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**HALF-INTEGRAL FLOWS IN A PLANAR GRAPH
WITH FOUR HOLES**

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HALF-INTEGRAL FLOWS IN A PLANAR GRAPH WITH FOUR HOLES

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Abstract. This paper contains an improved and shorter proof of the main theorem in [Ka3].

Suppose that $G = (VG, EG)$ is a planar graph embedded in the euclidean plane, that I, J, K, O are four of its faces (called *holes* in G), that $s_1, \dots, s_r, t_1, \dots, t_r$ are vertices of G such that each pair $\{s_i, t_i\}$ belongs to the boundary of some of I, J, K, O , and that the graph $(VG, EG \cup \{\{s_1, t_1\}, \dots, \{s_r, t_r\}\})$ is eulerian.

We prove that if the multi(commodity)flow problem in G with unit demands on the values of flows from s_i to t_i ($i = 1, \dots, r$) has a solution then it has a *half-integral* solution as well. In other words, there exist paths $P_1^1, P_1^2, P_2^1, P_2^2, \dots, P_r^1, P_r^2$ in G such that each P_i^j connects s_i and t_i , and each edge of G is covered at most twice by these paths. (It is known that in case of at most three holes there exist edge-disjoint paths connecting s_i and t_i , $i = 1, \dots, r$, provided that the corresponding multiflow problem has a solution, but this is, in general, false in case of four holes.)

1. Introduction

Throughout, we deal with an undirected planar graph G ; speaking of a planar graph we mean that some of its embeddings in the euclidean plane \mathbb{R}^2 (or the sphere S^2) is fixed. VG is the vertex set, EG is the edge set of G (multiple edges and loops are admitted), and $\mathcal{F} = \mathcal{F}_G$ is the set of faces of G . A subset $\mathcal{H} \subseteq \mathcal{F}$ of faces of G , called its *holes*, is distinguished. Let $U = \{\{s_1, t_1\}, \dots, \{s_r, t_r\}\}$ be a family of pairs (possibly repeated) of vertices of G such that each $\{s_i, t_i\}$ is contained in the boundary $bd(I)$ of some hole $I \in \mathcal{H}$.

Problem (G, U, k) : *given an integer $k \geq 1$, find paths $P_1^1, \dots, P_1^k, \dots, P_r^1, \dots, P_r^k$ in G such that each P_i^j connects s_i and t_i , and each edge of G occurs at most k times in these paths.*

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If no restriction on k is imposed, the problem is denoted as $(G, U)^*$; thus $(G, U)^*$ is the fractional relaxation of $(G, U, 1)$, or the *multi(commodity)flow* problem with unit capacities of the edges of G and unit demands on the values of flows connecting pairs in U . We prove the following theorem.

Theorem 1. *Let $|\mathcal{H}| = 4$, and let the graph $(VG, EG \cup U)$ be eulerian, that is,*

$$(1.1) \quad |\delta X| + |\{i : \delta X \text{ separates } s_i \text{ and } t_i\}| \text{ is even for any } X \subset VG.$$

Let $(G, U)^$ have a solution. Then $(G, U, 2)$ has a solution as well; in other words, there exist $P_1^1, P_1^2, P_2^1, P_2^2, \dots, P_r^1, P_r^2$ such that each P_i^j is a path in G connecting s_i and t_i , and each edge of G is covered at most twice by these paths.*

[For $X \subseteq V$, $\delta X = \delta^G X$ denotes the set of edges of G with one end in X and the other in $VG - X$; a nonempty set δX is called a *cut* in G ; we say that δX *separates* vertices x and y if exactly one of x, y is in X .] An obvious necessary condition for solvability of (G, U, k) for arbitrary G, U, k is the *cut condition*:

$$(1.2) \text{ each cut } \delta X \text{ in } G \text{ separates at most } |\delta X| \text{ pairs in } U.$$

The following theorem is well known.

Okamura's theorem [Ok]. *If $|\mathcal{H}| = 2$ and if (1.1) and (1.2) hold then $(G, U, 1)$ has a solution (that is, there exist edge-disjoint paths P_1, \dots, P_r in G such that P_i connects s_i and t_i).*

(A similar result for $|\mathcal{H}| = 1$ is proved in [OkS].) The cut condition is, in general, not sufficient for the solvability of (G, U, k) if $|\mathcal{H}| = 3$. Nevertheless, the following is true.

Theorem 2 [Ka2]. *Let $|\mathcal{H}| = 3$, and let (1.1) and (1.2) hold. The problem $(G, U, 1)$ has a solution if for any 2,3-metric m on VG the following inequality holds:*

$$(1.3) \quad \sum (m(e) : e \in EG) \geq \sum (m(s_i, t_i) : i = 1, \dots, r).$$

[By a *metric* on a set V we mean a real-valued function m on $V \times V$ satisfying $m(x, x) = 0$, $m(x, y) = m(y, x)$ and $m(x, y) + m(y, z) \geq m(x, z)$ for all $x, y, z \in V$. We say that m is *induced* by (H, σ) , where H is a graph and σ is a mapping of V into VH , if $m(x, y) = \text{dist}^H(\sigma(x), \sigma(y))$ for all $x, y \in V$. Here $\text{dist}^{G'}(x', y')$ denotes the distance in a graph G' between vertices x' and y' . When it is not confusing, we say that m is induced by H or m is induced by σ . If $\sigma(V) = VH$ and H is the complete

graph K_2 on two vertices (the complete bipartite graph $K_{2,3}$ with parts of two and three vertices) then m is called a *cut-metric* (respectively, a *2,3-metric*.)] Note that satisfying (1.3) with any metric m on VG is necessary for the solvability of (G, U, k) for arbitrary G, U, k because if P_i^j 's give a solution of (G, U, k) then

$$\sum_{e \in EG} m(e) \geq \frac{1}{k} \sum_{i=1}^r \sum_{j=1}^k \sum (m(e) : e \in P_i^j) \geq \sum_{i=1}^r m(s_i, t_i)$$

(we write $e \in P_i^j$ considering a path as an edge-set). Thus, if $|\mathcal{H}| \leq 3$, (1.1) holds, and $(G, U)^*$ has a solution then $(G, U, 1)$ has a solution as well. Such a property does not remain, in general, true for $|\mathcal{H}| = 4$, as shown in [Ka2]. Hence, for $|\mathcal{H}| = 4$ Theorem 1 provides the least (in terms of \mathcal{H}) value of k for which (G, U, k) has a solution in the eulerian case. Another feature of case $|\mathcal{H}| = 4$ is that more exotic metrics are involved in the solvability criterion for $(G, U)^*$. We say that a metric induced by a bipartite planar graph H with $|\mathcal{F}_H| = 4$ is a *4f-metric*.

Theorem 3 [Ka1]. *For $|\mathcal{H}| = 4$, $(G, U)^*$ is solvable if and only if (1.3) holds for every m that is a cut-metric or a 2,3-metric or a 4f-metric.*

The proof of Theorem 1 will rely essentially on a strengthening of the fractional version of Theorem 2 and a strengthening of Theorem 3 (Theorems 4 and 5 below); they describe classes of 2,3- and 4f-metrics sufficient for verification of solvability of $(G, U)^*$. To state these, we need some terminology, conventions and simple facts about multiflows and metrics.

First, the faces of a planar graph are considered as *open* regions in the plane. An edge e with end vertices x and y is identified with the corresponding curve in the plane (x and y are usually not included in the curve); when it leads to no confusion, e is denoted by xy . A path (circuit) $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ (where x_i 's are vertices and e_i 's are edges) is denoted by $x_0 x_1 \dots x_k$ and called an $x_0 - x_k$ path; P is often considered up to reversing and, if P is a circuit, shifting cyclically. $|P|$ is the number k of edges in P ; if $|P| = 0$, P is called *trivial*. A path P from x to y is called an $x - y$ path; if both x and y are in the boundary of a face F we say that P is an F -path. The boundary $\text{bd}(F)$ of a face F is identified with the corresponding (possibly not simple) circuit. For $g : E \rightarrow \mathbb{R}$ and $E' \subseteq E$, $g(E')$ denotes $\sum(g(e) : e \in E')$; in particular, we write $g(P)$ for a function g on the edges of a graph and a path (or circuit) P , considering P as an edge-set.

Second, consider a planar graph G' with a set \mathcal{H}' of holes. For $F \in \mathcal{F}_{G'}$ let W_F denote the set of pairs $\{s, t\}$ of vertices in $\text{bd}(F)$, and let $W_{\mathcal{H}'} := \cup(W_F : F \in \mathcal{H}')$. Suppose we are given a family U' of pairs in $W_{\mathcal{H}'}$, and functions $c' : EG' \rightarrow \mathbb{Q}_+$ (of

capacities of edges) and $d' : U' \rightarrow \mathbb{Q}_+$ (of demands). Denote by $\mathcal{P}(G', s, t)$ the set of simple paths in G' connecting vertices s and t . Let $\mathcal{P}(G', U') := \cup(\mathcal{P}(G', s, t) : \{s, t\} \in U')$. We denote by (c', d') the *multiflow problem*: find a function $f : \mathcal{P}(G', U') \rightarrow \mathbb{Q}_+$ satisfying:

$$(1.4) \quad f^e := \sum(f(P) : e \in P \in \mathcal{P}(G', U')) \leq c'(e) \quad \text{for all } e \in EG';$$

$$(1.5) \quad \sum(f(P) : P \in \mathcal{P}(G', s, t)) = d'(s, t) \quad \text{for all } (s, t) \in U'$$

(when c' and d' are all-unit, (c', d') turns into $(G', U')^*$). We say that f satisfying (1.4)-(1.5) is a (c', d') -*admissible multiflow*. Applying Farkas lemma to the system (1.4)-(1.5) and then making easy transformations, one can obtain the following criterion (this is valid for arbitrary G', U', c', d' [Lo]):

$$(1.6) \quad \text{Solvability criterion: } (c', d') \text{ is solvable if and only if the inequality } c'(m) \leq d'(m) \text{ holds for any metric } m \text{ on } VG', \text{ where } c'(m) := \sum_{e \in EG'} c'(e)m(e) \text{ and } d'(m) := \sum_{(s,t) \in U'} d'(s,t)m(s,t).$$

Third, we say that a metric m on VG' is *bipartite* if m is integer-valued and the length $m(C)$ of every circuit C in G' is even (in particular, every cut-, 2,3- or 4f-metric is bipartite). A bipartite m is called \mathcal{H}' -*primitive* if there are no non-zero bipartite metrics m' and m'' on VG' such that $m(e) \geq m'(e) + m''(e)$ for all $e \in EG'$ and $m(s, t) \leq m'(s, t) + m''(s, t)$ for all $\{s, t\} \in W_{\mathcal{H}'}$. A simple observation is that in criterion (1.6) it suffices to consider the \mathcal{H}' -primitive metrics rather than all metrics m on VG' ; in other words, if (c', d') is unsolvable then $c'(m) < d'(m)$ holds for some \mathcal{H}' -primitive m .

Fourth, let m be a metric induced by $\sigma : VG' \rightarrow VH$, where H is a bipartite planar graph with $|\mathcal{F}_H| = |\mathcal{H}'|$. As a rule, we shall deal with the situation when σ yields a certain topological correspondence of the face structures for G' and H . More precisely, σ can be extended to a continuous mapping of \mathbb{R}^2 into itself so that:

- (1.7) (i) for any point $x \in \mathbb{R}^2$ each of the sets $\sigma^{-1}(x)$ and $\mathbb{R}^2 - \sigma^{-1}(x)$ is connected, and $\sigma^{-1}(x)$ is compact;
- (ii) each hole $F \in \mathcal{H}'$ is mapped homeomorphically to a face of H ;
- (iii) for each edge $e = xy \in EG'$ the path (x, e, y) is mapped homeomorphically to a simple path in H unless it is mapped to a single point.

In this case we say that m is *consistent*. For convenience we also assume that σ preserves orientation clockwise in \mathbb{R}^2 . From (i)-(ii) and the fact that $|\mathcal{F}_H| = |\mathcal{H}'|$ it follows that σ gives a one-to-one correspondence of the holes in G' to the faces in H , and that the unbounded face of G' is a hole. It was shown in [Kal] that

(1.8) if $|\mathcal{H}'| = 3$ ($|\mathcal{H}'| = 4$) then any \mathcal{H}' -primitive 2,3-metric (respectively, 4f-metric) on VG' is consistent.

Suppose that $|\mathcal{H}'| = 3$. Let m be a consistent 2,3-metric induced by $\sigma : VG' \rightarrow VK_{2,3}$, and let $\{y_1, y_2\}$ and $\{x_1, x_2, x_3\}$ be the parts in $VK_{2,3}$. Denote by $\Pi(\sigma)$ the (ordered) partition $(S_1, S_2, S_3, T_1, T_2)$ of VG' , where $S_i := \sigma^{-1}(x_i) \cap VG'$ and $T_j := \sigma^{-1}(y_j) \cap VG'$. Let Φ_i denote the closed region $\sigma^{-1}(x_i)$ in \mathbb{R}^2 . One can see that there is a labelling $I_1, I_2, I_3 = I_0$ of the holes such that (see Fig. 1.1):

(1.9) $\Phi_i \cap \text{bd}(I_p) = \emptyset$ if and only if $p = i$; and the space $\Omega(\sigma) := \mathcal{R}^2 - (I_1 \cup I_2 \cup I_3 \cup \Phi_1 \cup \Phi_2 \cup \Phi_3)$ consists of two disjoint regions, one containing T_1 and the other containing T_2 .

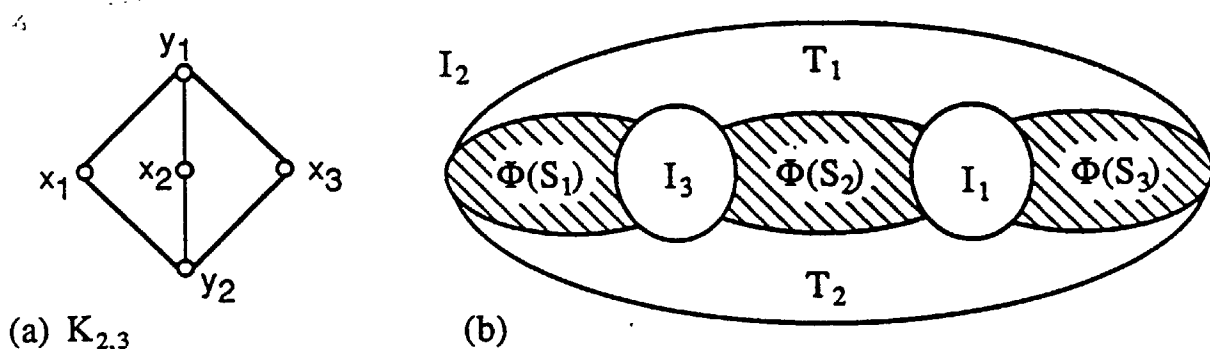


Fig. 1.1

Theorem 4 [Kal]. Let $|\mathcal{H}'| = 3$, and let m be an \mathcal{H}' -primitive 2,3-metric induced by $\sigma : VG' \rightarrow VK_{2,3}$. Let $\Pi(\sigma) = (S_1, S_2, S_3, T_1, T_2)$ and I_1, I_2, I_3 be defined as above (taking into account that m is consistent, by (1.8)). Then:

- (i) all sets in $\Pi(\sigma)$ are nonempty;
- (ii) for $i = 1, 2, 3$ the subgraph $\langle S_i \rangle$ in G' induced by S_i is connected, and S_i meets both $\text{bd}(I_{i-1})$ and $\text{bd}(I_{i+1})$.

In particular, no edge of G' connects T_1 and T_2 .

In Section 2 we show how Okamura's theorem and Theorem 4 are applied in order to prove that in the eulerian case a solvable problem $(G, U)^*$ with $|\mathcal{H}| = 4$ has a $1/4$ -integral solution; this proof is relatively easy. Using this, we then prove Theorem 1. This proof involves more intricate arguments and is given throughout Sections 3-5. In particular, at many stages of the proof we appeal to the fact that, besides being consistent, a primitive 4f-metric possesses a spectrum of structural properties and its value on an edge is at most four (compared with the cut-metrics and 2,3-metrics, which

take their values in $\{0, 1\}$ and $\{0, 1, 2\}$, respectively; note also that the set of graphs H inducing primitive 4f-metrics m is infinite, thus values of m on pairs of vertices that are not edges of G can be large). These properties are exposed in the following theorem.

Theorem 5. *Let $|\mathcal{H}'| = 4$, and let m be an \mathcal{H}' -primitive 4f-metric induced by $\sigma : VG' \rightarrow VH$. Then $m(e) \leq 4$ for each $e \in EG'$. Moreover, if $m(e) = 4$ for some edge $e = xy$ then:*

- (i) H is homeomorphic to K_4 ;
- (ii) the image by σ of the path (x, e, y) is a shortest path $L_e = b_0b_1b_2b_3b_4$ in H which belongs to the boundary of a unique face, \tilde{J} say, in H ;
- (iii) each shortest $x - y$ path in H with $x, y \in \text{bd}(\tilde{J}) - \{b_1, b_2, b_3\}$ lies in $\text{bd}(\tilde{J})$;
- (iv) for each $\tilde{I} \in \mathcal{F}_H - \{\tilde{J}\}$, no shortest \tilde{I} -path contains both b_0 and b_4 ;
- (v) if each of b_0, b_4 belongs to a shortest \tilde{I} -path for the same $\tilde{I} \in \mathcal{F}_H - \{\tilde{J}\}$, and $b \in \{b_0, b_4\}$ is not in $\text{bd}(\tilde{I})$, then (a) every shortest \tilde{I} -path containing b separates \tilde{J} from \tilde{K} and \tilde{O} , and (b) no shortest \tilde{I} -path contains an edge in $\text{bd}(\tilde{K}) \cap \text{bd}(\tilde{O})$, where $\mathcal{F}_H = \{\tilde{I}, \tilde{J}, \tilde{K}, \tilde{O}\}$.

Here we say that an I -path L separates faces J and K if they lie in different components of $\mathbb{R}^2 - (I \cup L)$. Though this result is very important to get Theorem 1, the proof of Theorem 5 is very technical and we do not give it here, referring the reader to [Ka3, Section 3]. Figure 1.2 illustrates an \mathcal{H}' -primitive metric m with $m(e) = 4$ for some e , and properties (i)-(v); here $\mathcal{H} = \{I, J, K, O\}$, m is induced by a mapping of VG' to VH and its values on the edges of G' are indicated.

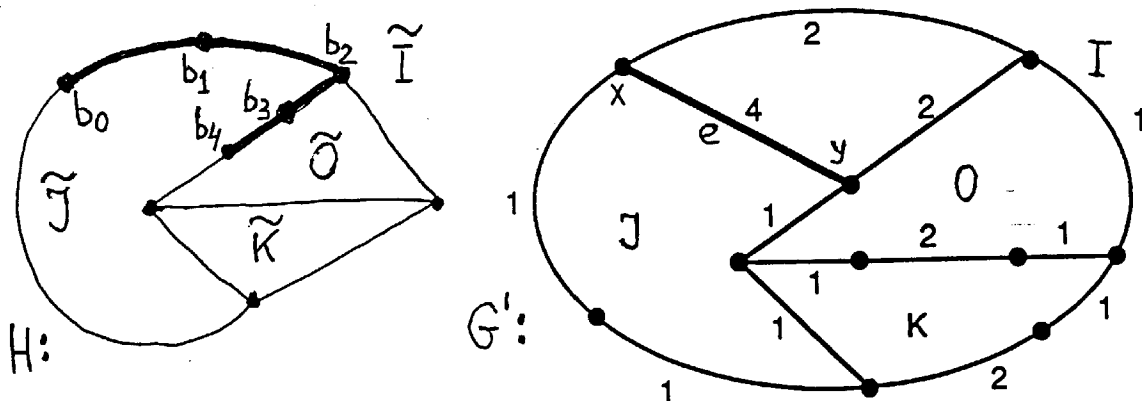


Fig. 1.2

2. EXISTENCE OF A QUARTER-INTEGRAL SOLUTION

Let $|\mathcal{H}| = 4$, and let $(G, U)^* = (c, d)$ have a solution $f : \mathcal{P}(G, U) \rightarrow \mathbb{Q}_+$, where c and d are the all-unit functions on EG and U , respectively. It is convenient to think of f as consisting of four flows f_F ($F \in \mathcal{H}$), where f_F is the restriction of f to the F -paths in $\mathcal{P}(G, U)$ (one may assume that no member of U belongs to the boundaries of two holes). Denote by $\mathcal{L} = \mathcal{L}(f)$ the set of paths $P \in \mathcal{P}(G, U)$ with $f(P) > 0$ (the *support* of f). Similarly, $\mathcal{L}_F = \mathcal{L}_F(f)$ denotes the support of f_F ; thus $\{\mathcal{L}_I, \mathcal{L}_J, \mathcal{L}_K, \mathcal{L}_O\}$ is a partition of \mathcal{L} .

A path $P \in \mathcal{L}_F$ ($F \in \mathcal{H}$) divides the space $\mathbb{R}^2 - F$ into a pair $\mathcal{R}(P)$ of closed regions whose intersection is P and union is $\mathbb{R}^2 - F$. We say that f is *non-crossing* if any two paths $P \in \mathcal{L}_F$ and $P' \in \mathcal{L}_{F'}$ for $F \neq F'$ do not cross, that is, P' is contained entirely in some of the members of $\mathcal{R}(P)$. Applying to f standard uncrossing techniques, it is easy to show that

- (2.1) if $(G, U)^*$ has a $1/k$ -integral solution then it has a $1/k$ -integral non-crossing solution.

In what follows we assume that f is non-crossing. Consider two different holes F and F' . Remove from the sphere \mathbb{S}^2 the hole F , its boundary and the paths in \mathcal{L}_F . Then F' occurs in a component Z of the resulting space. Define $D_{FF'} = D_{FF'}(f)$ to be $\mathbb{S}^2 - Z$. Easy topological observations using the fact that all paths in \mathcal{L}_F are simple show that $D_{FF'}$ is homeomorphic to a closed disc, i.e., the boundary $C_{FF'} = C_{FF'}(f)$ of $D_{FF'}$ is a closed non-self-intersecting curve. Moreover,

- (2.2) $C_{FF'}$ is a simple circuit in G , and $f_F^e > 0$ holds for each edge $e \in C_{FF'}$ that is not in $\text{bd}(F)$, where $f_F^e := \sum(f(P) : e \in P \in \mathcal{L}_F(f))$.

(An equivalent definition: $D_{FF'}$ is the largest region in \mathbb{S}^2 that does not contain F' and whose boundary is in the union of $\text{bd}(F)$ and $\cup(P \in \mathcal{L}_F)$.) Since f is non-crossing, $D_{FF'}$ and $D_{F'F}$ are obviously openly disjoint, i.e., $D_{FF'} \cap D_{F'F} = C_{FF'} \cap C_{F'F}$. Furthermore, for $F'' \in \mathcal{H} - \{F, F'\}$, if $F'' \subset D_{FF'}$ then $D_{FF''} \cup D_{FF'} = \mathbb{S}^2$, while if $F'' \cap D_{FF'} = \emptyset$ then $D_{FF''}$ and $D_{F''F}$ are openly disjoint and $D_{FF''} = D_{F''F}$. This justifies introducing the following notion, which plays the central role in the proof of Theorem 1.

Definition. Given a non-crossing f , a maximal subset $B \subseteq \mathcal{H}$ such that $D_{FF'}$ and $D_{F'F}$ are openly disjoint for any two distinct $F, F' \in B$ is called a *bunch*.

Clearly $2 \leq |B| \leq 4$. For $F \in B$ we denote $D_{FF'}$ and $C_{FF'}$ by D_F and C_F , respectively (these do not depend on $F' \in B - \{F\}$). The family of $|B|$ circuits C_F ($F \in B$) is denoted by $\mathcal{C}(B)$ (in case $B = \{F, F'\}$ the circuits C_F and $C_{F'}$ may coincide).

Also denote by G_F , \mathcal{H}_F , and U_F the subgraph of G contained in D_F , the set of holes $\widehat{F} \in \mathcal{H}$ in D_F , and the set of pairs $\{s, t\} \in U$ such that $\{s, t\} \in W_{\widehat{F}}$ for $\widehat{F} \in \mathcal{H}_F$, respectively. Obviously,

(2.3) for a bunch B , the space $\mathbf{S}^2 - \cup(D_F : F \in B)$ contains no hole, and each edge e of G occurs in at most two members of $\mathcal{C}(B)$.

Fix a bunch B . We may assume that for each $F \in B$, C_F has an edge with some $F' \in B - \{F\}$ in common. Indeed, if this is not so for some F , consider the problems $(G_F, U_F)^*$ and $(G', U')^*$, where $G' = (VG, EG - EG_F)$ and $U' = U - U_F$. Clearly every path in \mathcal{L} is entirely within some of G_F and G' , therefore the corresponding restrictions of f give solutions for these problems. Since $|\mathcal{H}_F| \leq 3$ and $|\mathcal{H} - \mathcal{H}_F| \leq 3$, by Okamura's theorem or Theorem 2, each problem has a *half-integral* solution (not necessarily integral as $(VG_F, EG_F \cup U_F)$ may not be eulerian). Combining these, we get a half-integral solution for $(G, U)^*$, and Theorem 1 follows. By similar arguments, we may assume that

(2.4) for any $\emptyset \neq B' \subset B$, $\cup_{F \in B'} C_F$ and $\cup_{F \in B - B'} C_F$ have an edge in common.

Later on we assume that a non-crossing f and a bunch B are chosen so that:

- (2.5) (i) $|B|$ is as great as possible;
(ii) $\sum((|\mathcal{H}_F|)^2 : F \in B)$ is minimum subject to (i);
(iii) the number of faces in $\cup(D_F : F \in B)$ is minimum subject to (i)-(ii).

In particular, a bunch B (for some f) with $\{|\mathcal{H}_F| : F \in B\} = \{1, 1, 1, 1\}$ is preferable to choose than one with $\{1, 1, 2\}$, and $\{2, 2\}$ is preferable than $\{1, 3\}$. Let \overline{f}_F^e stand for $\sum(f_{F'}^e : F' \in \mathcal{H}_F)$.

Lemma 2.1. For each $F \in B$ there exists a function h_F on EG_F such that:

- (i) $h_F(e) \in \{0, \frac{1}{2}, 1\}$ for each $e \in C_F$ and $h_F(e) = 1$ for the other edges e in G_F ;
(ii) if e is a common edge for C_F and $C_{F'}$ ($F, F' \in B$) then $h_F(e) + h_{F'}(e) \leq 1$;
(iii) each problem (h_F, d_F) is solvable; here d_F is the all-unit function on U_F .

This lemma shows the existence of a $1/4$ -integral solution for $(G, U)^*$. Indeed, for each $F \in B$ the function $2h_F$ is integral, hence the problem $(2h_F, 2d_F)$ has a half-integral solution. So (h_F, d_F) has a $1/4$ -integral solution. Taken together, these solutions form an admissible solution for $(G, U)^*$.

Proof of Lemma 2.1. Choose functions h_F ($F \in B$) so that (ii)-(iii) hold, h_F is

all-unit on $EG_F - C_F$, and the value $\gamma(h) := \sum_{F \in B} |Q_F|$ is as small as possible, where $Q_F := \{e \in C_F : h_F(e) \notin \{0, \frac{1}{2}, 1\}\}$. Such functions exist since we can take as $h_F(e)$ the value \bar{f}_F^e for $e \in C_F$, and 1 for the other edges e of G_F . One has to prove that $\gamma(h) = 0$. Suppose that $\gamma(h) > 0$.

For $F \in B$ let Q_F^+ (Q_F^-) be the set of edges $e \in Q_F$ with $h_F(e) > 1/2$ (respectively, $h_F(e) < 1/2$). We perform *balancing* h_F 's (simultaneously for all $F \in B$); this means that for some $\varepsilon \in \mathbb{R}_+$ each h_F is transformed to h_F^ε , where

$$(2.6) \quad \begin{aligned} h_F^\varepsilon(e) &:= h_F(e) - \varepsilon && \text{if } e \in Q_F^+; \\ &:= h_F(e) + \varepsilon && \text{if } e \in Q_F^-; \\ &:= h_F(e) && \text{for the remaining } e\text{'s in } G_F; \end{aligned}$$

Take ε to be maximum provided that for each $F \in B$, (a) $\varepsilon \leq h_F(e) - 1/2$ for $e \in Q_F^+$; (b) $\varepsilon \leq 1/2 - h_F(e)$ for $e \in Q_F^-$; and (c) (h_F^ε, d_F) has a solution g_F . Clearly $h_F^\varepsilon(e) + h_{F'}^\varepsilon(e) \leq 1$ for each edge e common for C_F and $C_{F'}$ ($F, F' \in B$). Also $\gamma(h^\varepsilon) \leq \gamma(h)$, whence $\gamma(h^\varepsilon) = \gamma(h)$, by the choice of h . Furthermore, one can see that combining the g_F 's we get a multiflow which has a bunch B' not worse than B in the sense of (2.5). By the maximality of ε , there is $F \in B$ such that for any $\Delta > \varepsilon$ the problem $(h_F^{\varepsilon'}, d_F)$ has no solution for some $\varepsilon < \varepsilon' \leq \Delta$. Two cases are possible.

Case 1. $|\mathcal{H}_F| \leq 2$. Applying Okamura's theorem, we observe that for every $\varepsilon' > \varepsilon$ there is $X' \subset VG_F$ such that $h_F^{\varepsilon'}(X') < d_F(X')$, where $h_F^{\varepsilon'}(X')$ stands for $\sum(h_F^{\varepsilon'}(e) : e \in \delta X')$ and $d_F(X')$ stands for $|\{\{s, t\} \in U_F : \delta X' \text{ separates } s \text{ and } t\}|$ (letting $\delta X' := \delta^{G_F} X'$). Hence, there is $X \subset VG_F$ such that

$$h_F^\varepsilon(X) = d_F(X) \text{ and } h_F^{\varepsilon'}(X) < d_F(X) \text{ for any } \varepsilon' > \varepsilon.$$

Without loss of generality, we may assume that δX is a *simple* cut, i.e., δX meets at most twice the boundary of every face in G_F . In particular, $|\delta X \cap C_F| \leq 2$ (as C_F is the boundary of a face in G_F). Then $|\delta X \cap C_F| = 2$; let $\delta X \cap C_F = \{e, e'\}$. Since d_F is an integer and $h_F^\varepsilon(e'')$ is an integer for each $e'' \in \delta X - \{e, e'\}$, $h_F^\varepsilon(X) = d_F(X)$ implies that $\phi := h_F^\varepsilon(e) + h_F^\varepsilon(e')$ is an integer. Hence, either $h_F^\varepsilon(e) + h_F^\varepsilon(e') = \frac{1}{2}$, or one of e, e' is in Q_F^+ and the other in Q_F^- . In both cases we have $h_F^{\varepsilon'}(X) = h_F^\varepsilon(X)$ for any ε' ; a contradiction.

Case 2. $|\mathcal{H}_F| = 3$. Then $|B| = 2$; let for definiteness $B = \{I, K\}$, $F = I$ and $\mathcal{H}_I = \{I, J, O\}$. Apply Theorem 4. Arguing as above, we conclude that there exists (i) $X \subset VG_I$ such that $h_I^\varepsilon(X) = d_I(X)$ and $h_I^{\varepsilon'}(X) < d_I(X)$ for any $\varepsilon' > \varepsilon$, or (ii) an \mathcal{H}_I -primitive 2,3-metric m on VG_I such that

$$h_I^\varepsilon(m) = d_I(m) \text{ and } h_I^{\varepsilon'}(m) < d_I(m) \text{ for any } \varepsilon' > \varepsilon,$$

where $h_I^\varepsilon(m) := \sum(h_I^\varepsilon(e)m(e) : e \in EG_I)$ and $d_I(m) := \sum(m(s,t) : \{s,t\} \in U_I)$ (cf. (1.6)). By arguments as in Case 1, (i) is impossible.

Thus (ii) takes place. Consider the partition $\Pi(\sigma) = (S_1, S_2, S_2, T_1, T_2)$ of VG_I as in Theorem 4 (where m is induced by σ). Since C_I is the boundary of some face \tilde{F} in G_I and each subgraph $\langle S_i \rangle$ is connected, C_I can pass across exactly one component of $\Omega(\sigma)$ (defined in (1.9)), say, the component Ω_1 that contains T_1 . Next, if there is an edge $e \in C_I$ connecting $u \in S_i$ and $v \in S_j$ ($i \neq j$), we could slightly transform G_I and m by replacing e by a pair of edges in series, $e' = uz$ and $e'' = zv$ say, and by adding z to T_1 (and, accordingly, placing z in the region Ω_1); it is easy to see that the new graph and 2,3-metric maintain the above properties. Thus, one may assume that each edge in C_I connecting different sets in $\Pi(\sigma)$ connects just T_1 and some S_i . Let $\xi = (e_1 = u_1v_1, \dots, e_k = u_kv_k)$ be the sequence of such edges in C_I , and let the vertices $u_1, v_1, \dots, u_k, v_k$ occur in this order in C_I . Note that there are no two consecutive edges e_j, e_{j+1} in ξ such that $v_j, u_{j+1} \in T_1$ and $u_j, v_{j+1} \in S_i$ for some $i \in \{1, 2, 3\}$. For otherwise, assuming for definiteness that $i = 1$ and letting Z to be the component in $\langle T_1 \rangle$ that contains the part of C_I from v_j to u_{j+1} , we observe that the partition $(T_1 - VZ, T_2, S_1 \cup VZ, S_2, S_3)$ corresponds to a 2,3-metric m' such that $h_I^\varepsilon(m') < h_I^\varepsilon(m)$ and $d_F(m') = d_F(m)$, which is impossible. Now the latter property together with the fact that each $\langle S_i \rangle$ is connected implies that $k \leq 6$ and for each $i = 1, 2, 3$ there is at most one j such that $u_j \in S_i$ and $v_j \in T_1$. Consider three cases.

(i) $k = 2$. Then a contradiction is shown in a similar way as in Case 1.

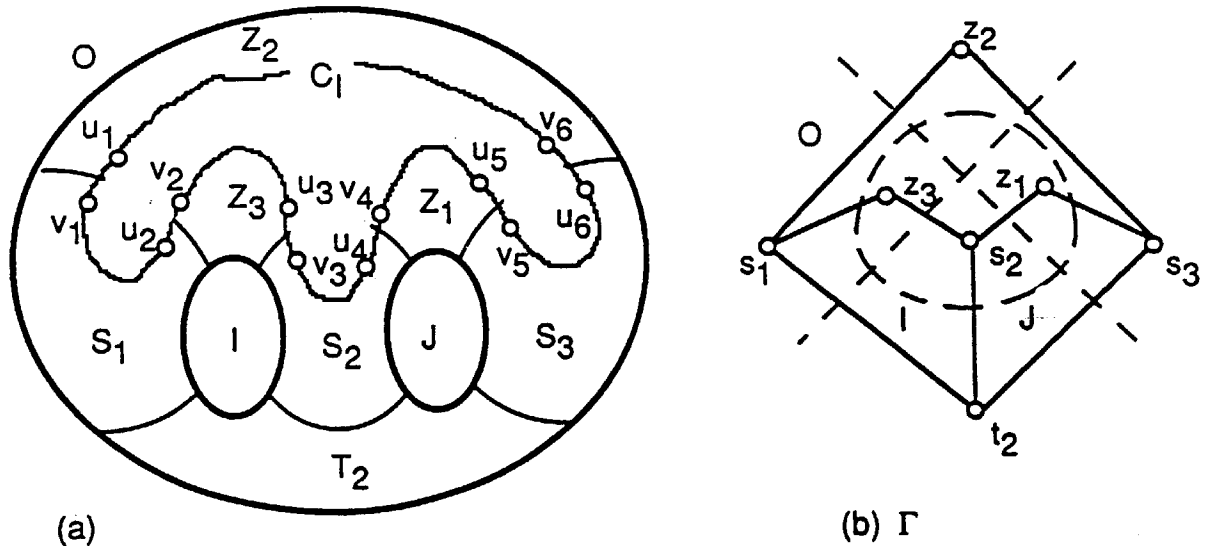


Fig. 2.1

(ii) $k = 6$. Let for definiteness $v_1, u_2 \in S_1$, $v_3, u_4 \in S_2$ and $v_5, u_6 \in S_3$; see Fig. 2.1a. Denote by Z_1 (Z_2 ; Z_3) the set of vertices in the component of the space $\Omega_1 - \tilde{F}$

that contains the part of C_I from v_4 to u_5 (respectively, from v_6 to u_1 ; from v_2 to u_3). Then $\{Z_1, Z_2, Z_3\}$ is a partition of T_1 . Shrink S_i to a single vertex s_i , Z_j to a vertex z_j , and T_2 to a vertex t_2 , obtaining the graph Γ drawn in Fig. 2.1b.

Let τ be the natural mapping of VG_I to $V\Gamma$, and let m' be the metric on VG_I induced by τ . It is easy to see that $m'(e) = m(e)$ for each $e \in EG_I$ and $m'(p, q) = m(p, q)$ for each $\{p, q\} \in U_I$. One can also check that $m' = \rho_{X(1)} + \rho_{X(2)} + \rho_{X(3)}$, where for $i = 1, 2, 3$, $X(i) := \tau^{-1}(\{s_i, z_{i-1}, z_{i+1}\})$ (letting $z_4 = z_1$ and $z_0 = z_3$), and $\rho = \rho_{X'}$ denotes the cut-metric on VG_I defined as $\rho(x, y) := 1$ if $|X' \cap \{x, y\}| = 1$, and $\rho(x, y) := 0$ otherwise. Then $h_I^\varepsilon(X(i)) = d_I(X(i))$, $i = 1, 2, 3$. Moreover, for at least one i we have $h_I^{\varepsilon'}(X(i)) < d_I(X(i))$ (for $\varepsilon' > \varepsilon$); a contradiction.

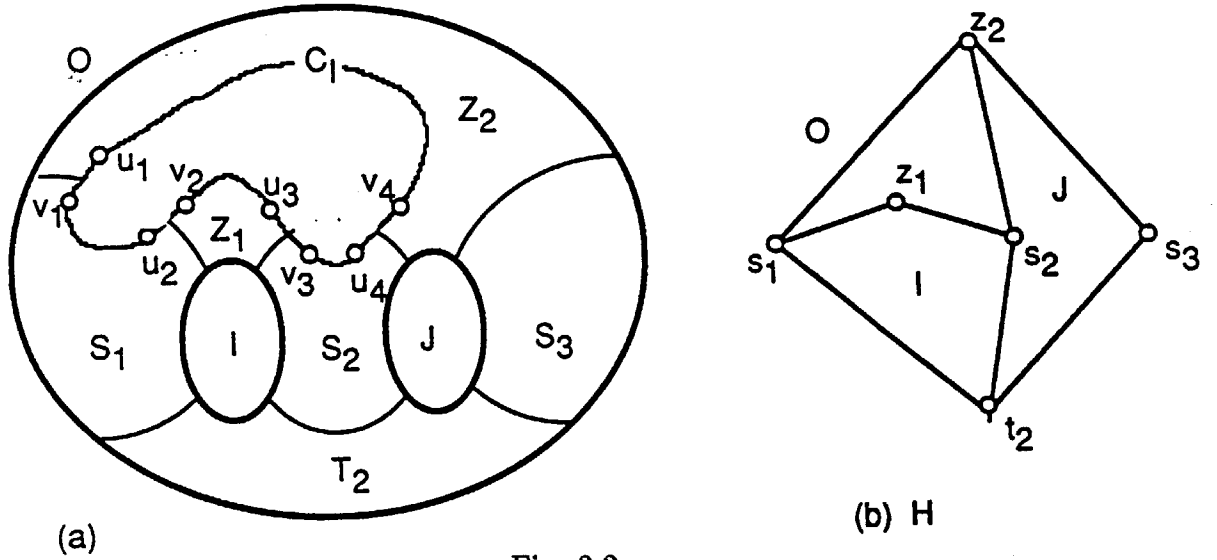


Fig. 2.2

(iii) $k = 4$. Fix a solution f' to (h_I^ε, d_I) (f' concerns G_I). Let for definiteness $v_1, u_2 \in S_1$ and $v_3, u_4 \in S_2$; see Fig. 2.2a. Let Z_1 (Z_2) be the set of vertices in the component of $\Omega_1 - \tilde{F}$ that contains the part of C_I from v_2 to u_3 (respectively, from v_4 to u_1). Consider the mapping $\tau : VG_I \rightarrow VH$ that brings the sets $S_1, S_2, S_3, T_2, Z_1, Z_2$ to the vertices $s_1, s_2, s_3, t_2, z_1, z_2$ (respectively) of the graph H drawn in Fig. 2.2b. Let m' be the metric on VG_I induced by τ . Then $m'(e) = m(e)$ for each $e \in EG_I$ and $m'(p, q) = m(p, q)$ for each $\{p, q\} \in U_I$. This implies that $h_I^\varepsilon(m') = d_I(m')$. An easy consequence of this equality is that if f' is a solution of (h_I^ε, d_I) (f' concerns G_I) then any path $P \in \mathcal{L}(f')$ must be shortest for m' .

On the other hand, it is easy to see that the vertex z_1 does not belong to any shortest path connecting vertices in $\tau(\text{bd}(J))$ or in $\tau(\text{bd}(O))$, while s_3 does not belong to any shortest path connecting vertices in $\tau(\text{bd}(I))$. This implies that the circuits $C_{JI}(f')$ and $C_{OI}(f')$ cannot separate I and K , while $C_{IJ}(f')$ cannot separate J and O . Form a solution \hat{f} for $(G, U)^*$ by combining the flows f' and f_K . From said above

it follows that for \hat{f} there is a bunch B' such that either $|B'| \geq 3$, or $|B'| = 2$ and $\{|\mathcal{H}_F| : F \in B'\} = \{2, 2\}$. In each case B' contradicts to the choice of B in (2.5).

This completes the proof of the lemma. •

For h_F and d_F as above a cut δX in G_F is called *tight* if $h_F(X) = d_F(X)$. Throughout the rest of the paper we assume that f, B and h_F 's as in Lemma 2.1 are chosen so that

$$(2.7) \quad \sum_{F \in B} h_F(C_F) \text{ is minimum subject to (2.5).}$$

In particular, (2.7) implies that

$$(2.8) \quad h_F(e) = \frac{1}{2} [2\bar{f}_F^e] \text{ for any } e \in C_F, F \in B$$

Statement 2.2. *Let $F \in B$ and $|\mathcal{H}_F| \leq 2$. Then for each $e \in C_F$ with $h_F(e) > 0$, (i) e belongs to a tight cut in G_F , and (ii) $\bar{f}_F^e = h_F(e)$, where \bar{f}_F is a solution to (h_F, d_F) .*

Proof. (ii) follows from (i) since $h_F(X) = d_F(X)$ implies that all edges of δX are "saturated" by \bar{f}_F . Suppose that (i) is false for some e . Decrease h_F by $1/2$ on this e , obtaining a new function h'_F on EG_F . Since h_F is half-integral, $h'_F(X) = h_F(X) - 1/2 \geq d_F(X)$ for any X such that $e \in \delta X$. Hence, (h'_F, d_F) has a solution (by Okamura's theorem), and we get a contradiction with (2.5) or (2.7). •

In the proof of Theorem 1 the functions h_F will play more important role than a multifold f behind them; roughly speaking, these functions provide a splitting of the graph (or the all-unit capacities on its edges) into two or more pieces in order to solve then the corresponding easier problem in each piece separately. In fact, throughout the proof we are trying to show the existence of some h_F 's with a "nice property" which enables us to find half-integral solutions for the corresponding pieces. The following expose a kind of such a property.

Statement 2.3. *Let some $F \in B$ be such that either $h_F(e) = 1/2$ for all $e \in C_F$ or $h_F(e) \in \{0, 1\}$ for all $e \in C_F$. Then $(G, U)^*$ has a half-integral solution.*

Proof. Consider the problems (h_F, d_F) and (c', d') , where $c'(e) := 1 - h_F(e)$ for $e \in EG_F$ and $d'(s, t) := 1 - d_F(s, t)$ for $\{s, t\} \in U$ (assuming that h_F and d_F are extended by zero to $EG - EG_F$ and $U - U_F$, respectively). Clearly both $2h_F(X) - 2d_F(X)$ and $2c'(X) - 2d'(X)$ are even for any $X \subseteq V$. Hence $(2h_F, 2d_F)$ and $(2c', 2d')$ have integral solutions, and the result follows. •

3. PROOF OF THEOREM 1. EXCLUSION OF $|B| = 2$

Similar to the proof of Theorem 2 given in [Ka2], the proof of Theorem 1 utilizes the integral and fractional variants of the so-called “splitting-off method”, but now in a more complicated context. We first discuss how such a method works in our case.

Without loss of generality we assume that: G is connected; all $s_1, \dots, s_r, t_1, \dots, t_r$ are distinct and of valency 1 (since one can add to G new vertices s'_i, t'_i and edges $\{s'_i, s_i\}, \{t'_i, t_i\}$ and consider the pairs $\{s'_i, t'_i\}$ instead of $\{s_i, t_i\}$'s). Let $T := \{s_1, \dots, s_r, t_1, \dots, t_r\}$. Also one may assume that each *inner* vertex x (i.e., $x \in VG - T$) is of valency 2 or 4 (otherwise one can repeatedly transform G at x as shown in Fig. 3.1; this does not change, in essence, our problem).



becomes

Fig. 3.1

We assume that Theorem 1 is false and consider (G, U) to be a counterexample to it with $|VG|$ minimum (under the above properties). Then G has neither loops nor inner vertices of valency 2.

For $x \in VG$ let $E(x)$ denote the set of edges of G incident to x and ordered clockwise in the plane. Consider $x \in VG - T$ and two consecutive edges $e = xy$ and $e' = xz$ in $E(x)$. The triple $\tau = (e, x, e')$ is called a *fork*. Denote by G_τ the graph obtained from G by adding a new edge (or a loop) e_τ connecting y and z . Define the function ω_τ on EG_τ by

$$\begin{aligned} \omega_\tau(u) &:= 1 && \text{for } u = e, e', \\ &:= -1 && \text{for } u = e_\tau, \\ &:= 0 && \text{otherwise.} \end{aligned}$$

For $0 \leq \varepsilon \leq 1$, let $c_{\tau, \varepsilon}$ denote the function on EG_τ taking the value $1 - \varepsilon$ on e and e' , ε on e_τ , and 1 on the edges in $EG - \{e, e'\}$. We say that ε is *feasible* if $(c_{\tau, \varepsilon}, d)$ has a solution; e.g., $\varepsilon = 0$ is feasible. The maximum feasible $\varepsilon \leq 1$ is denoted by $\alpha(\tau)$.

Suppose that there is a fork $\tau = (e, x, e')$ with $\alpha(\tau) = 1$. Then one can split off e, e' at x preserving solvability of the problem. More precisely, let G' arise from G by

deleting e, e' and adding e_τ . Since $|EG'| = |EG| - 1$ and $(G', U)^*$ is solvable, it has a half-integral solution; this is easily transformed into a half-integral solution to $(G, U)^*$.

Thus, $\alpha(\tau) < 1$ for all forks τ in G . Consider a fork $\tau = (e, x, e')$; let $e = xy$ and $e' = xz$. Since $(c_{\tau, \varepsilon}, d)$ has no solution for $\alpha(\tau) < \varepsilon \leq 1$, there is an \mathcal{H} -primitive cut-, 2,3-, or 4f-metric m on $VG_\tau = VG$ such that $c_{\tau, \varepsilon}(m) - d(m) < 0$ (by Theorem 3 and arguments in Section 1). Define $\omega_\tau(m) := m(e) + m(e') - m(e_\tau)$; then $\omega_\tau(m) \geq 0$ (since m is a metric). Clearly $c_{\tau, \varepsilon}(m) = c(m) - \varepsilon\omega_\tau(m)$, and now $c(m) \geq d(m)$ (as (c, d) is solvable) implies that $\omega_\tau(m) > 0$. Hence,

$$(3.1) \quad \alpha(\tau) = \min\{(c(m) - d(m))/\omega_\tau(m)\}, \text{ where the minimum is taken over all } \mathcal{H}\text{-primitive cut-, 2,3- and 4f-metrics } m \text{ for which } \omega_\tau(m) > 0.$$

An \mathcal{H} -primitive m that achieves the minimum in (3.1) is called *critical* for τ .

Statement 3.1. $c(m) - d(m)$ and $\omega_\tau(m)$ are even for any cut-, 2,3- or 4f-metric m .

Proof. Let C be the circuit formed by the edges e, e', e_τ . Since $\omega_\tau(m) \equiv m(C) \pmod{2}$ and m is bipartite, $\omega_\tau(m)$ is even. Next, the graph $(VG, EG \cup U)$ is eulerian, therefore it is represented as the union of pairwise edge-disjoint circuits C_1, \dots, C_k . Then $c(m) - d(m) \equiv \sum_{i=1}^k m(C_i) \pmod{2}$. Since each $m(C_i)$ is even, $c(m) - d(m)$ is even. •

We know that for any $u \in EG$, $m(u) \leq 1$ if m is a cut metric, $m(u) \leq 2$ if m is a 2,3-metric, and $m(u) \leq 4$ if m is an \mathcal{H} -primitive 4f-metric (by Theorem 5). Hence,

$$(3.2) \quad \begin{aligned} \omega_\tau(m) &\in \{0, 2\} && \text{if } m \text{ is a cut metric;} \\ &\in \{0, 2, 4\} && \text{if } m \text{ is a 2,3-metric;} \\ &\in \{0, 2, 4, 6, 8\} && \text{if } m \text{ is a 4f-metric.} \end{aligned}$$

Summing up (3.1),(3.2) and Statement 3.1, we observe the following.

Statement 3.2. Let $0 < \alpha(\tau) < 1$, and let m be a metric critical for τ . Then:

- (i) m is not a cut-metric;
- (ii) if m is a 2,3-metric then $\alpha(\tau) = 1/2$ (cf. [Ka2]);
- (iii) if m is a 4f-metric then $\alpha(\tau) \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$, and in case $\alpha(\tau) = 3/4$ the equalities $m(e) = m(e') = 4$ and $m(y, z) = 0$ hold. •

The case $\alpha(\tau) = 3/4$ will be of most interest for us in many stages of the proof.

Now we continue considerations begun in Section 2. Let us fix f, B and h_F ($F \in B$) satisfying (2.5),(2.7) and the properties as in Lemma 2.1. In view of (2.4) and

Statement 2.3, for any $F \in B$ the circuit C_F has at least one common edge with $C_{F'}$ for some $F' \in B - \{F\}$, and C_F has edges u, u' with $h_F(u) = 1/2$ and $h_F(u') \in \{0, 1\}$. We first eliminate one simple case.

Statement 3.3. *For distinct $F, F' \in B$ there are no edges $e \in C_F$ and $e' \in C_{F'}$ such that $h_F(e) = h_{F'}(e') = 0$ and either $e = e'$ or e and e' are adjacent.*

Proof. Suppose that such e, e' exist. By (2.2), $e \in \text{bd}(F)$ and $\text{bd}(F')$ (as $f_F^e = f_{F'}^{e'} = 0$). Let $e = e'$. Delete e from G , forming G' ; then the holes F and F' merge into one new face. Clearly f gives a solution for $(G', U)^*$. Since we get the (non-eulerian) three hole case, $(G', U)^*$ has a half-integral solution, whence $(G, U)^*$ has a half-integral solution; a contradiction.

Now let e and e' be distinct and incident to a vertex x . Clearly G can be splitted at x in such a way that the holes F and F' merge into one face of the resulting graph G' , and f gives a solution for $(G', U)^*$. Now apply arguments as above. •

In the rest of this section we show that case $|B| = 2$ is impossible for the minimal counterexample in question. Cases $|B| = 4$ and $|B| = 3$ will be excluded in Sections 4 and 5, respectively, and thus Theorem 1 will follow. We use the following two key lemmas (they will be important for next sections as well).

Lemma 3.4. *Let L be a maximal nontrivial path in $C_F \cap C_{F'}$ ($F, F' \in B$). Then either $h_F(e) = h_{F'}(e) = 1/2$ for all $e \in L$, or $h_F(e) = 0$ for all $e \in L$, or $h_{F'}(e) = 0$ for all $e \in L$.*

Lemma 3.5. *Let $F, F' \in B$, and let $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ and $P' = (v'_0, e'_1, v'_1, \dots, e'_q, v'_q)$ be paths (possibly circuits) in C_F and $C_{F'}$, respectively, such that $v_0 = v'_0$, $e_1 = e'_1$, $e_2 \neq e'_2$ and $v_k = v'_q$. Let the region bounded by P and P' (outside D_F and $D_{F'}$) contain no hole. Then at least one of $h_F(e_1)$ and $h_{F'}(e_1)$ is not $1/2$.*

Assuming that these lemmas are valid, consider case $|B| = 2$. Let for definiteness $B = \{I, J\}$. If $C_I = C_J$ then $(G, U)^*$ has a half-integral solution by Lemma 3.4 and Statement 2.3. So assume that C_I is different from C_J , and let $\{P_1, \dots, P_k\}$ be the set of maximal nontrivial paths in $C_I \cap C_J$. If for some $i \in \{1, \dots, k\}$ and $e \in P_i$, $h_I(e) = h_J(e) = 1/2$ then these equalities hold for all $e \in P_i$ (by Lemma 3.4) and now lemma 3.5 leads to a contradiction. Otherwise there is $N \subseteq \{1, \dots, k\}$ such that $h_I(e) = 0$ for all $e \in P_i$, $i \in N$, and $h_J(e) = 0$ for all $e \in P_i$, $i \notin N$. Define capacities c' on EG_I and capacities c'' on $EG - (EG_I - C_I)$ by

$$(3.3) \quad c'(e) := 0 \text{ if } e \in P_i \text{ and } i \in N, \text{ and } c'(e) := 1 \text{ otherwise,}$$

$$c''(e) := 0 \text{ if } e \in P_i \text{ and } i \notin N, \text{ and } c''(e) := 1 \text{ otherwise.}$$

Then $c'(e) \geq h_I(e)$ for $e \in C_I$, $c''(e) \geq h_J(e)$ for $e \in C_J$, and $c'(e) + c''(e) = 1$ for $e \in C_I \cap C_J$. Since c' is integral and $|\mathcal{H}_I| \leq 3$, the problem for c' and U_I has a half-integral solution, and similarly for c'' and U_J . Combining these we get a half-integral solution to $(G, U)^*$; a contradiction.

To prove Lemmas 3.4 and 3.5, we need some preliminary observations.

Following [Ka2], for a fork $\tau = (e, x, e')$ we introduce the number $\beta(\tau)$ which, as we shall see later, gives a lower bound for $\alpha(\tau)$ and is easier to handle than $\alpha(\tau)$:

$$(3.4) \quad \beta(\tau) := 1 + f^{e,e'} - \frac{1}{2}f^e - \frac{1}{2}f^{e'} \quad (= 1 - \frac{1}{2}(f^{e,u} + f^{e,u'} + f^{e',u} + f^{e',u'})),$$

where $E(x) = (e, e', u, u')$, and for edges p and q , $f^{p,q}$ denotes $\sum(f(L) : L \in \mathcal{L}, p, q \in L)$. By symmetry,

$$(3.5) \quad \beta(e, x, e') = \beta(u, x, u').$$

Statement 3.6 [Ka2]. $\beta(\tau) \leq \alpha(\tau)$.

Proof. Let for definiteness $f^e \geq f^{e'}$. Define the function c' on EG_τ as: $c'(e) := f^e - f^{e,e'}$; $c'(e') := f^{e'} - f^{e,e'}$; $c'(e_\tau) := 1 + f^{e,e'} - f^e$; and $c'(w) := c(w)$ for the other edges w . An easy transformation of f gives a solution to (c', d) . Put $c'' := c_{\tau, \beta(\tau)}$ and $\varepsilon := (f^e - f^{e'})/2$. One can check that $c''(w) - c'(w)$ is equal to ε for $w = e', e_\tau$; $-\varepsilon$ for $w = e$; and 0 for the other $w \in EG_\tau$. Since $\varepsilon > 0$, $c''(m) \geq c'(m)$ for any metric m , whence the solvability for (c', d) implies that for (c'', d) . •

Remark 3.7. Statements 3.2 and 3.6 imply that for a fork τ if $\beta(\tau) = 3/4$ then $\alpha(\tau) = \beta(\tau)$. Moreover, from the proof of Statement 3.6 one can see that in this case f can be transformed locally, within the edges e, e', e_τ , to give a solution f' to $(c_{\tau, 3/4}, d)$. More precisely, let $f^e \geq f^{e'}$ and $P \in \mathcal{L}$. If $e \notin P$, put $f'(P) := f(P)$. If $e, e' \in P$ then P is transformed into P' with $f'(P') := f(P)$ by replacing $\{e, e'\}$ by e_τ . If $e \in P \not\ni e'$, create the path P' from P by replacing e by $\{e', e_\tau\}$; put $f'(P) := f(P)(\frac{1}{2}f^e + \frac{1}{2}f^{e'} - f^{e,e'})/(f^e - f^{e,e'})$ and $f'(P') := f(P)(\frac{1}{2}f^e - \frac{1}{2}f^{e'})/(f^e - f^{e,e'})$. One can check that f' is $(c_{\tau, 3/4}, d)$ -admissible. By Statement 3.2, there is a primitive 4f-metric m critical for τ ; then $c_{\tau, 3/4}(m) = d(m)$. These observations yield two useful properties:

(3.6) each edge $w \in EG_\tau$ with $m(w) > 0$ is saturated by f' (i.e., $(f')^w = c_{\tau, 3/4}(w)$) and every path $P \in \mathcal{L}(f')$ is shortest for m ;

(3.7) if $f^e > f^{e'}$ then for $F \in \mathcal{H}$ with $f_F^e > f_F^{e'}$, each of $(f')_F^e$, $(f')_F^{e'}$, $(f')_F^{e_r}$ is nonzero; if, in addition, $f_F^{e_r} = 0$ then every path in $\mathcal{L}_F(f')$ passing e_r contains e' .

The following statement appeals to (1.7), evident topological observations and the fact that all paths in $\mathcal{L}(f')$ are shortest for m' ; we leave its proof to the reader.

Statement 3.8. *Let f' be a (non-crossing) solution for some G', c', d' , and let B be a bunch for f' . Let $c'(m') = d'(m')$ for some consistent 4f-metric m' on VG' induced by $\sigma : VG' \rightarrow VH$. Next, let C_F be a circuit in $\mathcal{C}(B)$, and let C be its image (by σ extended as in (1.7)) in H . Then C is a simple circuit, and C_F separates holes $F', F'' \in \mathcal{H}'$ in G' if and only if C separates the faces $\sigma(F'), \sigma(F'')$ in H . •*

This statement together with (1.7)(i) and (ii)-(iii) in Theorem 5 implies that

(3.8) for G', c', d', f', B, m' as in Statement 3.8, if m' is \mathcal{H}' -primitive, $m'(e) = 4$ for some $e = xy \in EG'$ and e lies in the region $D' = D_F(f')$ for $F \in B$, then D' contains no hole except F .

Indeed, let C be the image by σ of the boundary of D' . If D' contains a hole $F' \neq F$ then, by Statement 3.8, the circuit C does not follow the boundary of the face $\tilde{F} := \sigma(F)$ in H . This means that there is an $s-t$ path $P \in \mathcal{L}_F(f')$ such that its image $Q := \sigma(P)$ does not lie in $bd(\tilde{F})$. Since P is shortest for m' (as $c'(m') = d'(m')$), Q is a shortest path in H . Hence, some of the ends of Q , $\sigma(s)$ say, is b_i for $i \in \{1, 2, 3\}$, by (iii) in Theorem 5 (here $L_e = b_0 \dots b_5$ is the image of (x, e, y) as in (ii) of this theorem). Remove from \mathbb{R}^2 the set $F \cup e \cup X \cup Y$, where $X := \sigma^{-1}(x)$ and $Y := \sigma^{-1}(y)$. In the resulting space consider the component Ω containing s . Obviously σ brings Ω into L_e . This implies that P meets X or Y . Then the part of P outside Ω is a path P' such that $Q' := \sigma(P')$ has both ends in $bd(\tilde{F}) - \{b_1, b_2, b_3\}$. Furthermore, Q' is shortest and it does not lie in $bd(\tilde{F})$; a contradiction with (iii) in Theorem 5.

For a vertex x in C_F ($F \in B$) let $E_F(x)$ denote the set of edges incident to x and contained in $D_F - C_F$; then $|E_F(x)| \leq 2$.

Proof of Lemma 3.4. Let for definiteness $F = I$ and $F' = J$, and let each of h_I and h_J be not identically zero on L . One must prove that $h_I(e) = h_J(e) = 1/2$ for all $e \in L$. Suppose this is not so. Then for some of I, J , for I say, there are consecutive elements e, x, e' in L (where $x \in VG$ and $e, e' \in E(x)$) such that $h_I(e) \neq 0 = h_I(e')$. By Statement 3.3, $h_J(e) \neq 0 \neq h_J(e')$, hence $h_I(e) = h_J(e) = 1/2$ and $h_J(e') \in \{\frac{1}{2}, 1\}$. Since $h_I(e) \neq h_I(e')$, $E_I(x)$ is nonempty (in view of (2.8)). Consider two possible cases.

Case 1. $|E_I(x)| = 1$. Let for definiteness $E(x) = (e, u, e', u')$ and $E_I(x) = \{u\}$; see Fig. 3.2. Clearly $f^{u, e'} = f^{u, u'} = 0$. Also $f^{e, e'} + f^{e, u'} \leq 1/2$ (as any path in $\mathcal{L}(f)$ passing e and some of e', u' concerns the flow \bar{f}_J , and the total amount of flow on these paths is at most $h_J(e) = 1/2$). Hence, for the fork $\tau = (e, x, u)$ we have $\beta(\tau) \geq 3/4$ (cf. (3.4)), whence $\alpha(\tau) = \beta(\tau) = 3/4$.

Consider the solution f' for $G_\tau, c_{\tau, 3/4}, U$ obtained from f as in Remark 3.7 (for $\tau = (e, x, u)$), and a 4f-metric m critical for τ and induced by $\sigma : VG \rightarrow VH$. By Statement 3.2(iii), $m(e) = m(u) = 4$ and $m(y, z) = 0$, where y (z) is the end of e (u) different of x . Let for definiteness $\sigma(x) = b_4$, $\sigma(y) = \sigma(z) = b_0$ (cf. (ii) in Theorem 5). By (3.7) (for e, u), $(f')_J^u > 0$, therefore u belongs to a path P in $\mathcal{L}_J(f')$. By (3.6), P is shortest for m . So b_0 and b_4 belong to a shortest $\sigma(J)$ -path in H , whence $\sigma(J)$ is \tilde{J} as in Theorem 5. On the other hand, $e' \in \text{bd}(I)$ (as $h_I(e') = 0$), whence $u = xz \in \text{bd}(I)$ (as u is in D_I and u, e' are consecutive in $E(x)$). This implies that $b_0 = \sigma(z)$ and $b_4 = \sigma(x)$ belong to the boundary of $\sigma(I)$ in H . A contradiction with (iv) in Theorem 5.

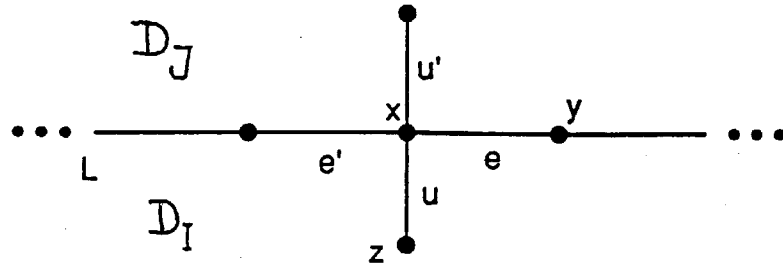


Fig. 3.2

Case 2. $|E_I(x)| = 2$. Let for definiteness $E(x) = (e, u, u', e')$; then $E_I(x) = \{u, u'\}$; see Fig. 3.3a. Since $E_J(x)$ is empty, $h_J(e) = h_J(e') = 1/2$ (in view of (2.8)). Obviously $f^{e', u} = f^{e', u'} = 0$ and $f^{e, u} + f^{e, u'} \leq h_I(e) = 1/2$, whence $\beta(\tau) = 3/4$ for $\tau = (e, x, e')$.

Consider the solution f' for $G_\tau, c_{\tau, 3/4}, U$ as in Remark 3.7, and a 4f-metric m critical for τ and induced by $\sigma : VG \rightarrow VH$. By Statement 3.2(iii), $m(e) = m(e') = 4$ and $m(y, z) = 0$, where y (z) is the end of e (e') different of x . Let for definiteness $\sigma(x) = b_4$ and $\sigma(y) = \sigma(z) = b_0$. By (3.7), $(f')_I^e = (f')_I^{e'} > 0$ (taking into account that $h_I(e) = 1/2$, $h_I(e') = 0$), whence $\sigma(I)$ coincides with \tilde{J} as in Theorem 5.

Next, let $D' := D_{IJ}(f')$. Clearly the boundary C' of D' is obtained from C_I by replacing e, e' by e_τ . Then e lies in D' , whence D' contains no hole except I (by (3.8)). Consider the regions $X := \sigma^{-1}(b_4)$ and $Y := \sigma^{-1}(b_0)$ (assuming that σ is extended as in (1.7)); then x is in X and y, z, e_τ are in Y . Since I is the only hole in D' and b_4 belongs to the boundaries of at least two faces of H , X meets C' . Moreover, some $v \in \{y, z\}$ belongs to a component Ω of $D_I - X$ that does not contain I ; see Fig. 3.3(b). Let for definiteness $v = y$, and let $Q = z_0 \dots z_r$ be the part of C_I such that $z_0 \in Y$,

$z_\tau \in X$, Q does not contain e , and all edges and inner vertices of Q are in $\Omega - Y$. From Statement 3.8 it follows that Q is mapped by σ to $L_e = b_0 \dots b_4$. Then $\sigma(z_1) = b_i$ for some $i \geq 1$.

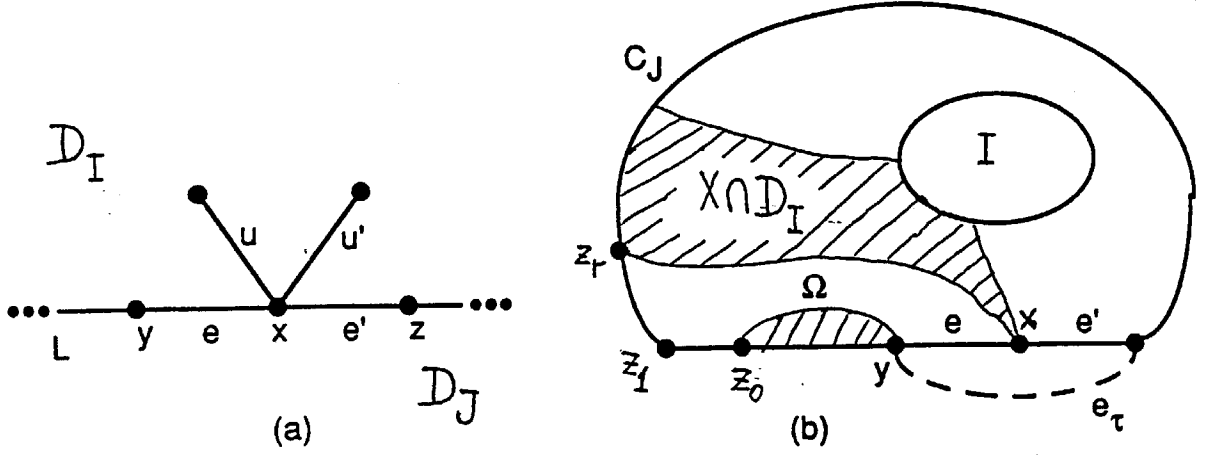


Fig. 3.3

Obviously the edge $w = z_0 z_1$ is not in $\text{bd}(I)$. Hence, w belongs to a path $P = y_0 \dots y_q \in \mathcal{L}_I(f')$; let for definiteness $z_1 = y_j$ and $z_0 = y_{j+1}$. We observe that the part P' of P from y_{j+1} to y_q does not meet X (otherwise $\sigma(P)$ would pass b_i, b_0, b_4 in this order, contrary to the fact that P is shortest for m). Hence, P' must pass through e_τ . Then P' contains e (by (3.7)), and therefore, P' contains $x \in X$; a contradiction. •

Proof of Lemma 3.5. Put $e := e_1$, $x := v_1$, $e' := e_2$, $u' := e'_2$. Let for definiteness $F = I$ and $F' = J$. Suppose that $h_I(e) = h_J(e) = 1/2$. Since e, e', u' are distinct, $|E_I(x)| + |E_J(x)| \leq 1$. Therefore, one may assume that $E_I(x) = \emptyset$; let $E(x) = \{e, e', u', u\}$ (in case $E(x) = \{e, e', u, u'\}$ arguments are similar). We observe that $h_I(e') = h_I(e) = 1/2$, that $f^{e',u} + f^{e,u'} = 0$ (taking into account that $f^{e'} = \bar{f}_I^{e'}$ and $f^{u'} = \bar{f}_J^{u'}$ since there is no hole between P and P'), and that $f^{e,u} + f^{e,u'} \leq h_J(e) = 1/2$. Hence, $\beta(\tau) = 3/4$ for $\tau = (e, x, e')$. Consider a solution f' for $G_\tau, c_\tau, 3/4, U$ as in Remark 3.7, and a 4f-metric m critical for τ and induced by $\sigma : VG \rightarrow VH$. Let for definiteness $\sigma(x) = b_4$ and $\sigma(y) = \sigma(z) = b_0$, where y (z) is the end of e (e') different of x , see Fig. 3.4.

By (3.7), $(f')_e^e, (f')_J^{e'}, (f')_J^{e_\tau} > 0$, whence the corresponding circuit C'_J for f' is formed from C_J by replacing e by e', e_τ . Also C'_I is formed from C_I by replacing e, e' by e_τ . Hence, $\sigma(J)$ is \tilde{J} as in Theorem 5. Clearly e lies in $D' := D_{JI}(f')$, whence J is the only hole in D' , by (3.8). In addition, b_0 belongs to a shortest \tilde{I} -path, where $\tilde{I} := \sigma(I)$ (since $y \in C'_I$).

Let $\tilde{I} := \sigma(I)$. Since y is in C'_I , b_0 belongs to a shortest \tilde{I} -path in H . Furthermore, the facts that $x \in X := \sigma^{-1}(b_4)$ and there is no hole between P and P' imply that X meets the part \tilde{P} of P from v_2 to v_k . Thus, there is a vertex x' in C'_I such that $\sigma(x') = b_4$, whence b_4 belongs to a shortest \tilde{I} -path in H .

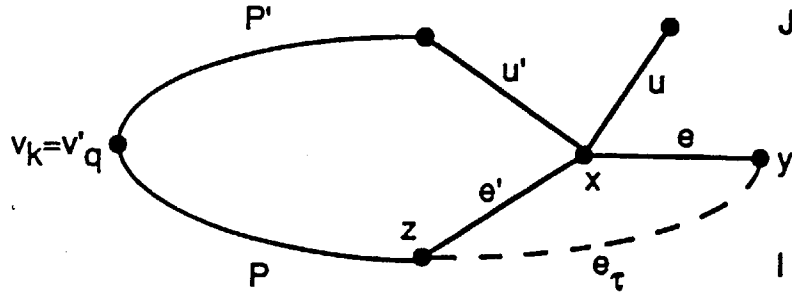


Fig. 3.4

By (iv) in Theorem 5, some $b \in \{b_0, b_4\}$ is not in $\text{bd}(\tilde{I})$. Then $v \in \{x', y\}$ such that $\sigma(v) = b$ is contained in a path $Q \in \mathcal{L}_I(f')$. By (v)(a) in Theorem 5, Q must separate J from K and O , where $\mathcal{H} = \{I, J, K, O\}$ (taking into account that $\sigma(Q)$ is a shortest \tilde{I} -path in H passing b). This means that C'_I (as well as C_I) separates J from I, K, O . Hence, $|\mathcal{H}_I| = 3$. On the other hand, by (v)(b) in Theorem 5, no I -path in $\mathcal{L}(f')$ (as well as in $\mathcal{L}(f)$) separates K and O . Thus, there is a bunch B' for f such that either $|B'| = 3$, or $|B'| = 2$ and $\{|\mathcal{H}_F| : F \in B'\} = \{2, 2\}$. A contradiction with the choice of B in (2.5). •

4. EXCLUSION OF $|B| = 4$

In this section our goal is to show that case $|B| = 4$ is impossible for the minimal counterexample in question. In fact, we show that the functions h_F can be transformed to some h'_F 's in such a way that at least one h'_F is integral, whence $(G, U)^*$ has a half-integral solution, by Statement 2.3. Our arguments will rely on Lemmas 4.1-4.4 below (they will be also used in the next section where we study case $|B| = 3$). These lemmas will be proved in the end of this section.

Let $F \in B$. A maximal nontrivial path P in C_F with $h_F(e) = 1/2$ for all $e \in P$ is called a $1/2$ -segment for F . By Statement 2.3, F has at least one $1/2$ -segment, and this segment is not C_F . Next, let δX be a cut in G_F . Obviously, if δX is tight (i.e., $h_F(X) = d_F(X)$) then δX is the union of simple tight cuts (see Section 2 for definitions). In what follows, speaking of a cut, we usually mean a simple cut of the

graph in question. Lemma 4.1 strengthens (i) in Statement 2.2 for case $|\mathcal{H}_F| = 1$, and Lemma 4.2 exhibit a relation between tight cuts and 1/2-segments.

Lemma 4.1. *Let $F \in B$ and $|\mathcal{H}_F| = 1$. Then each edge $e \in C_F$ belongs to a tight cut in G_F .*

Lemma 4.2. *Let $F \in B$ and $|\mathcal{H}_F| = 1$. Then:*

- (i) *for any tight cut δX in G_F and any 1/2-segment S for F , $|\delta X \cap S| \leq 1$;*
- (ii) *the number ω_F of 1/2-segments for F is even;*
- (iii) *if $S_0, S_1, \dots, S_{2k-1}$ are the 1/2-segments for F occurring in this order in C_F then every tight cut meeting some S_i meets the opposite 1/2-segment S_{i+k} (taking indices modulo $2k$).*

A face in G that is not a hole is called *intermediate*. We say that two elements $x, y \in VG \cup EG$ are *dually connected* if they belong to the boundary of the same intermediate face in G .

Lemma 4.3. *For distinct $F, F', F'' \in B$ let $P = x_1 \dots x_k, P' = y_1 \dots y_r, P'' = z_1 \dots z_q$ be 1-paths in $C_F, C_{F'}$ and $C_{F''}$, respectively, such that $x_1 = y_r, y_1 = z_q, z_1 = x_k$, and $x_2 \neq y_{r-1}$. Let C_F and $C_{F'}$ have a common edge e with an end at x_1 for which $h_F(e) = h_{F'}(e) = 1/2$. Let the region bounded by P, P', P'' contain no hole. Then for some edge $u = z_{i-1}z_i$ ($1 < i \leq q$) one holds:*

- (i) $h_{F''}(u) = 1$;
- (ii) u is dually connected with x_1 .

Lemma 4.4. *Let e and u be two consecutive edges in C_F such that $e \in C_{F'}$ and $u \in C_{F''}$ for distinct $F', F'' \in B - \{F\}$. Let e, u be incident to a vertex x . Let e' (u') be the edge in $C_{F'}$ ($C_{F''}$) incident to x and different of e (u). Then:*

- (i) $h_F(e) = h_{F'}(e) = h_F(u) = h_{F''}(u) = 1/2$;
- (ii) $e' = u'$ unless $|B| = 4$ and x is in $C_{\tilde{F}}$, where $B = \{F, F', F'', \tilde{F}\}$.

Assuming that the above lemmas are valid, we now begin to study case $|B| = 4$. Clearly $|\mathcal{H}_F| = 1$ for each $F \in B$. We need some additional terminology and notations. Consider some $F \in B$. We say that an edge $e \in C_F$ is a *1-edge* if $e \notin C_{F'}$ for any $F' \in B - \{F\}$, and a *2-edge* otherwise. A maximal nontrivial path in C_F of which all edges are 1-edges (respectively, 2-edges common for C_F and $C_{F'}$ for some fixed

