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Edge-disjoint T-paths of Minimum Total Cost

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Edge-disjoint T -paths of minimum total cost [†]

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Abstract. Suppose that $G = (V, E)$ is a graph and T is a subset of its vertices. Let ν be the maximum number of edge-disjoint T -paths (i.e. paths in G connecting distinct elements in T). A classical result in combinatorial optimization, due to Mader and, independently, Lomonosov, is that ν can be expressed, by use of a minimax relation, via a value determined by a family of certain cuts in G .

We consider a more general problem in which, given nonnegative costs $c(e)$ of edges $e \in E$, one requires to find ν edge-disjoint T -paths P_1, \dots, P_ν such that their total cost $\sum(c(e) : e \in P_i, i = 1, \dots, \nu)$ is as small as possible.

We prove a minimax relation for this problem. Moreover, being “constructive”, the proof provides a strongly polynomial algorithm to find paths as required.

1. Introduction

By a *graph* we mean an undirected graph with possible multiple edges. VG and EG denote the vertex-set and the edge-set, respectively, of a graph G . When it leads to no confusion, an edge with end vertices x and y is denoted by xy .

We deal with a graph G whose edges $e \in EG$ have nonnegative integer-valued costs $c(e) \in \mathbf{Z}_+$, and with a subset $T \subseteq VG$, called the set of *terminals* in G . A path P in G with both ends in T is called a T -*path*; the value $\sum_{e \in P} c(e)$, the “cost of P ”, is denoted by $c(P)$. [We will often consider a path as an edge-set; for $S' \subseteq S$ and $f : S \rightarrow \mathbf{R}$, $f(S')$ denotes $\sum(f(e) : e \in S')$.] Let $\nu = \nu(G, T)$ denote the greatest number of edge-disjoint T -paths in G . We consider the problem:

- (1.1) find a set \mathcal{P} consisting of ν edge-disjoint T -paths in G so that their total cost $c(\mathcal{P}) := \sum_{P \in \mathcal{P}} c(P)$ is as small as possible.

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E.g., if c is all-unit on EG , (1.1) is the problem of finding ν edge-disjoint T -paths in G that cover the least number of edges. If c is identically zero, (1.1) turns into the well-known problem on maximum packing of (edge-disjoint) T -paths; Mader [Ma1] and, independently, Lomonosov [Lom] showed that ν satisfies a certain minimax relation.

It turns out to be more convenient to pose a slightly more general problem, namely:

- (1.2) given a nonnegative real $p \in \mathbb{R}_+$, find a set \mathcal{P} of edge-disjoint T -paths in G that maximizes the objective function

$$\varphi(\mathcal{P}, p) := p|\mathcal{P}| - c(\mathcal{P}).$$

Evidently, if p is large enough, (1.2) becomes equivalent to (1.1); in particular, one can take $p = c(EG) + 1$.

The main aim of the present paper is to establish a minimax relation for (1.2). To state this, we need some definitions. Let $\phi = (X, U)$ be a pair consisting of a subset X of *inner* vertices in G (i.e. $X \subseteq VG - T$) and a subset $U \subseteq \delta X$ such that $|U|$ is odd. [Here $\delta X = \delta^G X$, the *cut* induced by X , is the set of edges of G connecting X and $VG - X$.] We say that ϕ is a *fragment* and denote X and U by X_ϕ and U_ϕ , respectively. Define

$$\begin{aligned} \chi_\phi(e) &:= 1 && \text{if } e \in U_\phi, \\ &:= -1 && \text{if } e \in \delta X_\phi - U_\phi, \\ &:= 0 && \text{for the other edges in } G \end{aligned}$$

(the *characteristic function* of ϕ). Let $\tilde{\mathcal{F}}$ be the set of all fragments for G, T . For $\alpha : \tilde{\mathcal{F}} \rightarrow \mathbb{R}_+$ and $\gamma : EG \rightarrow \mathbb{R}_+$, define the *amortized cost* function $c_{\alpha, \gamma}$ on EG to be

$$(1.3) \quad c_{\alpha, \gamma} := c + \gamma + \sum (\alpha(\phi)\chi_\phi : \phi \in \tilde{\mathcal{F}}).$$

We say that (α, γ) is *admissible* for $p \geq 0$ if:

- (1.4) $c_{\alpha, \gamma}$ is nonnegative;

- (1.5) $\lambda := c_{\alpha, \gamma}$ satisfies $\text{dist}_\lambda(s, s') \geq p$ for all distinct $s, s' \in T$.

[$\text{dist}_\lambda(x, y)$ is the distance between vertices x and y in G with the edge length λ .]

Statement 1.1. For any set \mathcal{P} of edge-disjoint T -paths and (α, γ) admissible for p ,

$$(1.6) \quad \varphi(\mathcal{P}, p) \leq \gamma(EG) + \sum (\alpha(\phi)(|U_\phi| - 1) : \phi \in \tilde{\mathcal{F}}).$$

Proof. By (1.3) and (1.5), for $P \in \mathcal{P}$ we have

$$p - c(P) \leq \gamma(P) + \sum_{\phi \in \tilde{\mathcal{F}}} \alpha(\phi) \chi_\phi(P).$$

Since the paths in \mathcal{P} are edge-disjoint, $\sum (\gamma(P) : P \in \mathcal{P}) \leq \gamma(EG)$. Hence,

$$\varphi(\mathcal{P}, p) = \sum_{P \in \mathcal{P}} (p - c(P)) \leq \sum_{P \in \mathcal{P}} (\gamma(P) + \sum_{\phi \in \tilde{\mathcal{F}}} \alpha(\phi) \chi_\phi(P)) \leq \gamma(EG) + \sum_{\phi} \alpha(\phi) \sum_P \chi_\phi(P).$$

From the fact that the paths in \mathcal{P} are edge-disjoint it follows that $\sum_{P \in \mathcal{P}} \chi_\phi(P) \leq |U_\phi|$. Moreover, since $\chi_\phi(P)$ is, obviously, even, while $|U_\phi|$ is odd, $\sum_{P \in \mathcal{P}} \chi_\phi(P)$ does not exceed $|U_\phi| - 1$. This implies (1.6) (as $\alpha(\phi)$ is nonnegative). •

We prove the following theorem.

Theorem 1. For any $p \geq 0$,

$$(1.7) \quad \max\{\varphi(\mathcal{P}, p)\} = \min\{\gamma(EG) + \sum (\alpha(\phi)(|U_\phi| - 1) : \phi \in \tilde{\mathcal{F}})\},$$

where the maximum is taken over the sets \mathcal{P} of edge-disjoint T -paths, while the minimum is taken over all (α, γ) admissible for p .

Like many proofs of minimax relations in combinatorial optimization, the proof of Theorem 1 utilizes the primal-dual method in linear programming. Also various combinatorial techniques are elaborated in the proof. Some tools involved appeal to the matching theory. Some other ones come from [Ka1] where it was proved that the fractional relaxation of (1.2) (i.e. the corresponding minimum cost maximum multi(commodity)flow problem) has a half-integral optimal solution (see also [Ka2]). Theorem 1 will follow from an auxiliary theorem discussed in the next section, where we also briefly outline the structure of the present paper and main stages of the proof.

In fact, the proof of Theorem 1 will provide a strongly polynomial algorithm to solve problems (1.2) and (1.1).

Remark 1.2. We shall see in Section 7 (Statement 7.3) that the minimum in (1.7) can be achieved with an α such that $|U_\phi| \geq 3$ for all ϕ with $\alpha(\phi) > 0$. In other words, in the above definition of a fragment ϕ we could add the condition that $|U_\phi| \geq 3$.

2. Auxiliary theorem

We call $\mathcal{P}, \alpha, \gamma$ *good* for p if they achieve the equality in (1.7). From the proof of Statement 1.1 it easily follows that $\mathcal{P}, \alpha, \gamma$ are good if and only if the following (“complementary slackness”) conditions hold:

- (2.1) $\lambda(P) = p$ for each $P \in \mathcal{P}$ (hence, P is shortest for $\lambda := c_{\alpha, \gamma}$, by (1.5));
- (2.2) for $e \in EG$, $\gamma(e) > 0$ implies that e is *covered* by \mathcal{P} (i.e. e belongs to some $P \in \mathcal{P}$);
- (2.3) for $\phi \in \tilde{\mathcal{F}}$, $\alpha(\phi) > 0$ implies $\sum_{P \in \mathcal{P}} \chi_\phi(P) = |U_\phi| - 1$.

If ϕ satisfies the equality in (2.3), we say that ϕ is *saturated* by \mathcal{P} . Observe that if ϕ is saturated then one of the two situations takes place:

- (2.4) \mathcal{P} covers exactly $|U_\phi| - 1$ edges in U_ϕ and no edge in $\delta X_\phi - U_\phi$; or
- (2.5) \mathcal{P} covers all edges in U_ϕ and exactly one edge in $\delta X_\phi - U_\phi$.

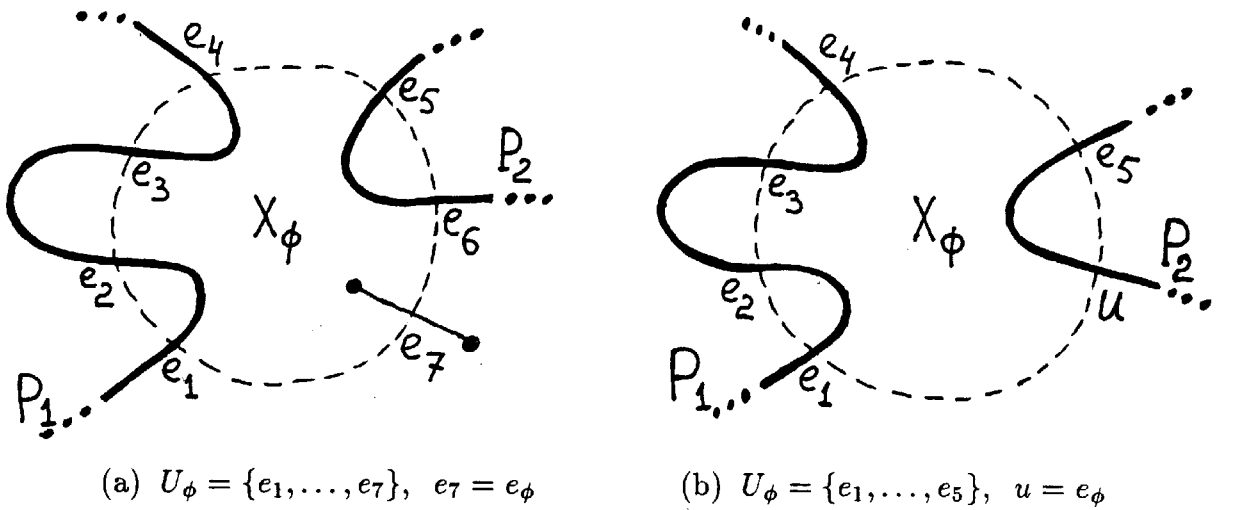


Fig. 2.1

(See Fig. 2.1.) The edge in U_ϕ that is not covered by \mathcal{P} in case (2.4) as well as the

edge in $\delta X_\phi - U_\phi$ that is covered by \mathcal{P} in case (2.5) is called the *root* of ϕ and denoted by e_ϕ .

Let us consider p as a parameter in the problem which increases from 0 to ∞ . Clearly, $\mathcal{P} = \emptyset, \alpha = 0, \gamma = 0$ are good for $p = 0$.

In what follows we refer to a set of edge-disjoint T -paths in G as a *packing*. Validity of Theorem 1 is provided by the following result.

Theorem 2. *Suppose that $\mathcal{P}, \alpha, \gamma$ are good for p . Then one of the following is true:*

- (i) *there exists a packing \mathcal{P}' such that $|\mathcal{P}'| = |\mathcal{P}| + 1$, and $\mathcal{P}', \alpha, \gamma$ are good for p ;*
- (ii) *there exist $p' > p, \alpha' : \tilde{\mathcal{F}} \rightarrow \mathbb{R}_+$ and $\gamma' : EG \rightarrow \mathbb{R}_+$ such that for any $0 \leq \xi \leq 1$, $\mathcal{P}, \alpha_\xi, \gamma_\xi$ are good for $(1 - \xi)p + \xi p'$, where $\alpha_\xi := (1 - \xi)\alpha + \xi\alpha'$ and $\gamma_\xi := (1 - \xi)\gamma + \xi\gamma'$.*

[In fact, the alternative takes place: either (i) or (ii) is true.]

Proof of Theorem 1 from Theorem 2. Supposing, for a contradiction, that Theorem 1 is false, let \bar{p} be the maximum number such that (1.7) holds for every $0 \leq p < \bar{p}$. Two cases are possible.

(a) The equality (1.7) holds for \bar{p} . Choose $\mathcal{P}, \alpha, \gamma$ so that they are good for \bar{p} and $|\mathcal{P}|$ is as large as possible. Then (ii) in Theorem 2 implies the existence of $\bar{p}' > \bar{p}$ such that (1.7) holds for any p'' in the segment $[\bar{p}, \bar{p}']$, contrary to the definition of \bar{p} .

(b) The equality (1.7) is wrong for \bar{p} . Then $\bar{p} > 0$, and (1.7) holds for an infinite sequence $p_1 < p_2 < \dots$ of numbers which tend to \bar{p} . Choose good $\mathcal{P}_i, \alpha_i, \gamma_i$ for p_i . Since the set of different packings consisting of simple paths is finite, we may assume that all the \mathcal{P}_i 's are the same packing \mathcal{P} .

For each i , (α_i, γ_i) is a solution of the system L_i formed by the linear constraints: (i) $\alpha_i \geq 0; \gamma_i \geq 0$; (ii) $\lambda := c_{\alpha_i, \gamma_i} \geq 0$; (iii) $\lambda(P) = p_i$ for each $P \in \mathcal{P}$; (iv) $\lambda(P) \geq p_i$ for each *simple* T -path P ; (v) $\gamma_i(EG) + \sum(\alpha_i(\phi)(|U_\phi| - 1) : \phi \in \tilde{\mathcal{F}}) = p_i|\mathcal{P}| - c(\mathcal{P})$. We observe that the constraint matrices of L_i are the same for all i while the right hand side vector linearly depends on p_i . As the p_i 's tend to \bar{p} , standard l.p. arguments imply that there are solutions (α_i, γ_i) of the L_i 's which tend to some $(\alpha, \gamma) \in \mathbb{R}_+^{\tilde{\mathcal{F}}} \times \mathbb{R}_+^{EG}$. Then $\mathcal{P}, \alpha, \gamma$ are good for \bar{p} ; a contradiction. •

Throughout the remaining part of the paper we assume that the cost function c is *positive*, i.e. $c(e) > 0$ for all $e \in EG$. This assumption will significantly simplify some details of the proof. On the other hand, it leads to no loss of generality, as it is easy to show by use of standard l.p. arguments.

The proof of Theorem 2 is presented in Sections 3-8. It starts in Section 3 with

introducing elementary, but important, notions and structures and describing their properties; some of them occurred in [Ka1,Ka2]. In Sections 4,5 we give a relatively simple proof of Theorem 2 under the assumption that the current α is identically zero (whence $\lambda := c_{\alpha,\gamma}$ is positive since c is positive and γ is nonnegative). We distinguish this special case to expose basic ideas of our combinatorial primal-dual approach to the problem. The general case is studied in Sections 6-8. In this case edges $e \in EG$ with $\lambda(e) = 0$ are possible (we cannot, in general, avoid appearance of such edges, even assuming the positivity of c); this makes the analysis more involved. In Section 6 we show validity of Theorem 2 provided that a number of additional conditions is imposed, which specify the current $\mathcal{P}, \alpha, \gamma$ as well as structures related to them (in fact, to be able to prove Theorem 2 we are forced to strengthen this theorem; in particular, we have to require that the graph induced by edges $e \in \cup(P \in \mathcal{P})$ with $\lambda(e) = 0$ is a forest). In Sections 7 and 8 we verify maintenance of these properties (i.e. satisfying the imposed conditions) after the transformation of \mathcal{P} or (α, γ) ; this part is most technical and tiresome in the proof.

As mentioned above, the proof provides an algorithm for finding good $\mathcal{P}, \alpha, \gamma$ for an arbitrary p . More precisely, in Section 9 we show that the algorithm finds, in strongly polynomial time, a sequence $0 \leq p_1 < \dots < p_N$ of numbers and objects $\mathcal{P}_i, \alpha_i, \gamma_i$ ($i = 1, \dots, N$), $\alpha_{N+1}, \gamma_{N+1}$ so that: (i) $\mathcal{P} = \emptyset, \alpha = 0, \gamma = 0$ are good for any $p \in [0, p_1]$; (ii) for $i = 1, \dots, N - 1$ and $p \in [p_i, p_{i+1}]$, the packing \mathcal{P}_i and functions α and γ are good for p , where α (γ) is the corresponding convex combination of α_i and α_{i+1} (γ_i and γ_{i+1}); and (iii) $|\mathcal{P}_N| = \nu$, and for any $p \geq p_N$ the packing \mathcal{P}_N and functions α and γ are good for p , where α (γ) is the corresponding nonnegative combination of α_N and α_{N+1} (γ_N and γ_{N+1}).

Finally, in Section 10 we explain that the final $\alpha_{N+1}, \gamma_{N+1}$ enable us to derive optimal dual objects figured in the following theorem describing a minimax relation for the “pure” (i.e. zero cost) problem (hereinafter for a subset A of vertices (edges) of a graph G' , $G' - A$ denotes the graph obtained from G by removing (deleting) A).

Theorem 3 [Ma1,Lom].

$$\nu(G, T) = \frac{1}{2} \min \left\{ \sum_{s \in T} |\delta Y_s| - \eta \right\},$$

where the minimum ranges over all families of pairwise disjoint sets $Y_s \subset VG$ ($s \in T$) such that $Y_s \cap T = \{s\}$, and η denotes the number of components K of the graph $G - (\cup_{s \in T} Y_s)$ such that $|\delta^G(VK)|$ is odd.

3. Lines and potentials

Let $\lambda : EG \rightarrow \mathbb{R}_+$ be a function. Define

$$(3.1) \quad p := p_\lambda := \min\{\text{dist}_\lambda(s, s') : s, s' \in T, s \neq s'\}.$$

A *potential* $\pi(v) = \pi_\lambda(v)$ of a vertex $v \in VG$ is the λ -distance from v to T , i.e. $\min\{\text{dist}_\lambda(v, s) : s \in T\}$. Denote by $T(v) = T_\lambda(v)$ the set of terminals s closest to v , i.e. such that $\text{dist}_\lambda(v, s) = \pi(v)$. A T -path (possibly non-simple) of λ -length exactly p is called a T, λ -*line*, or, briefly, a T -*line*. A path that is part of a T -line is called a *line*. Denote by $\Gamma = \Gamma^\lambda$ the subgraph of G formed by the set of terminals and all vertices and edges occurring in T -lines.

The vertices in Γ are naturally partitioned into sets V_s ($s \in T$) and V^\bullet . Here V_s consists of all $v \in V\Gamma$ such that $\text{dist}_\lambda(s, v) < p/2$; and $V^\bullet := V\Gamma - \cup_{s \in T} V_s$; a vertex in V^\bullet is called *central*. Clearly, $T(v) = \{s\}$ for $v \in V_s$, whereas $|T(v)| \geq 2$ and $\pi(v) = p/2$ for $v \in V^\bullet$. The following property is obvious.

Statement 3.1. *Let L be a path from x to y in G .*

- (i) *If $x, y \in V_s \cup V^\bullet$ then L is a line if and only if $\lambda(L) = |\pi(x) - \pi(y)|$.*
- (ii) *If $x \in V_s$ and $y \in V_{s'}$ for distinct $s, s' \in T$ then L is a line if and only if $\pi(x) + \pi(y) + \lambda(L) = p$. •*

For $x \in V\Gamma$ denote by $E(x) = E_\lambda(x)$ the set of edges in Γ incident to x . Consider an edge $e = xy \in E\Gamma$ with $\lambda(e) > 0$. We assign to (x, e) an *attachment* $l(x, e) = l_\lambda(x, e)$ by the following rule:

- $$(3.2) \quad \begin{aligned} \text{(i)} & \text{ if } x \in V_s \cup V^\bullet, y \in V_s \text{ and } \pi(y) < \pi(x), \text{ put } l(x, e) := s; \\ \text{(ii)} & \text{ if } x \in V_s \text{ and either } y \notin V_s, \text{ or } y \in V_s \text{ and } \pi(x) < \pi(y), \text{ put } l(x, e) := \bar{s}. \end{aligned}$$

It is easy to see that if $l(x, e) = s$ then for every T -line L from s_1 to s_2 that meets x, e, y in this order, s_2 is s ; while if $l(x, e) = \bar{s}$ then for L as above, s_1 is s . Note that $\lambda(e) > 0$ makes it impossible that $x, y \in V^\bullet$.

Next, it is convenient to assume that the terminals are numbered by the integers from 1 up to $t := |T|$. Then T is identified with $\langle 1, t \rangle$, and an attachment $s \in T$ for (x, e) means the corresponding number in $\langle 1, t \rangle$; while we assume that the attachment \bar{s} is identified with the number $-s \in \langle -t, -1 \rangle$. Hereinafter $\langle i, j \rangle$ denotes the set of integers $k \neq 0$ such that $i \leq k \leq j$. For $x \in V\Gamma$ and $s \in \langle -t, t \rangle$ define $E_s(x) := \{e \in$

$E(x) : l(x, e) = s$.

Suppose that the function λ is *positive*, i.e. $\lambda(e) > 0$ for all $e \in EG$. Then every pair (x, e) ($x \in V\Gamma$, $e \in E(x)$) has an attachment $s \in \langle -t, t \rangle$. Moreover, using Statement 3.1 one can easily obtain the following description of the lines in terms of attachments.

Statement 3.2. *A path $L = (x_0, e_1, x_1, \dots, e_k, x_k)$ in Γ is a line if and only if for $i = 1, \dots, k - 1$ the attachments $l(x_i, e_i)$ and $l(x_i, e_{i+1})$ are different. •*

Now return to consideration $\mathcal{P}, \alpha, \gamma$ as above. Put $\lambda := c_{\alpha, \gamma}$ (see (1.3)). Assume that λ is *positive* and $p = p_\lambda$ (see (3.1)). If $\mathcal{P}, \alpha, \gamma$ are good for p , and B is the set of edges covered by \mathcal{P} , then (2.1)-(2.2) imply that $B \subseteq E\Gamma$ and

$$(3.3) \quad \gamma(e) = 0 \quad \text{for all } e \in EG - B.$$

One of key ideas in the proof is that we can handle a set B of edges which can be covered by some optimal packing \mathcal{P} , rather than \mathcal{P} itself. More precisely, let $B \subseteq E\Gamma$ be a set. For $x \in V\Gamma$ and $s \in \langle -t, t \rangle$ define $B(x) := B \cap E(x)$ and $B_s(x) := B \cap E_s(x)$. We say that B is *regular* if

- (3.4) (i) $|B(x)|$ is even for all $x \in V\Gamma - T$;
(ii) B saturates each $\phi \in \tilde{\mathcal{F}}$ with $\alpha(\phi) > 0$, i.e. $\chi_\phi(B) = |U_\phi| - 1$;
(iii) B is *non-excessive* for each $x \in V\Gamma - T$; this means that $|B_s(x)| \leq \frac{1}{2}|B(x)|$ for any $s \in \langle -t, t \rangle$.

In particular, the set of edges covered by \mathcal{P} as above is, obviously, regular. It turns out that for a regular set B the converse takes place. More precisely, let $\mu(B)$ denote the number of pairs (s, e) such that $s \in T$ and $e \in B(s)$. We say that $s \in \langle -t, t \rangle$ is *tight* for $x \in V\Gamma - T$ if $|B_s(x)| = |B(x)|/2$.

Statement 3.3. *If B is regular then B has a representation of the form $B = \cup(P \in \mathcal{P})$, where \mathcal{P} is a packing consisting of $\mu(B)/2$ T -lines and satisfying (2.3).*

Proof. To prove the existence of \mathcal{P} consisting of $\mu(B)/2$ T -lines, we use induction on $|B|$. Let us design a path $P = (x_0, e_1, x_1, \dots)$, starting from some $x_0 \in V\Gamma$ and $e_1 \in B(x_0)$, as follows. Suppose a simple path $P' = (x_0, e_1, x_1, \dots, e_i, x_i)$ has been constructed. If $x := x_i \in T$, put $P := P'$. Otherwise extend P' by adding $e = e_{i+1} \in B(x)$ so that: (a) $e \neq e_i$; (b) if $s := l(x, e_i)$ is tight for x then $l(x, e) \neq s$ (e exists by (3.4)(i),(iii),

under a natural assumption about P'). Statement 3.2 implies that the resulting path P is a line. In particular, P is simple (as P is λ -shortest and λ is positive), whence P is finite and its final vertex belongs to T . Moreover, in view of the latter property, we may assume that $x_0 \in T$, i.e. P is a T -line. By construction of P , the set $B' := B - P$ obviously satisfies (i) and (iii) in (3.4), and now the result follows by induction. (2.3) obviously follows from (3.4)(ii). •

Thus, B, α, γ give an optimal solution for p_λ whenever $\lambda := c_{\alpha, \gamma}$ is positive, B is regular, and (3.3) holds. In general case, when edges $e \in EG$ with $\lambda(e) = 0$ are possible, it is a more complicated task to define an attachment $l(x, e)$ for such e 's as well as to generalize the notion of a regular set in such a way that it ensures the property as in Statement 3.3; we leave this up to Section 6.

In what follows we often call the edges in B *bold*, and the edges in $E\Gamma - B$ *thin*.

4. Augmenting paths

In this and the next sections we prove Theorem 2 for the simplest case when $\alpha = 0$. Let $\lambda := c_{\alpha, \gamma}$. Then $\lambda = c + \gamma$, hence λ is positive. Put $p := p_\lambda$ (see (3.1)). Let $B \subseteq E\Gamma_\lambda$ be regular, and (3.3) holds. As it was shown in Section 3, B and γ give an optimal solution for p .

[Observe that if $0 \leq p \leq p_c$ then $\mathcal{P} = \emptyset$ (or $B = \emptyset$), $\alpha = 0$ and $\gamma = 0$ are good for p . Furthermore, one can see that it suffices to prove Theorem 2 for $p, \mathcal{P}, \alpha, \gamma$ such that $p = p_\lambda$ for $\lambda := c_{\alpha, \gamma}$.]

We use notation as in Section 3. Define $B^0 := \{e \in B : \gamma(e) = 0\}$ and $Z := E\Gamma - B$. For $x \in V\Gamma$ and $s \in \langle -t, t \rangle$ define $Z(x) := Z \cap E(x)$, $Z_s(x) := Z \cap E_s(x)$, $B^0(x) := B^0 \cap E(x)$ and $B_s^0(x) := B^0 \cap E_s(x)$. An edge $e \in E\Gamma$ with $\gamma(e) = 0$ (i.e. $e \in Z \cup B^0$) is called *feasible* (for α, γ). Note that B can be changed only within the set of feasible edges in order to maintain the complementary slackness condition (2.2) (or (3.3)).

Consider a triple $\tau = (e, v, e')$, where $v \in V\Gamma - T$, and e, e' are distinct feasible edges incident to v . We say that τ is a *fork* if

$$(4.1) \text{ there is no } s \in \langle -t, t \rangle \text{ such that } s \text{ is tight for } v \text{ and } e, e' \in Z_s(v) \cup (B(v) - B_s(v)).$$

When it leads to no confusion, a fork (e, v, e') may be denoted as (e, e') . One can see that for (distinct) $e, e' \in Z(v) \cup B^0(v)$, (e, v, e') is a fork if and only if $|B'_s(v)| \leq |B'(v)|/2$

for any $s \in \langle -t, t \rangle$, where $B'(v) := B(v) \Delta \{e, e'\}$ and $B'_s(v) := B'(v) \cap E_s(v)$ ($X \Delta Y$ denotes the symmetric difference $(X - Y) \cup (Y - X)$). It is useful to list the cases when (e, v, e') is a fork, namely:

(C1) if $v \in V_s$ for some $s \in T$ then e, e' belong to different sets $Z_s(v)$ and $Z_{-s}(v)$, or different sets $B_s^0(v)$ and $B_{-s}^0(v)$, or different sets $Z_{s'}(v)$ and $B_{s'}^0(v)$ for $s' \in \{s, -s\}$ (as in this case both s and $-s$ are tight).

(C2) if $v \in V^\bullet$ and $B(v) = \emptyset$ then $e \in Z_s(v)$ and $e' \in Z_{s'}(v)$ for distinct $s, s' \in \langle -t, t \rangle$ (as in this case each $s \in \langle -t, t \rangle$ is tight);

(C3) if $v \in V^\bullet$, $B(v) \neq \emptyset$, and no s is tight for v , then e, e' are arbitrary;

(C4) if $v \in V^\bullet$, $B(v) \neq \emptyset$, and the set S of elements $s \in \langle -t, t \rangle$ tight for v is nonempty (clearly $|S| \leq 2$), then e, e' form a fork except the cases when either $e, e' \in Z_s(v)$ for $s \in S$, or $e, e' \in B(v) - B_s(v)$ for $s \in S$, or e, e' belong to different sets $Z_s(v)$ and $B(v) - B_s(v)$ for $s \in S$.

From (4.1) it easily follows that

(4.2) for any $v \in V\Gamma - T$ and feasible e, e', e'' incident to v , if neither (e, e') nor (e', e'') is a fork then (e, e'') is not a fork either.

A path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ in Γ is called *active* if: (i) all e_1, \dots, e_k are distinct, (ii) for $i = 1, \dots, k - 1$, $x_i \in V\Gamma - T$ and (e_i, x_i, e_{i+1}) is a fork, (iii) if $x_0 \in T$ then $e_1 \in Z$. We say that an active P is *primitive* if for any $1 \leq i < j < k$ such that $x_i = x_j$, the triple (e_i, x_i, e_{j+1}) is not a fork, and P meets each of x_0, x_k at most twice. Clearly if vertices x and y are connected by an active path P , they can be connected by a primitive path (e.g., consisting of certain parts of P). From (4.2) it follows that

(4.3) for a primitive path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ there are no $0 < i < j < r < k$ such that $x_i = x_j = x_r$.

Indeed, suppose that this is not so. Then (e_i, e_{j+1}) and (e_i, e_{r+1}) are not forks (otherwise P is not primitive). Hence, (e_{j+1}, e_{r+1}) is not a fork (by (4.2)). Now the fact that (e_j, e_{j+1}) is a fork implies that (e_j, e_{r+1}) is also a fork (by (4.2)). Thus, P is non-primitive.

A primitive T -path $(x_0, e_1, x_1, \dots, e_k, x_k)$ is called *augmenting* if $e_k \in Z$. We say that $B' \subseteq E\Gamma$ is obtained by the *alteration along a path P* if $B' = B \Delta P$ (considering P as an edge-set).

Statement 4.1. *If P is an augmenting path and $B' := B \Delta P$ then B' is regular, $\mu(B') = \mu(B) + 2$, and (3.3) holds for B' .*

Proof. $\mu(B') = \mu(B) + 2$ and (3.4)(i),(ii) for B' are obvious. (3.3) for B' is true since P uses only feasible edges. Let us prove (3.4)(iii) for B' .

Consider $x \in V\Gamma - T$. The inequality in (3.4)(iii) for B' and x is obvious if P passes x at most once. Otherwise P passes x twice, by (4.3); let $x = x_i = x_j$ for $0 < i < j < k$. Since P is primitive, $\tau = (e_i, e_{j+1})$ is not a fork. Then $e_i, e_{j+1} \in Z_s(x) \cup (B(x) - B_s(x))$ for some $s \in \langle -t, t \rangle$ tight for x, B , by (4.1). This implies that $e_{i+1} \in B_s(x) \cup (Z(x) - Z_s(x))$ since (e_i, x_i, e_{i+1}) is a fork. Similarly, $e_j \in B_s(x) \cup (Z(x) - Z_s(x))$ since (e_j, x_i, e_{j+1}) is a fork. Then s retains tightness for B' and x .

Now the result follows from the obvious fact that if some \tilde{s} is tight for B' and x , then B' is non-excessive for x . •

Thus, in case $\alpha = 0$, the existence of an augmenting path implies validity of (i) in Theorem 2. In the next section we show that lack of the augmenting path implies (ii) in this theorem.

Now we describe an approach to find an augmenting path or, if it does not exist, to construct the set of vertices reachable by active paths beginning at T ; such a set will play an important role in the transformation of (α, γ) . We apply techniques similar, in a sense, to that developed in the matching theory and even use terms from that area.

We grow in Γ , step by step, a digraph $D = (VD, AD)$ with $T \subseteq VD \subseteq V\Gamma$. For an arc $a = (x, y)$ in D the underlying edge e , denoted by e^a , is a feasible edge in Γ ; we may identify a with the edge e labelled from x to y . A vertex in Γ belonging to D is also called labelled. Let Q_1 be the set of edges in Γ labelled in one direction, or *1-labelled* edges, and let Q_2 be the set of edges in Γ labelled in both directions, or *2-labelled* ones. The components of the subgraph of Γ induced by Q_2 are called *blossoms*. Also a special blossom of the form $(\{v\}, \emptyset)$ is possible, where v is a certain labelled central vertex (see (4.4)(iv)); such a blossom is called *elementary*. The vertex-sets of the blossoms are pairwise-disjoint.

For a blossom F denote by AF the corresponding arc-set. A labelled vertex which does not belong to any blossom (belongs to a blossom) is called *1-labelled* (respectively, *2-labelled*). A blossom F satisfies the following conditions:

- (4.4) (i) F contains no terminal ($VF \subseteq V\Gamma - T$);
- (ii) there is an arc $a_F = (x, y) \in Q_1$ fixed that enters F (i.e. $x \notin VF \ni y$), called the *root* of F ;
- (iii) for each arc $a = (x, y) \in AF$ there is a directed path $P_{F,a} = (x_0, a_1, x_1, \dots, a_k, x_k)$ the part of which from x_1 to x_k is a path in F , $a_1 = a_F$, $a_k = a$, and $P_{F,a}$ is active (considering $P_{F,a}$ as a path in Γ);

- (iv) if F is an elementary blossom $(\{v\}, \emptyset)$, and $e = e^{aF}$, then there is no $s \in \langle -t, t \rangle$ such that s is tight for v , and $e \in Z_s(v) \cup (B(v) - B_s(v))$.

The digraph D satisfies the following conditions:

- (4.5) (i) if $a \in AD$ leaves T then $e^a \in Z$;
(ii) for each 1-labelled vertex $v \in VD - T$ there is an arc $a_v \in AD$ fixed that enters v ;
(iii) for each arc $a \in AD$ there is a directed path $P_a = (x_0, a_1, x_1, \dots, a_k, x_k)$ in D such that: (a) P_a is active, $x_0 \in T$ and $a_k = a$, (b) for each 1-labelled vertex x_i , $a_i = a_{x_i}$, (c) if a_i enters a blossom F then a_i is the root a_F ; (d) if some arc a_i belongs to a blossom F then P_a contains the path P_{F, a_i} as a part.

For a path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ let P^{-1} denote the reverse path (x_k, e_k, \dots, x_0) ; if $Q = (v_0, u_1, v_1, \dots, u_m, v_m)$ is a path with $v_0 = x_k$, let $P \cdot Q$ denote the concatenated path $(x_0, e_1, x_1, \dots, e_k, x_k, u_1, v_1, \dots, u_m, v_m)$.

If there exists an arc a in D entering T , and $e^a \in Z$, then P_a is an active T -path (by (4.5)(i),(iii)), hence, we can extract an augmenting path from P_a (we say that a *break-through* takes place). Otherwise we attempt to increase D by a natural way.

(A1) Suppose that there is an unlabelled feasible edge $e = xy$ such that: (a) $x \in VD$; (b) x and y do not belong to the same blossom; and (c) either $x \in T$, or $x \notin T$ and there is an arc $a = (z, x) \in AD$ such that (e^a, x, e) is a fork. Then we add (x, y) to D (so e becomes labelled from x to y). If y does not belong to the old D , we put $a_y := (x, y)$. If, in addition, the case as in (4.4)(iv) (with $v = y$) occurs, then we form $(\{y\}, \emptyset)$ to be a new elementary blossom rooted at (x, y) .

(A2) Suppose that there is a 1-labelled edge $e = xy$ such that: (a) e is labelled from y to x ; (b) x and y do not belong to the same blossom; (c) there is an arc $a = (z, x) \in AD$ such that (e^a, x, e) is a fork; and (d) the path P_a does not contain the arc a' , where a' is e labelled from y to x . Two cases are possible.

(i) P_a and $P_{a'}$ do not contain arcs b and b' , respectively, such that $e^b = e^{b'}$. Then $P_a \cdot (P_{a'})^{-1}$ is an active T -path, hence, a break-through happens.

(ii) $P_a = (x_0, a_1, x_1, \dots, a_k, x_k)$ and $P_{a'} = (y_0, b_1, y_1, \dots, b_m, y_m)$ contain arcs a_i and b_j , respectively, such that $e^{a_i} = e^{b_j}$. Let i be chosen maximum under this condition. Represent P_a as $P_1 \cdot L_1 \cdot \dots \cdot P_r \cdot L_r \cdot P_{r+1}$, where all vertices in L_q belong to the same blossom $F_{\sigma(q)}$, while all inner vertices and all arcs in P_q do not belong to any blossoms (here P_1, \dots, P_r are non-trivial paths, but P_{r+1} may be trivial). Similarly, represent $P_{a'}$ as $P'_1 \cdot L'_1 \cdot \dots \cdot P'_h \cdot L'_h \cdot P'_{h+1}$, where L'_q lies in a blossom $F_{\sigma'(q)}$. From (4.5)(iii) it follows

that if L_q and L'_q belong to the same blossom then $q = q'$ and $P_1 \cdot L_1 \cdot \dots \cdot P_{q-1} \cdot L_{q-1} \cdot P_q$ coincides with $P'_1 \cdot L'_1 \cdot \dots \cdot P'_{q-1} \cdot L'_{q-1} \cdot P'_q$. Let q be chosen maximum under such a property; if there is no common blossom for P_a and $P_{a'}$, put $q := 0$. Consider two cases.

(a) a_i belongs to L_q . Then $F_{\sigma(q)} = F_{\sigma'(q)}$. For each arc in P_{q+1}, \dots, P_{r+1} add to D the opposite arc; and similarly for each arc in $P'_{q+1}, \dots, P'_{h+1}$. As a result, $F_{\sigma(q)}, \dots, F_{\sigma(r)}, F_{\sigma'(q+1)}, \dots, F_{\sigma'(h)}$ together with the vertices and arcs of the paths $P_{q+1}, \dots, P_{r+1}, P'_{q+1}, \dots, P'_{h+1}$ merge into a new blossom rooted at $a_{F_{\sigma(q)}}$.

(b) a_i belongs to P_{q+1} . Then $a_i = b_j$. For $i' = i+1, \dots, k$, add to D the arc opposite to $a_{i'}$ (if it did not occur in D earlier). Similarly, for $j' = j+1, \dots, m$, add to D the arc opposite to $b_{j'}$. All these arcs together with the blossoms $F_{\sigma(q+1)}, \dots, F_{\sigma(r)}, F_{\sigma'(q+1)}, \dots, F_{\sigma'(h)}$ merge into a new blossom rooted at a_i .

One can see that the created blossom, formed in (a) or (b), satisfies (4.4)(i)-(iii). Suppose that neither D can be increased by the above rules nor a break-through happens. Then D has features similar, in a sense, to those occurring in the so-called ‘‘Hungarian tree with flowers’’ in the matching theory [Ed]. More precisely, the following are true:

(4.6) Each edge $e = xy \in Z$ with $x \in T$ is labelled as leaving (but not entering) x .

(4.7) Let $x \in VD - T$ be a 1-labelled vertex, and $E^+(x)$, $E^-(x)$, $E^{\text{un}}(x)$ be the sets of feasible edges $e = xy \in E(x)$ labelled as entering x , labelled as leaving x , and unlabelled, respectively; then:

- (i) no pair of distinct edges in $E^+(x) \cup E^{\text{un}}(x)$ forms a fork;
- (ii) each pair $e \in E^+(x)$, $e' \in E^-(x)$ forms a fork.

(4.8) for each blossom F all feasible edges in the cut $\delta^\Gamma(VF)$ are labelled as leaving F except $e_F := e^{a_F}$, and e_F is the only edge in this set that is labelled as entering F .

[One can see that (4.6)-(4.8) imply that no augmenting path in Γ exists.] Proofs of (4.6) and (4.7) are easy. To prove (4.8), we need the following statement.

Statement 4.2. *Let $e = xy$ be a feasible edge such that $x \in VF \not\cong y$ for some blossom F , and $e \neq e_F$. Then there exists an arc $a = (u, x) \in AF \cup \{a_F\}$ such that (e^a, x, e) is a fork.*

Proof. If F is an elementary blossom then (e^{a_F}, x, e) is a fork, by (4.4)(iv). For a non-elementary F there is an edge $e' = xz \in EF$. We know that e' is labelled in both

directions. If (e, x, e') is a fork, we are done. Otherwise take the arc $a' = (x, z) \in AD$ with $e^{a'} = e'$, and consider the path $P_{F, a'} = (x_0, a_1, x_1, \dots, a_k, x_k)$ as in (4.4)(iii); then $k \geq 2$ and $a' = a_k$. Let e'' be the underlying edge for the arc $a := a_{k-1}$. Since (e', x, e'') is a fork and (e, x, e') is not, (e'', x, e) is a fork (by (4.2)), whence a is as required. •

Suppose that there is a blossom F and a feasible edge $e = xy$ different from e_F so that $x \in VF \not\cong y$ and e is not labelled from x to y . Let $a = (u, x)$ be as in Statement 4.2. If e is unlabelled then the arc (x, y) can be added to D according to (A1). And if D contains the arc $a' = (y, x)$ then D can be increased according to (A2) (note that P_a does not contain a' since $e \neq e_F$ and P_a uses exactly one edge in $\delta^\Gamma(VF)$, namely, e_F). This contradiction proves (4.8).

Remark. One can show that search for an augmenting path in Γ can be reduced to the standard problem on finding an alternating path in a graph Q with a matching M in it. As a consequence, (4.7)-(4.8) are derived from properties of the ‘‘Hungarian tree with flowers’’ [Ed]. Such a Q is designed by replacing each vertex x in Γ by a special subgraph (depending on the set of forks for x). However, this approach would make our description more intricate, and it is preferable to argue explicitly in terms of Γ itself.

5. Transformation of (α, γ)

As before, we consider case $\alpha = 0$ and use notation as in the previous section. We assume that there is no augmenting path in Γ .

Each blossom F in Γ generates the fragment ϕ for G, T with $X_\phi := VF$ and

$$(5.1) \quad U_\phi := (B \cap \delta X_\phi) \cup \{e_F\} \text{ if } e_F \in Z, \text{ and } U_\phi := (B \cap \delta X_\phi) - \{e_F\} \text{ if } e_F \in B$$

(cf. (2.4)-(2.5)). Then $|U_\phi|$ is odd (by (3.4)(i)), B saturates ϕ , and e_F is the root e_ϕ of ϕ . Let \mathcal{F} be the set of these fragments.

The required α' and γ' will be assigned to be α^ε and γ^ε (for some $\varepsilon > 0$) defined below. Let us fix $\varepsilon \in \mathbb{R}_+$. We put $\alpha^\varepsilon(\phi)$ to be ε for all $\phi \in \mathcal{F}$ (and 0 for the remaining fragments ϕ for G, T). For brevity, the value $\sum(\alpha^\varepsilon(\phi)\chi_\phi(e) : \phi \in \mathcal{F})$ is denoted by $\hat{\alpha}^\varepsilon(e)$.

To define γ^ε is a more involved task. First of all we shall introduce (in (5.4)-(5.5) below) a certain value $\rho(x, e) = \rho^\varepsilon(x, e) \in \{0, \varepsilon, -\varepsilon\}$ for each $x \in V\Gamma$ and $e \in E(x)$. This gives the function $\rho = \rho^\varepsilon$ on $E\Gamma$ by

$$(5.2) \quad \rho(e) := \rho(x, e) + \rho(y, e) \quad \text{for } e = xy \in E\Gamma.$$

Then γ^ε is defined by

$$(5.3) \quad \begin{aligned} \gamma^\varepsilon(e) &:= \gamma(e) + \rho^\varepsilon(e) - \widehat{\alpha}^\varepsilon(e) + \widehat{\alpha}(e) & \text{for } e \in B, \\ &:= 0 & \text{for } e \in EG - B, \end{aligned}$$

where, for our case, $\widehat{\alpha}(e) := 0$ for all $e \in EG$.

Let L, N, M denote the sets of 1-labelled, 2-labelled and unlabelled vertices in Γ , respectively. Clearly $T \subseteq L$. Put

$$(5.4) \quad \begin{aligned} \rho(x, e) = \rho^\varepsilon(x, e) &:= \varepsilon & \text{for } x \in T \text{ and } e \in E(x), \\ &:= 0 & \text{for } x \in N \cup M \text{ and } e \in E(x). \end{aligned}$$

To define $\rho(x, e)$ for the vertices in L , we need the following notion. For a vertex $v \in V\Gamma - T$ and distinct edges $e, e' \in E(v)$, we say that (e, v, e') is a *pseudo-fork* if it satisfies (4.1) (thus, the difference with the definition of a fork is that e, e' are not required to be feasible).

Consider a 1-labelled vertex $x \in V\Gamma - T$, and fix some edge u labelled as entering x . For $e \in E(x)$ put

$$(5.5) \quad \begin{aligned} \rho(x, e) = \rho^\varepsilon(x, e) &:= -\varepsilon \text{ if either } e \in Z \text{ and } (u, x, e) \text{ is not a pseudo-fork,} \\ &\quad \text{or } e \in B \text{ and } (u, x, e) \text{ is a pseudo-fork;} \\ &:= \varepsilon \text{ if either } e \in Z \text{ and } (u, x, e) \text{ is a pseudo-fork,} \\ &\quad \text{or } e \in B \text{ and } (u, x, e) \text{ is not a pseudo-fork ;} \end{aligned}$$

letting by definition that (u, x, u) is not a pseudo-fork. To make the meaning of (5.5) clearer, consider possible cases.

(L1) $x \in V_s$ for some $s \in T$. Then both s and $-s$ are tight for x . Suppose that $u \in Z_s(x) \cup B_{-s}(x)$. Then $\rho(x, e) = -\varepsilon$ for all $e \in E_s(x)$ and $\rho(x, e) = \varepsilon$ for $e \in E_{-s}(x)$. If $u \in Z_{-s}(x) \cup B_s(x)$ then $\rho(x, e) = -\varepsilon$ for all $e \in E_{-s}(x)$ and $\rho(x, e) = \varepsilon$ for $e \in E_s(x)$.

(L2) $x \in V^\bullet$. Since x does not form an elementary blossom, we observe from (4.4)(iii) and (4.7) that there is $s \in T$ tight for x and such that each edge labelled as entering x belongs to $Z_s(x) \cup (B^0(x) - B_s^0(x))$, while each edge labelled from x belongs to $B_s^0(x) \cup (Z(x) - Z_s(x))$. Then $\rho(x, e) = -\varepsilon$ for $e \in E_s(x)$ and $\rho(x, e) = \varepsilon$ for $e \in E(x) - E_s(x)$.

Note that in definition (5.5) it does no matter what entering labelled edge is chosen as u (by (4.7) and (4.2)).

Now we study properties of functions ρ^ε , γ^ε and $\lambda^\varepsilon := c_{\alpha^\varepsilon, \gamma^\varepsilon}$. Our goal is to prove the following two statements.

Statement 5.1. *For a sufficiently small $\varepsilon > 0$, $\lambda^{\varepsilon'}(e)$ and $\gamma^{\varepsilon'}(e)$ are nonnegative for any $e \in EG$ and $0 \leq \varepsilon' \leq \varepsilon$.*

Let ε_1 (ε_2) denote the greatest ε such that $\lambda^{\varepsilon'}$ (respectively, $\gamma^{\varepsilon'}$) is nonnegative for any $0 \leq \varepsilon' \leq \varepsilon$.

Statement 5.2. *For a sufficiently small ε , $0 < \varepsilon \leq \varepsilon_1$, the equality $p_{\lambda^{\varepsilon'}} = p + 2\varepsilon'$ holds for any $0 \leq \varepsilon' \leq \varepsilon$.*

Properties (4.7)-(4.8) show that some sorts of feasible edges need not connect certain sets among $L, M, \{X_\phi : \phi \in \mathcal{F}\}$ or vertices in L . More precisely, for a feasible edge $e = xy$:

- (5.6) (i) if e is unlabelled and $x, y \in L$ then neither (u, x, e) nor (u', y, e) is a fork, where u (u') is an edge labelled as entering x (respectively, y);
- (ii) if e is unlabelled, $x \in L$ and $y \in M$ then (u, x, e) is not a fork, where u is an edge labelled as entering x ;
- (iii) e does not connect X_ϕ ($\phi \in \mathcal{F}$) and M ;
- (iv) if $x \in L$ and $y \in X_\phi$ ($\phi \in \mathcal{F}$) then e is labelled; moreover, if e is labelled from x to y then $e = e_\phi$;
- (v) if e connects X_ϕ and $X_{\phi'}$ ($\phi, \phi' \in \mathcal{F}$) then e is the root of exactly one of ϕ, ϕ' .

(For if any of (i)-(v) is violated, the labelled digraph D can be enlarged.)

Proof of Statement 5.1. The assertion for λ^ε is obvious since $\lambda(e) > 0$ and $|\lambda^\varepsilon(e) - \lambda(e)| = O(\varepsilon)$ for all $e \in EG$.

To prove the assertion for γ^ε , it suffices to examine an edge $e = xy \in B^0$ (since

$\gamma^\varepsilon(e) = 0$ for $e \in EG - B^0$, by (5.3), and $\gamma(e) > 0$ for $e \in B - B^0$. First of all we observe that

$$(5.7) \text{ for any labelled edge } e' = x'y', \quad \rho^\varepsilon(e') - \widehat{\alpha}^\varepsilon(e') + \widehat{\alpha}(e') = 0.$$

Indeed, this is trivial if $x', y' \in X_\phi$ for some $\phi \in \mathcal{F}$, and easily follows from (5.5) if both x', y' are 1-labelled (then $\widehat{\alpha}^\varepsilon(e') = 0$ and $\rho(x', e') = -\rho(y', e')$). If $x' \in L$ and $y' \in X_\phi$ for some $\phi \in \mathcal{F}$ then $e' = e_\phi$ (by (5.6)(iv)); this yields that $\rho(x', e') = \varepsilon$, $\rho(y', e') = 0$ and $\widehat{\alpha}^\varepsilon(e') = \varepsilon$ hold in case $e' \in Z$, and that $\rho(x', e') = -\varepsilon$, $\rho(y', e') = 0$ and $\widehat{\alpha}^\varepsilon(e') = -\varepsilon$ hold in case $e' \in B$, whence (5.7) follows. And if $x' \in X_\phi$, $y' \in X_{\phi'}$ and $e' = e_\phi$ for distinct $\phi, \phi' \in \mathcal{F}$ then $\rho(x', e') = \rho(y', e') = 0$ and $\widehat{\alpha}^\varepsilon(e') = 0$ (as $e' \in U_\phi$ if and only if $e' \notin U_{\phi'}$), and (5.7) is also true. Other cases for e' are impossible by (5.6).

Now consider $e = xy \in B^0$. (5.3) and (5.7) show that $\gamma^\varepsilon(e) = \gamma(e) = 0$ if e is labelled. The same equalities are obvious if $x, y \in M$. Assuming that e is unlabelled and x is labelled, only the following cases are possible.

(i) $x, y \in L$. By (5.6)(i) and (5.5), $\rho(x, e) = \rho(y, e) = \varepsilon$, whence $\gamma^\varepsilon(e) = 2\varepsilon \geq 0$.

(ii) $x \in L, y \in M$. By (5.6)(ii), $\rho(x, e) = \varepsilon$. Furthermore, $\rho(y, e) = 0$, by (5.4). Hence $\gamma^\varepsilon(e) = \varepsilon \geq 0$. •

Proof of Statement 5.2. Consider a simple T -path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$. We have to show that for any $\varepsilon \geq 0$, $\lambda^\varepsilon(P) \geq p + 2\varepsilon$, and that this inequality holds with equality for some P .

We say that a T -line $Q = (v_0, u_1, v_1, \dots, u_m, v_m)$ is *strong* if all u_i 's are bold, and for $i = 1, \dots, m-1$, if $s \in \langle -t, t \rangle$ is tight for v_i then exactly one of u_i, u_{i+1} belongs to $B_s(v_i)$ (in other words, Q is strong if and only if Q is a member of a packing consisting of $\mu(B)/2$ T -lines and covering exactly B , cf. Statement 3.3). We know that Γ contains at least one T -line (since $p = p_\lambda$) and no augmenting path, therefore, B is non-empty, whence, there is at least one strong T -line in Γ .

Next, if P is a T -line, put $\widehat{\rho}(x_i) = \widehat{\rho}^\varepsilon(x_i) := \rho^\varepsilon(x_i, e_i) + \rho^\varepsilon(x_i, e_{i+1})$, $i = 1, \dots, k-1$. Then $\rho(P)$ is equal to $\rho(x_0, e_1) + \sum_{i=1}^{k-1} (\rho(x_i, e_i) + \rho(x_i, e_{i+1})) + \rho(x_k, e_k)$. Hence, by (5.2) and (5.4),

$$(5.8) \quad \rho^\varepsilon(P) = 2\varepsilon + \sum (\widehat{\rho}^\varepsilon(x_i) : i = 1, \dots, k-1).$$

Claim 1. If P is a strong T -line, then $\widehat{\rho}^\varepsilon(x_i) \neq 0$ for $i = 1, \dots, k-1$. ✓

Proof. This immediately follows from (5.4) if x_i is unlabelled or 2-labelled. Let $x = x_i$ be 1-labelled. Put $l := l(x, e_i)$ and $l' := l(x, e_{i+1})$. If $x \in V_s$ for some $s \in T$ then the fact that one of l, l', l say, is s and the other, l' , is $-s$ implies that $\rho(x, e_i) = -\rho(x, e_{i+1})$ (cf. (L1) above). Hence $\widehat{\rho}(x) = 0$.

Now suppose that $x \in V^\bullet$. Then there is $s \in T$ such that s is tight for x and an edge labelled as entering x is in $Z_s(x) \cup (B(x) - B_s(x))$. The fact that P is strong implies that one of l, l', l say, is s and the other, l' , is different from s . This yields $\rho(x, e_i) = -\varepsilon$ and $\rho(x, e_{i+1}) = \varepsilon$ (cf. (L2)), whence $\widehat{\rho}(x) = 0$. •

Claim 2. Let P be a T -line, and $\varepsilon > 0$. For $i = 1, \dots, k-1$, $\widehat{\rho}^\varepsilon(x_i)$ is nonnegative, and $\widehat{\rho}^\varepsilon(x_i) > 0$ holds if and only if

(5.9) the triple (e_i, x_i, e_{i+1}) is such that: $x = x_i$ is an (ordinary) 1-labelled central vertex, and there is $s \in T$ such that s is tight for x , an edge labelled as entering x_i is in $Z_s(x) \cup (B(x) - B_s(x))$, and $e_i, e_{i+1} \in E(x) - E_s(x)$.

Proof. Again, it suffices to consider an ordinary 1-labelled $x = x_i$. As in the proof of Claim 1, $\widehat{\rho}(x) = 0$ holds if exactly one of e_i, e_{i+1} belongs to $E_s(x)$, where s is tight for x , and an edge labelled as entering x is in $Z_s(x) \cup (B(x) - B_s(x))$. Thus, it remains to consider the case as in (5.9) (taking into account that $e_i, e_{i+1} \in E_s(x)$ is impossible since P is a line). Then $\rho(x, e_i) = \rho(x, e_{i+1}) = \varepsilon$ (cf. (L1)-(L2)), whence $\widehat{\rho}(x) = 2\varepsilon > 0$. •

Define

(5.10) \mathcal{T} to be the set of unlabelled edges $e = xy \in Z$ such that either $x, y \in L$, or $x \in L$ and $y \in M$.

Claim 3. Let $e \in E\Gamma$, and $\varepsilon > 0$. Then

$$(5.11) \quad \begin{aligned} \lambda^\varepsilon(e) &= \lambda(e) + \rho^\varepsilon(e) && \text{if } e \in E\Gamma - \mathcal{T}; \\ &> \lambda(e) + \rho^\varepsilon(e) && \text{if } e \in \mathcal{T}. \end{aligned}$$

Proof. Put $b := \lambda^\varepsilon(e) - \lambda(e) - \rho(e)$. Then

$$\begin{aligned} b &= (c(e) + \gamma^\varepsilon(e) + \widehat{\alpha}^\varepsilon(e)) - (c(e) + \gamma(e) + \widehat{\alpha}(e)) - \rho(e) \\ &= \gamma^\varepsilon(e) - \gamma(e) + \widehat{\alpha}^\varepsilon(e) - \widehat{\alpha}(e) - \rho(e). \end{aligned}$$

Thus, $b = 0$ if $e \in B$ (by (5.3)) or if $e \in Z$ and e is labelled (by (5.7) and the fact that $\gamma^\varepsilon(e) = \gamma(e) = 0$). Also $b = 0$ holds if $e \in Z$ and both ends of e are unlabelled (since $\widehat{\alpha}^\varepsilon(e) = \widehat{\alpha}(e) = 0$). Let $e = xy \in Z$ be unlabelled and x be labelled. Then (5.6) shows that $e \in \mathcal{T}$; hence, $\gamma^\varepsilon(e) = \gamma(e) = \widehat{\alpha}^\varepsilon(e) = 0$ and $b = -\rho(e)$. If $x, y \in L$ then (5.6)(i) and (5.5) imply $\rho(x, e) = \rho(y, e) = -\varepsilon$. And if $x \in L, y \in M$ then, by (5.6)(ii) and (5.5), $\rho(x, e) = -\varepsilon$ and $\rho(y, e) = 0$. In both cases, $\rho(e) < 0$, whence the result follows. •

Thus, by Claim 3, $\lambda^\varepsilon(P) = \lambda(P) + \rho^\varepsilon(P) = p + \rho^\varepsilon(P)$ if P is a strong T -line, and $\lambda^\varepsilon(P) \geq p + \rho^\varepsilon(P)$ if P is a T -line. Now we conclude from Claims 1,2 and (5.8) that

$$(5.12) \quad \begin{aligned} \lambda^\varepsilon(P) &= p + 2\varepsilon && \text{if } P \text{ is a strong } T\text{-line,} \\ &\geq p + 2\varepsilon && \text{if } P \text{ is a } T\text{-line.} \end{aligned}$$

Finally, if P is a T -path in G that is not a T -line then $\lambda(P) > p$, hence, there is $\varepsilon > 0$ such that $\lambda^{\varepsilon'} > p + 2\varepsilon$ for $0 \leq \varepsilon' \leq \varepsilon$. Now the statement follows from finiteness of the set of simple T -paths. ••

Statements 5.1 and 5.2 enable us to determine α', γ' as required in (ii) in Theorem 2. More precisely, put

$$(5.13) \quad \varepsilon^* := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\},$$

where ε_1 and ε_2 were defined above (after Statement 5.1), and ε_3 is the largest ε as in Statement 5.2. Then $\varepsilon^* > 0$.

Arguments above imply that $B, \alpha' := \alpha^{\varepsilon^*}, \gamma' := \gamma^{\varepsilon^*}$ are good for $p' := p_{\lambda^{\varepsilon^*}}$ ($= p + 2\varepsilon^*$). Thus, Theorem 2 is valid for case $\alpha = 0$.

In conclusion of this section let us emphasize one result obtained in the proof of Statement 5.2; it will be used in what follows.

$$(5.14) \quad \text{For a } T, \lambda\text{-line } P = (x_0, e_1, x_1, \dots, e_k, x_k), \lambda^\varepsilon(P) = \lambda(P) + 2\varepsilon \text{ holds if and only if:}$$

- (i) $\widehat{\rho}^\varepsilon(x_i) = 0$ for $i = 1, \dots, k-1$, or, equivalently, P has no triple as in (5.9);
- (ii) $\lambda^\varepsilon(e_i) = \lambda(e_i) + \rho^\varepsilon(e_i)$ for $i = 1, \dots, k$, or, equivalently, P does not meet \mathcal{T} .

6. General case

We now consider the case when edges $e \in EG$ with $\lambda(e) = 0$ are possible, where $\lambda := c_{\alpha, \gamma}$. These edges are called *0-edges*, and their set is denoted by J . As before, $E\Gamma$ is partitioned into two subsets B (of bold edges) and Z (of thin edges). [Note that in our case a T -line may not be a simple path; the notion of regularity of B will be specified below.] We denote $\beta := B \cap J$ and $\zeta := Z \cap J$. Let \mathcal{F} denote the set $\{\phi \in \tilde{\mathcal{F}} : \alpha(\phi) > 0\}$ (the *support* of α), and let for $e \in EG$, $\hat{\alpha}(e)$ denote $\sum(\alpha(\phi)\chi_\phi(e) : \phi \in \mathcal{F})$.

As mentioned in Section 2, we need to impose a number of additional conditions on B, α, γ ; taken together, they enable us to prove the existence of \mathcal{P}' or (α', γ') as required in Theorem 2. First of all we assume that \mathcal{F} and J satisfy the following conditions:

$$(6.1) \text{ for each } \phi \in \mathcal{F}, \text{ either } e_\phi \notin B \text{ and } U_\phi = (B \cap \delta X_\phi) \cup \{e_\phi\}, \text{ or } e_\phi \in B \text{ and } U_\phi = (B \cap \delta X_\phi) - \{e_\phi\}$$

(this is equivalent to saying that B saturates ϕ , cf. (2.4)-(2.5), or (5.1));

$$(6.2) \text{ for each } \phi \in \mathcal{F}, e_\phi \text{ is feasible, i.e.}$$

- (i) e_ϕ is contained in Γ , and
- (ii) $\gamma(e_\phi) = 0$;

$$(6.3) \text{ the sets } X_\phi \text{ form a } \textit{nested} \text{ family, i.e. for distinct } \phi, \phi' \in \mathcal{F}, \text{ either } X_\phi \cap X_{\phi'} = \emptyset \text{ or } X_\phi \subset X_{\phi'} \text{ or } X_{\phi'} \subset X_\phi;$$

$$(6.4) \text{ (i) if } \phi_1, \dots, \phi_k \in \mathcal{F} \text{ and } e \in EG \text{ are such that } X_{\phi_1} \subset \dots \subset X_{\phi_k} \text{ and } e \in \delta X_{\phi_1}, \dots, \delta X_{\phi_k} \text{ then either } e \in U_{\phi_1}, \dots, U_{\phi_k} \text{ or } e \notin U_{\phi_1}, \dots, U_{\phi_k}; \text{ in particular, if } e = e_{\phi_i} \text{ for some } i \text{ then } e_{\phi_1} = \dots = e_{\phi_k};$$

- (ii) no sequence $\phi_1, \dots, \phi_k \in \mathcal{F}$ with $k > 1$ exists such that the sets X_{ϕ_i} are pairwise disjoint, and $e_{\phi_i} \in \delta X_{\phi_{i+1}}$ for $i = 1, \dots, k$ (letting $\phi_{k+1} := \phi_1$);

$$(6.5) \text{ for each } e \in J, e \text{ is contained in } \Gamma, \gamma(e) = 0, \text{ and both ends of } e \text{ are in } V\Gamma - T.$$

A component of the subgraph induced by β is called a *0-component*. We observe that each 0-component is a tree. Indeed, suppose that β contains a sequence $Q = (e_1, e_2, \dots, e_k)$ of edges forming a circuit. Then for any $\phi \in \mathcal{F}$, $|U_\phi \cap Q| \geq |(\delta X_\phi - U_\phi) \cap Q|$ (by (6.1) and the fact that $Q \subseteq B$), whence $\chi_\phi(Q) \geq 0$. Hence,

$$\lambda(Q) - c(Q) - \gamma(Q) = \hat{\alpha}(Q) = \sum(\alpha(\phi)\chi_\phi(Q) : \phi \in \mathcal{F}) \geq 0.$$

Since $\gamma(Q) \geq 0$ and $c(Q) > 0$ (as c is positive), $\lambda(Q) > 0$; a contradiction.

For $e \in J$ the facts that $c(e) > 0$, $\lambda(e) = 0$ and $\gamma(e) = 0$ (by (6.5)) imply that $\hat{\alpha}(e) < 0$, therefore, there is $\phi \in \mathcal{F}$ such that $e \in \delta X_\phi - U_\phi$. We impose a stronger condition, namely,

(6.6) for any $e \in J$ and $\phi \in \mathcal{F}$, $e \notin U_\phi$; in particular, for $e \in J \cap \delta X_\phi$, $e \in \beta$ if and only if $e = e_\phi$.

We say that $\phi' \in \mathcal{F}$ *precedes* $\phi \in \mathcal{F}$ (or ϕ' is a *predecessor* of ϕ) if $X_{\phi'} \subset X_\phi$ and there is no $\phi'' \in \mathcal{F}$ such that $X_{\phi'} \subset X_{\phi''} \subset X_\phi$. The set of predecessors of ϕ is denoted by \mathcal{F}_ϕ . Let \mathcal{F}^{\max} denote the set of $\phi \in \mathcal{F}$ preceding no fragment in \mathcal{F} .

For $\phi' \in \mathcal{F}$ let $\Gamma_{\phi'}$ denote the graph that is the union of the edge $e_{\phi'}$ and the subgraph of Γ induced by $X_{\phi'}$. Form the graph Γ^* from Γ by shrinking the set X_ϕ for each $\phi \in \mathcal{F}^{\max}$ into a single new vertex, denoted by f_ϕ , and then by deleting the loops if appeared. Similarly, for $\phi' \in \mathcal{F}$ form $\Gamma_{\phi'}^*$ from $\Gamma_{\phi'}$ by shrinking X_ϕ into a vertex f_ϕ for each $\phi \in \mathcal{F}_{\phi'}$, and then by deleting loops. We refer to such f_ϕ 's as *non-ordinary* vertices, while the other vertices in Γ^* ($\Gamma_{\phi'}^*$) are called *ordinary*. When it is not confusing, the image in Γ^* ($\Gamma_{\phi'}^*$) of an edge $e \in E\Gamma$ ($e \in E\Gamma_{\phi'}$) that does not turn into a loop under above shrinking is also denoted by e ; in particular, the image in Γ^* ($\Gamma_{\phi'}^*$) of the root e_ϕ of $\phi \in \mathcal{F}^{\max}$ ($\phi \in \mathcal{F}_{\phi'} \cup \{\phi'\}$) is also denoted by e_ϕ . (6.3) and (6.6) imply the following important property:

(6.7) let H be a 0-component, and let H^* (H_ϕ^*) be the image of H in Γ^* (respectively, Γ_ϕ^* for $\phi \in \mathcal{F}$); then H^* (H_ϕ^*) is either (i) a single vertex $f_{\phi'}$ for $\phi' \in \mathcal{F}^{\max}$ ($\phi' \in \mathcal{F}_\phi$), or (ii) an edge connecting $f_{\phi'}$ and $f_{\phi''}$ for some ϕ', ϕ'' in \mathcal{F}^{\max} (\mathcal{F}_ϕ), or (iii) a star induced by an ordinary vertex x in Γ^* (Γ_ϕ^*) and edges of the form $xf_{\phi_1}, \dots, xf_{\phi_k}$ for distinct ϕ_1, \dots, ϕ_k in \mathcal{F}^{\max} (\mathcal{F}_ϕ).

(See Fig. 6.1.)

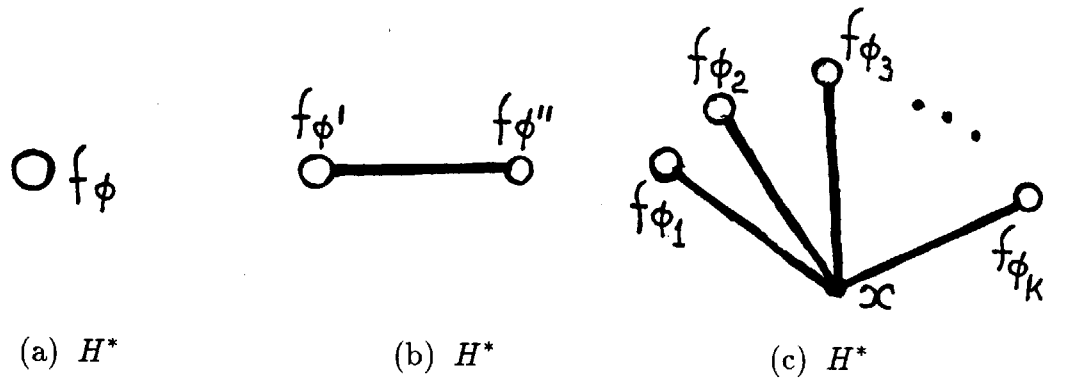


Fig. 6.1

Now we specify the notion of regularity for B . Consider a 0-component H . Clearly

the potential $\pi(x)$ is the same for all vertices x in H and the vertices x reachable from H by paths with all edges in J . Moreover, all these vertices are either in V^\bullet or in the same V_s for some $s \in T$.

Let $B^+ := B - \beta$. For a subgraph Q in Γ denote by $B(Q)$, $\beta(Q)$, $B^+(Q)$ the set of edges in B , β , B^+ , respectively, with exactly one end in Q ; we write $B(x)$, $\beta(x)$, $B^+(x)$ instead of $B(Q)$, $\beta(Q)$, $B^+(Q)$ for $Q = (\{x\}, \emptyset)$. Note that if Q is a 0-component then every edge e in B^+ has at most one end in Q (since e belongs to a λ -shortest path). We say that a set $B \subseteq E\Gamma$ (satisfying (6.1)) is regular if it satisfies (3.4)(i),(ii) and the following condition:

(6.8) if H' is either a vertex in $V\Gamma - T$ or a subtree in a 0-component then B is *non-excessive* for H' ; this means that for any $s \in \langle -t, t \rangle$:

$$|B_s^+(H')| \leq \frac{1}{2}|B(H')| \quad (= \frac{1}{2}|B^+(H')| + \frac{1}{2}|\beta(H')|).$$

Here $B_s^+(H')$ is the set of edges $e = xy \in B^+(H')$ with $x \in VH'$ and $l(x, e) = s$. [Recall that $\langle -t, t \rangle$ does not contain 0. For $e = xy \in E\Gamma$ with $\lambda(e) > 0$ the attachment $l(x, e)$ is defined as in (3.2).] Obviously, if $H' \in V\Gamma - T$ and $\beta(H') = \emptyset$ then (6.8) turns into (3.4)(iii) for $x := H'$. If the inequality in (6.8) holds with equality, s is called *tight* for H' . Note that if H' is a subtree in a 0-component H and $H' \neq H$ then at most one s can be tight for H' (since $|\beta(H')|$ is nonzero).

Statement 6.1. B is regular if and only if $B = \cup(P \in \mathcal{P})$, where \mathcal{P} consists of $\mu(B)/2$ simple edge-disjoint T, λ -lines. (Cf. Statement 3.3.)

Proof. The part “if” is easy. To prove “only if” part, it suffices to show the following:

(6.9) for a regular B and a 0-component H let \hat{H} be the subgraph of Γ induced by the set $EH \cup B^+(H)$; then \hat{H} is represented as the union of $m := |B^+(H)|/2$ simple edge-disjoint paths P_1, \dots, P_m such that for each $P_i = (x_0, e_1, x_1, \dots, e_k, x_k)$: $k \geq 2$, $e_1, e_k \in B^+(H)$, the part of P_i from x_1 to x_{k-1} is a path in H , and $l(x_1, e_1) \neq l(x_{k-1}, e_k)$.

[For contraction of every 0-component results in the case considered in the proof of Statement 3.3 in Section 3.] (6.9) can be proved by use the following result due to Lovász [Lov] and Cherkassky [Ch]: let Q be a graph and $T' \subseteq VQ$, let every vertex in $VQ - T'$ be of an even valency, and let U_v denote the set of edges in Q incident to $v \in VQ$; then Q has $\sum_{s \in T'} |U_s|/2$ edge-disjoint T' -paths if and only if

(6.10) $|\delta^Q X| \geq |U_s|$ holds for any $s \in T'$ and $X \subseteq VQ$ such that $X \cap T' = \{s\}$.

In our case we put T' to be the set of all distinct $s \in \langle -t, t \rangle$ such that $s = l(x, e)$ for some $x \in VH$ and $e = xy \in B^+(x)$, and form Q from \widehat{H} by identifying each y as above with the corresponding s . Then each vertex in $VQ - T'$ has an even valency, by (3.4)(i). It is easy to see that (6.8) implies (6.10), and now the existence of edge-disjoint paths P_1, \dots, P_m as in (6.9) follows from the above-mentioned result. Note that the facts that H is a tree and that $|B(x)|$ is even for all $x \in VH$ imply that P_1, \dots, P_m are simple and cover all edges in H . •

[It should be noted that paths as in (6.9) can be found in polynomial time.] The regularity of B implies the following properties (6.11) and (6.12) for a 0-component H .

(6.11) Let $s \in \langle -t, t \rangle$ be tight for a subtree $H' \subseteq H$, let $e \in EH'$, and let H_1, H_2 be the components of $H' - \{e\}$; then s is tight for exactly one H_i .

Indeed,

$$\begin{aligned} 0 &= 2|B_s^+(H')| - |B(H')| \\ &= (2|B_s^+(H_1)| + 2|B_s^+(H_2)|) - (|B(H_1)| + |B(H_2)| - 2) \\ &= 2 + \sum (2|B_s^+(H_i)| - |B(H_i)| : i = 1, 2) \end{aligned}$$

Now (6.11) follows from the facts that $2|B_s^+(H_i)| - |B(H_i)|$ is non-positive (by (6.8)) and even (by (3.4)(i) and (6.5)).

(6.12) For any $e = xy \in EH$ there is at most one $s \in \langle -t, t \rangle$ that is tight for some subtree $H' \subseteq H$ with $x \notin VH' \ni y$.

To see this, consider paths P_1, \dots, P_m for \widehat{H} as in (6.9). If s is tight for some H' as above then, obviously, there is a path P_i (considered up to reversing it) that has the first edge in $B_s^+(H)$ and passes y, e, x in this order. This shows that s is determined uniquely (independently of H').

Property (6.12) enables us to associate with (x, e) , $e = xy \in EH$, an attachment $l(x, e)$ by the following rule:

(6.13) put $l(x, e) := s$ if s is tight for some $H' \subseteq H$ with $x \notin VH' \ni y$, and put $l(x, e) := 0$ otherwise.

[So the special attachment 0 may appear for (x, e) with $e \in \beta$.] For each edge $e = xy \in \zeta$ we also put $l(x, e) := 0$. As before, for $x \in V\Gamma$ and $s \in \langle -t, t \rangle \cup \{0\}$ define $E_s(x) := \{e \in E(x) : l(x, e) = s\}$, $Z_s(x) := Z \cap E_s(x)$, $B_s(x) := B \cap E_s(x)$ and

