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HALF–INTEGRAL FLOWS IN A PLANAR GRAPH WITH FOUR HOLES

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Abstract. This paper contains an improved and shorter proof of the main theorem in [Ka3].

Suppose that $G = (VG, EG)$ is a planar graph embedded in the euclidean plane, that I, J, K, O are four of its faces (called *holes* in G), that $s_1, ..., s_r, t_1, ..., t_r$ are vertices of G such that each pair $\{s_i, t_i\}$ belongs to the boundary of some of I, J, K, O, and that the graph $(VG, EG \cup \{\{s_1, t_1\}, ..., \{s_r, t_r\}\})$ is eulerian.

We prove that if the multi(commodity)flow problem in G with unit demands on the values of flows from s_i to t_i ($i = 1, \ldots, r$) has a solution then it has a *half-integral* solution as well. In other words, there exist paths $P_1^1, P_1^2, P_2^1, P_2^2, ..., P_r^1, P_r^2$ in G such that each P_i^j connects s_i and t_i, and each edge of G is covered at most twice by these paths. (It is known that in case of at most three holes there exist edge-disjoint paths connecting s_i and t_i, $i = 1, \ldots, r$, provided that the corresponding multiflow problem has a solution, but this is, in general, false in case of four holes.)

1. Introduction

Throughout, we deal with an undirected planar graph G; speaking of a planar graph we mean that some of its embeddings in the euclidean plane $\mathbb{R}^2$ (or the sphere $\mathbf{S}^2$) is fixed. VG is the vertex set, EG is the edge set of G (multiple edges and loops are admitted), and $\mathcal{F} = \mathcal{F}_G$ is the set of faces of G. A subset $\mathcal{H} \subseteq \mathcal{F}$ of faces of G, called its *holes*, is distinguished. Let $U = \{\{s_1, t_1\}, ..., \{s_r, t_r\}\}$ be a family of pairs (possibly repeated) of vertices of G such that each $\{s_i, t_i\}$ is contained in the boundary $bd(I)$ of some hole $I \in \mathcal{H}$.

Problem (G, U, k): *given an integer* $k \geq 1$, *find paths* $P_1^1, \ldots, P_1^k, \ldots, P_r^1, \ldots, P_r^k$ *in* G *such that each* P_i^j *connects* s_i *and* t_i, *and each edge of* G *occurs at most* k *times in these paths.*

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If no restriction on k is imposed, the problem is denoted as $(G,U)^*$; thus $(G,U)^*$ is the fractional relaxation of $(G,U,1)$, or the *multi*(commodity)*flow* problem with unit capacities of the edges of G and unit demands on the values of flows connecting pairs in U. We prove the following theorem.

Theorem 1. *Let $|\mathcal{H}|=4$, and let the graph $(VG, EG\cup U)$ be eulerian, that is,*

$$(1.1)\qquad |\delta X|+|\{i\ :\ \delta X \text{ separates } s_i \text{ and } t_i\}| \text{ is even for any } X\subset VG.$$

Let $(G,U)^*$ have a solution. Then $(G,U,2)$ has a solution as well; in other words, there exist $P_1^1,P_1^2,P_2^1,P_2^2,\dots,P_r^1,P_r^2$ such that each P_i^j is a path in G connecting s_i and t_i, and each edge of G is covered at most twice by these paths.

[For $X\subseteq V$, $\delta X=\delta^G X$ denotes the set of edges of G with one end in X and the other in $VG-X$; a nonempty set δX is called a *cut* in G; we say that δX *separates* vertices x and y if exactly one of x,y is in X.] An obvious necessary condition for solvability of (G,U,k) for arbitrary G,U,k is the *cut condition*:

(1.2) each cut δX in G separates at most $|\delta X|$ pairs in U.

The following theorem is well known.

Okamura's theorem [Ok]. *If $|\mathcal{H}|=2$ and if (1.1) and (1.2) hold then $(G,U,1)$ has a solution (that is, there exist edge-disjoint paths $P_1,\dots,P_r$ in G such that P_i connects s_i and t_i).*

(A similar result for $|\mathcal{H}|=1$ is proved in [OkS].) The cut condition is, in general, not sufficient for the solvability of (G,U,k) if $|\mathcal{H}|=3$. Nevertheless, the following is true.

Theorem 2 [Ka2]. *Let $|\mathcal{H}|=3$, and let (1.1) and (1.2) hold. The problem $(G,U,1)$ has a solution if for any 2,3-metric m on VG the following inequality holds:*

$$(1.3)\qquad \sum(m(e)\ :\ e\in EG)\geq\sum(m(s_i,t_i)\ :\ i=1,\dots,r).$$

[By a *metric* on a set V we mean a real-valued function m on $V\times V$ satisfying $m(x,x)=0$, $m(x,y)=m(y,x)$ and $m(x,y)+m(y,z)\geq m(x,z)$ for all $x,y,z\in V$. We say that m is *induced* by (H,σ), where H is a graph and σ is a mapping of V into VH, if $m(x,y)=\mathrm{dist}^H(\sigma(x),\sigma(y))$ for all $x,y\in V$. Here $\mathrm{dist}^{G'}(x',y')$ denotes the distance in a graph G' between vertices x' and y'. When it is not confusing, we say that m is induced by H or m is induced by σ. If $\sigma(V)=VH$ and H is the complete graph K_2 on two vertices (the complete bipartite graph $K_{2,3}$ with parts of two and three vertices) then m is called a *cut-metric* (respectively, a 2,3-*metric*).] Note that satisfying (1.3) with any metric m on VG is necessary for the solvability of (G, U, k) for arbitrary G, U, k because if P_i^j's give a solution of (G, U, k) then

$$\sum_{e \in EG} m(e) \geq \frac{1}{k} \sum_{i=1}^{r} \sum_{j=1}^{k} \sum (m(e) \ : \ e \in P_i^j) \geq \sum_{i=1}^{r} m(s_i, t_i)$$

(we write $e \in P_i^j$ considering a path as an edge-set). Thus, if $|\mathcal{H}| \leq 3$, (1.1) holds, and $(G, U)^*$ has a solution then $(G, U, 1)$ has a solution as well. Such a property does not remain, in general, true for $|\mathcal{H}| = 4$, as shown in [Ka2]. Hence, for $|\mathcal{H}| = 4$ Theorem 1 provides the least (in terms of $\mathcal{H}$) value of k for which (G, U, k) has a solution in the eulerian case. Another feature of case $|\mathcal{H}| = 4$ is that more exotic metrics are involved in the solvability criterion for $(G, U)^*$. We say that a metric induced by a bipartite planar graph H with $|\mathcal{F}_H| = 4$ is a 4f-*metric*.

Theorem 3 [Ka1]. *For $|\mathcal{H}| = 4$, $(G, U)^*$ is solvable if and only if (1.3) holds for every m that is a cut-metric or a 2,3-metric or a 4f-metric.*

The proof of Theorem 1 will rely essentially on a strengthening of the fractional version of Theorem 2 and a strengthening of Theorem 3 (Theorems 4 and 5 below); they describe classes of 2,3- and 4f-metrics sufficient for verification of solvability of $(G, U)^*$. To state these, we need some terminilogy, conventions and simple facts about multiflows and metrics.

First, the faces of a planar graph are considered as *open* regions in the plane. An edge e with end vertices x and y is identified with the corresponding curve in the plane (x and y are usually not included in the curve); when it leads to no confusion, e is denoted by xy. A path (circuit) $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ (where x_i's are vertices and e_i's are edges) is denoted by $x_0 x_1 \ldots x_k$ and called an $x_0 - x_k$ path; P is often considered up to reversing and, if P is a circuit, shifting cyclically. $|P|$ is the number k of edges in P; if $|P| = 0$, P is called *trivial*. A path P from x to y is called an $x - y$ *path*; if both x and y are in the boundary of a face F we say that P is an F-*path*. The boundary bd(F) of a face F is identified with the corresponding (possibly not simple) circuit. For $g : E \to \mathbb{R}$ and $E' \subseteq E$, $g(E')$ denotes $\sum(g(e) \ : \ e \in E')$; in particular, we write $g(P)$ for a function g on the edges of a graph and a path (or circuit) P, considering P as an edge-set.

Second, consider a planar graph G' with a set $\mathcal{H}'$ of holes. For $F \in \mathcal{F}_{G'}$ let W_F denote the set of pairs $\{s, t\}$ of vertices in bd(F), and let $W_{\mathcal{H}'} := \cup(W_F \ : \ F \in \mathcal{H}')$. Suppose we are given a family U' of pairs in $W_{\mathcal{H}'}$, and functions $c' : EG' \to \mathbb{Q}_+$ (of *capacities* of edges) and $d' : U' \to \mathbb{Q}_+$ (of *demands*). Denote by $\mathcal{P}(G', s, t)$ the set of simple paths in G' connecting vertices s and t. Let $\mathcal{P}(G', U') := \cup(\mathcal{P}(G', s, t) : \{s, t\} \in U')$. We denote by (c', d') the *multiflow problem*: find a function $f : \mathcal{P}(G', U') \to \mathbb{Q}_+$ satisfying:

$$f^e := \sum(f(P) : e \in P \in \mathcal{P}(G', U')) \leq c'(e) \quad \text{for all } e \in EG'; \tag{1.4}$$

$$\sum(f(P) : P \in \mathcal{P}(G', s, t)) = d'(s, t) \quad \text{for all } (s, t) \in U' \tag{1.5}$$

(when c' and d' are all-unit, (c', d') turns into $(G', U')^*$). We say that f satisfying (1.4)-(1.5) is a (c', d')-*admissible multiflow*. Applying Farkas lemma to the system (1.4)-(1.5) and then making easy transformations, one can obtain the following criterion (this is valid for arbitrary G', U', c', d' [Lo]):

(1.6) *Solvability criterion*: (c', d') is solvable if and only if the inequality $c'(m) \leq d'(m)$ holds for any metric m on VG', where $c'(m) := \sum_{e \in EG'} c'(e)m(e)$ and $d'(m) := \sum_{(s,t) \in U'} d'(s, t)m(s, t)$.

Third, we say that a metric m on VG' is *bipartite* if m is integer-valued and the length $m(C)$ of every circuit C in G' is even (in particular, every cut-, 2,3- or 4f-metric is bipartite). A bipartite m is called $\mathcal{H}'$-*primitive* if there are no non-zero bipartite metrics m' and m'' on VG' such that $m(e) \geq m'(e) + m''(e)$ for all $e \in EG'$ and $m(s, t) \leq m'(s, t) + m''(s, t)$ for all $\{s, t\} \in W_{\mathcal{H}'}$. A simple observation is that in criterion (1.6) it suffices to consider the $\mathcal{H}'$-primitive metrics rather that all metrics m on VG'; in other words, if (c', d') is unsolvable then $c'(m) < d'(m)$ holds for some $\mathcal{H}'$-primitive m.

Fourth, let m be a metric induced by $\sigma : VG' \to VH$, where H is a bipartite planar graph with $|\mathcal{F}_H| = |\mathcal{H}'|$. As a rule, we shall deal with the situation when σ yields a certain topological correspondence of the face structures for G' and H. More precisely, σ can be extended to a continuous mapping of $\mathbb{R}^2$ into itself so that:

(1.7) (i) for any point $x \in \mathbb{R}^2$ each of the sets $\sigma^{-1}(x)$ and $\mathbb{R}^2 - \sigma^{-1}(x)$ is connected, and $\sigma^{-1}(x)$ is compact;

(ii) each hole $F \in \mathcal{H}'$ is mapped homeomorphically to a face of H;

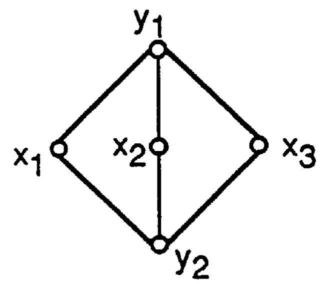
(iii) for each edge $e = xy \in EG'$ the path (x, e, y) is mapped homeomorphically to a simple path in H unless it is mapped to a single point.

In this case we say that m is *consistent*. For convenience we also assume that σ preserves orientation clockwise in $\mathbb{R}^2$. From (i)-(ii) and the fact that $|\mathcal{F}_H| = |\mathcal{H}'|$ it follows that σ gives a one-to-one correspondence of the holes in G' to the faces in H, and that the unbounded face of G' is a hole. It was shown in [Ka1] that

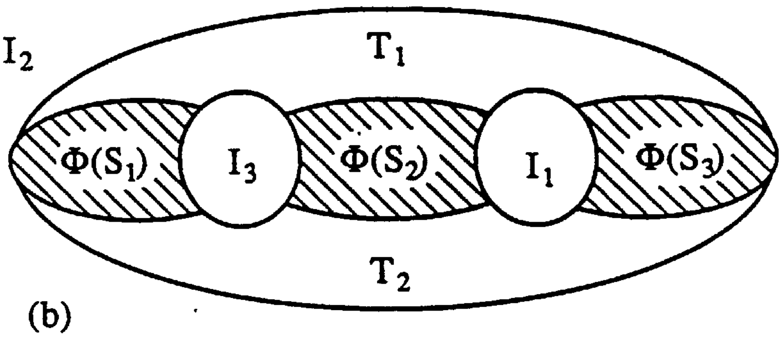
(1.8) if $|\mathcal{H}'| = 3$ ($|\mathcal{H}'| = 4$) then any $\mathcal{H}'$-primitive 2,3-metric (respectively, 4f-metric) on VG' is consistent.

Suppose that $|\mathcal{H}'| = 3$. Let m be a consistent 2,3-metric induced by $\sigma : VG' \to VK_{2,3}$, and let $\{y_1, y_2\}$ and $\{x_1, x_2, x_3\}$ be the parts in $VK_{2,3}$. Denote by $\Pi(\sigma)$ the (ordered) partition $(S_1, S_2, S_3, T_1, T_2)$ of VG', where $S_i := \sigma^{-1}(x_i) \cap VG'$ and $T_j := \sigma^{-1}(y_j) \cap VG'$. Let Φ_i denote the closed region $\sigma^{-1}(x_i)$ in $\mathbb{R}^2$. One can see that there is a labelling $I_1, I_2, I_3 = I_0$ of the holes such that (see Fig. 1.1):

(1.9) $\Phi_i \cap \mathrm{bd}(I_p) = \emptyset$ if and only if $p = i$; and the space $\Omega(\sigma) := \mathcal{R}^2 - (I_1 \cup I_2 \cup I_3 \cup \Phi_1 \cup \Phi_2 \cup \Phi_3)$ consists of two disjoint regions, one containing T_1 and the other containing T_2.



(a) $K_{2,3}$



(b)

Fig. 1.1

Theorem 4 [Ka1]. *Let $|\mathcal{H}'| = 3$, and let m be an $\mathcal{H}'$-primitive 2,3-metric induced by $\sigma : VG' \to VK_{2,3}$. Let $\Pi(\sigma) = (S_1, S_2, S_3, T_1, T_2)$ and I_1, I_2, I_3 be defined as above (taking into account that m is consistent, by (1.8)). Then:*

(i) *all sets in $\Pi(\sigma)$ are nonempty;*

(ii) *for $i = 1, 2, 3$ the subgraph $\langle S_i \rangle$ in G' induced by S_i is connected, and S_i meets both $\mathrm{bd}(I_{i-1})$ and $\mathrm{bd}(I_{i+1})$.*

In particular, no edge of G' connects T_1 and T_2.

In Section 2 we show how Okamura's theorem and Theorem 4 are applied in order to prove that in the eulerian case a solvable problem $(G, U)^*$ with $|\mathcal{H}| = 4$ has a 1/4-*integral* solution; this proof is relatively easy. Using this, we then prove Theorem 1. This proof involves more intricate arguments and is given throughout Sections 3-5. In particular, at many stages of the proof we appeal to the fact that, besides being consistent, a primitive 4f-metric possesses a spectrum of structural properties and its value on an edge is at most four (compared with the cut-metrics and 2,3-metrics, which take their values in $\{0,1\}$ and $\{0,1,2\}$, respectively; note also that the set of graphs H inducing primitive 4f-metrics m is infinite, thus values of m on pairs of vertices that are not edges of G can be large). These properties are exposed in the following theorem.

Theorem 5. *Let $|\mathcal{H}'| = 4$, and let m be an $\mathcal{H}'$-primitive 4f-metric induced by $\sigma : VG' \to VH$. Then $m(e) \leq 4$ for each $e \in EG'$. Moreover, if $m(e) = 4$ for some edge $e = xy$ then:*

(i) *H is homeomorphic to K_4;*

(ii) *the image by σ of the path (x, e, y) is a shortest path $L_e = b_0b_1b_2b_3b_4$ in H which belongs to the boundary of a unique face, $\tilde{J}$ say, in H;*

(iii) *each shortest $x - y$ path in H with $x, y \in \mathrm{bd}(\tilde{J}) - \{b_1, b_2, b_3\}$ lies in $\mathrm{bd}(\tilde{J})$;*

(iv) *for each $\tilde{I} \in \mathcal{F}_H - \{\tilde{J}\}$, no shortest $\tilde{I}$-path contains both b_0 and b_4;*

(v) *if each of b_0, b_4 belongs to a shortest $\tilde{I}$-path for the same $\tilde{I} \in \mathcal{F}_H - \{\tilde{J}\}$, and $b \in \{b_0, b_4\}$ is not in $\mathrm{bd}(\tilde{I})$, then (a) every shortest $\tilde{I}$-path containing b separates $\tilde{J}$ from $\tilde{K}$ and $\tilde{O}$, and (b) no shortest $\tilde{I}$-path contains an edge in $\mathrm{bd}(\tilde{K}) \cap \mathrm{bd}(\tilde{O})$, where $\mathcal{F}_H = \{\tilde{I}, \tilde{J}, \tilde{K}, \tilde{O}\}$.*

Here we say that an I-path L *separates* faces J and K if they lie in different components of $\mathbb{R}^2 - (I \cup L)$. Though this result is very important to get Theorem 1, the proof of Theorem 5 is very technical and we do not give it here, referring the reader to [Ka3, Section 3]. Figure 1.2 illustrates an $\mathcal{H}'$-primitive metric m with $m(e) = 4$ for some e, and properties (i)-(v); here $\mathcal{H} = \{I, J, K, O\}$, m is induced by a mapping of VG' to VH and its values on the edges of G' are indicated.

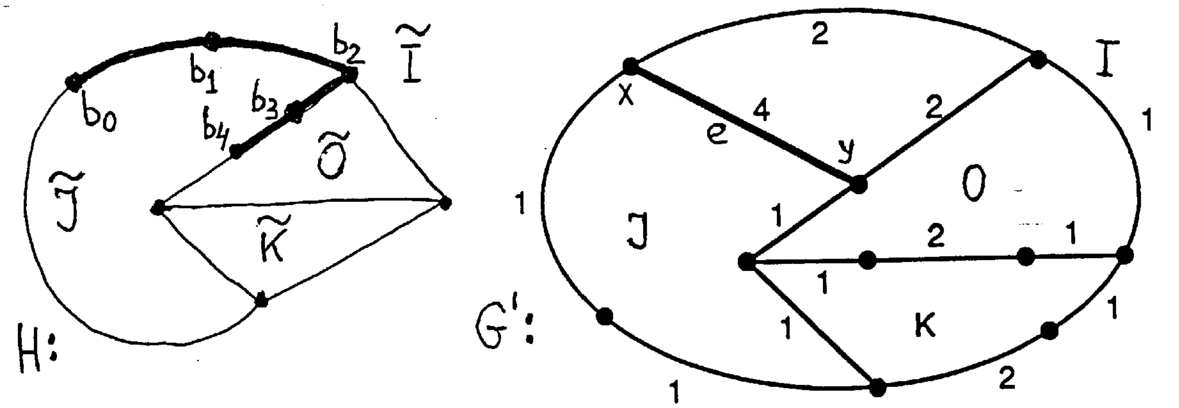


Fig. 1.2

2. EXISTENCE OF A QUARTER-INTEGRAL SOLUTION

Let $|\mathcal{H}| = 4$, and let $(G,U)^* = (c,d)$ have a solution $f : \mathcal{P}(G,U) \to \mathbb{Q}_+$, where c and d are the all-unit functions on EG and U, respectively. It is convenient to think of f as consisting of four flows f_F ($F \in \mathcal{H}$), where f_F is the restriction of f to the F-paths in $\mathcal{P}(G,U)$ (one may assume that no member of U belongs to the boundaries of two holes). Denote by $\mathcal{L} = \mathcal{L}(f)$ the set of paths $P \in \mathcal{P}(G,U)$ with $f(P) > 0$ (the *support* of f). Similarly, $\mathcal{L}_F = \mathcal{L}_F(f)$ denotes the support of f_F; thus $\{\mathcal{L}_I, \mathcal{L}_J, \mathcal{L}_K, \mathcal{L}_O\}$ is a partition of $\mathcal{L}$.

A path $P \in \mathcal{L}_F$ ($F \in \mathcal{H}$) divides the space $\mathbb{R}^2 - F$ into a pair $\mathcal{R}(P)$ of closed regions whose intersection is P and union is $\mathbb{R}^2 - F$. We say that f is *non-crossing* if any two paths $P \in \mathcal{L}_F$ and $P' \in \mathcal{L}_{F'}$ for $F \neq F'$ do not cross, that is, P' is contained entirely in some of the members of $\mathcal{R}(P)$. Applying to f standard uncrossing techniques, it is easy to show that

(2.1) if $(G,U)^*$ has a $1/k$-integral solution then it has a $1/k$-integral non-crossing solution.

In what follows we assume that f is non-crossing. Consider two different holes F and F'. Remove from the sphere $\mathbf{S}^2$ the hole F, its boundary and the paths in $\mathcal{L}_F$. Then F' occurs in a component Z of the resulting space. Define $D_{FF'} = D_{FF'}(f)$ to be $\mathbf{S}^2 - Z$. Easy topological observations using the fact that all paths in $\mathcal{L}_F$ are simple show that $D_{FF'}$ is homeomorphic to a closed disc, i.e., the boundary $C_{FF'} = C_{FF'}(f)$ of $D_{FF'}$ is a closed non-self-intersecting curve. Moreover,

(2.2) $C_{FF'}$ is a simple circuit in G, and $f_F^e > 0$ holds for each edge $e \in C_{FF'}$ that is not in $\mathrm{bd}(F)$, where $f_F^e := \sum(f(P) \,:\, e \in P \in \mathcal{L}_F(f))$.

(An equivalent definition: $D_{FF'}$ is the largest region in $\mathbf{S}^2$ that does not contain F' and whose boundary is in the union of $\mathrm{bd}(F)$ and $\cup(P \in \mathcal{L}_F)$.) Since f is non-crossing, $D_{FF'}$ and $D_{F'F}$ are obviously openly disjoint, i.e., $D_{FF'} \cap D_{F'F} = C_{FF'} \cap C_{F'F}$. Furthermore, for $F'' \in \mathcal{H} - \{F, F'\}$, if $F'' \subset D_{FF'}$ then $D_{FF''} \cup D_{FF'} = \mathbf{S}^2$, while if $F'' \cap D_{FF'} = \emptyset$ then $D_{FF''}$ and $D_{F''F}$ are openly disjoint and $D_{FF''} = D_{FF'}$. This justify introducing the following notion, which plays the central role in the proof of Theorem 1.

Definition. Given a non-crossing f, a maximal subset $B \subseteq \mathcal{H}$ such that $D_{FF'}$ and $D_{F'F}$ are openly disjoint for any two distinct $F, F' \in B$ is called a *bunch*.

Clearly $2 \le |B| \le 4$. For $F \in B$ we denote $D_{FF'}$ and $C_{FF'}$ by D_F and C_F, respectively (these do not depend on $F' \in B - \{F\}$). The family of $|B|$ circuits C_F ($F \in B$) is denoted by $\mathcal{C}(B)$ (in case $B = \{F, F'\}$ the circuits C_F and $C_{F'}$ may coincide).

Also denote by G_F, $\mathcal{H}_F$, and U_F the subgraph of G contained in D_F, the set of holes $\widehat{F} \in \mathcal{H}$ in D_F, and the set of pairs $\{s,t\} \in U$ such that $\{s,t\} \in W_{\widehat{F}}$ for $\widehat{F} \in \mathcal{H}_F$, respectively. Obviously,

(2.3) for a bunch B, the space $\mathbf{S}^2 - \cup(D_F : F \in B)$ contains no hole, and each edge e of G occurs in at most two members of $\mathcal{C}(B)$.

Fix a bunch B. We may assume that for each $F \in B$, C_F has an edge with some $F' \in B - \{F\}$ in common. Indeed, if this is not so for some F, consider the problems $(G_F, U_F)^*$ and $(G', U')^*$, where $G' = (VG, EG - EG_F)$ and $U' = U - U_F$. Clearly every path in $\mathcal{L}$ is entirely within some of G_F and G', therefore the corresponding restrictions of f give solutions for these problems. Since $|\mathcal{H}_F| \leq 3$ and $|\mathcal{H} - \mathcal{H}_F| \leq 3$, by Okamura's theorem or Theorem 2, each problem has a ***half-integral*** solution (not necessarily integral as $(VG_F, EG_F \cup U_F)$ may not be eulerian). Combining these, we get a half-integral solution for $(G,U)^*$, and Theorem 1 follows. By similar arguments, we may assume that

(2.4) for any $\emptyset \neq B' \subset B$, $\cup_{F \in B'} C_F$ and $\cup_{F \in B-B'} C_F$ have an edge in common.

Later on we assume that a non-crossing f and a bunch B are chosen so that:

(2.5) (i) $|B|$ is as great as possible;

(ii) $\sum((|\mathcal{H}_F|)^2 : F \in B)$ is minimum subject to (i);

(iii) the number of faces in $\cup(D_F : F \in B)$ is minimum subject to (i)-(ii).

In particular, a bunch B (for some f) with $\{|\mathcal{H}_F| : F \in B\} = \{1,1,1,1\}$ is preferable to choose than one with $\{1,1,2\}$, and $\{2,2\}$ is preferable than $\{1,3\}$. Let $\overline{f}^e_F$ stand for $\sum(f^e_{F'} : F' \in \mathcal{H}_F)$.

Lemma 2.1. *For each $F \in B$ there exists a function h_F on EG_F such that:*

(i) $h_F(e) \in \{0, \frac{1}{2}, 1\}$ for each $e \in C_F$ and $h_F(e) = 1$ for the other edges e in G_F;

(ii) if e is a common edge for C_F and $C_{F'}$ $(F, F' \in B)$ then $h_F(e) + h_{F'}(e) \leq 1$;

(iii) each problem (h_F, d_F) is solvable; here d_F is the all-unit function on U_F.

This lemma shows the existence of a 1/4-integral solution for $(G,U)^*$. Indeed, for each $F \in B$ the function $2h_F$ is integral, hence the problem $(2h_F, 2d_F)$ has a half-integral solution. So (h_F, d_F) has a 1/4-integral solution. Taken together, these solutions form an admissible solution for $(G,U)^*$.

Proof of Lemma 2.1. Choose functions h_F $(F \in B)$ so that (ii)-(iii) hold, h_F is all-unit on $EG_F - C_F$, and the value $\gamma(h) := \sum_{F\in B} |Q_F|$ is as small as possible, where $Q_F := \{e \in C_F \ : \ h_F(e) \notin \{0, \frac{1}{2}, 1\}\}$. Such functions exist since we can take as $h_F(e)$ the value $\overline{f}^e_F$ for $e \in C_F$, and 1 for the other edges e of G_F. One has to prove that $\gamma(h) = 0$. Suppose that $\gamma(h) > 0$.

For $F \in B$ let Q^+_F (Q^-_F) be the set of edges $e \in Q_F$ with $h_F(e) > 1/2$ (respectively, $h_F(e) < 1/2$). We perform *balancing* h_F's (simultaneously for all $F \in B$); this means that for some $\varepsilon \in \mathbb{R}_+$ each h_F is transformed to h^ε_F, where

$$
\begin{aligned}
(2.6) \qquad\qquad h^\varepsilon_F(e) &:= h_F(e) - \varepsilon \quad \text{ if } e \in Q^+_F;\\
&:= h_F(e) + \varepsilon \quad \text{ if } e \in Q^-_F;\\
&:= h_F(e) \quad \text{ for the remaining } e\text{'s in } G_F;
\end{aligned}
$$

Take ε to be maximum provided that for each $F \in B$, (a) $\varepsilon \le h_F(e) - 1/2$ for $e \in Q^+_F$; (b) $\varepsilon \le 1/2 - h_F(e)$ for $e \in Q^-_F$; and (c) (h^ε_F, d_F) has a solution g_F. Clearly $h^\varepsilon_F(e) + h^\varepsilon_{F'}(e) \le 1$ for each edge e common for C_F and $C_{F'}$ ($F, F' \in B$). Also $\gamma(h^\varepsilon) \le \gamma(h)$, whence $\gamma(h^\varepsilon) = \gamma(h)$, by the choice of h. Furthermore, one can see that combining the g_F's we get a multiflow which has a bunch B' not worse than B in the sense of (2.5). By the maximality of ε, there is $F \in B$ such that for any $\Delta > \varepsilon$ the problem $(h^{\varepsilon'}_F, d_F)$ has no solution for some $\varepsilon < \varepsilon' \le \Delta$. Two cases are possible.

Case 1. $|\mathcal{H}_F| \le 2$. Applying Okamura's theorem, we observe that for every $\varepsilon' > \varepsilon$ there is $X' \subset VG_F$ such that $h^{\varepsilon'}_F(X') < d_F(X')$, where $h^{\varepsilon'}_F(X')$ stands for $\sum(h^{\varepsilon'}_F(e) \ : \ e \in \delta X')$ and $d_F(X')$ stands for $|\{\{s,t\} \in U_F \ : \ \delta X' \text{ separates } s \text{ and } t\}|$ (letting $\delta X' := \delta^{G_F} X'$). Hence, there is $X \subset VG_F$ such that

$$h^\varepsilon_F(X) = d_F(X) \text{ and } h^{\varepsilon'}_F(X) < d_F(X) \text{ for any } \varepsilon' > \varepsilon.$$

Without loss of generality, we may assume that δX is a *simple* cut, i.e., δX meets at most twice the boundary of every face in G_F. In particular, $|\delta X \cap C_F| \le 2$ (as C_F is the boundary of a face in G_F). Then $|\delta X \cap C_F| = 2$; let $\delta X \cap C_F \doteq \{e, e'\}$. Since d_F is an integer and $h^\varepsilon_F(e'')$ is an integer for each $e'' \in \delta X - \{e, e'\}$, $h^\varepsilon_F(X) = d_F(X)$ implies that $\phi := h^\varepsilon_F(e) + h^\varepsilon_F(e')$ is an integer. Hence, either $h^\varepsilon_F(e) + h^\varepsilon_F(e') = \frac{1}{2}$, or one of e, e' is in Q^+_F and the other in Q^-_F. In both cases we have $h^{\varepsilon'}_F(X) = h^\varepsilon_F(X)$ for any ε'; a contradiction.

Case 2. $|\mathcal{H}_F| = 3$. Then $|B| = 2$; let for definiteness $B = \{I, K\}$, $F = I$ and $\mathcal{H}_I = \{I, J, O\}$. Apply Theorem 4. Arguing as above, we conclude that there exists (i) $X \subset VG_I$ such that $h^\varepsilon_I(X) = d_I(X)$ and $h^{\varepsilon'}_I(X) < d_I(X)$ for any $\varepsilon' > \varepsilon$, or (ii) an $\mathcal{H}_I$-primitive 2,3-metric m on VG_I such that

$$h^\varepsilon_I(m) = d_I(m) \text{ and } h^{\varepsilon'}_I(m) < d_I(m) \text{ for any } \varepsilon' > \varepsilon,$$

where $h^\varepsilon_I(m) := \sum(h^\varepsilon_I(e)m(e) \ : \ e \in EG_I)$ and $d_I(m) := \sum(m(s,t) \ : \ \{s,t\} \in U_I)$ (cf. (1.6)). By arguments as in Case 1, (i) is impossible.

Thus (ii) takes place. Consider the partition $\Pi(\sigma) = (S_1, S_2, S_2, T_1, T_2)$ of VG_I as in Theorem 4 (where m is induced by σ). Since C_I is the boundary of some face $\widetilde{F}$ in G_I and each subgraph $\langle S_i\rangle$ is connected, C_I can pass across exactly one component of $\Omega(\sigma)$ (defined in (1.9)), say, the component Ω_1 that contains T_1. Next, if there is an edge $e \in C_I$ connecting $u \in S_i$ and $v \in S_j$ $(i \neq j)$, we could slightly transform G_I and m by replacing e by a pair of edges in series, $e' = uz$ and $e'' = zv$ say, and by adding z to T_1 (and, accordingly, placing z in the region Ω_1); it is easy to see that the new graph and 2,3-metric maintain the above properties. Thus, one may assume that each edge in C_I connecting different sets in $\Pi(\sigma)$ connects just T_1 and some S_i. Let $\xi = (e_1 = u_1v_1, \ldots, e_k = u_kv_k)$ be the sequence of such edges in C_I, and let the vertices $u_1, v_1, \ldots, u_k, v_k$ occur in this order in C_I. Note that there are no two consecutive edges e_j, e_{j+1} in ξ such that $v_j, u_{j+1} \in T_1$ and $u_j, v_{j+1} \in S_i$ for some $i \in \{1,2,3\}$. For otherwise, assuming for definiteness that $i = 1$ and letting Z to be the component in $\langle T_1\rangle$ that contains the part of C_I from v_j to u_{j+1}, we observe that the partition $(T_1 - VZ, T_2, S_1 \cup VZ, S_2, S_3)$ corresponds to a 2,3-metric m' such that $h^\varepsilon_I(m') < h^\varepsilon_I(m)$ and $d_F(m') = d_F(m)$, which is impossible. Now the latter property together with the fact that each $\langle S_i\rangle$ is connected implies that $k \leq 6$ and for each $i = 1,2,3$ there is at most one j such that $u_j \in S_i$ and $v_j \in T_1$. Consider three cases.

(i) $k = 2$. Then a contradiction is shown in a similar way as in Case 1.

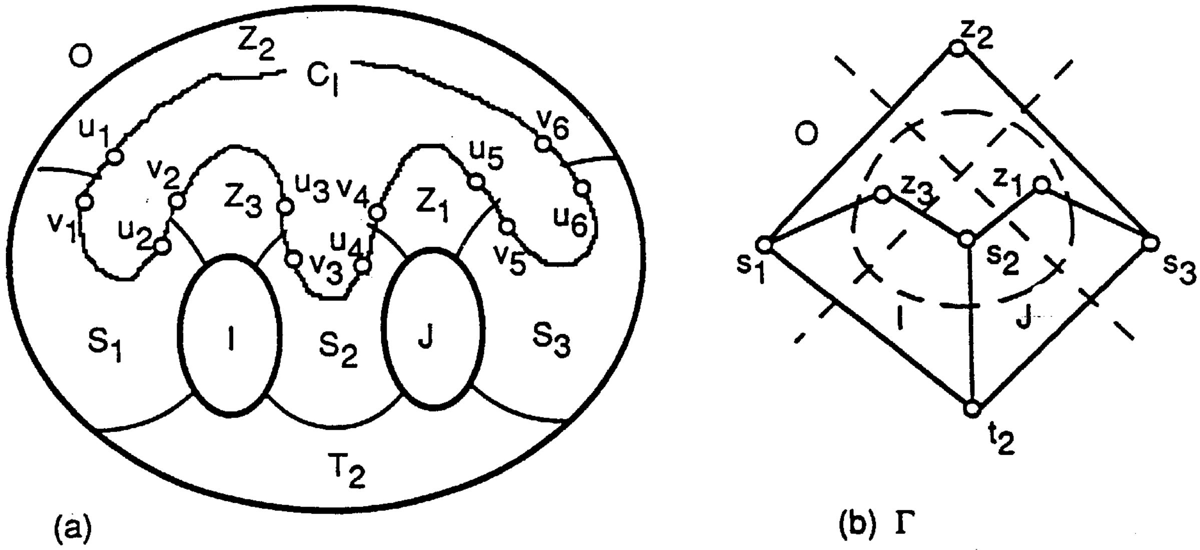


Fig. 2.1

(ii) $k = 6$. Let for definiteness $v_1, u_2 \in S_1$, $v_3, u_4 \in S_2$ and $v_5, u_6 \in S_3$; see Fig. 2.1a. Denote by Z_1 $(Z_2; Z_3)$ the set of vertices in the component of the space $\Omega_1 - \widetilde{F}$

that contains the part of C_I from v_4 to u_5 (respectively, from v_6 to u_1; from v_2 to u_3). Then $\{Z_1, Z_2, Z_3\}$ is a partition of T_1. Shrink S_i to a single vertex s_i, Z_j to a vertex z_j, and T_2 to a vertex t_2, obtaining the graph Γ drawn in Fig. 2.1b.

Let τ be the natural mapping of VG_I to $V\Gamma$, and let m' be the metric on VG_I induced by τ. It is easy to see that $m'(e) = m(e)$ for each $e \in EG_I$ and $m'(p,q) = m(p,q)$ for each $\{p,q\} \in U_I$. One can also check that $m' = \rho_{X(1)} + \rho_{X(2)} + \rho_{X(3)}$, where for $i = 1,2,3$, $X(i) := \tau^{-1}(\{s_i, z_{i-1}, z_{i+1}\})$ (letting $z_4 = z_1$ and $z_0 = z_3$), and $\rho = \rho_{X'}$ denotes the cut-metric on VG_I defined as $\rho(x,y) := 1$ if $|X' \cap \{x,y\}| = 1$, and $\rho(x,y) := 0$ otherwise. Then $h_I^\varepsilon(X(i)) = d_I(X(i))$, $i = 1,2,3$. Moreover, for at least one i we have $h_I^{\varepsilon'}(X(i)) < d_I(X(i))$ (for $\varepsilon' > \varepsilon$); a contradiction.

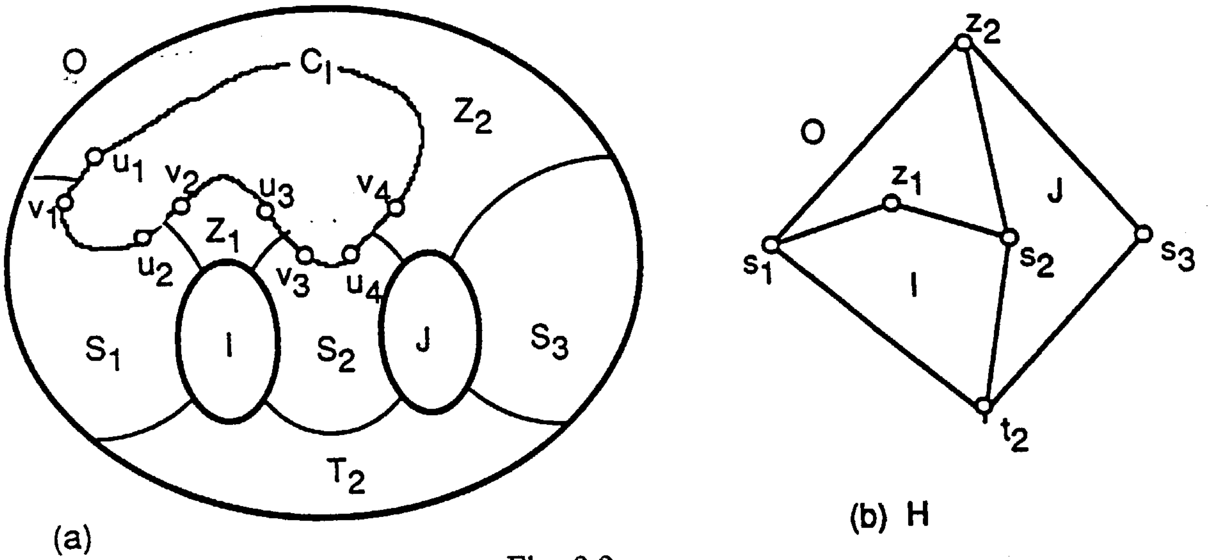


Fig. 2.2

(iii) $k = 4$. Fix a solution f' to (h_I^ε, d_I) (f' concerns G_I). Let for definiteness $v_1, u_2 \in S_1$ and $v_3, u_4 \in S_2$; see Fig. 2.2a. Let Z_1 (Z_2) be the set of vertices in the component of $\Omega_1 - \widetilde{F}$ that contains the part of C_I from v_2 to u_3 (respectively, from v_4 to u_1). Consider the mapping $\tau : VG_I \to VH$ that brings the sets $S_1, S_2, S_3, T_2, Z_1, Z_2$ to the vertices $s_1, s_2, s_3, t_2, z_1, z_2$ (respectively) of the graph H drawn in Fig. 2.2b. Let m' be the metric on VG_I induced by τ. Then $m'(e) = m(e)$ for each $e \in EG_I$ and $m'(p,q) = m(p,q)$ for each $\{p,q\} \in U_I$. This implies that $h_I^\varepsilon(m') = d_I(m')$. An easy consequence of this equality is that if f' is a solution of (h_I^ε, d_I) (f' concerns G_I) then any path $P \in \mathcal{L}(f')$ must be shortest for m'.

On the other hand, it is easy to see that the vertex z_1 does not belong to any shortest path connecting vertices in $\tau(\text{bd}(J))$ or in $\tau(\text{bd}(O))$, while s_3 does not belong to any shortest path connecting vertices in $\tau(bd(I))$. This implies that the circuits $C_{JI}(f')$ and $C_{OI}(f')$ cannot separate I and K, while $C_{IJ}(f')$ cannot separate J and O. Form a solution $\hat{f}$ for $(G,U)^*$ by combining the flows f' and f_K. From said above it follows that for $\hat{f}$ there is a bunch B' such that either $|B'| \geq 3$, or $|B'| = 2$ and $\{|\mathcal{H}_F| : F \in B\} = \{2, 2\}$. In each case B' contradicts to the choice of B in (2.5).

This completes the proof of the lemma. •

For h_F and d_F as above a cut δX in G_F is called *tight* if $h_F(X) = d_F(X)$. Throughout the rest of the paper we assume that f, B and h_F's as in Lemma 2.1 are chosen so that

$$\text{(2.7)} \qquad \sum_{F \in B} h_F(C_F) \text{ is minimum subject to (2.5).}$$

In particular, (2.7) implies that

$$\text{(2.8)} \qquad h_F(e) = \frac{1}{2}\lceil 2\overline{f}^e_F \rceil \text{ for any } e \in C_F,\ F \in B$$

Statement 2.2. *Let $F \in B$ and $|\mathcal{H}_F| \leq 2$. Then for each $e \in C_F$ with $h_F(e) > 0$, (i) e belongs to a tight cut in G_F, and (ii) $\overline{f}^e_F = h_F(e)$, where $\overline{f}_F$ is a solution to (h_F, d_F).*

Proof. (ii) follows from (i) since $h_F(X) = d_F(X)$ implies that all edges of δX are "saturated" by $\overline{f}_F$. Suppose that (i) is false for some e. Decrease h_F by 1/2 on this e, obtaining a new function h'_F on EG_F. Since h_F is half-integral, $h'_F(X) = h_F(X) - 1/2 \geq d_F(X)$ for any X such that $e \in \delta X$. Hence, (h'_F, d_F) has a solution (by Okamura's theorem), and we get a contradiction with (2.5) or (2.7). •

In the proof of Theorem 1 the functions h_F will play more important role than a multiflow f behind them; roughly speaking, these functions provide a splitting of the graph (or the all-unit capacities on its edges) into two or more pieces in order to solve then the corresponding easier problem in each piece separately. In fact, throughout the proof we are trying to show the existence of some h_F's with a "nice property" which enables us to find half-integral solutions for the corresponding pieces. The following expose a kind of such a property.

Statement 2.3. *Let some $F \in B$ be such that either $h_F(e) = 1/2$ for all $e \in C_F$ or $h_F(e) \in \{0, 1\}$ for all $e \in C_F$. Then $(G, U)^*$ has a half-integral solution.*

Proof. Consider the problems (h_F, d_F) and (c', d'), where $c'(e) := 1 - h_F(e)$ for $e \in EG_F$ and $d'(s,t) := 1 - d_F(s,t)$ for $\{s,t\} \in U$ (assuming that h_F and d_F are extended by zero to $EG - EG_F$ and $U - U_F$, respectively). Clearly both $2h_F(X) - 2d_F(X)$ and $2c'(X) - 2d'(X)$ are even for any $X \subseteq V$. Hence $(2h_F, 2d_F)$ and $(2c', 2d')$ have integral solutions, and the result follows. •

3. PROOF OF THEOREM 1. EXCLUSION OF $|B| = 2$

Similar to the proof of Theorem 2 given in [Ka2], the proof of Theorem 1 utilizes the integral and fractional variants of the so-called "splitting-off method", but now in a more complicated context. We first discuss how such a method works in our case.

Without loss of generality we assume that: G is connected; all $s_1, \dots, s_r, t_1, \dots, t_r$ are distinct and of valency 1 (since one can add to G new vertices s'_i, t'_i and edges $\{s'_i, s_i\}, \{t'_i, t_i\}$ and consider the pairs $\{s'_i, t'_i\}$ instead of $\{s_i, t_i\}$'s). Let $T := \{s_1, \dots, s_r, t_1, \dots, t_r\}$. Also one may assume that each *inner* vertex x (i.e., $x \in VG - T$) is of valency 2 or 4 (otherwise one can repeatedly transform G at x as shown in Fig. 3.1; this does not change, in essense, our problem).

Fig. 3.1

We assume that Theorem 1 is false and consider (G, U) to be a counterexample to it with $|VG|$ minimum (under the above properties). Then G has neither loops nor inner vertices of valency 2.

For $x \in VG$ let $E(x)$ denote the set of edges of G incident to x and ordered clockwise in the plane. Consider $x \in VG - T$ and two consecutive edges $e = xy$ and $e' = xz$ in $E(x)$. The triple $\tau = (e, x, e')$ is called a *fork*. Denote by G_τ the graph obtained from G by adding a new edge (or a loop) e_τ connecting y and z. Define the function ω_τ on EG_τ by

$$\begin{aligned} \omega_\tau(u) &:= 1 \quad \text{for } u = e, e', \\ &:= -1 \quad \text{for } u = e_\tau, \\ &:= 0 \quad \text{otherwise.} \end{aligned}$$

For $0 \le \varepsilon \le 1$, let $c_{\tau,\varepsilon}$ denote the function on EG_τ taking the value $1 - \varepsilon$ on e and e', ε on e_τ, and 1 on the edges in $EG - \{e, e'\}$. We say that ε is *feasible* if $(c_{\tau,\varepsilon}, d)$ has a solution; e.g., $\varepsilon = 0$ is feasible. The maximum feasible $\varepsilon \le 1$ is denoted by $\alpha(\tau)$.

Suppose that there is a fork $\tau = (e, x, e')$ with $\alpha(\tau) = 1$. Then one can split off e, e' at x preserving solvability of the problem. More precisely, let G' arise from G by deleting e,e' and adding e_τ. Since $|EG'| = |EG| - 1$ and $(G', U)^*$ is solvable, it has a half-integral solution; this is easily transformed into a half-integral solution to $(G, U)^*$.

Thus, $\alpha(\tau) < 1$ for all forks τ in G. Consider a fork $\tau = (e, x, e')$; let $e = xy$ and $e' = xz$. Since $(c_{\tau,\varepsilon}, d)$ has no solution for $\alpha(\tau) < \varepsilon \leq 1$, there is an $\mathcal{H}$-primitive cut-, 2,3-, or 4f-metric m on $VG_\tau = VG$ such that $c_{\tau,\varepsilon}(m) - d(m) < 0$ (by Theorem 3 and arguments in Section 1). Define $\omega_\tau(m) := m(e) + m(e') - m(e_\tau)$; then $\omega(m) \geq 0$ (since m is a metric). Clearly $c_{\tau,\varepsilon}(m) = c(m) - \varepsilon\omega_\tau(m)$, and now $c(m) \geq d(m)$ (as (c, d) is solvable) implies that $\omega_\tau(m) > 0$. Hence,

(3.1) $\alpha(\tau) = \min\{(c(m) - d(m))/\omega_\tau(m)\}$, where the minimum is taken over all $\mathcal{H}$-primitive cut-, 2,3- and 4f-metrics m for which $\omega_\tau(m) > 0$.

An $\mathcal{H}$-primitive m that achives the minimum in (3.1) is called *critical* for τ.

Statement 3.1. *$c(m) - d(m)$ and $\omega_\tau(m)$ are even for any cut-, 2,3- or 4f-metric m.*

Proof. Let C be the circuit formed by the edges e, e', e_τ. Since $\omega_\tau(m) \equiv m(C) \pmod 2$ and m is bipartite, $\omega_\tau(m)$ is even. Next, the graph $(VG, EG \cup U)$ is eulerian, therefore it is represented as the union of pairwise edge-disjoint circuits $C_1, \ldots, C_k$. Then $c(m) - d(m) \equiv \sum_{i=1}^{k} m(C_i) \pmod 2$. Since each $m(C_i)$ is even, $c(m) - d(m)$ is even. •

We know that for any $u \in EG$, $m(u) \leq 1$ if m is a cut metric, $m(u) \leq 2$ if m is a 2,3-metric, and $m(u) \leq 4$ if m is an $\mathcal{H}$-primitive 4f-metric (by Theorem 5). Hence,

$$
\begin{aligned}
(3.2) \qquad \omega_\tau(m) &\in \{0, 2\} && \text{if } m \text{ is a cut metric;} \\
&\in \{0, 2, 4\} && \text{if } m \text{ is a 2,3-metric;} \\
&\in \{0, 2, 4, 6, 8\} && \text{if } m \text{ is a 4f-metric.}
\end{aligned}
$$

Summing up (3.1),(3.2) and Statement 3.1, we observe the following.

Statement 3.2. *Let $0 < \alpha(\tau) < 1$, and let m be a metric critical for τ. Then:*

(i) *m is not a cut-metric;*

(ii) *if m is a 2,3-metric then $\alpha(\tau) = 1/2$ (cf. [Ka2]);*

(iii) *if m is a 4f-metric then $\alpha(\tau) \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$, and in case $\alpha(\tau) = 3/4$ the equalities $m(e) = m(e') = 4$ and $m(y, z) = 0$ hold.* •

The case $\alpha(\tau) = 3/4$ will be of most interest for us in many stages of the proof.

Now we continue considerations begun in Section 2. Let us fix f, B and h_F ($F \in B$) satisfying (2.5),(2.7) and the properties as in Lemma 2.1. In view of (2.4) and

Statement 2.3, for any $F \in B$ the circuit C_F has at least one common edge with $C_{F'}$ for some $F' \in B - \{F\}$, and C_F has edges u, u' with $h_F(u) = 1/2$ and $h_F(u') \in \{0,1\}$. We first eliminate one simple case.

Statement 3.3. *For distinct $F, F' \in B$ there are no edges $e \in C_F$ and $e' \in C_{F'}$ such that $h_F(e) = h_{F'}(e') = 0$ and either $e = e'$ or e and e' are adjacent.*

Proof. Suppose that such e, e' exist. By (2.2), $e \in \mathrm{bd}(F)$ and $\mathrm{bd}(F')$ (as $f_F^e = f_{F'}^{e'} = 0$). Let $e = e'$. Delete e from G, forming G'; then the holes F and F' merge into one new face. Clearly f gives a solution for $(G', U)^*$. Since we get the (non-eulerian) three hole case, $(G', U)^*$ has a half-integral solution, whence $(G, U)^*$ has a half-integral solution; a contradiction.

Now let e and e' be distinct and incident to a vertex x. Clearly G can be splitted at x in such a way that the holes F and F' merge into one face of the resulting graph G', and f gives a solution for $(G', U)^*$. Now apply arguments as above. •

In the rest of this section we show that case $|B| = 2$ is impossible for the minimal counterexample in question. Cases $|B| = 4$ and $|B| = 3$ will be excluded in Sections 4 and 5, respectively, and thus Theorem 1 will follow. We use the following two key lemmas (they will be important for next sections as well).

Lemma 3.4. *Let L be a maximal nontrivial path in $C_F \cap C_{F'}$ $(F, F' \in B)$. Then either $h_F(e) = h_{F'}(e) = 1/2$ for all $e \in L$, or $h_F(e) = 0$ for all $e \in L$, or $h_{F'}(e) = 0$ for all $e \in L$.*

Lemma 3.5. *Let $F, F' \in B$, and let $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ and $P' = (v'_0, e'_1, v'_1, \ldots, e'_q, v'_q)$ be paths (possibly circuits) in C_F and $C_{F'}$, respectively, such that $v_0 = v'_0$, $e_1 = e'_1$, $e_2 \neq e'_2$ and $v_k = v'_q$. Let the region bounded by P and P' (outside D_F and $D_{F'}$) contain no hole. Then at least one of $h_F(e_1)$ and $h_{F'}(e_1)$ is not $1/2$.*

Assuming that these lemmas are valid, consider case $|B| = 2$. Let for definiteness $B = \{I, J\}$. If $C_I = C_J$ then $(G, U)^*$ has a half-integral solution by Lemma 3.4 and Statement 2.3. So assume that C_I is different from C_J, and let $\{P_1, \ldots, P_k\}$ be the set of maximal nontrivial paths in $C_I \cap C_J$. If for some $i \in \{1, \ldots, k\}$ and $e \in P_i$, $h_I(e) = h_J(e) = 1/2$ then these equalities hold for all $e \in P_i$ (by Lemma 3.4) and now lemma 3.5 leads to a contradiction. Otherwise there is $N \subseteq \{1, \ldots, k\}$ such that $h_I(e) = 0$ for all $e \in P_i$, $i \in N$, and $h_J(e) = 0$ for all $e \in P_i$, $i \notin N$. Define capacities c' on EG_I and capacities c'' on $EG - (EG_I - C_I)$ by

(3.3) $\quad c'(e) := 0 \;$ if $e \in P_i$ and $i \in N$, and $c'(e) := 1 \;$ otherwise,

$$c''(e) := 0 \text{ if } e \in P_i \text{ and } i \notin N, \quad \text{and } c''(e) := 1 \quad \text{otherwise.}$$

Then $c'(e) \geq h_I(e)$ for $e \in C_I$, $c''(e) \geq h_J(e)$ for $e \in C_J$, and $c'(e) + c''(e) = 1$ for $e \in C_I \cap C_J$. Since c' is integral and $|\mathcal{H}_I| \leq 3$, the problem for c' and U_I has a half-integral solution, and similarly for c'' and U_J. Combining these we get a half-integral solution to $(G, U)^*$; a contradiction.

To prove Lemmas 3.4 and 3.5, we need some preliminary observations.

Following [Ka2], for a fork $\tau = (e, x, e')$ we introduce the number $\beta(\tau)$ which, as we shall see later, gives a lower bound for $\alpha(\tau)$ and is easier to handle than $\alpha(\tau)$:

$$(3.4) \qquad \beta(\tau) := 1 + f^{e,e'} - \frac{1}{2}f^e - \frac{1}{2}f^{e'} \quad \left(= 1 - \frac{1}{2}(f^{e,u} + f^{e,u'} + f^{e',u} + f^{e',u'})\right),$$

where $E(x) = (e, e', u, u')$, and for edges p and q, $f^{p,q}$ denotes $\sum(f(L) : L \in \mathcal{L}, p, q \in L)$. By symmetry,

$$(3.5) \qquad \beta(e, x, e') = \beta(u, x, u').$$

Statement 3.6 [Ka2]. $\beta(\tau) \leq \alpha(\tau)$.

Proof. Let for definiteness $f^e \geq f^{e'}$. Define the function c' on EG_τ as: $c'(e) := f^e - f^{e,e'}$; $c'(e') := f^{e'} - f^{e,e'}$; $c'(e_\tau) := 1 + f^{e,e'} - f^e$; and $c'(w) := c(w)$ for the other edges w. An easy transformation of f gives a solution to (c', d). Put $c'' := c_{\tau,\beta(\tau)}$ and $\varepsilon := (f^e - f^{e'})/2$. One can check that $c''(w) - c'(w)$ is equal to ε for $w = e', e_\tau$; $-\varepsilon$ for $w = e$; and 0 for the other $w \in EG_\tau$. Since $\varepsilon > 0$, $c''(m) \geq c'(m)$ for any metric m, whence the solvability for (c', d) implies that for (c'', d). •

Remark 3.7. Statements 3.2 and 3.6 imply that for a fork τ if $\beta(\tau) = 3/4$ then $\alpha(\tau) = \beta(\tau)$. Moreover, from the proof of Statement 3.6 one can see that in this case f can be transformed locally, within the edges e, e', e_τ, to give a solution f' to $(c_{\tau,3/4}, d)$. More precisely, let $f^e \geq f^{e'}$ and $P \in \mathcal{L}$. If $e \notin P$, put $f'(P) := f(P)$. If $e, e' \in P$ then P is transformed into P' with $f'(P') := f(P)$ by replacing $\{e, e'\}$ by e_τ. If $e \in P \not\ni e'$, create the path P' from P by replacing e by $\{e', e_\tau\}$; put $f'(P) := f(P)(\frac{1}{2}f^e + \frac{1}{2}f^{e'} - f^{e,e'})/(f^e - f^{e,e'})$ and $f'(P') := f(P)(\frac{1}{2}f^e - \frac{1}{2}f^{e'})/(f^e - f^{e,e'})$. One can check that f' is $(c_{\tau,3/4}, d)$-admissible. By Statement 3.2, there is a primitive 4f-metric m critical for τ; then $c_{\tau,3/4}(m) = d(m)$. These observations yield two useful properties:

(3.6) each edge $w \in EG_\tau$ with $m(w) > 0$ is saturated by f' (i.e., $(f')^w = c_{\tau,3/4}(w)$) and every path $P \in \mathcal{L}(f')$ is shortest for m;

(3.7) if $f^e > f^{e'}$ then for $F \in \mathcal{H}$ with $f_F^e > f_F^{e'}$, each of $(f')_F^e$, $(f')_F^{e'}$, $(f')_F^{e_\tau}$ is nonzero; if, in addition, $f_F^{e,e'} = 0$ then every path in $\mathcal{L}_F(f')$ passing e_τ contains e'.

The following statement appeals to (1.7), evident topological observations and the fact that all paths in $\mathcal{L}(f')$ are shortest for m'; we leave its proof to the reader.

Statement 3.8. *Let f' be a (non-crossing) solution for some G', c', d', and let B be a bunch for f'. Let $c'(m') = d'(m')$ for some consistent 4f-metric m' on VG' induced by $\sigma : VG' \to VH$. Next, let C_F be a circuit in $\mathcal{C}(B)$, and let C be its image (by σ extended as in (1.7)) in H. Then C is a simple circuit, and C_F separates holes $F', F'' \in \mathcal{H}'$ in G' if and only if C separates the faces $\sigma(F'), \sigma(F'')$ in H.* •

This statement together with (1.7)(i) and (ii)-(iii) in Theorem 5 implies that

(3.8) for G', c', d', f', B, m' as in Statement 3.8, if m' is $\mathcal{H}'$-primitive, $m'(e) = 4$ for some $e = xy \in EG'$ and e lies in the region $D' = D_F(f')$ for $F \in B$, then D' contains no hole except F.

Indeed, let C be the image by σ of the boundary of D'. If D' contains a hole $F' \neq F$ then, by Statement 3.8, the circuit C does not follow the boundary of the face $\widetilde{F} := \sigma(F)$ in H. This means that there is an $s-t$ path $P \in \mathcal{L}_F(f')$ such that its image $Q := \sigma(P)$ does not lie in *bd*$(\widetilde{F})$. Since P is shortest for m' (as $c'(m') = d'(m')$), Q is a shortest path in H. Hence, some of the ends of Q, $\sigma(s)$ say, is b_i for $i \in \{1,2,3\}$, by (iii) in Theorem 5 (here $L_e = b_0 \dots b_5$ is the image of (x, e, y) as in (ii) of this theorem). Remove from $\mathbb{R}^2$ the set $F \cup e \cup X \cup Y$, where $X := \sigma^{-1}(x)$ and $Y := \sigma^{-1}(y)$. In the resulting space consider the component Ω containing s. Obviously σ brings Ω into L_e. This implies that P meets X or Y. Then the part of P outside Ω is a path P' such that $Q' := \sigma(P')$ has both ends in $\mathrm{bd}(\widetilde{F}) - \{b_1, b_2, b_3\}$. Furthermore, Q' is shortest and it does not lie in $\mathrm{bd}(\widetilde{F})$; a contradiction with (iii) in Theorem 5.

For a vertex x in C_F ($F \in B$) let $E_F(x)$ denote the set of edges incident to x and contained in $D_F - C_F$; then $|E_F(x)| \leq 2$.

Proof of Lemma 3.4. Let for definiteness $F = I$ and $F' = J$, and let each of h_I and h_J be not identically zero on L. One must prove that $h_I(e) = h_J(e) = 1/2$ for all $e \in L$. Suppose this is not so. Then for some of I, J, for I say, there are consecutive elements e, x, e' in L (where $x \in VG$ and $e, e' \in E(x)$) such that $h_I(e) \neq 0 = h_I(e')$. By Statement 3.3, $h_J(e) \neq 0 \neq h_J(e')$, hence $h_I(e) = h_J(e) = 1/2$ and $h_J(e') \in \{\frac{1}{2}, 1\}$. Since $h_I(e) \neq h_I(e')$, $E_I(x)$ is nonempty (in view of (2.8)). Consider two possible cases.

Case 1. $|E_I(x)| = 1$. Let for definiteness $E(x) = (e, u, e', u')$ and $E_I(x) = \{u\}$; see Fig. 3.2. Clearly $f^{u,e'} = f^{u,u'} = 0$. Also $f^{e,e'} + f^{e,u'} \le 1/2$ (as any path in $\mathcal{L}(f)$ passing e and some of e', u' concerns the flow $\overline{f}_J$, and the total amount of flow on these paths is at most $h_J(e) = 1/2$). Hence, for the fork $\tau = (e, x, u)$ we have $\beta(\tau) \ge 3/4$ (cf. (3.4)), whence $\alpha(\tau) = \beta(\tau) = 3/4$.

Consider the solution f' for $G_\tau, c_{\tau,3/4}, U$ obtained from f as in Remark 3.7 (for $\tau = (e, x, u)$), and a 4f-metric m critical for τ and induced by $\sigma : VG \to VH$. By Statement 3.2(iii), $m(e) = m(u) = 4$ and $m(y, z) = 0$, where y (z) is the end of e (u) different of x. Let for definiteness $\sigma(x) = b_4$, $\sigma(y) = \sigma(z) = b_0$ (cf. (ii) in Theorem 5). By (3.7) (for e, u), $(f')^u_{\widetilde{J}} > 0$, therefore u belongs to a path P in $\mathcal{L}_J(f')$. By (3.6), P is shortest for m. So b_0 and b_4 belong to a shortest $\sigma(J)$-path in H, whence $\sigma(J)$ is $\widetilde{J}$ as in Theorem 5. On the other hand, $e' \in \mathrm{bd}(I)$ (as $h_I(e') = 0$), whence $u = xz \in \mathrm{bd}(I)$ (as u is in D_I and u, e' are consecutive in $E(x)$). This implies that $b_0 = \sigma(z)$ and $b_4 = \sigma(x)$ belong to the boundary of $\sigma(I)$ in H. A contradiction with (iv) in Theorem 5.

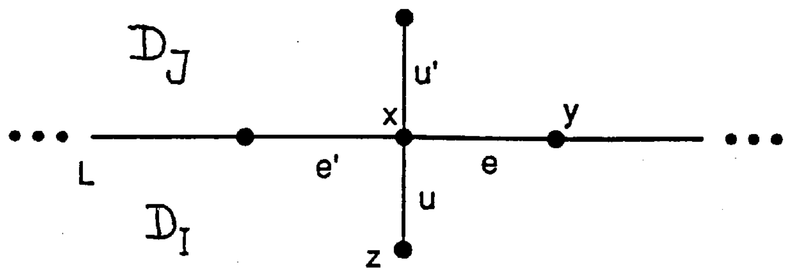


Fig. 3.2

Case 2. $|E_I(x)| = 2$. Let for definiteness $E(x) = (e, u, u', e')$; then $E_I(x) = \{u, u'\}$; see Fig. 3.3a. Since $E_J(x)$ is empty, $h_J(e) = h_J(e') = 1/2$ (in view of (2.8)). Obviously $f^{e',u} = f^{e',u'} = 0$ and $f^{e,u} + f^{e,u'} \le h_I(e) = 1/2$, whence $\beta(\tau) = 3/4$ for $\tau = (e, x, e')$.

Consider the solution f' for $G_\tau, c_{\tau,3/4}, U$ as in Remark 3.7, and a 4f-metric m critical for τ and induced by $\sigma : VG \to VH$. By Statement 3.2(iii), $m(e) = m(e') = 4$ and $m(y, z) = 0$, where y (z) is the end of e (e') different of x. Let for definiteness $\sigma(x) = b_4$ and $\sigma(y) = \sigma(z) = b_0$. By (3.7), $(f')^e_{\widetilde{I}} = (f')^{e'}_{\widetilde{I}} > 0$ (taking into account that $h_I(e) = 1/2$, $h_I(e') = 0$), whence $\sigma(I)$ coincides with $\widetilde{J}$ as in Theorem 5.

Next, let $D' := D_{IJ}(f')$. Clearly the boundary C' of D' is obtained from C_I by replacing e, e' by e_τ. Then e lies in D', whence D' contains no hole except I (by (3.8)). Consider the regions $X := \sigma^{-1}(b_4)$ and $Y := \sigma^{-1}(b_0)$ (assuming that σ is extended as in (1.7)); then x is in X and y, z, e_τ are in Y. Since I is the only hole in D' and b_4 belongs to the boundaries of at least two faces of H, X meets C'. Moreover, some $v \in \{y, z\}$ belongs to a component Ω of $D_I - X$ that does not contain I; see Fig. 3.3(b). Let for definiteness $v = y$, and let $Q = z_0 \dots z_r$ be the part of C_I such that $z_0 \in Y$,

$z_r \in X$, Q does not contain e, and all edges and inner vertices of Q are in $\Omega - Y$. From Statement 3.8 it follows that Q is mapped by σ to $L_e = b_0 \dots b_4$. Then $\sigma(z_1) = b_i$ for some $i \geq 1$.

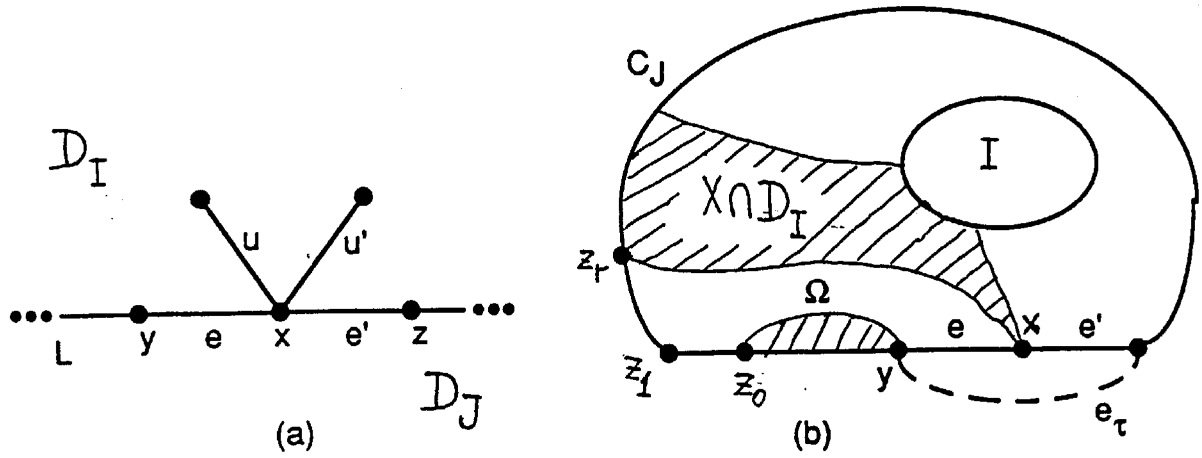


Fig. 3.3

Obviously the edge $w = z_0 z_1$ is not in bd(I). Hence, w belongs to a path $P = y_0 \dots y_q \in \mathcal{L}_I(f')$; let for definiteness $z_1 = y_j$ and $z_0 = y_{j+1}$. We observe that the part P' of P from y_{j+1} to y_q does not meet X (otherwise $\sigma(P)$ would pass b_i, b_0, b_4 in this order, contrary to the fact that P is shortest for m). Hence, P' must pass through e_τ. Then P' contains e (by (3.7)), and therefore, P' contains $x \in X$; a contradiction. •

Proof of Lemma 3.5. Put $e := e_1$, $x := v_1$, $e' := e_2$, $u' := e'_2$. Let for definiteness $F = I$ and $F' = J$. Suppose that $h_I(e) = h_J(e) = 1/2$. Since e, e', u' are distinct, $|E_I(x)| + |E_J(x)| \leq 1$. Therefore, one may assume that $E_I(x) = \emptyset$; let $E(x) = \{e, e', u', u\}$ (in case $E(x) = \{e, e'u, u'\}$ arguments are similar). We observe that $h_I(e') = h_I(e) = 1/2$, that $f^{e',u} + f^{e',u'} = 0$ (taking into account that $f^{e'} = \overline{f}_I^{e'}$ and $f^{u'} = \overline{f}_J^{u'}$ since there is no hole between P and P'), and that $f^{e,u} + f^{e,u'} \leq h_J(e) = 1/2$. Hence, $\beta(\tau) = 3/4$ for $\tau = (e, x, e')$. Consider a solution f' for $G_\tau, c_{\tau,3/4}, U$ as in Remark 3.7, and a 4f-metric m critical for τ and induced by $\sigma : VG \to VH$. Let for definiteness $\sigma(x) = b_4$ and $\sigma(y) = \sigma(z) = b_0$, where y (z) is the end of e (e') different of x, see Fig. 3.4.

By (3.7), $(f')^e_J, (f')^{e'}_J, (f')^{e_\tau}_J > 0$, whence the corresponding circuit C'_J for f' is formed from C_J by replacing e by e', e_τ. Also C'_I is formed from C_I by replacing e, e' by e_τ. Hence, $\sigma(J)$ is $\widetilde{J}$ as in Theorem 5. Clearly e lies in $D' := D_{JI}(f')$, whence J is the only hole in D', by (3.8). In addition, b_0 belongs to a shortest $\widetilde{I}$-path, where $\widetilde{I} := \sigma(I)$ (since $y \in C'_I$).

Let $\widetilde{I} := \sigma(I)$. Since y is in C'_I, b_0 belongs to a shortest $\widetilde{I}$-path in H. Furthermore, the facts that $x \in X := \sigma^{-1}(b_4)$ and there is no hole between P and P' imply that X meets the part $\widetilde{P}$ of P from v_2 to v_k. Thus, there is a vertex x' in C'_I such that $\sigma(x') = b_4$, whence b_4 belongs to a shortest $\widetilde{I}$-path in H.

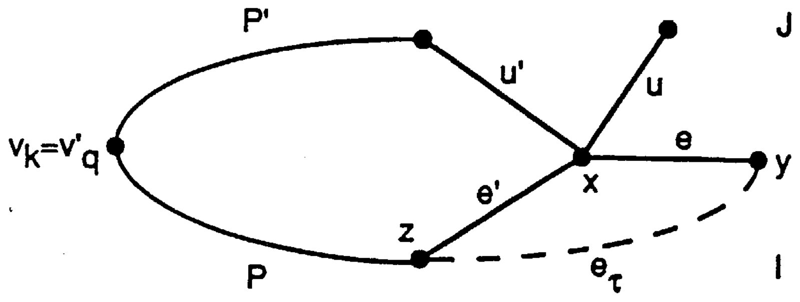


Fig. 3.4

By (iv) in Theorem 5, some $b \in \{b_0, b_4\}$ is not in $\mathrm{bd}(\widetilde{I})$. Then $v \in \{x', y\}$ such that $\sigma(v) = b$ is contained in a path $Q \in \mathcal{L}_I(f')$. By (v)(a) in Theorem 5, Q must separate J from K and O, where $\mathcal{H} = \{I, J, K, O\}$ (taking into account that $\sigma(Q)$ is a shortest $\widetilde{I}$-path in H passing b). This means that C'_I (as well as C_I) separates J from I, K, O. Hence, $|\mathcal{H}_I| = 3$. On the other hand, by (v)(b) in Theorem 5, no I-path in $\mathcal{L}(f')$ (as well as in $\mathcal{L}(f)$) separates K and O. Thus, there is a bunch B' for f such that either $|B'| = 3$, or $|B'| = 2$ and $\{|\mathcal{H}_F| : F \in B'\} = \{2, 2\}$. A contradiction with the choice of B in (2.5). •

4. EXCLUSION OF $|B| = 4$

In this section our goal is to show that case $|B| = 4$ is impossible for the minimal counterexample in question. In fact, we show that the functions h_F can be transformed to some h'_F's in such a way that at least one h'_F is integral, whence $(G, U)^*$ has a half-integral solution, by Statement 2.3. Our arguments will rely on Lemmas 4.1-4.4 below (they will be also used in the next section where we study case $|B| = 3$). These lemmas will be proved in the end of this section.

Let $F \in B$. A maximal nontrivial path P in C_F with $h_F(e) = 1/2$ for all $e \in P$ is called a 1/2-*segment* for F. By Statement 2.3, F has at least one 1/2-segment, and this segment is not C_F. Next, let δX be a cut in G_F. Obviously, if δX is tight (i.e., $h_F(X) = d_F(X)$) then δX is the union of simple tight cuts (see Section 2 for definitions). In what follows, speaking of a cut, we usually mean a simple cut of the graph in question. Lemma 4.1 strengthens (i) in Statement 2.2 for case $|\mathcal{H}_F| = 1$, and Lemma 4.2 exhibit a relation between tight cuts and 1/2-segments.

Lemma 4.1. *Let $F \in B$ and $|\mathcal{H}_F| = 1$. Then each edge $e \in C_F$ belongs to a tight cut in G_F.*

Lemma 4.2. *Let $F \in B$ and $|\mathcal{H}_F| = 1$. Then:*

(i) *for any tight cut δX in G_F and any 1/2-segment S for F, $|\delta X \cap S| \leq 1$;*

(ii) *the number ω_F of 1/2-segments for F is even;*

(iii) *if $S_0, S_1, \ldots, S_{2k-1}$ are the 1/2-segments for F occurring in this order in C_F then every tight cut meeting some S_i meets the opposite 1/2-segment S_{i+k} (taking indices modulo $2k$).*

A face in G that is not a hole is called *intermediate*. We say that two elements $x, y \in VG \cup EG$ are *dually connected* if they belong to the boundary of the same intermediate face in G.

Lemma 4.3. *For distinct $F, F', F'' \in B$ let $P = x_1 \ldots x_k$, $P' = y_1 \ldots y_r$, $P'' = z_1 \ldots z_q$ be 1-paths in $C_F, C_{F'}$ and $C_{F''}$, respectively, such that $x_1 = y_r$, $y_1 = z_q$, $z_1 = x_k$, and $x_2 \neq y_{r-1}$. Let C_F and $C_{F'}$ have a common edge e with an end at x_1 for which $h_F(e) = h_{F'}(e) = 1/2$. Let the region bounded by P, P', P'' contain no hole. Then for some edge $u = z_{i-1}z_i$ $(1 < i \leq q)$ one holds:*

(i) *$h_{F''}(u) = 1$;*

(ii) *u is dually connected with x_1.*

Lemma 4.4. *Let e and u be two consecutive edges in C_F such that $e \in C_{F'}$ and $u \in C_{F''}$ for distinct $F', F'' \in B - \{F\}$. Let e, u be incident to a vertex x. Let e' (u') be the edge in $C_{F'}$ $(C_{F''})$ incident to x and different of e (u). Then:*

(i) *$h_F(e) = h_{F'}(e) = h_F(u) = h_{F''}(u) = 1/2$;*

(ii) *$e' = u'$ unless $|B| = 4$ and x is in $C_{\widetilde{F}}$, where $B = \{F, F', F'', \widetilde{F}\}$.*

Assuming that the above lemmas are valid, we now begin to study case $|B| = 4$. Clearly $|\mathcal{H}_F| = 1$ for each $F \in B$. We need some additional terminilogy and notations. Consider some $F \in B$. We say that an edge $e \in C_F$ is a *1-edge* if $e \notin C_{F'}$ for any $F' \in B - \{F\}$, and a *2-edge* otherwise. A maximal nontrivial path in C_F of which all edges are 1-edges (respectively, 2-edges common for C_F and $C_{F'}$ for some fixed

$F' \in B - \{F\}$) is called a 1-*path* (respectively, a 2-*path*).

We classify 2-paths P as follows. We say that $P \subseteq C_F \cap C_{F'}$ is *strong* if for some (or, in view of Lemma 3.4, for any) edge $e \in P$ one has $h_F(e) = h_{F'}(e) = 1/2$; and P is *weak* otherwise. By Lemma 3.4 and Statement 3.3, if P is weak then either $h_F(e) = 0$ and $h_{F'}(e) > 0$ for all $e \in P$, or $h_F(e) > 0$ and $h_{F'}(e) = 0$ for all $e \in P$. Clearly a strong path P is contained in some 1/2-segment S (but P and S may not coincide). A strong path P in C_F is called *reducible* for F if it belongs to a 1/2-segment S for F such that the opposite segment (see Lemma 4.2) contains no strong path. Otherwise P is called *non-reducible* (for F). Thus if a 1/2-segment contains a non-reducible path then the opposite segment does so as well. Define the function $\overline{h}_F$ on EG_F by

$$(4.1) \qquad \begin{aligned} \overline{h}_F(e) &:= 0 \quad \text{if } e \text{ belongs to a reducible path for } F, \\ &\qquad\;\; \text{or } e \text{ belongs to a weak path and } h_F(e) = 0, \\ &:= 1/2 \quad \text{if } e \text{ belongs to a non-reducible path for } F, \\ &:= 1 \quad \text{otherwise.} \end{aligned}$$

Note that if an edge e belongs to a reducible path in C_F, and δX is a tight cut in G_F containing e then for the other edge e' in $\delta X \cap C_F$ we have $h_F(e') = 1/2$, and e' belongs to either a 1-path or a weak path in C_F (in the latter case, $h_F(e') + h_{F'}(e') = 1/2$, where $F' \in B - \{F\}$ is such that $e' \in C_{F'}$). This implies that for each $F \in B$ the problem $(\overline{h}_F, d_F)$ is solvable, and the collection of $\overline{h}_F$'s is *admissible* (i.e. $\overline{h}_F(e) + \overline{h}_{F'}(e) \le 1$ for distinct $F, F' \in B$ and $e \in C_F \cap C_{F'}$). If for some $F \in B$ every strong path in C_F is reducible then $\overline{h}_F$ is integral, whence $(G, U)^*$ has a half-integral solution. Thus, each C_F contains a non-reducible path. Moreover,

(4.2) each $F \in B$ has two non-reducible paths contained in opposite 1/2-segments.

Denote by Q the graph that is the union of the circuits C_F ($F \in B$) and denote by Q' the graph that is obtained from Q by shrinking each 1-edge; let μ be the natural mapping of Q to Q'. Let $\mathcal{R}(F)$ denote the set of all maximal nontrivial paths $P = v_0 v_1 \ldots v_k$ in C_F such that: (i) $v_0 v_1$ and $v_{k-1} v_k$ are 2-edges, (ii) there is $F' \in B - \{F\}$ such that each 2-edge in P belongs to $C_{F'}$, and (iii) each 1-edge $e \in P$ (if any) belongs to a simple circuit C in $P \cup C_{F'}$ such that one component in $\mathbb{R}^2 - C$ contains no hole. In view of Lemma 3.5,

(4.3) every strong path in C_F is a member of $\mathcal{R}(F)$, and for each $P \in \mathcal{R}(F)$ either P is a strong path or every 2-path in P is weak.

The fact that $|\mathcal{R}(F)| \ge 2$ for any $F \in B$ (by (4.2) and (4.3)) easily implies that Q' is 2-connected, whence Q' is homeomorphic to some of the graphs Q'_1, Q'_2, Q'_3, Q'_4

drawn in Fig. 4.1. Let us call a vertex of degree at least three *essential*. One can see that $|\mathcal{R}(F)|$ is equal to the number of essential vertices in $\mu(C_F)$. Let Z denote the set of essential vertices in Q', and Z^0 denote the set of $x \in Z$ for which $\mu^{-1}(x)$ consists of a unique vertex in Q. For $F \in B$ we keep the notation F for the corresponding faces in Q and Q'.

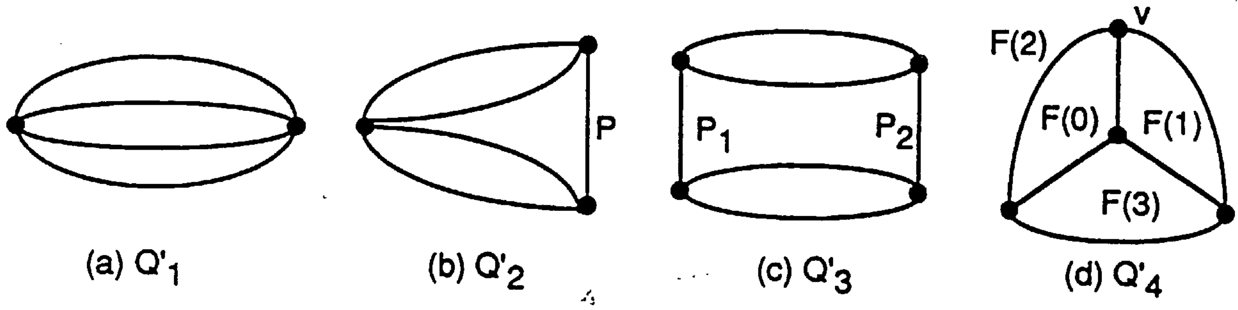


Fig. 4.1

Now we describe one more sort of transformations of functions h_F. Namely, for a sequence $\pi = (P_1, \dots, P_k)$ of paths in C_F and a sequence $\rho = (*_1, \dots, *_k)$ of signs $+$ or $-$, define the function h' on EG_F by

$$(4.4) \qquad \begin{aligned} h'(e) &:= 1 \quad \text{if } e \in P_i \text{ and } *_i = +, \\ &:= 0 \quad \text{if } e \in P_i \text{ and } *_i = -, \\ &:= \overline{h}_F(e) \quad \text{otherwise,} \end{aligned}$$

where $\overline{h}_F$ is defined in (4.1). h' as in (4.4) is called the (π, ρ)-*transformation* of h_F.

Statement 4.5. *For some $F \in B$ there exist two non-reducible paths contained in the same 1/2-segment.*

Proof. Suppose that this is not so. Consider the set Ξ of maximal sequences $\xi = (L_0, F_1, L_1, \dots, F_r, L_r)$ such that for $i = 1, \dots, r$: (i) $F_i \in B$, (ii) L_{i-1} and L_i are non-reducible paths for F_i which are contained in opposite 1/2-segments for F_i, and (iii) $L_0, \dots, L_{r-1}$ are different (assuming that none of the members of Ξ is obtained from another one by reversing and/or shifting cyclically (when $L_0 = L_k$)). Since for each $F \in B$ no two non-reducible paths are contained in the same 1/2-segment, Ξ is well-defined and each non-reducible path belongs to a unique member of Ξ.

Next, for each $F \in B$ fix a sequence $\pi_F = (P_1, \dots, P_{k(F)})$ of all non-reducible paths in C_F. Define $\rho_F = (*_1, \dots, *_{k(F)})$ as follows. For $i = 1, \dots, k(F)$ take $(L_0, F_1, L_1, \dots, F_r, L_r) \in \Xi$ such that $P_i = L_j$ for some j. Put $*_i := +$ if $F = F_{j+1}$ and put $*_i := -$ if $F = F_j$.

Finally, for each $F \in B$ let h'_F be the (π_F, ρ_F)-transformation of h_F. One can check that each problem (h'_F, d_F) is solvable, the collection of h'_F's is admissible, and each h'_F is integral. Hence $(G, U)^*$ has a half-integral solution; a contradiction. •

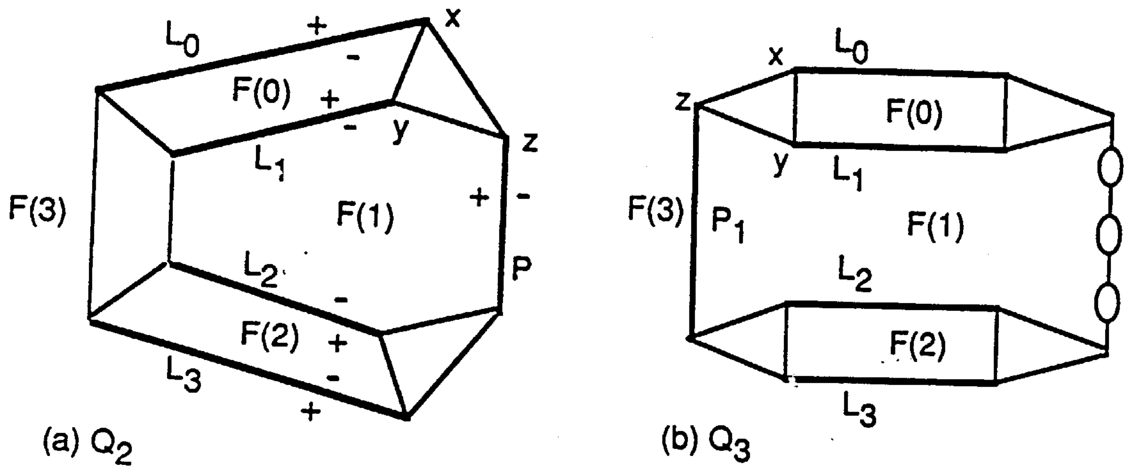


Fig. 4.2

By (4.2) and Statement 4.5, there is $F \in B$ such that $|\mathcal{R}(F)| \geq 3$, and $\mathcal{R}(F)$ contains three strong paths P_1, P_2, P_3 such that P_1 and P_2 belong to the same 1/2-segment for F that is opposite to the 1/2-segment containing P_3. In particular, this shows that Q' is not homeomorphic to Q'_1. Also if $|\mathcal{R}(F)| = 2$ then each of the two essential vertices in Q' belonging to bd(F) cannot be in Z^0 (otherwise the non-reducible paths for F would be contained in the same 1/2-segment, contrary to (4.2)). Hence, if Q' is homeomorphic to Q'_2 or Q'_3 then $Z^0 = \emptyset$, and therefore Q is "of type" Q_2 or Q_3 as drawn in Fig. 4.2. Now we consider possible types for Q (or Q'), using notations as in Fig. 4.2-4.3 (here $B = \{F(i) \ : \ i = 1, \ldots, 4\}$).

A. Q is of type Q_2. For definiteness let the paths L_1, L_2, P in $\mathcal{R}(F(1))$ be strong, and let S_1, S_2, S be the 1/2-segments for $F(1)$ containing them, respectively. We know that two of these segments are the same and opposite to the third one. We observe that $S \neq S_1, S_2$. Indeed, by Lemma 4.3 applied to the 1-paths connecting x, y, z as in Fig. 4.2a, the 1-path connecting the vertices y and z contains an edge u with $h_{F(1)}(u) = 1$, whence $S \neq S_1$; and similarly, $S \neq S_2$. Thus $S_1 = S_2$. A similar property holds for $F(3)$ if two of paths L_0, L_3, P belong to the same 1/2-segment for $F(3)$. For $i = 0, 1, 2, 3$ let h_i be the (π_i, ρ_i)-transformation of $h_{F(i)}$, where

$$\pi_0 = (L_0, L_1) \text{ and } \rho_0 = (-, +),$$
$$\pi_1 = (L_1, L_2, P) \text{ and } \rho_1 = (-, -, +),$$
$$\pi_2 = (L_2, L_3) \text{ and } \rho_2 = (+, -),$$
$$\pi_3 = (L_3, L_0) \text{ and } \rho_3 = (+, +, -),$$

see Fig. 4.2a. One can check that each $(h_i, d_{F(i)})$ is solvable, the collection $\{h_1, \ldots, h_4\}$ is admissible, and each h_i is integral. Hence, $(G,U)^*$ has a half-integral solution.

B. Q is of type Q_3. Without loss of generality one may assume that P_1 is a non-reducible path for $F(1)$, and that P_1 and L_1 belong to the same 1/2-segment for $F(1)$. On the other hand, by Lemma 4.3 (applied to the 1-paths connecting the vertices x, y, z as in Fig. 4.2b) the 1-path connecting y and z must contain an edge u with $h_{F(1)}(u) = 1$. Hence, P_1 and L_1 belong to different 1/2-segments; a contradiction.

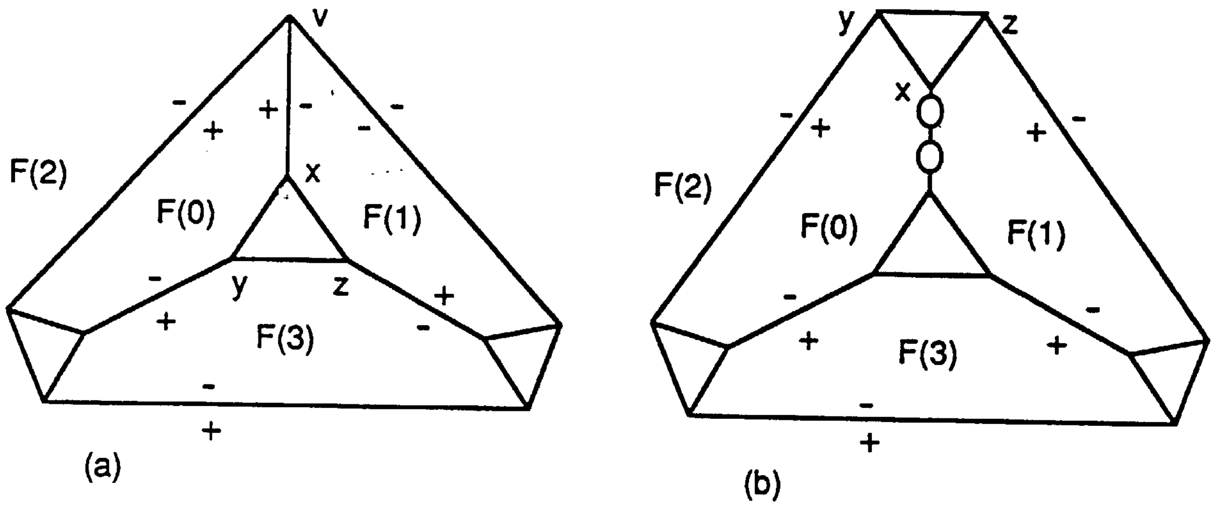


Fig. 4.3

C. Q' is of type Q'_4. Then $|\mathcal{R}(F)| = 3$ for all $F \in B$. Let $P_{ij} = P_{ji}$ denote the maximal path in Q'_4 common for $\mathrm{bd}(F(i))$ and $\mathrm{bd}(F(j))$. Consider two cases.

Case 1. $Z^0 \neq \emptyset$. Let for definiteness $v \in Z^0$, where v is the vertex indicated in Fig. 4.1d. By Lemma 4.4, for $j = 0,1,2$ the paths $P_{j-1,j}$ and $P_{j,j+1}$ belong to the same 1/2-segment for $F(j)$ (indices are taken modulo 3); therefore $P_{j,3}$ must belong to the opposite segment for $F(j)$. In particular, $|Z^0| = 1$, and Q is as in Fig. 4.3a.

Next, the path P_{01} is strong, so by Lemma 4.3 (applied to the 1-paths connecting the vertices x, y, z as in Fig. 4.3a) the 1-path connecting y and z contains an edge u with $h_{F(3)}(u) = 1$. Hence, P_{30} and P_{31} belong to different 1/2-segments in $C_{F(3)}$, and similarly for P_{30}, P_{32} and for P_{31}, P_{32}. Then some $P_{3,j}$, say P_{32}, is reducible for $F(3)$. For $i = 0,1,2,3$ let h_i be the (π_i, ρ_i)-transformation of $h_{F(i)}$, where

$$\pi_0 = (P_{01}, P_{02}, P_{03}) \text{ and } \rho_0 = (+,+,-),$$
$$\pi_1 = (P_{10}, P_{12}, P_{13}) \text{ and } \rho_1 = (-,-,+),$$
$$\pi_2 = (P_{20}, P_{21}, P_{23}) \text{ and } \rho_2 = (-,-,+),$$
$$\pi_3 = (P_{30}, P_{31}, P_{32}) \text{ and } \rho_3 = (+,-,-).$$

Case 2. $Z^0 = \emptyset$. Then Q is of type as in Fig. 4.3b. Let for definiteness P_{20} and P_{21}

belong to the same 1/2-segment for $F(2)$. Then Lemma 4.3 (for the 1-paths connecting x, y, z as in Fig. 4.3b) implies that P_{01} is not a strong path. Hence, all paths $P_{3,j}$ for $j = 0, 1, 2$ are strong. Next, applying Lemma 4.3, we observe that the paths P_{30} and P_{32} belong to different 1/2-segments for $F(3)$, and similarly for P_{31} and P_{32}. A similar property is true with respect to $F(2)$. So we may assume that P_{30} and P_{31} are in the same 1/2-segment for $F(3)$. For $i = 0, 1, 2, 3$ let h_i be the (π_i, ρ_i)-transformation of $h_{F(i)}$, where

$$\pi_0 = (P_{02}, P_{03}) \text{ and } \rho_0 = (+, -),$$
$$\pi_1 = (P_{12}, P_{13}) \text{ and } \rho_1 = (+, -),$$
$$\pi_2 = (P_{20}, P_{21}, P_{23}) \text{ and } \rho_2 = (-, -, +),$$
$$\pi_3 = (P_{30}, P_{31}, P_{32}) \text{ and } \rho_3 = (+, +, -).$$

A straightforward check-up shows that in both cases each problem $(h_i, d_{F(i)})$ is solvable, the functions h_i are integral, and the collection of h_i's is admissible, whence $(G, U)^*$ has a half-integral solution.

To complete consideration of case $|B| = 4$, it remains to prove Lemmas 4.1-4.4.

Proof of Lemma 4.1. In view of Statement 2.2, it suffices to consider $e \in C_F$ with $h_F(e) = 0$. Then $e \in \mathrm{bd}(F)$. Suppose that the statement for e is wrong. Then

$$(4.5) \qquad h_F(X) - d_F(X) \geq \frac{1}{2} \quad \text{for any } X \subseteq VG_F \text{ such that } e \in \delta X.$$

Let x and y be the ends of e. Add to U_F one more demand pair $w = \{x, y\}$ for which we put demand $d_F(w) = 1/2$. In view of (4.5), from Okamura's theorem it follows that the problem (h_F, d') (where d' denotes the demand function on $U_F \cup \{w\}$) has a solution f'. Let L be a path with $f'(L) > 0$ connecting x and y. Since $e \in \mathrm{bd}(F) \cap C_F$, every cut δX which meets both $\mathrm{bd}(F)$ and C_F must have a common edge with L, therefore $h_F(X) > d_F(X)$. This implies that no edge in C_F belongs to a tight cut for h_F and d_F, whence, by Statement 2.2, $h_F(e') = 0$ for all $e' \in C_F$. Then $(G, U)^*$ has a half-integral solution; a contradiction. •

Proof of Lemma 4.2. Let for definiteness $F = I$. Consider a tight cut δX with $|\delta X \cap C_I| = 2$; let $\{e, e'\} = \delta X \cap C_I$. This cut is naturally associated with the dual circuit (or the circuit of the dual graph) $D_X = (F_0, e_1, F_1, \ldots, e_k, F_k)$, where $\delta X = \{e_1, \ldots, e_k\}$, each e_i is a common edge in the boundaries of the faces F_{i-1} and F_i of G_I, $e_1 = e$, $e_k = e'$, and $F_0 = F_k$ is the face $\widetilde{F}$ in G_I surrounded by C_I. Since δX is tight, some F_i is I. Such a D_X has a natural partition into two dual paths:

$$D_X(e) := (F_0, e_1, \ldots, F_i) \quad \text{and} \quad D_X(e') := (F_i, e_{i+1}, \ldots, F_k).$$

Next, since $d_I(X) \in \mathbb{Z}$ and $h_I(e_j) \in \mathbb{Z}$ for $j = 2, \ldots, k-1$, we have

$$\text{(4.6)} \qquad \text{either } h_I(e), h_I(e') = \frac{1}{2} \quad \text{or } h_I(e), h_I(e') \in \{0,1\}.$$

First of all we prove two claims.

Claim 1. *Let $\delta X, \delta Y$ be two tight cuts such that $\delta X \cap C_I = \{u, u'\}$, $\delta Y \cap C_I = \{z, z'\}$, $h_I(u), h_I(u') = \frac{1}{2}$ and $h_I(z), h_I(z') \in \{0,1\}$. Then D_X and D_Y have no common faces except I and $\widetilde{F}$.*

Proof. Consider the dual paths $D_X(u), D_X(u')$ in $D_X = (F_0, e_1, F_1, \ldots, e_k, F_k)$ and the dual paths $D_Y(z), D_Y(z')$ in $D_Y = (F'_0, e'_1, F'_1, \ldots, e'_{k'}, F'_{k'})$. For definiteness assume that $D_X(u) = (F_0, e_1, \ldots, e_j, F_j)$ and $D(z) = (F'_0, e'_1, \ldots, e_{j'} F'_{j'})$ have a common face $F_i = F'_{i'}$ different from I and $\widetilde{F}$. Put $E_1 := \{e_1, \ldots, e_i, e'_{i'+1}, \ldots, e'_{k'}\}$ and $E_2 := \{e'_1, \ldots, e'_{i'}, e_{i+1}, \ldots, e_k\}$. One can see that there are tight cuts $\delta X' \subseteq E_1$ and $\delta Y' \subseteq E_2$ such that $\delta X'$ contains $e_1 = u$ and $e'_{k'} = z'$, while $\delta Y'$ contains $e'_1 = z$ and $e_k = u'$. But $h_I(u) = 1/2$ and $h_I(z') \in \{0,1\}$; a contradiction with (4.6) (for $\delta X', u, z'$). •

Claim 2. *Let $\delta X, \delta Y, u, u', z, z'$ be as in the hypotheses of Claim 1. Then the pairs $\{u, u'\}$ and $\{z, z'\}$ are crossing in C_I (that is, up to permutation of u, u' and permutation of z, z', these edges occur in C_I in order u, z, u', z').*

Proof. Assume that these edges occur in C_I in order u, u', z, z' (clockwise from a point a in $\widetilde{F}$). Let $\overline{u}, \overline{u}', \overline{z}, \overline{z}'$ be the edges in bd(I) that belong to $D_X(u), D_X(u'), D_Y(z), D_Y(z')$, respectively. From Claim 1 it follows that the latter edges occur in bd(I) in order $\overline{u}, \overline{u}', \overline{z}, \overline{z}'$ (clockwise from a). Let $\delta X'$ ($\delta Y'$) be the cut formed by the edges in $D_X(u) \cup D_Y(z)$ (in $D_X(u') \cup D_Y(z')$). One can see that $d_I(X') + d_I(Y') \geq d_I(X) + d_I(Y)$, whence we conclude that $\delta X'$ and $\delta Y'$ are tight. A contradiction with (4.6). •

Now suppose that there are a tight cut δX and a 1/2-segment S having two common edges u, u'. Since $S \neq C_I$ (by Statement 2.3), there is an edge $z \in C_I$ with $h_I(z) \in \{0,1\}$. By Lemma 4.1, z belongs to a tight cut δY; let $\delta Y \cap C_I = \{z, z'\}$. By (4.6), $h_I(z')$ is an integer, so $z' \notin S$. This contradicts Claim 2 and proves (i).

Let us prove (ii)-(iii). From (i) and (4.7) it follows that $\omega_I \geq 2$ and that (iii) is true if $\omega_I = 2$. Let $\omega_I \geq 3$. Split C_I as $S_0 \cdot L_0 \cdot S_1 \cdot L_1 \cdot \ldots \cdot S_{k'} \cdot L_{k'}$ ($k' = \omega_I - 1$), where each S_i is a 1/2-segment. It is easy to see from (i) that if (ii) or (iii) is not true then there are indices (up to a cyclical shift) $0 \leq i \leq i' < j' < j \leq k'$ and tight cuts $\delta X, \delta X'$ so that δX meets S_i and S_j while $\delta X'$ meets $S_{i'}$ and $S_{j'}$. Choose an edge $z \in L_{j'}$ and a tight cut δY containing z. Clearly at least one of the pairs $\{\delta X, \delta Y\}$ and $\{\delta X', \delta Y\}$ contradicts Claim 2. • •

Proof of Lemma 4.3. Let $F = I$ and $F' = J$. One may assume that e and $e' = xx_2$ are consecutive edges in $E(x)$, where $x := x_1$; see Fig. 4.4. Then $h_I(e') = 1/2$ and $\beta(\tau) = 3/4$ for $\tau = (e, x, e')$. Consider the solution f' for $G_\tau, c_{\tau,3/4}$ obtained from f as in Remark 3.7, and a 4f-metric critical for τ and induced by $\sigma : VG \to VH$. The corresponding circuit C'_J for G_τ, f' is formed from C_J by replacing e by e', e_τ. Since $f_J^{e'} > 0$, $\sigma(J)$ is $\widetilde{J}$ as in Theorem 5. Furthermore, the region $\Omega \subset \mathbb{R}^2$ bounded by P, P', P'' contains no hole; so from Statement 3.8 it follows the closed path that is the image by σ of the circuit $P \cdot P'' \cdot P'$ separate no faces of H. This implies that there is a vertex $x' \in \{x_3, \ldots, x_k, z_2, \ldots, z_{q-1}\}$ such that $\sigma(x') = \sigma(x)$. Note that $x' = x_j$ (for some j) is impossible; otherwise we get a contradiction using arguments as in the proof of Lemma 3.5.

Hence, $x' = z_i$ for some $1 < i < q$. Choose i to be minimum subject to $\sigma(z_i) = b_0$ (letting for definiteness that $\sigma(x) = b_0$). Then for the edge $u = z_{i-1}z_i$ we have $m(u) > 0$. Now the result follows from the facts that each edge $w \in EG_\tau$ with $m(w) > 0$ must be saturated by f', that $(f')^{w'} = 0$ for any edge w' in the interior of Ω, and that the image of each of $\widetilde{P}, P', P''$ is a simple path in H, where $\widetilde{P}$ is the part of P from x_2 to x_k (the latter follows from Statement 3.8). •

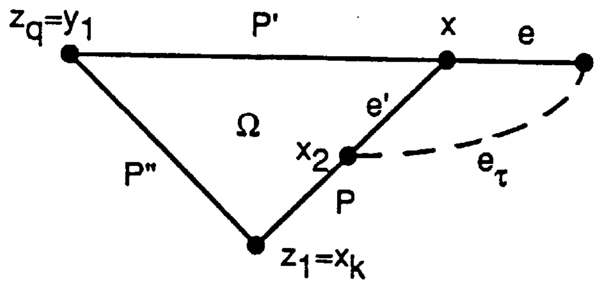


Fig. 4.4

Proof of Lemma 4.4. Suppose that (i) is false. Let for definiteness $F = I$, $F' = J$, $F'' = K$. By Statement 3.3, among $h_I(e), h_I(u), h_J(e), h_J(e'), h_K(u), h_K(u')$ there are no zero numbers $h_Q(q), h_{Q'}(q')$ with $Q \neq Q'$. In particular, if $h_J(e) = 0$ then $h_I(u) > 0$ and $h_K(u) > 0$, whence it is impossible that $h_I(e) = h_I(u) = 1$, or $h_I(e) = 0$ and $h_I(u) = 1$. Consider the other cases for $h_I(e)$ and $h_I(u)$ (omitting symmetric cases).

(a) $h_I(e) = h_I(u) = 0$. Then $e' = u'$ (otherwise we would have $f^{e,u} = f^{e,u'} = f^{e',u} = f^{e',u'} = 0$, whence $\beta(\tau) = 1$ for $\tau = (e, x, e')$). Let for definiteness $E_J(x) = \emptyset$. Since $h_J(e') \neq 0$ and $h_K(e') \neq 0$, $h_J(e') = h_K(e') = 1/2$. This implies that $h_J(e) = 1/2$ and $\beta(\tau) = 3/4$ for $\tau = (e, x, e')$. Consider the solution f' for $G_\tau, c_{\tau,3/4}, U$ obtained from f as in Remark 3.7, and a 4f-metric m critical for τ and induced by $\sigma : VG \to VH$. Since $e \in \mathrm{bd}(I)$, $\sigma(I)$ is $\widetilde{J}$ as in Theorem 5. On the other hand, by (3.7) for $F = K$, e' belongs to a path in $\mathcal{L}_K(f')$. A contradiction with (iv) in Theorem 5.

(b) $h_I(e) = 1/2$ and $h_I(u) = 0$. Then $|E_I(x)| = 1$; let $E_I(x) = \{e''\}$. It is easy to see that $\beta(\tau) = 3/4$ for $\tau = (e, x, e'')$. A contradiction is shown in a similar way as in Case 1 in the proof of Lemma 3.4.

(c) $h_I(e) = 1$ and $h_I(u) = 1/2$. Then $|E_I(x)| = 1$, whence $e' = u'$. Moreover, from the facts that $h_I(e) = 1$ and $E_J(x) = \emptyset$ it follows that $h_J(e) = h_J(e') = 0$. Hence, this case is similar to (a).

(d) $h_I(e) = h_I(u) = 1/2$ and $h_J(e) = 0$. If $E_I(x) \neq \emptyset$ then $E_J(x) = \emptyset$, $e' = u'$ and $h_J(e') = 0$; so this case is similar to (a). Let $E_I(x) = \emptyset$. From $h_J(e) = 0$ it follows that $\beta(\tau) = 3/4$ for $\tau = (e, x, u)$; further arguments are similar to those applied in case (a) (for I instead of J).

To see (ii), suppose that $e' \neq u'$, and consider the fork $\tau = (e, x, e')$. If $|B| = 3$ or if $|B| = 4$ but x is not in $C_{\widetilde{F}}$ (where $B = \{F, F', F'', \widetilde{F}\}$) then it is easy to see that $\beta(\tau) = 3/4$. Now to get a contradiction we apply arguments as in Case 1 in the proof of Lemma 3.4. •

5. EXCLUSION OF $|B| = 3$

We show that in this case either $(G, U)^*$ has a half-integral solution, or there is a reduction to case $|B| = 2$ or $|B| = 4$. The following strengthens, in a sense, Lemma 4.3.

Lemma 5.1. *Let $B = \{F(0), F(1), F(2)\}$. For $i = 0, 1, 2$ let $P_i = x_1^i x_2^i \dots x_{k(i)}^i$ be a nontrivial 1-path in $C_{F(i)}$, and let x_1^i coincide with $x_{k(i+1)}^{i+1}$. Let $C_{F(i)}$ and $C_{F(i+1)}$ have a common edge e_i with an end at x_1^i, and let $h_{F(i)}(e_i) = h_{F(i+1)}(e_i) = 1/2$ (indices are taken modulo 3). Next, let $r(i)$ and $l(i)$ be the minimum and maximum indices such that $h_{F(i)}(u_i), h_{F(i)}(u_i') \in \{0, 1\}$ for $u_i := x_{r(i)}^i x_{r(i)+1}^i$ and $u_i' := x_{l(i)}^i x_{l(i)-1}^i$. Then:*

(i) all the edges u_i, u_i' ($i = 0, 1, 2$) belong to the boundary of the same intermediate face of G in the region $\Omega \subset \mathbb{R}^2$ bounded by P_1, P_2, P_3;

(ii) $h_{F(i)}(u_i) = h_{F(i)}(u_i') = 1$.

Proof. Observe that no vertex $x \in VG$ lies in the interior of Ω (otherwise there would exist a fork $\tau = (e, x, e')$ with $f^e = f^{e'} = 0$, whence $\beta(\tau) = 1$). Hence, every edge lying in the interior of Ω connects vertices in $P_1 \cup P_2 \cup P_3$.

By Lemma 4.3, there is an intermediate face containing the vertices x_1^i and some edges in P_i for $i = 1, 2, 3$. Suppose that some $w \in \{u_i, u_i'\}$ and $w' \in \{u_{i'}, u_{i'}'\}$ ($i \neq i'$)

are not dually connected. Without loss of generality one may assume that $w = u_1$. Then in the interior of Ω there is an edge e with ends $x = x^1_j$ and $y = x^1_{j'}$ for some $1 \le j \le r(1) < j' \le k(1)$. Consider the edge e' different from xx^1_{j+1} and such that $\tau = (e, x, e')$ is a fork, see Fig. 5.1. We observe that $\beta(\tau) = 3/4$. Indeed, e' does not lie in the interior of Ω (otherwise we would have $\beta(\tau) = 1$ since $f^e = f^{e'} = 0$). The following two cases are possible.

(i) $j > 1$ and $e' = xx^1_{j-1}$. Then $f^e = 0$ and $f^{e'} = 1/2$ (as $j \le r(1)$) imply $\beta(\tau) = 3/4$.

(ii) $j = 1$ and $e' = xx^2_{k(2)-1}$. Then $E_{F(2)}(x) = \emptyset$. Therefore $f^{e'}_{F(2)} = f^{e_1}_{F(2)} = 1/2$, and we again obtain $\beta(\tau) = 3/4$.

Let $E(x) = \{e, e', u, u'\}$ and $\tau' = (u, x, u')$; then $u' = xx^1_{j+1}$ and $\beta(\tau') = 3/4$ (by (3.5)). Denote $z := x^1_{j+1}$. Consider the solution f' for $G_{\tau'}, c_{\tau', 3/4}$ obtained from f as in Remark 3.7, and a 4f-metric m critical for τ' and induced by $\sigma : VG \to VH$. Let $C' := C_{F(1)}(f')$. Note that $\sigma(y) = \sigma(x)$ (as $(f')^e = 0$), $y, z \in C'$ and $\{\sigma(x), \sigma(z)\} = \{b_0, b_4\}$. Hence, $\sigma(F(1))$ is $\widetilde{J}$ as in Theorem 5. This shows that the case (ii) as above is impossible (otherwise we would have $(f'_{F(2)})^{u'} > 0$, whence $\sigma(F(2)) = \widetilde{J}$). Hence, $j > 1$. Now the fact that $f^{e'}_{F(1)} > 0$ easily implies that $(f')^{u'}_{F(1)} > 0$, whence $u' \in C'$ and $C' = C_{F(1)}$.

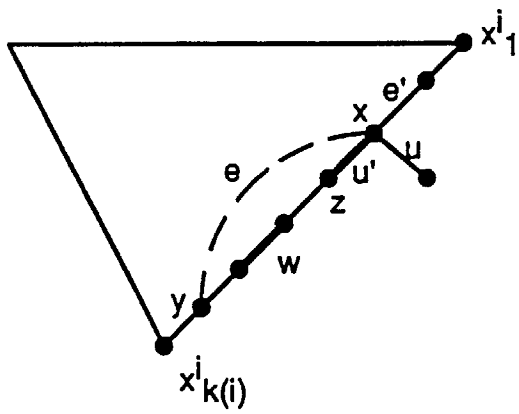


Fig. 5.1

Let for definiteness $\sigma(x) = b_0$. The vertices x, z, y occur in this order in C'. Therefore, in view of Statement 3.8, $\sigma(x') = b_0$ for all vertices x' in the part of C' between x and y that does not contain z. But then the whole circuit $\widetilde{C}$ formed from C' by replacing the path $x^1_j x^1_{j+1} \dots x^1_{j'}$ by the edge e is mapped by σ into the unique point b_0, which is impossible (since, e.g., $\widetilde{C}$ separates some holes).

Thus, w and w' are dually connected, whence (i) follows.

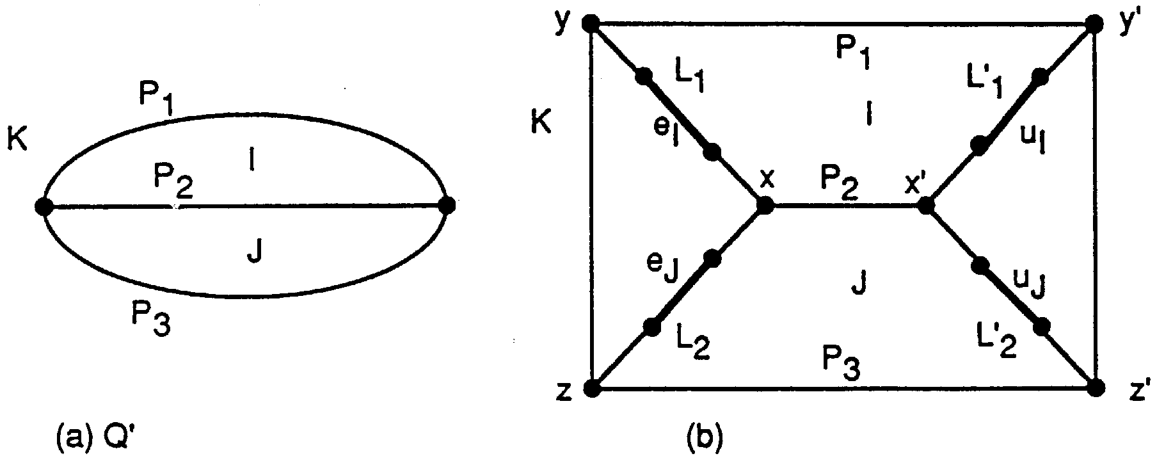
Now suppose that $h_{F(1)}(u) = 0$ for some $u \in \{u_i, u'_i\}$; let for definiteness $u = u_1$. Consider the fork $\tau = (u, x, e)$ belonging to the boundary of some face in Ω, where $x := x^1_{r(1)}$. Since $f^u = 0$, $f^e > 0$. Hence, either (i) $r(1) > 1$ and $e = x^1_{r(1)-1}x$, or

(ii) $r(1) = 1$ and $e = xx^2_{k(2)-1}$. One can see that in both cases, $f^e = 1/2$, whence $\beta(\tau) = 3/4$. In case (i), we get a contradiction using arguments as above (with τ instead of τ'). In case (ii), e belongs to both circuits $C_{F(1)}(f')$ and $C_{F(2)}(f')$ (for f' defined as in Remark 3.7); a contradiction with (iv) in Theorem 5. •

Now we begin to consider case $|B| = 3$. Let $B = \{I, J, K\}$ and $\mathcal{H}_K = \{K, O\}$. The graph Q' (defined as in Section 4) can be only as drawn in Fig. 5.2a.

By (4.2) (for $F = I, J$), the paths P_1, P_2, P_3 are strong, P_1, P_2 are non-reducible for I, while P_2, P_3 are non-reducible for J. Moreover, the graph Q is as in Fig. 5.2b. Let e_I be the first edge with $h_I(e_I) \in \{0,1\}$ contained in the 1-path L_1 from x to y in C_I, and e_J be the first edge with $h_J(e_J) \in \{0,1\}$ contained in the 1-path L_2 from x to z in C_J. Let u_I be the last edge with $h_I(u_I) \in \{0,1\}$ contained in the 1-path L'_1 from x' to y' in C_I, and u_J be the last edge with $h_J(u_J) \in \{0,1\}$ contained in the 1-path L'_2 from x' to z' in C_J, see Fig. 5.2b. By Lemma 5.1,

(5.1) $h_I(e_I) = h_I(u_I) = h_J(e_J) = h_J(u_J) = 1$; e_I and e_J are dually connected; u_I and u_J are dually connected.



(a) Q' (b)

Fig. 5.2

Statement 5.2. *e_I and u_I belong to a tight cut δX_I for G_I, h_I, U_I (and similarly, e_J and u_J belong to a tight cut δX_J for G_J, h_J, U_J).*

Proof. Let $L_1 = x_1 \dots x_k$ and $L'_1 = y_1 \dots y_r$, where $x_1 = x$ and $y_1 = x'$, and let $e_I = x_i x_{i+1}$ and $u_I = y_j y_{j+1}$. From Claim 2 in the proof of Lemma 4.2 and the fact that the 1/2-segments containing P_1 and P_2 are opposite in C_I it follows that every tight cut δX containing e_I meets L'_1 in some edge $w = y_{j'} y_{j'+1}$ with $h_I(w) \in \{0,1\}$. Similarly, every tight cut δY containing u_I meets L_1 in some edge $z = x_{i'} x_{i'+1}$ with $h_I(z) \in \{0,1\}$. Let δX (δY) be chosen so that j' is maximum (respectively, i' is minimum). Suppose that $j' < j$; then $i' > i$. Consider the dual paths

$$D = (\widetilde{F}, e_1, F_1, \dots, e_{p-1}, F_{p-1}, e_p, \widetilde{F}) \quad \text{and} \quad D' = (\widetilde{F}, e'_1, F'_1, \dots, e'_{q-1}, F'_{q-1}, e'_q, \widetilde{F}),$$

where $\{e_1, \dots, e_p\} = \delta X$, $\{e'_1, \dots, e'_q\} = \delta Y$, $e_1 = e_I$, $e_p = w$, $e'_1 = z$, $e'_q = u_I$, and $\widetilde{F}$ is the face in G_I surrounded by C_I. Let $e_s, e_{s+1}, e'_t, e'_{t+1}$ be in bd(I). Arguing as in the proof of Lemma 4.2 and taking into account the choice of i', j', we deduce that D and D' have no common face different of I and $\widetilde{F}$. Hence, $e_s, e'_t, e'_{t+1}, e_{s+1}$ occur in this order in bd(I). Then $\delta X' := \{e_1, \dots, e_s, e'_{t+1}, \dots, e'_q\}$ and $\delta Y' := \{e'_1, \dots, e'_t, e_{s+1}, \dots, e_p\}$ are tight cuts. A contradiction with the maximality of i'. •

By (5.1) and Statement 5.2, $\delta X_I \cup \delta X_J$ forms a tight cut δZ in $G_I \cup G_J$ (with all-unit capacities of the edges), that is, $|\delta Z| = d_I(Z) + d_J(Z)$. Hence, for *any* solution f' of $(G, U)^*$ the edges in δZ must be saturated by $f'_I + f'_J$. This implies that

(5.2) for any solution f' to $(G, U)^*$, I and J belong to a bunch, i.e., no circuit $C_{FF'}(f')$ with $F \neq I, J$ separates $C_{IJ}(f')$ and $C_{JI}(f')$.

Now we consider the graph G_K. Let $\mathcal{R}$ be the set of (simple) cuts in G_K that are tight for h_K, U_K and meet twice $P_1 \cup P_3$. Suppose that some of P_1 and P_3, say P_1, has the property that no cut δX in $\mathcal{R}$ meets twice P_1. Then defining the function h'_K on EG_K by

$$h'_K(e) := 0 \text{ if } e \in P_1, \qquad \text{and } h'_K(e) := 1 \text{ otherwise;}$$

and defining h'_I, h'_J on EG_I, EG_J, respectively, by

$$h'_I(e) := 0 \text{ if } e \in P_2, \qquad \text{and } h'_I(e) := 1 \text{ otherwise;}$$
$$h'_J(e) := 0 \text{ if } e \in P_3, \qquad \text{and } h'_J(e) := 1 \text{ otherwise;}$$

we observe that each (h'_F, d_F) $(F \in \{I, J, K\})$ is solvable, and the collection $\{h'_I, h'_J, h'_K\}$ is admissible. Therefore, $(G, U)^*$ has a half-integral solution.

Thus, there is a cut $\delta X \in \mathcal{R}$ that meets P_1 twice, and similarly, there is a cut $\delta X'$ that meets P_3 twice. Let L (L') be the 1-path in C_K from z to y (respectively, from z' to y'), and let $\widetilde{F}$ be the face in G_K surrounded by C_K, see Fig. 5.3a.

Next, denote by Q the set of edges w in $L \cup L'$ with $h_K(w) \in \{0, 1\}$. Let a (b) be the first edge in L (resp., in L') belonging to Q. By Lemma 5.1,

(5.3) $h_K(a) = h_K(b) = 1$; a is dually connected with e_I and e_J; and b is dually connected with u_I and u_J.

Let $\mathcal{A}$ be the set of all tight cuts in G_K that meet Q. Arguing as in the proof of Lemma 4.2, we conclude that

(5.4) for any $\delta Y \in \mathcal{A}$ and $\delta Z \in \mathcal{R}$ their corresponding dual paths D_Y and D_Z in G_K have no common face different from $\widetilde{F}, K, O$, and if they have a common face $F \in \{K, O\}$ then they are crossing at this face.

Statement 5.3. *There exists $\delta Z \in \mathcal{A}$ that meets both* bd(K) *and* bd(O) *and contains the edges a and b.*

Proof. Suppose that some of δX and $\delta X'$, δX say, meets only one of bd(K) and bd(O), bd(K) say. From (5.4) it follows that each cut in $\mathcal{A}$ meets only bd(O). Then $\delta X'$ does not meet bd(O) (by (5.4)), whence $\delta X'$ meets bd(K). But then for at least one $\widetilde{L} \in \{L, L'\}$ the dual path D_Z corresponding to a cut $\delta Z \in \mathcal{A}$ meeting $\widetilde{L}$ must have a common face $F \neq \widetilde{F}$ with D_X or $D_{X'}$; a contradiction with (5.4).

Hence, each of $\delta X, \delta X'$ meets both bd(K) and bd(O); see Fig. 5.3b. Applying (5.4), one can see that every cut in $\mathcal{A}$ meets L, L', bd(K) and bd(O). Now we use arguments as in the proof of Statement 5.2. •

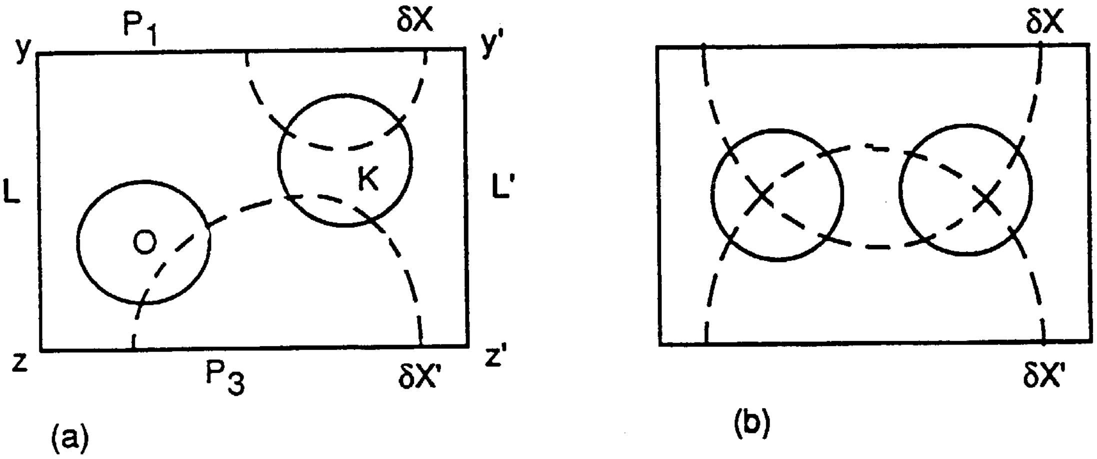


Fig. 5.3

From (5.1),(5.3) and Statements 5.2 and 5.3 it follows that

(5.5) if f' is an arbitrary solution of $(G, U)^*$ then all edges in the set δX_I are saturated by the flow f'_I, all edges in δX_J are saturated by f'_J, and all edges in δZ are saturated by $f'_K + f'_O$.

In particular, (5.5) shows that

(5.6) for any solution f' to $(G, U)^*$, $C_{IJ}(f')$ does not separate J, K, O, and $C_{JI}(f')$ does not separate I, K, O.

Return to the flow f, and consider the bunch $B' = \{K, O\}$. Apply the operation of "balancing" to C_{KO} and C_{OK} (see (2.6)). From the proof of Lemma 2.2 one can see that as a result we get a solution f' for $(G, U)^*$ and a bunch $\widetilde{B}$ satisfying the statement of this lemma and such that $K, O \in \widetilde{B}$. Two cases are possible.

(i) $|\widetilde{B}| = 2$. Then $(G, U)^*$ has a half-integral solution by arguments in Section 3.

(ii) $|\widetilde{B}| > 2$. Then (5.2) and (5.6) imply that $\widetilde{B} = \{I, J, K, O\}$, whence $(G, U)^*$ has a half-integral solution by arguments in Section 4.

This completes the proof of Theorem 1. • • •

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