$, provided $H=K_5$ or H is the union of K_3 and a star [5]. On the other hand, it is shown in [8] that $k_1(H)=\infty$ for $H=3K_2$. One can verify that the only scheme H such that $k_1(H)$ is not defined by the above results and Claim 1.1 is $2K_3$. In this case the demand problem can be reduced easily to the maximum multiflow problem for the same scheme $2K_3$ (observe that $H=2K_3$ violates the assumptions of Theorem 1); there are reasons to conjecture that $k_1(2K_3)=4$.

2) It is known that (1) $k_2(H) = 1$, provided H is a complete bipartite graph (a multi-terminal version of the maximum flow theorem [1]); (2) $k_2(H) = 2$, provided $|\mathscr{A}(H)| > 2$ and $\mathscr{A}(H)$ admits a partition $\{\mathscr{A}_1, \mathscr{A}_2\}$ such that each family $\mathscr{A}_i$ consists of pairwise disjoint anticliques [6] (see also [8] and [4]); and (3) $k_2(H) = 4$, provided the latter condition is violated, yet H does not contain three distinct pairwise intersecting anticliques [6].

Therefore, each scheme H such that $k_2(H)$ is not defined by the above results and Theorem 1 must contain three distinct pairwise intersecting anticliques, and for any such triple A, B, C one must have $A \cap B = B \cap C = C \cap A$. For such schemes H, the values $k_2(H)$ are not established yet (presumably, $k_2(H) \leqslant 4$); recently, the author proved that for such schemes H the fractionality for the problem dual to $\mathrm{M}(G, H, c)$ does not exceed 4.

3) It is known that $k_3(H) = 1$, provided H is a complete bipartite graph [1], and that $k_3(H) = 2$, provided $H = K_n$, $n \geqslant 3$ [3].

The latter result can be extended easily to the case of complete multipartite graphs H. Together with Theorem 2, this provides a complete description of the values of $k_3(H)$ for all schemes H.

The solution of fractionality problems for directed multiflows is significantly simpler. Classical results of Ford and Fulkerson [1] imply that $k_1(H) = 1$, provided $H = (T, U)$ is a directed star (that is, either $U = \{(s, t) \mid t \in T \setminus \{s\}\}$, or $U = \{(t, s) \mid t \in T \setminus \{s\}\}$ for some $s \in T$), and $k_2(H) = k_3(H) = 1$, provided H is a complete directed bipartite graph (that is, $U = \{(s, t) \mid s \in S,\ t \in T \setminus S\}$ for some $S \subset T$). Easy examples show that in all the other cases the fractionalities are infinite.

2. Proof of Theorem 1

The following proposition is true (the proof is left to the reader).

CLAIM 2. *A scheme H satisfies the assumptions of Theorem 1 if and only if H possesses an induced subgraph H' such that $3K_2 \subseteq H' \subseteq H^1$, where H^1 is the graph in Figure 1(b).*

According to Claims 1 and 2, to prove Theorem 1, it suffices to consider only schemes $H = (T, U)$ such that $3K_2 \subseteq H \subseteq H^1$. One must show that for an arbitrary natural k^* there exist a graph $G = (V, E)$, $V \supseteq T$, and a function $c \colon E \to \mathbb{Z}_+$ such that for any optimal solution f of the problem

(a) $3K_2$ (b) H^1

FIGURE 1

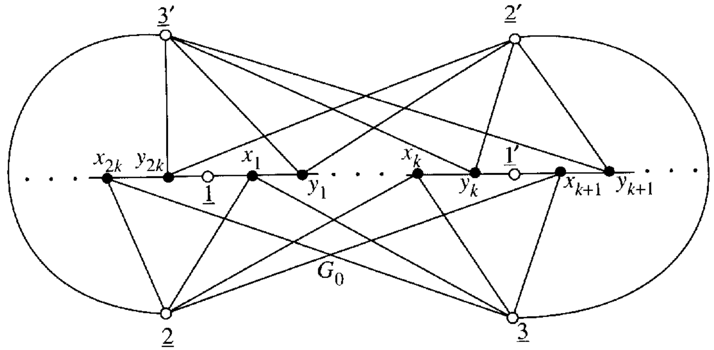

FIGURE 2

$\mathrm{D}(G, H, c)$ the vector $k'f$ has at least one noninteger entry for any natural $k' < k^*$. The construction providing such networks (G, c) for the above described schemes H relies on the construction for the "simplest" scheme $H_0 = 3K_2$. For the case of H_0, one can take the counterexample from §12 of [8]; however, below we present a simpler example. Namely, consider the graph $G_0 = (V_0, E_0)$ shown in Figure 2; it consists of the cycle

$$\underline{1}x_1y_1\cdots x_ky_k\underline{1}'x_{k+1}y_{k+1}\cdots x_{2k}y_{2k}\underline{1}$$

plus vertices $\underline{2}$, $\underline{2}'$, $\underline{3}$, $\underline{3}'$, edges $\underline{2}x_i$, $\underline{3}x_i$, $\underline{2}'y_i$, $\underline{3}'y_i$, $i = 1, \dots, 2k$, and two distinguished edges: $\underline{23}'$ and $\underline{32}'$. The capacities $c_0(e)$ for all the nondistinguished edges $e \in E_0$ equal 1, while those for the distinguished edges equal 2. Put $H_0 = (T_0, U_0)$, where

$$T_0 := \{\underline{1}, \underline{1}', \underline{2}, \underline{2}', \underline{3}, \underline{3}'\}, \qquad U_0 := \{\underline{11}', \underline{22}', \underline{33}'\}.$$

Define in G_0 the following multiflow f_0:

$$\begin{aligned} f_0(L_i) &:= f_0(L_i') = (k-1)/k, & i &= 1, \dots, 2k, \\ f_0(P_1) &:= f_0(P_2) = 1/k, \\ f_0(Q_i) &:= f_0(Q_i') = 1/k, & i &= 1, \dots, 2k \end{aligned}$$

where

$$\begin{aligned} L_i &= \underline{2}x_iy_i\underline{2}', & L_j' &= \underline{3}x_jy_{j-1}\underline{3}' \quad (j \neq 1, k+1), \\ L_1' &= \underline{3}x_1\underline{1}y_{2k}\underline{3}', & L_{k+1}' &= \underline{3}x_{k+1}\underline{1}'y_k\underline{3}', \\ P_1 &= \underline{1}x_1y_1\cdots x_ky_k\underline{1}', & P_2 &= \underline{1}y_{2k}x_{2k}\cdots y_{k+1}x_{k+1}\underline{1}', \\ Q_i &= \underline{2}x_i\underline{32}', & Q_i' &= \underline{23}'y_i\underline{2}'; \end{aligned}$$

for all the other chains belonging to $\mathscr{L}(G_0, H_0)$ the multiflow f_0 vanishes. A direct calculation shows that all the edges of G_0 are saturated by f_0 (that is, $\zeta^{f_0}(e) = c_0(e)$ for all $e \in E_0$), and that

$$v(f_0, \underline{11}') = 2/k, \qquad v(f_0, \underline{22}') = 2k+2, \qquad v(f_0, \underline{33}') = 2k-2;$$

hence f_0 is c_0-admissible, and its total value $v(f_0)$ equals $4k+2/k$.

Let us prove that f_0 is an optimal solution for $\mathrm{M}(G_0, H_0, c_0)$; this fact would imply that the fractionality for the above problem is at least $k/2$, since the denominator of $v(f_0)$ equals $k/2$. To do this, consider the dual problem $\mathrm{M}^*(G_0, H_0, c_0)$. Observe that the admissible solutions for the problem $\mathrm{M}^*(G, H, c)$ dual (in the sense of linear programming) to the problem $\mathrm{M}(G, H, c)$ are exactly the functions $l\colon E \to \mathbb{R}_+$ satisfying the restrictions

$$l(EL) \geqslant 1, \quad L \in \mathscr{L}(G, H).$$

The above restrictions can be rewritten as

$$\mathrm{dist}_l(st) \geqslant 1, \quad st \in U,$$

where $\mathrm{dist}_l(xy)$ denotes the distance between the vertices x and y in the graph G with edge lengths $l(e)$; in other words,

$$\mathrm{dist}_l(xy) = \min_{L \in \mathscr{L}(G, xy)} l(EL).$$

By the linear programming duality theorem, admissible solutions f and l are optimal if and only if

$$v(f) = c \cdot l \left(= \sum_{e \in E} c(e) l(e) \right);$$

this is equivalent to the complementary slackness conditions

$$L \in \mathscr{L}(G, H),\ f(L) > 0 \Rightarrow l(EL) = 1, \qquad \text{(C1)}$$

$$e \in E,\ l(e) > 0 \Rightarrow \zeta^f(e) = c(e). \qquad \text{(C2)}$$

In our case, put

$$l_0(\underline{1}x_1) := l_0(\underline{1}y_{2k}) := l_0(\underline{1}'x_{k+1}) := l_0(\underline{1}'y_k) := 1/4k;$$

$$l_0(\underline{23}') := l_0(\underline{32}') := l_0(x_i y_i) := l_0(y_j x_{j+1}) := 1/2k,$$

$$i = 1, \dots, 2k, \quad j = 1, \dots, k-1, k+1, \dots, 2k-1;$$

$$l_0(\underline{2}x_i) := l_0(\underline{3}x_i) := l_0(\underline{2}'y_i) := l_0(\underline{3}'y_i) := 1/2 - 1/4k, \quad i = 1, \dots, 2k.$$

It is easy to check that l_0 is an admissible solution for $\mathrm{M}^*(G_0, H_0, c_0)$, and that both f_0 and c_0 satisfy (C1) and (C2), as required.

Therefore, we have constructed the proper example for the scheme $H_0 = 3K_2$. To design the required examples for the other schemes H, $H_0 \subset H \subseteq H^1$, we reform G_0, c_0, f_0 in such a way that the values of each of the three flows involved remain integer.

1. Paste together k copies $(G_1, c_1), \dots, (G_k, c_k)$ of the network (G_0, c_0) by identifying the vertices $\underline{1}_j$ (denote the vertex obtained by $\tilde{1}$) and by identifying the vertices $\underline{1}'_j$ (denote the vertex obtained by $\tilde{1}'$); here w_j stands for the copy of the vertex $w \in V_0$ in the graph G_j.

2. Add to the graph obtained above new vertices 1, $1'$, 2, $2'$, 3, $3'$ and new edges $1\tilde{1}$, $1'\tilde{1}'$, $2\underline{2}_j$, $2'\underline{2}'_j$, $3\underline{3}_j$, $3'\underline{3}'_j$, $j = 1, \dots, k$, with

$$c(1\tilde{1}) := c(1'\tilde{1}') := 2, \qquad c(2\underline{2}_j) := c(2'\underline{2}'_j) := 2k+2,$$

$$c(3\underline{3}_j) := c(3'\underline{3}'_j) := 2k-2, \qquad j = 1, \dots, k.$$

Denote the resulting network by $(G = (V, E), c)$, and let $\tilde{H} = (\tilde{T}, \tilde{U})$ be the scheme whose edges are $11'$, $22'$, $33'$.

One can naturally "extend" the multiflow f_0 to a multiflow f for G, $\tilde{H}$, c. Namely, given a chain $L = \underline{i}v \cdots w\underline{i}'$, $i \in \{1, 2, 3\}$, take k chains $L_j = i\underline{i}_j v_j \cdots w_j \underline{i}'_j i'$, $j = 1, \dots, k$, in the graph G, and put $f(L_j) := f_0(L)$. Evidently, all the edges of G are saturated by f, and f is an optimal solution for G, c and $\tilde{H}$ as above.

Let us prove that the fractionality for any maximum flow f' for G, c and $\tilde{H}$ as above equals at least $k/4$. Denote by $E(x)$ the set of edges in G that are incident to $x \in V$; put $E' = \bigcup_{s \in \tilde{T}} E(s)$. Consider the following functions l^1 and l^2 defined on E:

$$l^1(e) = \begin{cases} 0 & \text{for } e \in E', \\ l_0(e') & \text{for } e \in E \setminus E', \end{cases} \qquad l^2(e) = \begin{cases} 1/2 & \text{for } e \in E', \\ 0 & \text{for } e \in E \setminus E', \end{cases}$$

where e' is the edge of G_0 such that e is a copy of e'. It is easy to see that both l^1 and l^2 are optimal solutions for the problem $\mathrm{M}^*(G, H, c)$. Assume that $L = i \cdots i'$ is a chain in $\mathscr{L}(G, \tilde{H})$ with $f'(L) > 0$. Relation (C1) for f' and l^2 implies that L passes through no vertices from $\tilde{T}$ except i or i'. Next, if $i \in \{2, 3\}$ and L contains a vertex $s \in \{\tilde{1}, \tilde{1}'\}$, then the relations $l^1(EL) = 1$ and $\mathrm{dist}_{l_1}(is) = \mathrm{dist}_{l^1}(si') = 1/2$ (these are easy to verify) yield $l^1(EL') = l^1(EL'') = 1/2$, where L' and L'' are the parts of the chain L from i to s and from s to i', respectively. Denote by $G'_j = (V'_j, E'_j)$ the subgraph of G obtained by adding vertices 2, $2'$, 3, $3'$ and edges $2\underline{2}_j$, $2'\underline{2}'_j$, $3\underline{3}_j$, $3'\underline{3}'_j$ to G_j. Let g be the multiflow in the graph G'_j with terminals at $T' = \{\tilde{1}, \tilde{1}', 2, 2', 3, 3'\}$ that is induced by f'; that is, $g(L') = \sum f'(L)$ for any chain L' in G'_j such that both its endpoints belong to T', the sum being taken over all chains $L \in \mathscr{L}(G, \tilde{H})$ containing L' as a part. Denote by $\mathscr{L}'$ the set of all chains in G'_j having both endpoints in T'. Since f' saturates all the edges of G, g must saturate all the edges of G'_j; thus

$$\sum_{L' \in \mathscr{L}'} l^1(EL') g(L') = \sum_{e \in E'_j} c(e) l^1(e) = c_0 \cdot l_0 = 4k + \frac{2}{k}.$$

On the other hand, the above arguments imply that for any chain $L' \in \mathscr{L}'$, the value $l^1(EL')$ equals either 1 or $1/2$, provided $g(L') > 0$. Hence the fractionality of g is at least $k/4$.

(Observe that the above example proves also that the fractionality of the demand problem for the scheme $3K_2$ is unbounded.)

Finally, consider an arbitrary scheme $H = (T, U)$ such that $3K_2 \subset H \subseteq H^1$. It is easy to see that H^1 contains exactly one subgraph isomorphic to $3K_2$. Thus, one can assume that $T = \tilde{T}$, $U \supset \tilde{U} = \{11', 22', 33'\}$, and that the edge set of H^1 is $U^1 = \tilde{U} \cup \{1'2, 1'2', 1'3, 1'3', 2'3, 2'3'\}$. Let us prove that for G and c as above, any optimal solution f^* for the problem $\mathrm{M}(G, H, c)$ satisfies $v(f^*, st) = 0$ for all $st \in U \setminus \tilde{U}$; this would yield that the restriction of f^* on $\mathscr{L}(G, \tilde{H})$ is an optimal solution for the problem $\mathrm{M}(G, \tilde{H}, c)$, and thus the fractionality of f^* is at least $k/4$. To do this, it suffices to present an optimal solution l for the dual problem $\mathrm{M}^*(G, \tilde{H}, c)$ such that $\mathrm{dist}_l(st) > 1$ for all $st \in U \setminus \tilde{U}$ (by (C1); this would imply $v(f^*, st) = 0$). The required solution l can be defined as follows:

$$l(1\tilde{1}) = 0, \qquad l(1'\tilde{1}') = 1,$$

$$l(2\underline{2}_j) = \tfrac{1}{4}, \qquad l(2'\underline{2}'_j) = \tfrac{3}{4}, \qquad l(3\underline{3}_j) = l(3'\underline{3}'_j) = \tfrac{1}{2}, \quad j = 1, \dots, k,$$

and $l(e) = 0$ for all the other edges of G. It is easy to verify that

$$\mathrm{dist}_l(st) = 1 \quad \text{for} \quad st \in \tilde{U},$$
$$\mathrm{dist}_l(st) > 1 \quad \text{for} \quad st \in U^1 \setminus \tilde{U};$$

besides, relations (C1) and (C2) are valid for the solution l and the multiflow f defined as above (for G, c and $\tilde{H}$); hence l and f (extended by zero to $\mathscr{L}(G, H - \tilde{H})$) are optimal solutions for $\mathrm{M}^*(G, H, c)$ and $\mathrm{M}(G, H, c)$, as required.

Theorem 1 is proved.

3. Proof of Theorem 2

The following proposition is rather trivial (its verification is left to the reader).

Claim 3. *Let H be a graph without isolated vertices. Then H possesses two distinct intersecting anticliques if and only if it contains a four-vertex subset $T' = \{1, 1', 2, 2'\}$ such that the subgraph H' induced by T' satisfies $H_0 \subseteq H' \subseteq H_1$, where $H_0 = (T', U_0)$, $H_1 = (T', U_1)$, $U_0 = \{11', 22'\}$, $U_1 = \{12, 12'\} \cup U_0$.*

By Claim 1.2, it suffices to consider only schemes $H = (T, U)$ satisfying the relation $H_0 \subseteq H \subseteq H_1$. Assume that $T = \{1, 1', 2, 2'\}$, and the edge sets U_0 and U_1 for the graphs H_0 and H_1 are defined as in Claim 3.

Given an arbitrary positive odd k, we construct a simple example of the problem $\mathrm{C}(G, H, c, a)$ such that it has exactly one optimal solution f, and the fractionality of this solution is just k. The graph $G = (V, E)$ is shown in Figure 3. Here all the edges have unit capacities, except the edges $2w$

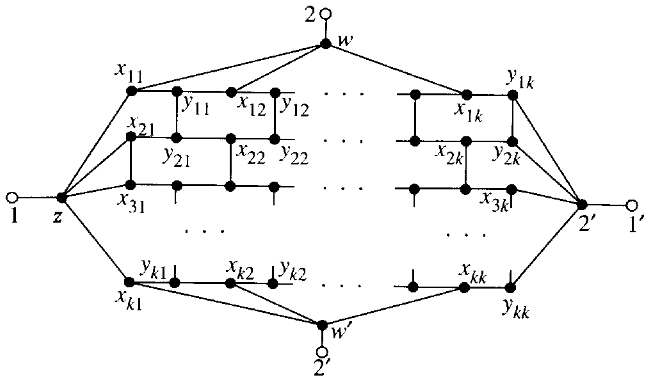

Figure 3

and $2'w'$; these two edges are of capacity $k - 1$. Edge costs are defined as follows:

$$a(x_{ij}y_{ij}) := 0, \quad a(1z) := 2k, \quad a(1'z') := 0, \qquad a(2w) := a(2'w') := k,$$
$$a(y_{ij}x_{i\,j+1}) := a(y_{pq}y_{p+1\,q}) := a(x_{rm}x_{r+1\,m}) := 1,$$
$$a(zx_{i1}) := a(z'y_{ik}) := a(wx_{1j}) := a(w'x_{kj}) := k.$$

Introduce the following chains:

$$L_i = 1zx_{i1}y_{i1}x_{i2}\dots x_{ik}y_{ik}z'1', \qquad\qquad i = 1, \dots, k,$$
$$P_i = 2wx_{1i}y_{1i}y_{2i}x_{2i}x_{3i}y_{3i}\dots y_{ki}x_{ki}w'2',$$

and define a multiflow f by $f(P_i) = (k-1)/k$, $f(L_i) = 1/k$, $i = 1, \dots, k$, $f(L) = 0$ for all the other chains in $\mathscr{L}(G, H)$. One can verify the following facts:

(1) $v(f) = k$, and for any c-admissible multiflow $f' \colon \mathscr{L}(G, H) \to \mathbb{R}_+$ one has $v(f') \leqslant (c(1z) + c(1'z') + c(2w) + c(2'w'))/2 = k$, whence f is a maximum multiflow.

(2) The costs for the chains L_i and P_i equal $5k - 1$, while the cost $a(EL)$ for any other chain L in $\mathscr{L}(G, H)$ is at least $5k$; therefore, the cost of f is minimal among all the multiflows (for G, c and H) of the same total value.

(3) The multiflow f is the unique multiflow of total value k that involves only chains L_i and P_i.

Properties (1)–(3) immediately imply the assertion of the theorem.

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