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RAPPORT DE RECHERCHE

DETERMINING THE DISTANCE TO THE PERFECT MATCHING POLYTOPE OF A BIPARTITE GRAPH

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DETERMINING THE DISTANCE TO THE PERFECT MATCHING POLYTOPE OF A BIPARTITE GRAPH

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Abstract. We consider the problem of finding the euclidean distance from the origin to the perfect matching polytope of a bipartite graph. Being a quadratic program with linear constraints, this problem can be solved in polynomial time by use of a version of the ellipsoid method. We develop a combinatorial algorithm which solves it in polynomial time. The algorithm as well as the proof of its correctness uses ideas involved in the cancelling minimum mean circuit algorithm for the minimum cost circulation problem due to Goldberg and Tarjan.

1. INTRODUCTION

Let $G = (V, E)$ be a bipartite graph with parts V_1 and V_2. Denote by $P = P_G$ the convex hull of the incidence vectors of all perfect matchings of G, the *perfect matching polytope* for G. It is known that P consists of the vectors $x \in \mathbf{R}_+^E$ such that

(1) $$x(E_v) = 1 \quad \text{for all } v \in V.$$

Here E_v is the set of edges of G incident to v, and for a subset X of a set S and a mapping $g : S \to \mathbf{R}$, $g(X)$ denotes $\sum_{e \in X} g(e)$.

We assume that P is nonempty and consider the problem of determining the point in P closest to the origin; in other words, we wish to

(2) *find* $x \in P$ *with* $\sum_{e \in E}(x(e))^2$ *minimum.*

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(Clearly (2) has a unique optimal solution.) Since (2) is a quadratic program with linear constrants, it can be solved in polynomial time by use of a version of the ellipsoid method [KTK]. In the present paper we construct a polynomial algorithm to solve (2) which exploits purely combinatorial techniques and is based on the idea of cancelling minimum mean circuits in a certain digraph (such an idea was applied in [GT] in connection with the minimum cost circulation problem).

The polytope P is the set $\{x \in \mathbb{R}^E \mid x \geq 0, Ax = 1\}$, where A is the matrix whose rows a_v are the incidence vectors of the sets E_v, $v \in V$. Hence, x is the optimal solution of (2) if and only if the gradient vector of the function $\sum_{e\in E}(x(e))^2$ at x can be represented as $\sum_{v\in V}\rho(v)a_v + \sum_{e\in E}\gamma(e)\chi_e$ for some numbers $\rho(v) \in \mathbb{R}$ $(v \in V)$ and $\gamma(e) \in \mathbb{R}_+$ $(e \in E)$ such that $x(e) = 0$ whenever $\gamma(e) > 0$; here χ_e is e-th unit basis vector in $\mathbb{R}^E$. It gives the following

Optimality criterion: $x \in P$ is optimal if and only if there exists $\rho : V \to \mathbb{R}$ such that for any $e = uv \in E$:

(3)
$$\rho(u) + \rho(v) = x(e) \text{ if } x(e) > 0$$
$$\leq 0 \quad \text{if } x(e) = 0.$$

(Though we allow G to have parallel edges, an edge $e \in E$ with end vertices u and v may be denoted by uv when it leads to no confusion; a similar convention will be admitted for the directed graphs below.) It should be noted that this criterion can be easily established directly, not appealing to general duality theorems in convex programming, as we explain in the Remark 1 below.

It is convenient to associate with $x \in P$ a pair (D, f), where $D = D^x$ is a digraph (V, A) and $f : A \to \mathbb{R}$ is a function on its arcs defined as follows. D is obtained from G by replacing each edge $e = uv \in E$ with $u \in V_1$ by a directed edge (an *arc*) (u, v) if $x(e) = 0$, and by two oppositely directed edges $(u, v), (v, u)$ if $x(e) > 0$. For $uv \in E$ with $u \in V_1$ put

(4)
$$f(u,v) := x(uv) \text{ and } f(v,u) := -x(uv) \quad \text{if } x(uv) > 0,$$
$$f(u,v) := 0 \quad \text{if } x(uv) = 0.$$

Conversely, if f is a function on A such that

(5)
$$f(u,v) = -f(v,u) > 0 \quad \text{if } u \in V_1 \text{ and } (u,v)(v,u) \in A,$$

$$f(u,v) = 0 \quad \text{if } (u,v) \in A \text{ and } (v,u) \notin A,$$
$$\sum_{v \in V_2} f(u,v) = 1 \text{ for } u \in V_1 \quad \text{and} \quad \sum_{u \in V_1} f(u,v) = 1 \text{ for } v \in V_2,$$

then the vector $x_f \in \mathbf{R}^E$, defined by $x_f(uv) = f(u,v)$ for $uv \in E$ with $u \in V_1$, belongs to P. From the above optimality criterion it follows that for D and f satisfying (5) the vector x_f is the optimal solution for (2) if and only if the weight $f(C) := \sum_{e \in C} f(e)$ of every (directed) circuit C in D is nonnegative (regarding a circuit as an arc-set). In other words, x_f is optimal if and only if there exists a function $\pi : V \to \mathbf{R}$ (a *potential*) so that

(6) $$\Delta_\pi(e) := \pi(v) - \pi(u) \le f(e) \quad \text{for any } e = (u,v) \in A.$$

Indeed, putting $\rho(u) := -\pi(u)$ for $u \in V_1$ and $\rho(v) := \pi(v)$ for $v \in V_2$, we observe that (6) is equivalent to (3).

Remark 1. The above criterion can be proved directly. Suppose that f admits a circuit C of negative weight $f(C)$. For $\varepsilon > 0$ let f^ε be the function on A obtained from f by increasing $f(u,v)$ by ε for $(u,v) \in C$ with $u \in V_1$ and by decreasing it by ε for $(u,v) \in C$ with $u \in V_2$. It is easy to see that for a sufficiently small ε the vector x_{f^ε} belongs to P and has smaller euclidean norm than x_f. The part "if" in the criterion follows, in particular, from the lemma below.

Notice that each entry of the optimal solution x^* of (2) is a fraction whose denominator is at most 2^{cn}, where $n := |V|$ and c is a constant. The algorithm developed in the present paper relies on the following lemma the matter of which is that if inequalities in (6) are possibly violated but by a rather small number then x_f is close to the optimal vector x^* (in fact, this lemma is a refinement to our case of known general results on certain optimization problems on polyhedra). Let $m := |E|$.

Lemma. *Let f satisfy (5), and let $\gamma \ge 0$ be such that*

(7) $$\overline{f}(C) := f(C)/|C| > -\gamma \quad \textit{for every circuit } C \textit{ in } D.$$

Then

(8) $$|x_f(e) - x^*(e)| < \gamma m \quad \textit{for any } e \in E.$$

We shall prove this lemma in Section 4. It shows that if we have succeeded in finding f (for some D) such that (7) is satisfied with $\gamma < 2^{-2cn-1}m^{-1}$, then $\|x_f - x^*\|_\infty < 2^{-2cn-1}$, so we can determine x^* from x_f using standard techniques on continuous fractions. (For a vector $a = (a_1, \ldots, a_m)$, $\|a\|_\infty$ denotes the norm $\max\{|a_i| \mid i = 1, \ldots, m\}$.)

2. ALGORITHM

Consider $D = (V, A)$ and f as in (5). Define $\lambda = \lambda(f)$ to be the least real for which there exists a potential $\pi : V \to \mathbb{R}$ satisfying

(9) $$\Delta_\pi(e) - f(e) \leq \lambda \quad \text{for any } e \in A.$$

Then for every circuit C in D one has $f(C) \geq -\lambda|C|$, whence the *mean* weight $\overline{f}(C) := f(C)/|C|$ of C is at least $-\lambda$.

Moreover, obviously (i) there exists C for which $\overline{f}(C) = -\lambda$, and (ii) a circuit C is a *minimum mean circuit* for f (that is, $\overline{f}(C) = -\lambda$) if and only if for π satisfying (9) we have

(10) $$\Delta_\pi(e) - f(e) = \lambda \quad \text{for each arc } e \in C.$$

Note that the value $\lambda(f)$, a potential π as in (9) and, therefore, a minimum mean circuit can be found in strongly polynomial time; for example, an algorithm of Karp [Ka] has running time $O(n^3)$.

Now we describe an algorithm to solve (2). It starts with choosing an arbitrary $x \in P$ (e.g., with x to be the incidence vector of a perfect matching in G) and the corresponding D_1 and f_1 for this x. At ith iteration of the algorithm there is a digraph $D = D_i = (V, A_i)$ and a function $f = f_i : A_i \to \mathbb{R}$ satisfying (5) (for $A := A_i$).

Determine $\lambda = \lambda_i = \lambda(f_i)$, $\pi = \pi_i$ satisfying (9), and a minimum mean circuit $C = C_i$. Let

(11) $$\varepsilon = \varepsilon_i := \min\{\lambda, \min\{-f(e) \mid e \in C^-\}\};$$

here and later on C^+ (C^-) denotes the set of arcs $(u,v) \in C$ with $u \in V_1$ (respectively, $u \in V_2$). We transform f into $f' = f_{i+1}$ by "pushing the flow of value ε along C". Namely, put

$$(12) \qquad f'(e) := f(e) + \varepsilon \quad \text{for } e \in C,$$

and accordingly correct the current digraph D and the current "flow" f on the other arcs. More precisely, we may consider D as a subgraph of the digraph $\widetilde{D} = (V, \widetilde{A})$ whose arc-set $\widetilde{A}$ consists of the arcs (u,v) and (v,u) for all $uv \in E$; extend f by zero on the arcs in $\widetilde{A} - A$ (then f is a "skew-symmetric" function on the arc-set of the symmetric digraph $\widetilde{D}$). Put

$$(13) \qquad \begin{aligned} f'(v,u) &:= -f'(u,v) \quad \text{for } (u,v) \in C; \\ &:= f(v,u) \quad \text{for the other } (v,u) \text{ in } \widetilde{A}. \end{aligned}$$

Finally, the new digraph $D' = D_{i+1}$ is formed from $\widetilde{D}$ by deleting the arcs (v,u) such that $v \in V_2$ and $f'(v,u) = 0$.

Clearly the new function f_{i+1} satisfies (5) (concerning $D_{i+1} = (V, A_{i+1})$). Algorithm finishs when the current λ becomes less than $\frac{1}{2}n^{-2cn}m^{-1}$ (then we are done by Lemma).

3. CORRECTNESS OF THE ALGORITHM

We use terminology and notation as in the previous section. Finiteness and, moreover, strong polynomiality of the above algorithm are provided by the following two claims, which are analogous to corresponding statements in [GT].

Claim 1. *Let λ and λ' be the corresponding numbers on two consecutive iterations. Then $\lambda' \leq \lambda$.*

Claim 2. *Let $m := |E|$. Then for any two iterations i and $i+m$, $\lambda_{i+m} \leq \left(\frac{2n-1}{2n}\right)\lambda_i$.*

By Claims 1 and 2, after each $2nm$ iterations the current λ decreases at least by a factor of $e = 2.71\ldots$. Since for any $x \in P$, $\|x\|_\infty \leq 1$ holds, the number of iterations of the algorithm is at most $2nm \log\left(2^{-2cn-1}m^{-1}\right)^{-1}$, or $O(nm(n + \log m))$.

Remark 2. In fact, arithmetical operations involved in the algorithm are only addition (or subtraction, or comparison) of fractions and division of a fraction by a natural number not exceeding n (when λ is being determined). This implies that if we start with an initial x to be the incident vector of a perfect matching then we can organize the procedure of computing each of intermediate data in the algorithm in such a way that, after a polynomial (in n, m) number of iteration, the size (that is, the number of digits in binary notation) of the numerator and the denominator of each currently appeared fraction would be bounded by a polynomial in n, m. This gives a polynomial (in n, m) boundary for the running time (calculated in operations over bits) of the whole algorithm.

Proof of Claim 1.

Consider π and C occurring on ith iteration. Let $e = (u, v) \in A'$ (A' is the arc-set of $D' = D_{i+1}$). We assert that

$$\Delta_\pi(e) - f'(e) \leq \lambda, \tag{14}$$

which obviously implies that $f'(C') \geq -\lambda|C'|$ for every circuit C' in D', and the result follows.

If $e \notin C$ and $(v, u) \notin C$ then $e \in A$ and $f'(e) = f(e)$, whence (14) follows from (9). If $e \in C$ then $f'(e) \geq f(e)$ (by (12)), and (14) is obvious. Finally, let $(v, u) \in C$. Then $\Delta_\pi(v, u) - f(v, u) = \lambda$ (by (10)) and $f'(v, u) = f(v, u) + \varepsilon$. Hence

$$\begin{aligned}\Delta_\pi(e) - f'(e) = f'(v, u) - \Delta_\pi(v, u) &= f(v, u) + \varepsilon - \Delta_\pi(v, u) \\ &= -\lambda + \varepsilon \leq 0,\end{aligned}$$

(since $\varepsilon \leq \lambda$, by (11)), and (14) follows. •

Proof of Claim 2.

Consider iterations $j = i, i+1, \ldots, i+m$. Fix π and λ occurring on ith iteration. For $i \leq j \leq i+m$ let $A^1 = A^1_j$ denote the set of arcs $e \in A_j$ such that

$$\Delta_\pi(e) - f_j(e) > \lambda/2,$$

and let $A^2 = A^2_j := A_j - A^1_j$. Two cases are possible.

Case 1. For all $j = i, \ldots, i+m-1$ each arc $e \in C_j$ belongs to A^1. Consider some j. By Claim 1, $\lambda_j \leq \lambda$, hence there is an arc $e \in C_j$ such that $\Delta_\pi(e) - f_j(e) \leq \lambda$. One may assume that $\lambda_j \geq \lambda/2$ (otherwise $\lambda_{i+m} \leq \lambda_j < \lambda/2$, and we are done). If $\varepsilon_j = \lambda_j$ then for e as above we have

$$\Delta_\pi(e) - f_{j+1}(e) = \Delta_\pi(e) - f_j(e) - \lambda_j \leq \lambda - \lambda_j \leq \lambda/2;$$

hence $e \notin A^1_{j+1}$. If $\varepsilon_j = -f_j(e')$ for some $e' \in C_j$ then $f_{j+1}(e') = 0$, which implies $e' \notin A_{j+1}$. On the other hang, no new edge e'' can appear in A^1_{j+1} in comparison with A^1_j. Indeed, if for $e'' \in A_{j+1}$ its opposite arc does not belong to C_j then $f_{j+1}(e) \geq f_j(e)$. And if e'' is opposite to an arc $(u,v) \in C_j$ then $\varepsilon_j \leq \lambda_j \leq \lambda$ implies that $-f_{j+1}(e'') = f_j(u,v) + \varepsilon_j \leq f_j(u,v) + \lambda$, whence

$$\begin{aligned}\Delta_\pi(e'') - f_{j+1}(e'') \leq \Delta_\pi(e'') + f_j(u,v) + \lambda = -\Delta_\pi(u,v) + f_j(u,v) + \lambda \\ < -\lambda/2 + \lambda = \lambda/2\end{aligned}$$

(taking into account that $(u,v) \in A^1_j$), that is, $e'' \in A^2_{j+1}$. Thus, $m \geq |A^1_i| > |A^1_{i+1}| > \ldots > |A^1_{i+m}|$, which implies that $A_{i+m} = A^2_{i+m}$ and $\lambda_{i+m} \leq \lambda/2$.

Case 2. For some $j \in \{i, \ldots, i+m-1\}$ there is an edge $e \in C_j$ in A^2_j. Consider the least j having such a property. Note that the fact that for $j' = i, \ldots, j-1$ all edges in $C_{j'}$ belong to $A^1_{j'}$ implies $\Delta_\pi(e') - f_{j'+1}(e') \leq \lambda$ for all $e' \in A_{j'+1}$ (this can be easily shown by induction on j', using arguments as in the previous case). Hence,

$$\begin{aligned}f_j(C_j) = f_j(C_j) - \sum_{e' \in C_j} \Delta_\pi(e') = -\sum(\Delta_\pi(e') - f_j(e') \mid e' \in C_j - \{e\}) \\ -(\Delta_\pi(e) - f_j(e)) \geq -\lambda(|C_j| - 1) - \lambda/2 = -\lambda(2|C_j| - 1)/2,\end{aligned}$$

whence $\lambda_j = -\overline{f}_j(C_j) \leq \lambda(2|C_j| - 1)/(2|C_j|)$, and the result follows. •

4. PROOF OF LEMMA

Let x be an arbitrary vector in P, and x^* be the optimal solution of (2). Denote by $D = (V, A)$ and f the digraph D^x and the function on A defined by (4), respectively. Similarly, denote by $D^* = (V, \vec{A}^*)$ and f^* the corresponding objects for x^*. Put

$\delta := \|x - x^*\|_\infty$. We show that there exists a circuit C in D such that $\overline{f}(C) \leq -\delta/m$: this will immediately imply the lemma.

As above, we may think of D and D^* as subgraphs of the symmetric digraph $\widetilde{D} = (V, \widetilde{A})$ ($\widetilde{D}$ was defined in Section 2). Extend f and f^* by zero on the arcs in $\widetilde{A} - A$ and $\widetilde{A} - A^*$, respectively. Let E_1 (E_2) be the set of arcs $e \in \widetilde{A}$ such that $f^*(e) > f(e) \geq 0$ (respectively, $f(e) < f^*(e) \leq 0$). For $e \in E_1 \cup E_2$ put $h(e) := f^*(e) - f(e)$. From the facts that $x, x^* \in P$ and that both f and f^* are skew symmetric it follows that for any $v \in V$:

$$\sum(h(u,v) \mid (u,v) \in E_1 \cup E_2) = \sum(h(v,u) \mid (v,u) \in E_1 \cup E_2),$$

that is, h is a circulation. Then h can be represented as

$$h = g_1\chi_{C(1)} + g_2\chi_{C(2)} + \ldots + g_k\chi_{C(k)}$$

with $k \leq |E_1 \cup E_2| \leq m$, where $g_1, \ldots, g_k$ are positive reals, and $\chi_{C(i)} \in \mathbb{R}^{E_1 \cup E_2}$ is the incidence vector of the arc-set of a circuit $C(i)$ in the digraph $(V, E_1 \cup E_2)$. Choose $C = C(i)$ with $g = g_i$ maximum. It is clear that $g \geq \delta/k \geq \delta/m$.

Let $C^+ := C \cap E_1$ and $C^- := C \cap E_2$. From the definition of E_1 and E_2 it follows that for $e = (u,v) \in C$, $e \in C^+$ if and only if $u \in V_1$.

Since x^* is optimal, there is a potential $\pi := V \rightarrow \mathbb{R}$ such that $\Delta_\pi(e) = f^*(e)$ holds for any $e \in A^*$ with $f^*(e) \neq 0$, and $\Delta_\pi(e) \leq 0$ for any $e = (u,v) \in A^*$ with $f^*(e) = 0$ (in the latter case u belongs to V_1). Hence, $\Delta_\pi(e) = f^*(e)$ for $e \in C^+$ (as $f^*(e') > 0$ for each $e' \in E_1$) and $\Delta_\pi(e) \geq f^*(e)$ for $e = (v,u) \in C^-$ (as $u \in V_1$, whence $(u,v) \in A^*$). Thus,

$$f^*(C) \leq \sum_{e \in C} \Delta_\pi(e) = 0.$$

Finally, since each arc $(u,v) \in \widetilde{A}$ with $u \in V_1$ belongs to A, and $f(e) \neq 0$ for any $e \in E_2$, we observe that C is a circuit in D. For each $e \in C$ we have $f^*(e) - f(e) = h(e) \geq g$, whence $f(C) \leq f^*(C) - g|C|$. Now $f^*(C) \leq 0$ implies $f(C) \leq -g|C|$. Thus, $\overline{f}(C) \leq -g \leq -\delta/m$, as required. •

Remark 3. The above algorithm can be easily extended to solve the problem of determining the distance to the perfect matching polytope of a bipartite graph from an *arbitrary* (rational) point a in $\mathbb{R}^E$. This is equivalent to the problem of finding the distance from the *origin* to the "shifted" polytope $P_{G,a}$ consisting of the vectors

$x \in \mathbb{R}^E$ such that $x(e) \geq -a(e)$ for $e \in E$ and $x(E_v) = 1 - a(E_v)$ for $v \in V$. In this case the number of iterations of the algorithm similar to that described above becomes $O(nm(n + \log m + \max\{s(a(e)) \mid e \in E\}))$, where for a rational number $r = p/q$, $s(r)$ denotes $1 + \lceil \log(|p| + 1) \rceil + \lceil \log(|q| + 1) \rceil$.

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