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Edge-disjoint T-paths of Minimum Total Cost

by

Alexander Karzanov

Department of Computer Science

Stanford University
Stanford, California 94305

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Alexander V. Karzanov♯

Abstract. Suppose that $G = (V, E)$ is a graph and T is a subset of its vertices. Let ν be the maximum number of edge-disjoint T-paths (i.e. paths in G connecting distinct elements in T). A classical result in combinatorial optimization, due to Mader and, independently, Lomonosov, is that ν can be expressed, by use of a minimax relation, via a value determined by a family of certain cuts in G.

We consider a more general problem in which, given nonnegative costs $c(e)$ of edges $e \in E$, one requires to find ν edge-disjoint T-paths $P_1, \dots, P_\nu$ such that their total cost $\sum(c(e) : e \in P_i, i = 1, \dots, \nu)$ is as small as possible.

We prove a minimax relation for this problem. Moreover, being "constructive", the proof provides a strongly polynomial algorithm to find paths as required.

1. Introduction

By a *graph* we mean an undirected graph with possible multiple edges. VG and EG denote the vertex-set and the edge-set, respectively, of a graph G. When it leads to no confusion, an edge with end vertices x and y is denoted by xy.

We deal with a graph G whose edges $e \in EG$ have nonnegative integer-valued *costs* $c(e) \in \mathbb{Z}_+$, and with a subset $T \subseteq VG$, called the set of *terminals* in G. A path P in G with both ends in T is called a *T-path*; the value $\sum_{e \in P} c(e)$, the "cost of P", is denoted by $c(P)$. [We will often consider a path as an edge-set; for $S' \subseteq S$ and $f : S \to \mathbb{R}$, $f(S')$ denotes $\sum(f(e) : e \in S')$.] Let $\nu = \nu(G, T)$ denote the greatest number of edge-disjoint T-paths in G. We consider the problem:

(1.1) *find a set $\mathcal{P}$ consisting of ν edge-disjoint T-paths in G so that their total cost $c(\mathcal{P}) := \sum_{P \in \mathcal{P}} c(P)$ is as small as possible.*

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♯ At present visiting Department of Computer Science, Stanford University, Stanford, California 94305–2140, U.S.A. On leave from Institute for Systems Analysis of Russian Academy of Sciences, 9, Prospect 60 Let Oktyabrya, 117312 Moscow, Russia.

E.g., if c is all-unit on EG, (1.1) is the problem of finding ν edge-disjoint T-paths in G that cover the least number of edges. If c is identically zero, (1.1) turns into the well-known problem on maximum packing of (edge-disjoint) T-paths; Mader [Ma1] and, independently, Lomonosov [Lom] showed that ν satisfies a certain minimax relation.

It turns out to be more convenient to pose a slightly more general problem, namely:

(1.2) *given a nonnegative real $p \in \mathbb{R}_+$, find a set $\mathcal{P}$ of edge-disjoint T-paths in G that maximizes the objective function*

$$\varphi(\mathcal{P}, p) := p|\mathcal{P}| - c(\mathcal{P}).$$

Evidently, if p is large enough, (1.2) becomes equivalent to (1.1); in particular, one can take $p = c(EG) + 1$.

The main aim of the present paper is to establish a minimax relation for (1.2). To state this, we need some definitions. Let $\phi = (X, U)$ be a pair consisting of a subset X of *inner* vertices in G (i.e. $X \subseteq VG - T$) and a subset $U \subseteq \delta X$ such that $|U|$ is odd. [Here $\delta X = \delta^G X$, the *cut* induced by X, is the set of edges of G connecting X and $VG - X$.] We say that ϕ is a *fragment* and denote X and U by X_ϕ and U_ϕ, respectively. Define

$$\begin{aligned} \chi_\phi(e) &:= 1 \quad \text{if } e \in U_\phi, \\ &:= -1 \quad \text{if } e \in \delta X_\phi - U_\phi, \\ &:= 0 \quad \text{for the other edges in } G \end{aligned}$$

(the *characteristic function* of ϕ). Let $\widetilde{\mathcal{F}}$ be the set of all fragments for G, T. For $\alpha : \widetilde{\mathcal{F}} \to \mathbb{R}_+$ and $\gamma : EG \to \mathbb{R}_+$, define the *amortized cost* function $c_{\alpha,\gamma}$ on EG to be

$$c_{\alpha,\gamma} := c + \gamma + \sum(\alpha(\phi)\chi_\phi \; : \; \phi \in \widetilde{\mathcal{F}}). \tag{1.3}$$

We say that (α, γ) is *admissible* for $p \geq 0$ if:

(1.4) $c_{\alpha,\gamma}$ is nonnegative;

(1.5) $\lambda := c_{\alpha,\gamma}$ satisfies $\text{dist}_\lambda(s, s') \geq p$ for all distinct $s, s' \in T$.

[$\text{dist}_\lambda(x, y)$ is the distance between vertices x and y in G with the edge length λ.]

Statement 1.1. *For any set $\mathcal{P}$ of edge-disjoint T-paths and (α, γ) admissible for p,* (1.6) $$\varphi(\mathcal{P},p) \le \gamma(EG) + \sum(\alpha(\phi)(|U_\phi|-1) \, : \, \phi \in \widetilde{\mathcal{F}}).$$

Proof. By (1.3) and (1.5), for $P \in \mathcal{P}$ we have

$$p - c(P) \le \gamma(P) + \sum_{\phi \in \widetilde{\mathcal{F}}} \alpha(\phi)\chi_\phi(P).$$

Since the paths in $\mathcal{P}$ are edge-disjoint, $\sum(\gamma(P) \, : \, P \in \mathcal{P}) \le \gamma(EG)$. Hence,

$$\varphi(\mathcal{P},p) = \sum_{P\in\mathcal{P}} (p - c(P)) \le \sum_{P\in\mathcal{P}} (\gamma(P) + \sum_{\phi\in\widetilde{\mathcal{F}}} \alpha(\phi)\chi_\phi(P)) \le \gamma(EG) + \sum_{\phi} \alpha(\phi) \sum_{P} \chi_\phi(P).$$

From the fact that the paths in $\mathcal{P}$ are edge-disjoint it follows that $\sum_{P\in\mathcal{P}} \chi_\phi(P) \le |U_\phi|$. Moreover, since $\chi_\phi(P)$ is, obviously, even, while $|U_\phi|$ is odd, $\sum_{P\in\mathcal{P}} \chi_\phi(P)$ does not exceed $|U_\phi| - 1$. This implies (1.6) (as $\alpha(\phi)$ is nonnegative). •

We prove the following theorem.

Theorem 1. *For any* $p \ge 0$,

(1.7) $$\max\{\varphi(\mathcal{P},p)\} = \min\{\gamma(EG) + \sum(\alpha(\phi)(|U_\phi|-1) \, : \, \phi \in \widetilde{\mathcal{F}})\},$$

where the maximum is taken over the sets $\mathcal{P}$ *of edge-disjoint* T*-paths, while the minimum it taken over all* (α,γ) *admissible for* p.

Like many proofs of minimax relations in combinatorial optimization, the proof of Theorem 1 utilizes the primal-dual method in linear programming. Also various combinatorial techniques are elaborated in the proof. Some tools involved appeal to the matching theory. Some other ones come from [Ka1] where it was proved that the fractional relaxation of (1.2) (i.e. the corresponding minimum cost maximum multi(commodity)flow problem) has a half-integral optimal solution (see also [Ka2]). Theorem 1 will follow from an auxiliary theorem discussed in the next section, where we also briefly outline the structure of the present paper and main stages of the proof.

In fact, the proof of Theorem 1 will provide a strongly polynomial algorithm to solve problems (1.2) and (1.1).

Remark 1.2. We shall see in Section 7 (Statement 7.3) that the minimum in (1.7) can be achieved with an α such that $|U_\phi| \geq 3$ for all ϕ with $\alpha(\phi) > 0$. In other words, in the above definition of a fragment ϕ we could add the condition that $|U_\phi| \geq 3$.

2. Auxiliary theorem

We call $\mathcal{P}, \alpha, \gamma$ *good* for p if they achieve the equality in (1.7). From the proof of Statement 1.1 it easily follows that $\mathcal{P}, \alpha, \gamma$ are good if and only if the following ("complementary slackness") conditions hold:

(2.1) $\lambda(P) = p$ for each $P \in \mathcal{P}$ (hence, P is shortest for $\lambda := c_{\alpha,\gamma}$, by (1.5));

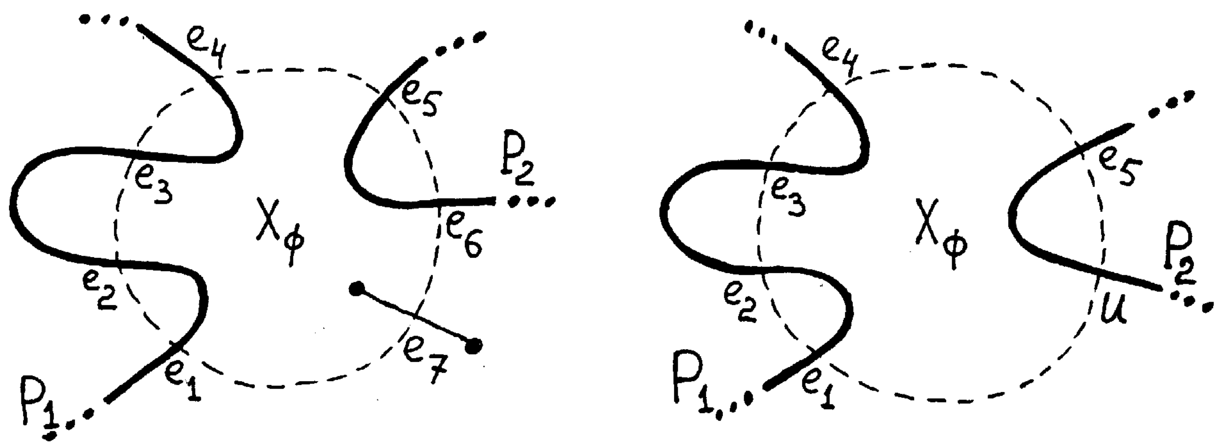
(2.2) for $e \in EG$, $\gamma(e) > 0$ implies that e is *covered* by $\mathcal{P}$ (i.e. e belongs to some $P \in \mathcal{P}$);

(2.3) for $\phi \in \widetilde{\mathcal{F}}$, $\alpha(\phi) > 0$ implies $\sum_{P \in \mathcal{P}} \chi_\phi(P) = |U_\phi| - 1$.

If ϕ satisfies the equality in (2.3), we say that ϕ is *saturated* by $\mathcal{P}$. Observe that if ϕ is saturated then one of the two situations takes place:

(2.4) $\mathcal{P}$ covers exactly $|U_\phi| - 1$ edges in U_ϕ and no edge in $\delta X_\phi - U_\phi$; or

(2.5) $\mathcal{P}$ covers all edges in U_ϕ and exactly one edge in $\delta X_\phi - U_\phi$.



(a) $U_\phi = \{e_1, \ldots, e_7\}$, $e_7 = e_\phi$ (b) $U_\phi = \{e_1, \ldots, e_5\}$, $u = e_\phi$

Fig. 2.1

(See Fig. 2.1.) The edge in U_ϕ that is not covered by $\mathcal{P}$ in case (2.4) as well as the edge in $\delta X_\phi - U_\phi$ that is covered by $\mathcal{P}$ in case (2.5) is called the *root* of ϕ and denoted by e_ϕ.

Let us consider p as a parameter in the problem which increases from 0 to ∞. Clearly, $\mathcal{P} = \emptyset, \alpha = 0, \gamma = 0$ are good for $p = 0$.

In what follows we refer to a set of edge-disjoint T-paths in G as a *packing*. Validity of Theorem 1 is provided by the following result.

Theorem 2. *Suppose that $\mathcal{P}, \alpha, \gamma$ are good for p. Then one of the following is true:*

(i) there exists a packing $\mathcal{P}'$ such that $|\mathcal{P}'| = |\mathcal{P}| + 1$, and $\mathcal{P}', \alpha, \gamma$ are good for p;

(ii) there exist $p' > p$, $\alpha' : \widetilde{\mathcal{F}} \to \mathbb{R}_+$ and $\gamma' : EG \to \mathbb{R}_+$ such that for any $0 \le \xi \le 1$, $\mathcal{P}, \alpha_\xi, \gamma_\xi$ are good for $(1-\xi)p + \xi p'$, where $\alpha_\xi := (1-\xi)\alpha + \xi\alpha'$ and $\gamma_\xi := (1-\xi)\gamma + \xi\gamma'$.

[In fact, the alternative takes place: either (i) or (ii) is true.]

Proof of Theorem 1 from Theorem 2. Supposing, for a contradiction, that Theorem 1 is false, let $\overline{p}$ be the maximum number such that (1.7) holds for every $0 \le p < \overline{p}$. Two cases are possible.

(a) The equality (1.7) holds for $\overline{p}$. Choose $\mathcal{P}, \alpha, \gamma$ so that they are good for $\overline{p}$ and $|\mathcal{P}|$ is as large as possible. Then (ii) in Theorem 2 implies the existence of $\overline{p}' > \overline{p}$ such that (1.7) holds for any p'' in the segment $[\overline{p}, \overline{p}']$, contrary to the definition of $\overline{p}$.

(b) The equality (1.7) is wrong for $\overline{p}$. Then $\overline{p} > 0$, and (1.7) holds for an infinite sequence $p_1 < p_2 < \ldots$ of numbers which tend to $\overline{p}$. Choose good $\mathcal{P}_i, \alpha_i, \gamma_i$ for p_i. Since the set of different packings consisting of simple paths is finite, we may assume that all the $\mathcal{P}_i$'s are the same packing $\mathcal{P}$.

For each i, (α_i,γ_i) is a solution of the system L_i formed by the linear constraints: (i) $\alpha_i \ge 0$; $\gamma_i \ge 0$; (ii) $\lambda := c_{\alpha_i,\gamma_i} \ge 0$; (iii) $\lambda(P) = p_i$ for each $P \in \mathcal{P}$; (iv) $\lambda(P) \ge p_i$ for each *simple* T-path P; (v) $\gamma_i(EG) + \sum(\alpha_i(\phi)(|U_\phi| - 1) \,:\, \phi \in \widetilde{\mathcal{F}}) = p_i|\mathcal{P}| - c(\mathcal{P})$. We observe that the constraint matrices of L_i are the same for all i while the right hand side vector linearly depends on p_i. As the p_i's tend to $\overline{p}$, standard l.p. arguments imply that there are solutions (α_i, γ_i) of the L_i's which tend to some $(\alpha, \gamma) \in \mathbb{R}_+^{\widetilde{\mathcal{F}}} \times \mathbb{R}_+^{EG}$. Then $\mathcal{P}, \alpha, \gamma$ are good for $\overline{p}$; a contradiction. •

Throughout the remaining part of the paper we assume that the cost function c is *positive*, i.e. $c(e) > 0$ for all $e \in EG$. This assumption will significantly simplify some details of the proof. On the other hand, it leads to no loss of generality, as it is easy to show by use of standard l.p. arguments.

The proof of Theorem 2 is presented in Sections 3-8. It starts in Section 3 with introducing elementary, but important, notions and structures and describing their properties; some of them occurred in [Ka1,Ka2]. In Sections 4,5 we give a relatively simple proof of Theorem 2 under the assumption that the current α is identically zero (whence $\lambda := c_{\alpha,\gamma}$ is positive since c is positive and γ is nonnegative). We distinguish this special case to expose basic ideas of our combinatorial primal-dual approach to the problem. The general case is studied in Sections 6-8. In this case edges $e \in EG$ with $\lambda(e) = 0$ are possible (we cannot, in general, avoid appearance of such edges, even assuming the positivity of c); this makes the analysis more involved. In Section 6 we show validity of Theorem 2 provided that a number of additional conditions is imposed, which specify the current $\mathcal{P}, \alpha, \gamma$ as well as structures related to them (in fact, to be able to prove Theorem 2 we are forced to strengthen this theorem; in particular, we have to require that the graph induced by edges $e \in \cup(P \in \mathcal{P})$ with $\lambda(e) = 0$ is a forest). In Sections 7 and 8 we verify maintenance of these properties (i.e. satisfying the imposed conditions) after the transformation of $\mathcal{P}$ or (α, γ); this part is most technical and tiresome in the proof.

As mentioned above, the proof provides an algorithm for finding good $\mathcal{P}, \alpha, \gamma$ for an arbitrary p. More precisely, in Section 9 we show that the algorithm finds, in strongly polynomial time, a sequence $0 \le p_1 < \ldots < p_N$ of numbers and objects $\mathcal{P}_i, \alpha_i, \gamma_i$ ($i = 1, \ldots, N$), $\alpha_{N+1}, \gamma_{N+1}$ so that: (i) $\mathcal{P} = \emptyset, \alpha = 0, \gamma = 0$ are good for any $p \in [0, p_1]$; (ii) for $i = 1, \ldots, N-1$ and $p \in [p_i, p_{i+1}]$, the packing $\mathcal{P}_i$ and functions α and γ are good for p, where α (γ) is the corresponding convex combination of α_i and α_{i+1} (γ_i and γ_{i+1}); and (iii) $|\mathcal{P}_N| = \nu$, and for any $p \ge p_N$ the packing $\mathcal{P}_N$ and functions α and γ are good for p, where α (γ) is the corresponding nonnegative combination of α_N and α_{N+1} (γ_N and γ_{N+1}).

Finally, in Section 10 we explain that the final $\alpha_{N+1}, \gamma_{N+1}$ enable us to derive optimal dual objects figured in the following theorem describing a minimax relation for the "pure" (i.e. zero cost) problem (hereinafter for a subset A of vertices (edges) of a graph G', $G' - A$ denotes the graph obtained from G by removing (deleting) A).

Theorem 3 [Ma1,Lom].

$$\nu(G,T) = \frac{1}{2}\min\{\sum_{s\in T} |\delta Y_s| - \eta\},$$

where the minimum ranges over all families of pairwise disjoint sets $Y_s \subset VG$ ($s \in T$) such that $Y_s \cap T = \{s\}$, and η denotes the number of components K of the graph $G - (\cup_{s\in T} Y_s)$ such that $|\delta^G(VK)|$ is odd.

3. Lines and potentials

Let $\lambda : EG \to \mathbb{R}_+$ be a function. Define

$$(3.1) \qquad p := p_\lambda := \min\{\mathrm{dist}_\lambda(s, s') \ : \ s, s' \in T, s \neq s'\}.$$

A *potential* $\pi(v) = \pi_\lambda(v)$ of a vertex $v \in VG$ is the λ-distance from v to T, i.e. $\min\{\mathrm{dist}_\lambda(v, s) \ : \ s \in T\}$. Denote by $T(v) = T_\lambda(v)$ the set of terminals s closest to v, i.e. such that $\mathrm{dist}_\lambda(v, s) = \pi(v)$. A T-path (possibly non-simple) of λ-length exactly p is called a *T, λ-line*, or, briefly, a *T-line*. A path that is part of a T-line is called a *line*. Denote by $\Gamma = \Gamma^\lambda$ the subgraph of G formed by the set of terminals and all vertices and edges occurring in T-lines.

The vertices in Γ are naturally partitioned into sets V_s $(s \in T)$ and $V^\bullet$. Here V_s consists of all $v \in V\Gamma$ such that $\mathrm{dist}_\lambda(s, v) < p/2$; and $V^\bullet := V\Gamma - \cup_{s \in T} V_s$; a vertex in $V^\bullet$ is called *central.* Clearly, $T(v) = \{s\}$ for $v \in V_s$, whereas $|T(v)| \geq 2$ and $\pi(v) = p/2$ for $v \in V^\bullet$. The following property is obvious.

Statement 3.1. *Let L be a path from x to y in G.*

(i) *If $x, y \in V_s \cup V^\bullet$ then L is a line if and only if $\lambda(L) = |\pi(x) - \pi(y)|$.*

(ii) *If $x \in V_s$ and $y \in V_{s'}$ for distinct $s, s' \in T$ then L is a line if and only if $\pi(x) + \pi(y) + \lambda(L) = p$.* $\bullet$

For $x \in V\Gamma$ denote by $E(x) = E_\lambda(x)$ the set of edges in Γ incident to x. Consider an edge $e = xy \in E\Gamma$ with $\lambda(e) > 0$. We assign to (x, e) an *attachment* $l(x, e) = l_\lambda(x, e)$ by the following rule:

(3.2) (i) if $x \in V_s \cup V^\bullet$, $y \in V_s$ and $\pi(y) < \pi(x)$, put $l(x, e) := s$;

(ii) if $x \in V_s$ and either $y \notin V_s$, or $y \in V_s$ and $\pi(x) < \pi(y)$, put $l(x, e) := \overline{s}$.

It is easy to see that if $l(x, e) = s$ then for every T-line L from s_1 to s_2 that meets x, e, y in this order, s_2 is s; while if $l(x, e) = \overline{s}$ then for L as above, s_1 is s. Note that $\lambda(e) > 0$ makes it impossible that $x, y \in V^\bullet$.

Next, it is convenient to assume that the termunals are numbered by the integers from 1 up to $t := |T|$. Then T is identified with $\langle 1, t \rangle$, and an attachment $s \in T$ for (x, e) means the corresponding number in $\langle 1, t \rangle$; while we assume that the attachment $\overline{s}$ is identified with the number $-s \in \langle -t, -1 \rangle$. Hereinafter $\langle i, j \rangle$ denotes the set of integers $k \neq 0$ such that $i \leq k \leq j$. For $x \in V\Gamma$ and $s \in \langle -t, t \rangle$ define $E_s(x) := \{e \in$

$E(x) \,:\, l(x,e) = s\}$.

Suppose that the function λ is *positive*, i.e. $\lambda(e) > 0$ for all $e \in EG$. Then every pair (x,e) ($x \in V\Gamma$, $e \in E(x)$) has an attachment $s \in \langle -t, t\rangle$. Moreover, using Statement 3.1 one can easily obtain the following description of the lines in terms of attachments.

Statement 3.2. *A path $L = (x_0, e_1, x_1 \ldots, e_k, x_k)$ in Γ is a line if and only if for $i = 1, \ldots, k-1$ the attachments $l(x_i, e_i)$ and $l(x_i, e_{i+1})$ are different.* •

Now return to consideration $\mathcal{P}, \alpha, \gamma$ as above. Put $\lambda := c_{\alpha,\gamma}$ (see (1.3)). Assume that λ is *positive* and $p = p_\lambda$ (see (3.1)). If $\mathcal{P}, \alpha, \gamma$ are good for p, and B is the set of edges covered by $\mathcal{P}$, then (2.1)-(2.2) imply that $B \subseteq E\Gamma$ and

$$\gamma(e) = 0 \quad \text{for all } e \in EG - B. \tag{3.3}$$

One of key ideas in the proof is that we can handle a set B of edges which can be covered by some optimal packing $\mathcal{P}$, rather than $\mathcal{P}$ itself. More precisely, let $B \subseteq E\Gamma$ be a set. For $x \in V\Gamma$ and $s \in \langle -t, t\rangle$ define $B(x) := B \cap E(x)$ and $B_s(x) := B \cap E_s(x)$. We say that B is *regular* if

(3.4) (i) $|B(x)|$ is even for all $x \in V\Gamma - T$;

(ii) B saturates each $\phi \in \widetilde{\mathcal{F}}$ with $\alpha(\phi) > 0$, i.e. $\chi_\phi(B) = |U_\phi| - 1$;

(iii) B is *non-excessive* for each $x \in V\Gamma - T$; this means that $|B_s(x)| \le \frac{1}{2}|B(x)|$ for any $s \in \langle -t, t\rangle$.

In particular, the set of edges covered by $\mathcal{P}$ as above is, obviously, regular. It turns out that for a regular set B the converse takes place. More precisely, let $\mu(B)$ denote the number of pairs (s,e) such that $s \in T$ and $e \in B(s)$. We say that $s \in \langle -t, t\rangle$ is *tight* for $x \in V\Gamma - T$ if $|B_s(x)| = |B(x)|/2$.

Statement 3.3. *If B is regular then B has a representation of the form $B = \cup(P \in \mathcal{P})$, where $\mathcal{P}$ is a packing consisting of $\mu(B)/2$ T-lines and satisfying (2.3).*

Proof. To prove the existence of $\mathcal{P}$ consisting of $\mu(B)/2$ T-lines, we use induction on $|B|$. Let us design a path $P = (x_0, e_1, x_1, \ldots)$, starting from some $x_0 \in V\Gamma$ and $e_1 \in B(x_0)$, as follows. Suppose a simple path $P' = (x_0, e_1, x_1, \ldots, e_i, x_i)$ has been constructed. If $x := x_i \in T$, put $P := P'$. Otherwise extend P' by adding $e = e_{i+1} \in B(x)$ so that: (a) $e \ne e_i$; (b) if $s := l(x, e_i)$ is tight for x then $l(x,e) \ne s$ (e exists by(3.4)(i),(iii), under a natural assumption about P'). Statement 3.2 implies that the resulting path P is a line. In particular, P is simple (as P is λ-shortest and λ is positive), whence P is finite and its final vertex belongs to T. Moreover, in view of the latter property, we may assume that $x_0 \in T$, i.e. P is a T-line. By construction of P, the set $B' := B - P$ obviously satisfies (i) and (iii) in (3.4), and now the result follows by induction. (2.3) obviously follows from (3.4)(ii). •

Thus, B, α, γ give an optimal solution for p_λ whenever $\lambda := c_{\alpha,\gamma}$ is positive, B is regular, and (3.3) holds. In general case, when edges $e \in EG$ with $\lambda(e) = 0$ are possible, it is a more complicated task to define an attachment $l(x, e)$ for such e's as well as to generalize the notion of a regular set in such a way that it ensures the property as in Statement 3.3; we leave this up to Section 6.

In what follows we often call the edges in B *bold*, and the edges in $E\Gamma - B$ *thin*.

4. Augmenting paths

In this and the next sections we prove Theorem 2 for the simplest case when $\alpha = 0$. Let $\lambda := c_{\alpha,\gamma}$. Then $\lambda = c + \gamma$, hence λ is positive. Put $p := p_\lambda$ (see (3.1)). Let $B \subseteq E\Gamma_\lambda$ be regular, and (3.3) holds. As it was shown in Section 3, B and γ give an optimal solution for p.

[Observe that if $0 \le p \le p_c$ then $\mathcal{P} = \emptyset$ (or $B = \emptyset$), $\alpha = 0$ and $\gamma = 0$ are good for p. Furthermore, one can see that it suffices to prove Theorem 2 for $p, \mathcal{P}, \alpha, \gamma$ such that $p = p_\lambda$ for $\lambda := c_{\alpha,\gamma}$.]

We use notation as in Section 3. Define $B^0 := \{e \in B \, : \, \gamma(e) = 0\}$ and $Z := E\Gamma - B$. For $x \in V\Gamma$ and $s \in \langle -t, t \rangle$ define $Z(x) := Z \cap E(x)$, $Z_s(x) := Z \cap E_s(x)$, $B^0(x) := B^0 \cap E(x)$ and $B^0_s(x) := B^0 \cap E_s(x)$. An edge $e \in E\Gamma$ with $\gamma(e) = 0$ (i.e. $e \in Z \cup B^0$) is called *feasible* (for α, γ). Note that B can be changed only within the set of feasible edges in order to maintain the complementary slackness condition (2.2) (or (3.3)).

Consider a triple $\tau = (e, v, e')$, where $v \in V\Gamma - T$, and e, e' are distinct feasible edges incident to v. We say that τ is a *fork* if

(4.1) there is no $s \in \langle -t, t \rangle$ such that s is tight for v and $e, e' \in Z_s(v) \cup (B(v) - B_s(v))$.

When it leads to no confusion, a fork (e, v, e') may be denoted as (e, e'). One can see that for (distinct) $e, e' \in Z(v) \cup B^0(v)$, (e, v, e') is a fork if and only if $|B'_s(v)| \le |B'(v)|/2$

for any $s \in \langle -t, t\rangle$, where $B'(v) := B(v)\triangle\{e, e'\}$ and $B'_s(v) := B'(v) \cap E_s(v)$ ($X\triangle Y$ denotes the symmetric difference $(X - Y) \cup (Y - X)$). It is useful to list the cases when (e, v, e') is a fork, namely:

(C1) if $v \in V_s$ for some $s \in T$ then e, e' belong to different sets $Z_s(v)$ and $Z_{-s}(v)$, or different sets $B^0_s(v)$ and $B^0_{-s}(v)$, or different sets $Z_{s'}(v)$ and $B^0_{s'}(v)$ for $s' \in \{s, -s\}$ (as in this case both s and $-s$ are tight).

(C2) if $v \in V^\bullet$ and $B(v) = \emptyset$ then $e \in Z_s(v)$ and $e' \in Z_{s'}(v)$ for distinct $s, s' \in \langle -t, t\rangle$ (as in this case each $s \in \langle -t, t\rangle$ is tight);

(C3) if $v \in V^\bullet$, $B(v) \neq \emptyset$, and no s is tight for v, then e, e' are arbitrary;

(C4) if $v \in V^\bullet$, $B(v) \neq \emptyset$, and the set S of elements $s \in \langle -t, t\rangle$ tight for v is nonempty (clearly $|S| \leq 2$), then e, e' form a fork except the cases when either $e, e' \in Z_s(v)$ for $s \in S$, or $e, e' \in B(v) - B_s(v)$ for $s \in S$, or e, e' belong to different sets $Z_s(v)$ and $B(v) - B_s(v)$ for $s \in S$.

From (4.1) it easily follows that

(4.2) for any $v \in V\Gamma - T$ and feasible e, e', e'' incident to v, if neither (e, e') nor (e', e'') is a fork then (e, e'') is not a fork either.

A path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ in Γ is called *active* if: (i) all $e_1, \dots, e_k$ are distinct, (ii) for $i = 1, \dots, k-1$, $x_i \in V\Gamma - T$ and (e_i, x_i, e_{i+1}) is a fork, (iii) if $x_0 \in T$ then $e_1 \in Z$. We say that an active P is *primitive* if for any $1 \leq i < j < k$ such that $x_i = x_j$, the triple (e_i, x_i, e_{j+1}) is not a fork, and P meets each of x_0, x_k at most twice. Clearly if vertices x and y are connected by an active path P, they can be connected by a primitive path (e.g., consisting of certain parts of P). From (4.2) it follows that

(4.3) for a primitive path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ there are no $0 < i < j < r < k$ such that $x_i = x_j = x_r$.

Indeed, suppose that this is not so. Then (e_i, e_{j+1}) and (e_i, e_{r+1}) are not forks (otherwise P is not primitive). Hence, (e_{j+1}, e_{r+1}) is not a fork (by (4.2)). Now the fact that (e_j, e_{j+1}) is a fork implies that (e_j, e_{r+1}) is also a fork (by (4.2)). Thus, P is non-primitive.

A primitive T-path $(x_0, e_1, x_1, \dots, e_k, x_k)$ is called *augmenting* if $e_k \in Z$. We say that $B' \subseteq E\Gamma$ is obtained by the *alteration along a path* P if $B' = B\triangle P$ (considering P as an edge-set).

Statement 4.1. *If P is an augmenting path and $B' := B\triangle P$ then B' is regular, $\mu(B') = \mu(B) + 2$, and (3.3) holds for B'.*

Proof. $\mu(B') = \mu(B) + 2$ and (3.4)(i),(ii) for B' are obvious. (3.3) for B' is true since P uses only feasible edges. Let us prove (3.4)(iii) for B'.

Consider $x \in V\Gamma - T$. The inequality in (3.4)(iii) for B' and x is obvious if P passes x at most once. Otherwise P passes x twice, by (4.3); let $x = x_i = x_j$ for $0 < i < j < k$. Since P is primitive, $\tau = (e_i, e_{j+1})$ is not a fork. Then $e_i, e_{j+1} \in Z_s(x) \cup (B(x) - B_s(x))$ for some $s \in \langle -t, t\rangle$ tight for x, B, by (4.1). This implies that $e_{i+1} \in B_s(x) \cup (Z(x) - Z_s(x))$ since (e_i, x_i, e_{i+1}) is a fork. Similarly, $e_j \in B_s(x) \cup (Z(x) - Z_s(x))$ since (e_j, x_i, e_{j+1}) is a fork. Then s retains tightness for B' and x.

Now the result follows from the obvious fact that if some $\widetilde{s}$ is tight for B' and x, then B' is non-excessive for x. •

Thus, in case $\alpha = 0$, the existence of an augmenting path implies validity of (i) in Theorem 2. In the next section we show that lack of the augmenting path implies (ii) in this theorem.

Now we describe an approach to find an augmenting path or, if it does not exist, to construct the set of vertices reachable by active paths beginning at T; such a set will play an important role in the transformation of (α, γ). We apply techniques similar, in a sense, to that developed in the matching theory and even use terms from that area.

We grow in Γ, step by step, a digraph $D = (VD, AD)$ with $T \subseteq VD \subseteq V\Gamma$. For an *arc* $a = (x, y)$ in D the underlying edge e, denoted by e^a, is a feasible edge in Γ; we may identify a with the edge e *labelled from x to y*. A vertex in Γ belonging to D is also called labelled. Let Q_1 be the set of edges in Γ labelled in one direction, or *1-labelled* edges, and let Q_2 be the set of edges in Γ labelled in both directions, or *2-labelled* ones. The components of the subgraph of Γ induced by Q_2 are called *blossoms*. Also a special blossom of the form $(\{v\}, \emptyset)$ is possible, where v is a certain labelled central vertex (see (4.4)(iv)); such a blossom is called *elementary*. The vertex-sets of the blossoms are pairwise-disjoint.

For a blossom F denote by AF the corresponding arc-set. A labelled vertex which does not belong to any blossom (belongs to a blossom) is called *1-labelled* (respectively, *2-labelled*). A blossom F satisfies the following conditions:

(4.4) (i) F contains no terminal ($VF \subseteq V\Gamma - T$);

(ii) there is an arc $a_F = (x, y) \in Q_1$ fixed that enters F (i.e. $x \notin VF \ni y$), called the *root* of F;

(iii) for each arc $a = (x, y) \in AF$ there is a directed path $P_{F,a} = (x_0, a_1, x_1, \ldots, a_k, x_k)$ the part of which from x_1 to x_k is a path in F, $a_1 = a_F$, $a_k = a$, and $P_{F,a}$ is active (considering $P_{F,a}$ as a path in Γ);

(iv) if F is an elementary blossom $(\{v\}, \emptyset)$, and $e = e^{a_F}$, then there is no $s \in \langle -t, t\rangle$ such that s is tight for v, and $e \in Z_s(v) \cup (B(v) - B_s(v))$.

The digraph D satisfies the following conditions:

(4.5) (i) if $a \in AD$ leaves T then $e^a \in Z$;

(ii) for each 1-labelled vertex $v \in VD - T$ there is an arc $a_v \in AD$ fixed that enters v;

(iii) for each arc $a \in AD$ there is a directed path $P_a = (x_0, a_1, x_1, \ldots, a_k, x_k)$ in D such that: (a) P_a is active, $x_0 \in T$ and $a_k = a$, (b) for each 1-labelled vertex x_i, $a_i = a_{x_i}$, (c) if a_i enters a blossom F then a_i is the root a_F; (d) if some arc a_i belongs to a blossom F then P_a contains the path P_{F,a_i} as a part.

For a path $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ let P^{-1} denote the reverse path $(x_k, e_k, \ldots, x_0)$; if $Q = (v_0, u_1, v_1, \ldots, u_m, v_m)$ is a path with $v_0 = x_k$, let $P \cdot Q$ denote the concatenated path $(x_0, e_1, x_1, \ldots, e_k, x_k, u_1, v_1, \ldots, u_m, v_m)$.

If there exists an arc a in D entering T, and $e^a \in Z$, then P_a is an active T-path (by (4.5)(i),(iii)), hence, we can extract an augmenting path from P_a (we say that a *break-through* takes place). Otherwise we attempt to increase D by a natural way.

(A1) Suppose that there is an unlabelled feasible edge $e = xy$ such that: (a) $x \in VD$; (b) x and y do not belong to the same blossom; and (c) either $x \in T$, or $x \notin T$ and there is an arc $a = (z, x) \in AD$ such that (e^a, x, e) is a fork. Then we add (x, y) to D (so e becomes labelled from x to y). If y does not belong to the old D, we put $a_y := (x, y)$. If, in addition, the case as in (4.4)(iv) (with $v = y$) occurs, then we form $(\{y\}, \emptyset)$ to be a new elementary blossom rooted at (x, y).

(A2) Suppose that there is a 1-labelled edge $e = xy$ such that: (a) e is labelled from y to x; (b) x and y do not belong to the same blossom; (c) there is an arc $a = (z, x) \in AD$ such that (e^a, x, e) is a fork; and (d) the path P_a does not contain the arc a', where a' is e labelled from y to x. Two cases are possible.

(i) P_a and $P_{a'}$ do not contain arcs b and b', respectively, such that $e^b = e^{b'}$. Then $P_a \cdot (P_{a'})^{-1}$ is an active T-path, hence, a break-through happens.

(ii) $P_a = (x_0, a_1, x_1, \ldots, a_k, x_k)$ and $P_{a'} = (y_0, b_1, y_1, \ldots, b_m, y_m)$ contain arcs a_i and b_j, respectively, such that $e^{a_i} = e^{b_j}$. Let i be chosen maximum under this condition. Represent P_a as $P_1 \cdot L_1 \cdot \ldots \cdot P_r \cdot L_r \cdot P_{r+1}$, where all vertices in L_q belong to the same blossom $F_{\sigma(q)}$, while all inner vertices and all arcs in P_q do not belong to any blossoms (here $P_1, \ldots, P_r$ are non-trivial paths, but P_{r+1} may be trivial). Similarly, represent $P_{a'}$ as $P'_1 \cdot L'_1 \cdot \ldots \cdot P'_h \cdot L'_h \cdot P'_{h+1}$, where L'_q lies in a blossom $F_{\sigma'(q)}$. From (4.5)(iii) it follows that if L_q and $L'_{q'}$ belong to the same blossom then $q = q'$ and $P_1 \cdot L_1 \cdot \ldots \cdot P_{q-1} \cdot L_{q-1} \cdot P_q$ coincides with $P'_1 \cdot L'_1 \cdot \ldots \cdot P'_{q-1} \cdot L'_{q-1} \cdot P'_q$. Let q be chosen maximum under such a property; if there is no common blossom for P_a and $P_{a'}$, put $q := 0$. Consider two cases.

(a) a_i belongs to L_q. Then $F_{\sigma(q)} = F_{\sigma'(q)}$. For each arc in $P_{q+1}, \ldots, P_{r+1}$ add to D the opposite arc; and similarly for each arc in $P'_{q+1}, \ldots, P'_{h+1}$. As a result, $F_{\sigma(q)}, \ldots, F_{\sigma(r)}, F_{\sigma'(q+1)}, \ldots, F_{\sigma'(h)}$ together with the vertices and arcs of the paths $P_{q+1}, \ldots, P_{r+1}, P'_{q+1}, \ldots, P'_{h+1}$ merge into a new blossom rooted at $a_{F_{\sigma(q)}}$.

(b) a_i belongs to P_{q+1}. Then $a_i = b_j$. For $i' = i+1, \ldots, k$, add to D the arc opposite to $a_{i'}$ (if it did not occur in D earlier). Similarly, for $j' = j+1, \ldots, m$, add to D the arc opposite to $b_{j'}$. All these arcs together with the blossoms $F_{\sigma(q+1)}, \ldots, F_{\sigma(r)}, F_{\sigma'(q+1)}, \ldots, F_{\sigma'(h)}$ merge into a new blossom rooted at a_i.

One can see that the created blossom, formed in (a) or (b), satisfies (4.4)(i)-(iii). Suppose that neither D can be increased by the above rules nor a break-through happens. Then D has features similar, in a sense, to those occurring in the so-called "Hungarian tree with flowers" in the matching theory [Ed]. More precisely, the following are true:

(4.6) Each edge $e = xy \in Z$ with $x \in T$ is labelled as leaving (but not entering) x.

(4.7) Let $x \in VD - T$ be a 1-labelled vertex, and $E^+(x)$, $E^-(x)$, $E^{\rm un}(x)$ be the sets of feasible edges $e = xy \in E(x)$ labelled as entering x, labelled as leaving x, and unlabelled, respectively; then:

(i) no pair of distinct edges in $E^+(x) \cup E^{\rm un}(x)$ forms a fork;

(ii) each pair $e \in E^+(x)$, $e' \in E^-(x)$ forms a fork.

(4.8) for each blossom F all feasible edges in the cut $\delta^\Gamma(VF)$ are labelled as leaving F except $e_F := e^{a_F}$, and e_F is the only edge in this set that is labelled as entering F.

[One can see that (4.6)-(4.8) imply that no augmenting path in Γ exists.] Proofs of (4.6) and (4.7) are easy. To prove (4.8), we need the following statement.

Statement 4.2. *Let $e = xy$ be a feasible edge such that $x \in VF \not\ni y$ for some blossom F, and $e \neq e_F$. Then there exists an arc $a = (u, x) \in AF \cup \{a_F\}$ such that (e^a, x, e) is a fork.*

Proof. If F is an elementary blossom then (e^{a_F}, x, e) is a fork, by (4.4)(iv). For a non-elementary F there is an edge $e' = xz \in EF$. We know that e' is labelled in both directions. If (e, x, e') is a fork, we are done. Otherwise take the arc $a' = (x, z) \in AD$ with $e^{a'} = e'$, and consider the path $P_{F,a'} = (x_0, a_1, x_1, \ldots, a_k, x_k)$ as in (4.4)(iii); then $k \geq 2$ and $a' = a_k$. Let e'' be the underlying edge for the arc $a := a_{k-1}$. Since (e', x, e'') is a fork and (e, x, e') is not, (e'', x, e) is a fork (by (4.2)), whence a is as required. •

Suppose that there is a blossom F and a feasible edge $e = xy$ different from e_F so that $x \in VF \not\ni y$ and e is not labelled from x to y. Let $a = (u, x)$ be as in Statement 4.2. If e is unlabelled then the arc (x, y) can be added to D according to (A1). And if D contains the arc $a' = (y, x)$ then D can be increased according to (A2) (note that P_a does not contain a' since $e \neq e_F$ and P_a uses exactly one edge in $\delta^\Gamma(VF)$, namely, e_F). This contradiction proves (4.8).

Remark. One can show that search for an augmenting path in Γ can be reduced to the standard problem on finding an alternating path in a graph Q with a matching M in it. As a consequence, (4.7)-(4.8) are derived from properties of the "Hungarian tree with flowers" [Ed]. Such a Q is designed by replacing each vertex x in Γ by a special subgraph (depending on the set of forks for x). However, this approach would make our description more intricate, and it is preferable to argue explicitly in terms of Γ itself.

5. Transformation of (α, γ)

As before, we consider case $\alpha = 0$ and use notation as in the previous section. We assume that there is no augmenting path in Γ.

Each blossom F in Γ generates the fragment ϕ for G, T with $X_\phi := VF$ and

(5.1) $U_\phi := (B \cap \delta X_\phi) \cup \{e_F\}$ if $e_F \in Z$, and $U_\phi := (B \cap \delta X_\phi) - \{e_F\}$ if $e_F \in B$

(cf. (2.4)-(2.5)). Then $|U_\phi|$ is odd (by (3.4)(i)), B saturates ϕ, and e_F is the root e_ϕ of ϕ. Let $\mathcal{F}$ be the set of these fragments.

The required α' and γ' will be assigned to be α^ε and γ^ε (for some $\varepsilon > 0$) defined below. Let us fix $\varepsilon \in \mathbb{R}_+$. We put $\alpha^\varepsilon(\phi)$ to be ε for all $\phi \in \mathcal{F}$ (and 0 for the remaining fragments ϕ for G, T). For brevity, the value $\sum(\alpha^\varepsilon(\phi)\chi_\phi(e) \ : \ \phi \in \mathcal{F})$ is denoted by $\widehat{\alpha}^\varepsilon(e)$.

To define γ^ε is a more involved task. First of all we shall introduce (in (5.4)-(5.5) below) a certain value $\rho(x, e) = \rho^\varepsilon(x, e) \in \{0, \varepsilon, -\varepsilon\}$ for each $x \in V\Gamma$ and $e \in E(x)$. This gives the function $\rho = \rho^\varepsilon$ on $E\Gamma$ by

$$(5.2) \qquad \rho(e) := \rho(x,e) + \rho(y,e) \qquad \text{for } e = xy \in E\Gamma.$$

Then γ^ε is defined by

$$(5.3) \qquad \begin{aligned} \gamma^\varepsilon(e) &:= \gamma(e) + \rho^\varepsilon(e) - \widehat{\alpha}^\varepsilon(e) + \widehat{\alpha}(e) \qquad \text{for } e \in B, \\ &:= 0 \qquad\qquad \text{for } e \in EG - B, \end{aligned}$$

where, for our case, $\widehat{\alpha}(e) := 0$ for all $e \in EG$.

Let L, N, M denote the sets of 1-labelled, 2-labelled and unlabelled vertices in Γ, respectively. Clearly $T \subseteq L$. Put

$$(5.4) \qquad \begin{aligned} \rho(x,e) = \rho^\varepsilon(x,e) &:= \varepsilon \qquad \text{for } x \in T \text{ and } e \in E(x), \\ &:= 0 \qquad \text{for } x \in N \cup M \text{ and } e \in E(x). \end{aligned}$$

To define $\rho(x,e)$ for the vertices in L, we need the following notion. For a vertex $v \in V\Gamma - T$ and distinct edges $e, e' \in E(v)$, we say that (e, v, e') is a *pseudo-fork* if it satisfies (4.1) (thus, the difference with the definition of a fork is that e, e' are not required to be feasible).

Consider a 1-labelled vertex $x \in V\Gamma - T$, and fix some edge u labelled as entering x. For $e \in E(x)$ put

$$(5.5) \quad \begin{aligned} \rho(x,e) = \rho^\varepsilon(x,e) &:= -\varepsilon \ \text{ if either } e \in Z \text{ and } (u,x,e) \text{ is not a pseudo-fork,} \\ &\qquad\qquad \text{or } e \in B \text{ and } (u,x,e) \text{ is a pseudo-fork;} \\ &:= \varepsilon \ \text{ if either } e \in Z \text{ and } (u,x,e) \text{ is a pseudo-fork,} \\ &\qquad\qquad \text{or } e \in B \text{ and } (u,x,e) \text{ is not a pseudo-fork ;} \end{aligned}$$

letting by definition that (u,x,u) is not a pseudo-fork. To make the meaning of (5.5) clearer, consider possible cases.

(L1) $x \in V_s$ for some $s \in T$. Then both s and $-s$ are tight for x. Suppose that $u \in Z_s(x) \cup B_{-s}(x)$. Then $\rho(x,e) = -\varepsilon$ for all $e \in E_s(x)$ and $\rho(x,e) = \varepsilon$ for $e \in E_{-s}(x)$. If $u \in Z_{-s}(x) \cup B_s(x)$ then $\rho(x,e) = -\varepsilon$ for all $e \in E_{-s}(x)$ and $\rho(x,e) = \varepsilon$ for $e \in E_s(x)$.

(L2) $x \in V^\bullet$. Since x does not form an elementary blossom, we observe from (4.4)(iii) and (4.7) that there is $s \in T$ tight for x and such that each edge labelled as entering x belongs to $Z_s(x) \cup (B^0(x) - B^0_s(x))$, while each edge labelled from x belongs to $B^0_s(x) \cup (Z(x) - Z_s(x))$. Then $\rho(x,e) = -\varepsilon$ for $e \in E_s(x)$ and $\rho(x,e) = \varepsilon$ for $e \in E(x) - E_s(x)$.

Note that in definition (5.5) it does no matter what entering labelled edge is chosen as u (by (4.7) and (4.2)).

Now we study properties of functions ρ^ε, γ^ε and $\lambda^\varepsilon := c_{\alpha^\varepsilon,\gamma^\varepsilon}$. Our goal is to prove the following two statements.

Statement 5.1. *For a sufficiently small $\varepsilon > 0$, $\lambda^{\varepsilon'}(e)$ and $\gamma^{\varepsilon'}(e)$ are nonnegative for any $e \in EG$ and $0 \le \varepsilon' \le \varepsilon$.*

Let ε_1 (ε_2) denote the greatest ε such that $\lambda^{\varepsilon'}$ (respectively, $\gamma^{\varepsilon'}$) is nonnegative for any $0 \le \varepsilon' \le \varepsilon$.

Statement 5.2. *For a sufficiently small ε, $0 < \varepsilon \le \varepsilon_1$, the equality $p_{\lambda^{\varepsilon'}} = p + 2\varepsilon'$ holds for any $0 \le \varepsilon' \le \varepsilon$.*

Properties (4.7)-(4.8) show that some sorts of feasible edges need not connect certain sets among $L, M, \{X_\phi : \phi \in \mathcal{F}\}$ or vertices in L. More precisely, for a feasible edge $e = xy$:

(5.6) (i) if e is unlabelled and $x, y \in L$ then neither (u, x, e) nor (u', y, e) is a fork, where u (u') is an edge labelled as entering x (respectively, y);

(ii) if e is unlabelled, $x \in L$ and $y \in M$ then (u, x, e) is not a fork, where u is an edge labelled as entering x;

(iii) e does not connect X_ϕ ($\phi \in \mathcal{F}$) and M;

(iv) if $x \in L$ and $y \in X_\phi$ ($\phi \in \mathcal{F}$) then e is labelled; moreover, if e is labelled from x to y then $e = e_\phi$;

(v) if e connects X_ϕ and $X_{\phi'}$ ($\phi, \phi' \in \mathcal{F}$) then e is the root of exactly one of ϕ, ϕ'.

(For if any of (i)-(v) is violated, the labelled digraph D can be enlarged.)

Proof of Statement 5.1. The assertion for λ^ε is obvious since $\lambda(e) > 0$ and $|\lambda^\varepsilon(e) - \lambda(e)| = O(\varepsilon)$ for all $e \in EG$.

To prove the assertion for γ^ε, it suffices to examine an edge $e = xy \in B^0$ (since

$\gamma^\varepsilon(e) = 0$ for $e \in EG - B^0$, by (5.3), and $\gamma(e) > 0$ for $e \in B - B^0$). First of all we observe that

(5.7) for any labelled edge $e' = x'y'$, $\rho^\varepsilon(e') - \widehat{\alpha}^\varepsilon(e') + \widehat{\alpha}(e') = 0$.

Indeed, this is trivial if $x', y' \in X_\phi$ for some $\phi \in \mathcal{F}$, and easily follows from (5.5) if both x', y' are 1-labelled (then $\widehat{\alpha}^\varepsilon(e') = 0$ and $\rho(x', e') = -\rho(y', e')$). If $x' \in L$ and $y' \in X_\phi$ for some $\phi \in \mathcal{F}$ then $e' = e_\phi$ (by (5.6)(iv)); this yields that $\rho(x', e') = \varepsilon$, $\rho(y', e') = 0$ and $\widehat{\alpha}^\varepsilon(e') = \varepsilon$ hold in case $e' \in Z$, and that $\rho(x', e') = -\varepsilon$, $\rho(y', e') = 0$ and $\widehat{\alpha}^\varepsilon(e') = -\varepsilon$ hold in case $e' \in B$, whence (5.7) follows. And if $x' \in X_\phi$, $y' \in X_{\phi'}$ and $e' = e_\phi$ for distinct $\phi, \phi' \in \mathcal{F}$ then $\rho(x', e') = \rho(y', e') = 0$ and $\widehat{\alpha}^\varepsilon(e') = 0$ (as $e' \in U_\phi$ if and only if $e' \notin U_{\phi'}$), and (5.7) is also true. Other cases for e' are impossible by (5.6).

Now consider $e = xy \in B^0$. (5.3) and (5.7) show that $\gamma^\varepsilon(e) = \gamma(e) = 0$ if e is labelled. The same equalities are obvious if $x, y \in M$. Assuming that e is unlabelled and x is labelled, only the following cases are possible.

(i) $x, y \in L$. By (5.6)(i) and (5.5), $\rho(x, e) = \rho(y, e) = \varepsilon$, whence $\gamma^\varepsilon(e) = 2\varepsilon \geq 0$.

(ii) $x \in L$, $y \in M$. By (5.6)(ii), $\rho(x, e) = \varepsilon$. Furthermore, $\rho(y, e) = 0$, by (5.4). Hence $\gamma^\varepsilon(e) = \varepsilon \geq 0$. •

Proof of Statement 5.2. Consider a simple T-path $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$. We have to show that for any $\varepsilon \geq 0$, $\lambda^\varepsilon(P) \geq p + 2\varepsilon$, and that this inequality holds with equality for some P.

We say that a T-line $Q = (v_0, u_1, v_1, \ldots, u_m, v_m)$ is *strong* if all u_i's are bold, and for $i = 1, \ldots, m-1$, if $s \in \langle -t, t\rangle$ is tight for v_i then exactly one of u_i, u_{i+1} belongs to $B_s(v_i)$ (in other words, Q is strong if and only if Q is a member of a packing consisting of $\mu(B)/2$ T-lines and covering exactly B, cf. Statement 3.3). We know that Γ contains at least one T-line (since $p = p_\lambda$) and no augmenting path, therefore, B is non-empty, whence, there is at least one strong T-line in Γ.

Next, if P is a T-line, put $\widehat{\rho}(x_i) = \widehat{\rho}^\varepsilon(x_i) := \rho^\varepsilon(x_i, e_i) + \rho^\varepsilon(x_i, e_{i+1})$, $i = 1, \ldots, k-1$. Then $\rho(P)$ is equal to $\rho(x_0, e_1) + \sum_{i=1}^{k-1}(\rho(x_i, e_i) + \rho(x_i, e_{i+1})) + \rho(x_k, e_k)$. Hence, by (5.2) and (5.4),

$$\rho^\varepsilon(P) = 2\varepsilon + \sum(\widehat{\rho}^\varepsilon(x_i) \ : \ i = 1, \ldots, k-1). \tag{5.8}$$

Claim 1. *If P is a strong T-line, then $\widehat{\rho}^\varepsilon(x_i) \neq 0$ for $i = 1, \ldots, k-1$.*

Proof. This immediately follows from (5.4) if x_i is unlabelled or 2-labelled. Let $x = x_i$ be 1-labelled. Put $l := l(x, e_i)$ and $l' := l(x, e_{i+1})$. If $x \in V_s$ for some $s \in T$ then the fact that one of l, l', l say, is s and the other, l', is $-s$ implies that $\rho(x, e_i) = -\rho(x, e_{i+1})$ (cf. (L1) above). Hence $\widehat{\rho}(x) = 0$.

Now suppose that $x \in V^{\bullet}$. Then there is $s \in T$ such that s is tight for x and an edge labelled as entering x is in $Z_s(x) \cup (B(x) - B_s(x))$. The fact that P is strong implies that one of l, l', l say, is s and the other, l', is different from s. This yields $\rho(x, e_i) = -\varepsilon$ and $\rho(x, e_{i+1}) = \varepsilon$ (cf. (L2)), whence $\widehat{\rho}(x) = 0$. •

Claim 2. *Let P be a T-line, and $\varepsilon > 0$. For $i = 1, \ldots, k-1$, $\widehat{\rho}^{\varepsilon}(x_i)$ is nonnegative, and $\widehat{\rho}^{\varepsilon}(x_i) > 0$ holds if and only if*

(5.9) *the triple (e_i, x_i, e_{i+1}) is such that: $x = x_i$ is an (ordinary) 1-labelled central vertex, and there is $s \in T$ such that s is tight for x, an edge labelled as entering x_i is in $Z_s(x) \cup (B(x) - B_s(x))$, and $e_i, e_{i+1} \in E(x) - E_s(x)$.*

Proof. Again, it suffices to consider an ordinary 1-labelled $x = x_i$. As in the proof of Claim 1, $\widehat{\rho}(x) = 0$ holds if exactly one of e_i, e_{i+1} belongs to $E_s(x)$, where s is tight for x, and an edge labelled as entering x is in $Z_s(x) \cup (B(x) - B_s(x))$. Thus, it remains to consider the case as in (5.9) (taking into account that $e_i, e_{i+1} \in E_s(x)$ is impossible since P is a line). Then $\rho(x, e_i) = \rho(x, e_{i+1}) = \varepsilon$ (cf. (L1)-(L2)), whence $\widehat{\rho}(x) = 2\varepsilon > 0$.
•

Define

(5.10) $\mathcal{T}$ to be the set of unlabelled edges $e = xy \in Z$ such that either $x, y \in L$, or $x \in L$ and $y \in M$.

Claim 3. *Let $e \in E\Gamma$, and $\varepsilon > 0$. Then*

(5.11)
$$\begin{aligned} \lambda^{\varepsilon}(e) &= \lambda(e) + \rho^{\varepsilon}(e) \quad && \text{if } e \in E\Gamma - \mathcal{T}; \\ &> \lambda(e) + \rho^{\varepsilon}(e) \quad && \text{if } e \in \mathcal{T}. \end{aligned}$$

Proof. Put $b := \lambda^{\varepsilon}(e) - \lambda(e) - \rho(e)$. Then

$$\begin{aligned} b &= (c(e) + \gamma^{\varepsilon}(e) + \widehat{\alpha}^{\varepsilon}(e)) - (c(e) + \gamma(e) + \widehat{\alpha}(e)) - \rho(e) \\ &= \gamma^{\varepsilon}(e) - \gamma(e) + \widehat{\alpha}^{\varepsilon}(e) - \widehat{\alpha}(e) - \rho(e). \end{aligned}$$

Thus, $b = 0$ if $e \in B$ (by (5.3)) or if $e \in Z$ and e is labelled (by (5.7) and the fact that $\gamma^\varepsilon(e) = \gamma(e) = 0$). Also $b = 0$ holds if $e \in Z$ and both ends of e are unlabelled (since $\widehat{\alpha}^\varepsilon(e) = \widehat{\alpha}(e) = 0$). Let $e = xy \in Z$ be unlabelled and x be labelled. Then (5.6) shows that $e \in \mathcal{T}$; hence, $\gamma^\varepsilon(e) = \gamma(e) = \widehat{\alpha}^\varepsilon(e) = 0$ and $b = -\rho(e)$. If $x, y \in L$ then (5.6)(i) and (5.5) imply $\rho(x,e) = \rho(y,e) = -\varepsilon$. And if $x \in L$, $y \in M$ then, by (5.6)(ii) and (5.5), $\rho(x,e) = -\varepsilon$ and $\rho(y,e) = 0$. In both cases, $\rho(e) < 0$, whence the result follows. •

Thus, by Claim 3, $\lambda^\varepsilon(P) = \lambda(P) + \rho^\varepsilon(P) = p + \rho^\varepsilon(P)$ if P is a strong T-line, and $\lambda^\varepsilon(P) \geq p + \rho^\varepsilon(P)$ if P is a T-line. Now we conclude from Claims 1,2 and (5.8) that

(5.12)
$$\begin{aligned} \lambda^\varepsilon(P) &= p + 2\varepsilon \quad \text{if } P \text{ is a strong } T\text{-line}, \\ &\geq p + 2\varepsilon \quad \text{if } P \text{ is a } T\text{-line}. \end{aligned}$$

Finally, if P is a T-path in G that is not a T-line then $\lambda(P) > p$, hence, there is $\varepsilon > 0$ such that $\lambda^{\varepsilon'} > p + 2\varepsilon$ for $0 \leq \varepsilon' \leq \varepsilon$. Now the statement follows from finiteness of the set of simple T-paths. • •

Statements 5.1 and 5.2 enable us to determine α', γ' as required in (ii) in Theorem 2. More precisely, put

(5.13)
$$\varepsilon^* := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\},$$

where ε_1 and ε_2 were defined above (after Statement 5.1), and ε_3 is the largest ε as in Statement 5.2. Then $\varepsilon^* > 0$.

Arguments above imply that $B, \alpha' := \alpha^{\varepsilon^*}, \gamma' := \gamma^{\varepsilon^*}$ are good for $p' := p_{\lambda^{\varepsilon^*}}$ $(= p + 2\varepsilon^*)$. Thus, Theorem 2 is valid for case $\alpha = 0$.

In conclusion of this section let us emphasize one result obtained in the proof of Statement 5.2; it will be used in what follows.

(5.14) For a T, λ-line $P = (x_0, e_1, x_1, \dots, e_k, x_k)$, $\lambda^\varepsilon(P) = \lambda(P) + 2\varepsilon$ holds if and only if:

(i) $\widehat{\rho}^\varepsilon(x_i) = 0$ for $i = 1, \dots, k-1$, or, equivalently, P has no triple as in (5.9);

(ii) $\lambda^\varepsilon(e_i) = \lambda(e_i) + \rho^\varepsilon(e_i)$ for $i = 1, \dots, k$, or, equivalently, P does not meet $\mathcal{T}$.

6. General case

We now consider the case when edges $e \in EG$ with $\lambda(e) = 0$ are possible, where $\lambda := c_{\alpha,\gamma}$. These edges are called *0-edges*, and their set is denoted by J. As before, $E\Gamma$ is partitioned into two subsets B (of bold edges) and Z (of thin edges). [Note that in our case a T-line may not be a simple path; the notion of regularity of B will be specified below.] We denote $\beta := B \cap J$ and $\zeta := Z \cap J$. Let $\mathcal{F}$ denote the set $\{\phi \in \widetilde{\mathcal{F}} : \alpha(\phi) > 0\}$ (the *support* of α), and let for $e \in EG$, $\widehat{\alpha}(e)$ denote $\sum(\alpha(\phi)\chi_\phi(e) \ : \ \phi \in \mathcal{F})$.

As mentioned in Section 2, we need to impose a number of additional conditions on B, α, γ; taken together, they enable us to prove the existence of $\mathcal{P}'$ or (α', γ') as required in Theorem 2. First of all we assume that $\mathcal{F}$ and J satisfy the following conditions:

(6.1) for each $\phi \in \mathcal{F}$, either $e_\phi \notin B$ and $U_\phi = (B \cap \delta X_\phi) \cup \{e_\phi\}$, or $e_\phi \in B$ and $U_\phi = (B \cap \delta X_\phi) - \{e_\phi\}$

(this is equivalent to saying that B saturates ϕ, cf. (2.4)-(2.5), or (5.1));

(6.2) for each $\phi \in \mathcal{F}$, e_ϕ is feasible, i.e.

- (i) e_ϕ is contained in Γ, and
- (ii) $\gamma(e_\phi) = 0$;

(6.3) the sets X_ϕ form a *nested* family, i.e. for distinct $\phi, \phi' \in \mathcal{F}$, either $X_\phi \cap X_{\phi'} = \emptyset$ or $X_\phi \subset X_{\phi'}$ or $X_{\phi'} \subset X_\phi$;

(6.4) (i) if $\phi_1, \ldots, \phi_k \in \mathcal{F}$ and $e \in EG$ are such that $X_{\phi_1} \subset \ldots \subset X_{\phi_k}$ and $e \in \delta X_{\phi_1}, \ldots, \delta X_{\phi_k}$ then either $e \in U_{\phi_1}, \ldots, U_{\phi_k}$ or $e \notin U_{\phi_1}, \ldots, U_{\phi_k}$; in particular, if $e = e_{\phi_i}$ for some i then $e_{\phi_1} = \ldots = e_{\phi_k}$;

- (ii) no sequence $\phi_1, \ldots, \phi_k \in \mathcal{F}$ with $k > 1$ exists such that the sets X_{ϕ_i} are pairwise disjoint, and $e_{\phi_i} \in \delta X_{\phi_{i+1}}$ for $i = 1, \ldots, k$ (letting $\phi_{k+1} := \phi_1$);

(6.5) for each $e \in J$, e is contained in Γ, $\gamma(e) = 0$, and both ends of e are in $V\Gamma - T$.

A component of the subgraph induced by β is called a *0-component*. We observe that each 0-component is a tree. Indeed, suppose that β contains a sequence $Q = (e_1, e_2, \ldots, e_k)$ of edges forming a circuit. Then for any $\phi \in \mathcal{F}$, $|U_\phi \cap Q| \geq |(\delta X_\phi - U_\phi) \cap Q|$ (by (6.1) and the fact that $Q \subseteq B$), whence $\chi_\phi(Q) \geq 0$. Hence,

$$\lambda(Q) - c(Q) - \gamma(Q) = \widehat{\alpha}(Q) = \sum(\alpha(\phi)\chi_\phi(Q) \ : \ \phi \in \mathcal{F}) \geq 0.$$

Since $\gamma(Q) \geq 0$ and $c(Q) > 0$ (as c is positive), $\lambda(Q) > 0$; a contradiction.

For $e \in J$ the facts that $c(e) > 0$, $\lambda(e) = 0$ and $\gamma(e) = 0$ (by (6.5)) imply that $\widehat{\alpha}(e) < 0$, therefore, there is $\phi \in \mathcal{F}$ such that $e \in \delta X_\phi - U_\phi$. We impose a stronger condition, namely,

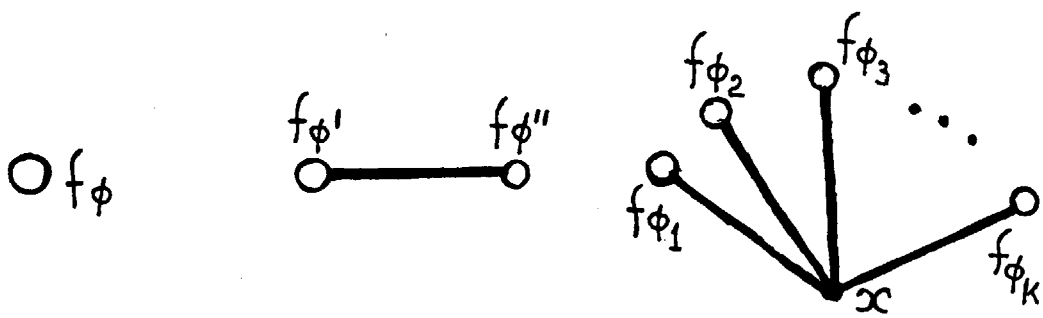
(6.6) for any $e \in J$ and $\phi \in \mathcal{F}$, $e \notin U_\phi$; in particular, for $e \in J \cap \delta X_\phi$, $e \in \beta$ if and only if $e = e_\phi$.

We say that $\phi' \in \mathcal{F}$ *precedes* $\phi \in \mathcal{F}$ (or ϕ' is a *predecessor* of ϕ) if $X_{\phi'} \subset X_\phi$ and there is no $\phi'' \in \mathcal{F}$ such that $X_{\phi'} \subset X_{\phi''} \subset X_\phi$. The set of predecessors of ϕ is denoted by $\mathcal{F}_\phi$. Let $\mathcal{F}^{\max}$ denote the set of $\phi \in \mathcal{F}$ preceding no fragment in $\mathcal{F}$.

For $\phi' \in \mathcal{F}$ let $\Gamma_{\phi'}$ denote the graph that is the union of the edge $e_{\phi'}$ and the subgraph of Γ induced by $X_{\phi'}$. Form the graph Γ^* from Γ by shrinking the set X_ϕ for each $\phi \in \mathcal{F}^{\max}$ into a single new vertex, denoted by f_ϕ, and then by deleting the loops if appeared. Similarly, for $\phi' \in \mathcal{F}$ form $\Gamma^*_{\phi'}$ from $\Gamma_{\phi'}$ by shrinking X_ϕ into a vertex f_ϕ for each $\phi \in \mathcal{F}_{\phi'}$, and then by deleting loops. We refer to such f_ϕ's as *non-ordinary* vertices, while the other vertices in Γ^* ($\Gamma^*_{\phi'}$) are called *ordinary*. When it is not confusing, the image in Γ^* ($\Gamma^*_{\phi'}$) of an edge $e \in E\Gamma$ ($e \in E\Gamma_{\phi'}$) that does not turn into a loop under above shrinking is also denoted by e; in particular, the image in Γ^* ($\Gamma^*_{\phi'}$) of the root e_ϕ of $\phi \in \mathcal{F}^{\max}$ ($\phi \in \mathcal{F}_{\phi'} \cup \{\phi'\}$) is also denoted by e_ϕ. (6.3) and (6.6) imply the following important property:

(6.7) let H be a 0-component, and let H^* (H^*_ϕ) be the image of H in Γ^* (respectively, Γ^*_ϕ for $\phi \in \mathcal{F}$); then H^* (H^*_ϕ) is either (i) a single vertex $f_{\phi'}$ for $\phi' \in \mathcal{F}^{\max}$ ($\phi' \in \mathcal{F}_\phi$), or (ii) an edge connecting $f_{\phi'}$ and $f_{\phi''}$ for some ϕ', ϕ'' in $\mathcal{F}^{\max}$ ($\mathcal{F}_\phi$), or (iii) a star induced by an ordinary vertex x in Γ^* (Γ^*_ϕ) and edges of the form $xf_{\phi_1}, \ldots, xf_{\phi_k}$ for distinct $\phi_1, \ldots, \phi_k$ in $\mathcal{F}^{\max}$ ($\mathcal{F}_\phi$).

(See Fig. 6.1.)



(a) H^* (b) H^* (c) H^*

Fig. 6.1

Now we specify the notion of regularity for B. Consider a 0-component H. Clearly the potential $\pi(x)$ is the same for all vertices x in H and the vertices x reachable from H by paths with all edges in J. Moreover, all these vertices are either in $V^\bullet$ or in the same V_s for some $s \in T$.

Let $B^+ := B - \beta$. For a subgraph Q in Γ denote by $B(Q)$, $\beta(Q)$, $B^+(Q)$ the set of edges in B, β, B^+, respectively, with exactly one end in Q; we write $B(x))$, $\beta(x)$, $B^+(x)$ instead of $B(Q)$, $\beta(Q)$, $B^+(Q)$ for $Q = (\{x\}, \emptyset)$. Note that if Q is a 0-component then every edge e in B^+ has at most one end in Q (since e belongs to a λ-shortest path). We say that a set $B \subseteq E\Gamma$ (satisfying (6.1)) is regular if it satisfies (3.4)(i),(ii) and the following condition:

(6.8) if H' is either a vertex in $V\Gamma - T$ or a subtree in a 0-component then B is *non-excessive* for H'; this means that for any $s \in \langle -t, t\rangle$:

$$|B_s^+(H')| \le \frac{1}{2}|B(H')| \quad (= \frac{1}{2}|B^+(H')| + \frac{1}{2}|\beta(H')|).$$

Here $B_s^+(H')$ is the set of edges $e = xy \in B^+(H')$ with $x \in VH'$ and $l(x,e) = s$. [Recall that $\langle -t, t\rangle$ does not contain 0. For $e = xy \in E\Gamma$ with $\lambda(e) > 0$ the attachment $l(x,e)$ is defined as in (3.2).] Obviously, if $H' \in V\Gamma - T$ and $\beta(H') = \emptyset$ then (6.8) turns into (3.4)(iii) for $x := H'$. If the inequality in (6.8) holds with equality, s is called *tight* for H'. Note that if H' is a subtree in a 0-component H and $H' \ne H$ then at most one s can be tight for H' (since $|\beta(H')|$ is nonzero).

Statement 6.1. *B is regular if and only if $B = \cup(P \in \mathcal{P})$, where $\mathcal{P}$ consists of $\mu(B)/2$ simple edge-disjoint T, λ-lines.* (Cf. Statement 3.3.)

Proof. The part "if" is easy. To prove "only if" part, it suffices to show the following:

(6.9) for a regular B and a 0-component H let $\widehat{H}$ be the subgraph of Γ induced by the set $EH \cup B^+(H)$; then $\widehat{H}$ is represented as the union of $m := |B^+(H)|/2$ simple edge-disjoint paths $P_1, \dots, P_m$ such that for each $P_i = (x_0, e_1, x_1, \dots, e_k, x_k)$: $k \ge 2$, $e_1, e_k \in B^+(H)$, the part of P_i from x_1 to x_{k-1} is a path in H, and $l(x_1, e_1) \ne l(x_{k-1}, e_k)$.

[For contraction of every 0-component results in the case considered in the proof of Statement 3.3 in Section 3.] (6.9) can be proved by use the following result due to Lovász [Lov] and Cherkassky [Ch]: let Q be a graph and $T' \subseteq VQ$, let every vertex in $VQ - T'$ be of an even valency, and let U_v denote the set of edges in Q incident to $v \in VQ$; then Q has $\sum_{s \in T'} |U_s|/2$ edge-disjoint T'-paths if and only if

(6.10) $|\delta^Q X| \ge |U_s|$ holds for any $s \in T'$ and $X \subseteq VQ$ such that $X \cap T' = \{s\}$.

In our case we put T' to be the set of all distinct $s \in \langle -t, t\rangle$ such that $s = l(x, e)$ for some $x \in VH$ and $e = xy \in B^+(x)$, and form Q from $\widehat{H}$ by identifying each y as above with the corresponding s. Then each vertex in $VQ - T'$ has an even valency, by (3.4)(i). It is easy to see that (6.8) implies (6.10), and now the existence of edge-disjoint paths $P_1, \ldots, P_m$ as in (6.9) follows from the above-mentioned result. Note that the facts that H is a tree and that $|B(x)|$ is even for all $x \in VH$ imply that $P_1, \ldots, P_m$ are simple and cover all edges in H. $\bullet$

[It should be noted that paths as in (6.9) can be found in polynomial time.] The regularity of B implies the following properties (6.11) and (6.12) for a 0-component H.

(6.11) Let $s \in \langle -t, t\rangle$ be tight for a subtree $H' \subseteq H$, let $e \in EH'$, and let H_1, H_2 be the components of $H' - \{e\}$; then s is tight for exactly one H_i.

Indeed,

$$
\begin{aligned}
0 &= 2|B_s^+(H')| - |B(H')| \\
&= (2|B_s^+(H_1)| + 2|B_s^+(H_2)|) - (|B(H_1)| + |B(H_2)| - 2) \\
&= 2 + \sum (2|B_s^+(H_i)| - |B(H_i)| \; : \; i = 1, 2)
\end{aligned}
$$

Now (6.11) follows from the facts that $2|B_s^+(H_i)| - |B(H_i)|$ is non-positive (by (6.8)) and even (by (3.4)(i) and (6.5)).

(6.12) For any $e = xy \in EH$ there is at most one $s \in \langle -t, t\rangle$ that is tight for some subtree $H' \subseteq H$ with $x \notin VH' \ni y$.

To see this, consider paths $P_1, \ldots, P_m$ for $\widehat{H}$ as in (6.9). If s is tight for some H' as above then, obviously, there is a path P_i (considered up to reversing it) that has the first edge in $B_s^+(H)$ and passes y, e, x in this order. This shows that s is determined uniquely (independently of H').

Property (6.12) enables us to associate with (x, e), $e = xy \in EH$, an attachment $l(x, e)$ by the following rule:

(6.13) put $l(x, e) := s$ if s is tight for some $H' \subseteq H$ with $x \notin VH' \ni y$, and put $l(x, e) := 0$ otherwise.

[So the special attachment 0 may appear for (x, e) with $e \in \beta$.] For each edge $e = xy \in \zeta$ we also put $l(x, e) := 0$. As before, for $x \in V\Gamma$ and $s \in \langle -t, t\rangle \cup \{0\}$ define $E_s(x) := \{e \in E(x) \; : \; l(x, e) = s\}$, $Z_s(x) := Z \cap E_s(x)$, $B_s(x) := B \cap E_s(x)$ and

$B_s^0(x) := B^0 \cap E_s(x)$ (regarding the resulting attachments $l(x,e)$). It is easy to check that

(6.14) (i) for $e = xy \in E\Gamma$, $l(x,e)$ and $l(y,e)$ are different unless $l(x,e) = l(y,e) = 0$;

(ii) $|B_s(x)| \le \frac{1}{2}|B(x)|$ for any $x \in V\Gamma - T$ and $s \in \langle -t, t\rangle$

(cf. (3.4)(iii)). To illustrate rule (6.13), consider two possible cases for $x \in VH$.

(Q1) $x \in V_s$ for $s \in T$. Then $l(x',e') \in \{s,-s\}$ for any $e' = x'y' \in B^+(H)$ with $x' \in VH$, and both s and $-s$ are tight for H. From (6.11),(6.12) it follows that for any $e = xy \in EH$ one of $l(x,e), l(y,e)$ is s and the other is $-s$. Moreover, $|B_s(x)| = |B_{-s}(x)|$, i.e. both s and $-s$ are tight for x.

(Q2) $x \in VH$ is central. Then for $e = xy \in EH$, (x,e) receives an attachment in $S \cup \{0\}$, where $S := \{s \ : \ s = l(x',e')$ for some $e' = x'y' \in B^+(H)$ with $x' \in VH\}$. Moreover, $s \in S$ is tight for x if and only if s is tight (in the sense of the original attachments) for some subtree $H' \subseteq H$ containing x.

Some of above observations can be summarized as follows:

(6.15) let $\mathcal{P}$ be a set of $\mu(B)/2$ edge-disjoint T,λ-lines such that $B = \cup(P \in \mathcal{P})$, let $P = (x_0, e_1, x_1, \ldots, e_k, x_k) \in \mathcal{P}$, let $i \in \{1, \ldots, k-1\}$, and let $s := l(x_i, e_i)$; then $s > 0$ (i.e. $s \in T$) implies $x_0 = s$, and $s < 0$ (i.e. $-s \in T$) implies $x_k = -s$.

Now our aim is to specify the notion of an active path. One trick enables us to define such a path in a way quite similar to that described in Section 4. More precisely, we extend the set J by adding for each $e \in J$ a parallel edge e' of length $\lambda(e') = 0$, considering e' as a thin edge (i.e. setting $e' \in \zeta$). The edges e, e' are called *mates* (to each other). Thus,

(6.16) for any pair $\{e, e'\}$ of mates at most one of e, e' belongs to B.

Speaking of Γ^* ($\Gamma^*_{\phi'}$) we keep notations $E(x), Z(x), B(x), E_s(x)$ and etc. for $x \in V\Gamma^*$ ($x \in V\Gamma^*_{\phi'}$) and the corresponding sets of edges in Γ^* ($\Gamma^*_{\phi'}$). For an ordinary vertex $x \in V\Gamma^*$ ($x \in V\Gamma^*_{\phi'}$) the set of forks is defined as in (4.1). In particular, if $e \in E(x)$ is such that $l(x,e) = 0$ (e.g., if $e \in \zeta$) then (e, x, e') is a fork for any feasible $e' \in E(x) - \{e\}$ (taking into account that $0 \notin \langle -t, t\rangle$). For $\phi \in \mathcal{F}^{\max}$ ($\phi \in \mathcal{F}_{\phi'}$) and distinct edges $e, e' \in Z(f_\phi) \cup B^0(f_\phi)$ in Γ^* ($\Gamma^*_{\phi'}$) we define

(6.17) (e, f_ϕ, e') to be a fork if and only if one of e, e' is e_ϕ.

A path $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ in Γ^* is called *active* if: all edges e_i are different and feasible; $x_1, \dots, x_{k-1} \in V\Gamma^* - T$; and $e_1 \in Z$ if $x_0 \in T$ (cf. the definition in Section 4). A primitive, augmenting path (in Γ^*) is defined as before. Note that (4.2) remains true; hence, each vertex of Γ^* occurs in a primitive path P at most twice. Furthermore, (6.17) implies that each non-ordinary vertex occurs in a primitive path at most once.

For $\phi \in \mathcal{F}$ define an active, primitive path in Γ^*_ϕ in a similar way. We say that an active path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ in Γ^*_ϕ is *rooted* if $e_1 = e_\phi$ and $x_1, \dots, x_k \in X^*_\phi$; hereinafter X^*_ϕ denotes the set obtained from X_ϕ by replacing each subset $X_{\phi'}$ ($\phi' \in \mathcal{F}_\phi$) by the element $f_{\phi'}$. We impose the following important condition on reachibility in Γ^*_ϕ:

(6.18) (i) for each $\phi' \in \mathcal{F}_\phi$ there is a rooted primitive path $Q_{\phi'}$ in Γ^*_ϕ with the last edge $e_{\phi'}$ and the last vertex $f_{\phi'}$;

(ii) for each ordinary $x \in X^*_\phi$ there is a rooted primitive path Q_x in Γ^*_ϕ with the last vertex x;

(iii) for each ordinary $x \in X^*_\phi$ and $s \in \langle -t, t \rangle$ such that s is tight for x there is a rooted primitive paths Q^s_x in Γ^*_ϕ with the last vertex x such that $|B'_s(x)| = (|B'(x)| - 1)/2$, where $B' := B \triangle Q^s_x$;

(iv) in addition, in case (ii), Q_x meets x exactly once, and in case (iii), if the path Q^s_x meets x twice, and Q' is its part to the first occurrence of x, then Q' does not satisfy the above equality (i.e. the paths in (ii)-(iii) are chosen to be minimal, in a sense, for x).

(Here sets $B'_q(x)$ are defined with respect to the attacment l.)

Let Γ^* have an augmenting path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$. Suppose that P contains a non-ordinary vertex, $x_i = f_\phi$ say, where $\phi \in \mathcal{F}^{\max}$. Let for definiteness $e_\phi = e_i$, and let x be the end of e_{i+1} in X^*_ϕ. Then we replace the part (e_i, x_i) of P by a certain rooted primitive path Q in Γ^*_ϕ which has the last vertex x; such a Q is chosen as in (i),(ii) or (iii) in (6.18). Repeating such a procedure while the current path has a non-ordinary vertex, we eventually obtain a path $\widetilde{P}$ in Γ. In Section 7 we prove the following key statement.

Statement 6.2. *For an augmenting path P in Γ^*, the non-ordinary vertices can be replaced, step by step, by rooted primitive paths as in (6.18) so that for the resulting path $\widetilde{P}$ in Γ the set $B' := B \triangle \widetilde{P}$ is regular.*

Thus, the existence of an augmenting path in Γ^* will imply validity of (i) in Theorem 2.

Suppose that $\widetilde{P}$ meets X_ϕ for some $\phi \in \mathcal{F}$. In view of (6.17) and the fact that all edges in $\widetilde{P}$ are different, $\widetilde{P}$ intersects δX_ϕ in exactly two edges u, u' one of them,

u say, is e_ϕ. This easily implies that (6.1) remains true for B' (instead of B) and for u' (instead of e_ϕ); u' becomes the root of ϕ with respect to the new set B' of bold edges. Next, let $\{e, e'\}$ be a pair of mates in J such that $e, e' \in \delta X_\phi$. We observe that $|\{e, e'\} \cap B'| \leq 1$, i.e. (6.16) holds for B'. Indeed, if $e = u$ $(= e_\phi)$ then $e \in B$ (by (6.6)), whence $e \notin B'$. And if $e, e' \neq u$ then $e, e' \in Z$ (again by (6.6)); hence, $e, e' \in B'$ would imply that $e, e' \in \widetilde{\mathcal{P}}$, and therefore $\{e, e'\} = \{u, u'\}$; a contradiction.

Note also that (6.6) trivially holds for B' since U_ϕ remains the same for each $\phi \in \mathcal{F}$.

To search for an augmenting path in Γ^*, we grow a digraph D of labelled vertices and edges in Γ^* in the same way as it was done for Γ in Section 4, i.e. according to rules (A1)-(A2) applied to Γ^* with the set of forks in Γ^* defined above. As a result, either an augmenting path is found, or D together a set of blossoms satisfying (4.6)-(4.8) (for Γ^*) is constructed. As before, a labelled vertex x is called 1-labelled if it belongs to no blossom (such an x can be ordinary or non-ordinary).

In the remaining part of this section we assume that no augmenting path in Γ^* exists (this is equivalent to lack of a break-through in the process of growing D). We show how (α, γ) should be transformed, using properties (4.6)-(4.8) (for Γ^*) (note that proofs of (4.7)-(4.8) are provided, in fact, by validity of (4.2)).

Let $\mathcal{Q}$ be the set of blossoms for D, and let $\mathcal{F}^{\rm new}$ denote the set of new fragments that correspond to blossoms in $\mathcal{Q}$. More precisely, each $F \in \mathcal{Q}$ generates a new fragment ϕ' for G, T, where $X_{\phi'}$ is the pre-image of VF in VG, $e_{\phi'} := e_F$ and $U_{\phi'}$ is defined as in (5.1) (or (6.1)).

Suppose that for some $\phi \in \mathcal{F}^{\max}$, f_ϕ belongs to a blossom $F \in \mathcal{Q}$. Note that the definition of an elementary blossom (see (4.4)(iv)) concerns ordinary vertices only. Hence, there is an edge $e \in E(f_\phi)$ labelled in both directions. Furthermore, (6.17) shows that if e_ϕ is labelled as leaving but not entering e_ϕ then each edge $e' \in E(f_\phi) - \{e_\phi\}$ is either unlabelled or labelled as entering but not leaving f_ϕ; so in this case f_ϕ cannot belong to any blossom. Thus, either e_ϕ is labelled in both directions (whence e_ϕ is in F), or e_ϕ is labelled as entering but not leaving f_ϕ; in the latter case, e_ϕ is just the root e_F of F. This shows validity of (6.4) for $\mathcal{F}' := \mathcal{F} \cup \mathcal{F}^{\rm new}$. (6.2) and (6.3) are obvious for $\mathcal{F}'$.

Denote by $\mathcal{F}^+$ ($\mathcal{F}^-$) the set of fragments $\phi \in \mathcal{F}^{\max}$ such that f_ϕ is 1-labelled and e_ϕ is labelled as entering (respectively, leaving) f_ϕ.

As before, α' and γ' required in Theorem 2 will be assigned to be α^ε and γ^ε for a certain $\varepsilon > 0$. For $\varepsilon \in \mathbb{R}_+$, ε-transformation of α consists of increasing $\alpha(\phi)$ by ε for $\phi \in \mathcal{F}^+ \cup \mathcal{F}^{\rm new}$ (recall that $\alpha(\phi) = 0$ for $\phi \in \mathcal{F}^{\rm new}$), decreasing it by ε for $\phi \in \mathcal{F}^-$, and preserving it for the other ϕ's in $\mathcal{F}$. The resulting function is just α^ε. The first requirement for ε is to provide α^ε to be nonnegative, i.e.

V

(6.19) $$\varepsilon \leq \varepsilon_0 := \min\{\alpha(\phi) : \phi \in \mathcal{F}^-\}.$$

Let L be the set of 1-labelled *ordinary* vertices x in Γ^*. Let M consist of unlabelled ~~1-labelled~~ vertices and vertices $x \in X_\phi$ such that $\phi \in \mathcal{F}^{\max}$ and f_ϕ is unlabelled. Let $N := V\Gamma - (L \cup M)$; then the sets X_ϕ for $\phi \in \mathcal{F}^{\text{new}} \cup \mathcal{F}^+ \cup \mathcal{F}^-$ give a partition of N.

Put $\rho^\varepsilon(x,e)$ as in (5.4) for $x \in T \cup N \cup M$ and as in (5.5) for $x \in L - T$, and then put $\rho^\varepsilon(e)$ and $\gamma^\varepsilon(e)$ as in (5.2) and (5.3), respectively. We shall show that Statements 5.1 and 5.2 remain true for our case. The required α' and γ' are α^ε and γ^ε, respectively, where ε is defined so that

(6.20) $$0 < \varepsilon \leq \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\};$$

here ε_0 is as in (6.19); and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ were defined in Section 5. This provides validity of (ii) in Theorem 2. [The reason why, in contrast to (5.13), we do not define, in general, ε to be equal to $\min\{\varepsilon_0, \ldots, \varepsilon_3\}$ is explained in Remark 6.5 below.]

To prove Statements 5.1-5.2, we first point out some properties of edges in δX_ϕ for $\phi \in \mathcal{F}^-$; they follow from (4.7) and (6.17). We know that for $\phi \in \mathcal{F}^-$ ($\phi \in \mathcal{F}^+ \cup \mathcal{F}^{\text{new}}$) the only edge labelling as leaving (entering) X_ϕ is the root e_ϕ. This implies that for a feasible $e \in E\Gamma^*$:

(6.21) (i) if e connects $x \in L$ and f_ϕ for $\phi \in \mathcal{F}^-$ and e is labelled from f_ϕ to x then $e = e_\phi$;

(ii) if e connects f_ϕ and $f_{\phi'}$ for $\phi \in \mathcal{F}^-$ and $\phi' \in \mathcal{F}^+ \cup \mathcal{F}^{\text{new}}$, then $e = e_\phi$ if and only if $e = e_{\phi'}$; hence, for each $e' \in EG$ connecting X_ϕ and $X_{\phi'}$, $e' \in U_\phi$ if and only if $e \in U_{\phi'}$;

(iii) if e is a labelled edge connecting f_ϕ and $f_{\phi'}$ for distinct $\phi, \phi' \in \mathcal{F}^-$ then e is the root of exactly one of ϕ, ϕ'; hence, $e \in U_\phi$ if and only if $e \notin U_{\phi'}$.

Note also that (5.6) holds if we replace there $\mathcal{F}$ by $\mathcal{F}^+ \cup \mathcal{F}^{\text{new}}$.

To prove the part of Statement 5.1 concerning γ^ε, it suffices to examine edges $e \in B^0$ in Γ^* which are incident to f_ϕ for some $\phi \in \mathcal{F}^-$. First of all, we observe that (5.7) remains true. Indeed, this is so if $e' = x'y'$ belongs to no cut δX_ϕ for $\phi \in \mathcal{F}^-$ (by arguments as in the proof of (5.7) in Section 5 where $\mathcal{F}$ should be replaced by $\mathcal{F}^+ \cup \mathcal{F}^{\text{new}}$). And if $x' = f_\phi$ for some $\phi \in \mathcal{F}^-$, consider three possible cases (taking into account that $\rho(x', e') = 0$ since $x' \in N$).

(i) Let $y' \in L$. If e' is labelled from x' to y' then $e' = e_\phi$ (by (6.21)(i)). Hence, $q := \widehat{\alpha}^\varepsilon(e') - \widehat{\alpha}(e') = -\varepsilon$, $\rho(y', e') = -\varepsilon$ in case $e' \in Z$; and $q = \varepsilon$, $\rho(y', e') = \varepsilon$ in case $e' \in B$. If e' is labelled from y' to x' then $e' \neq e_\phi$. Hence, $q = \varepsilon$, $\rho(y', e') = \varepsilon$ in case $e' \in Z$, and $q = -\varepsilon$, $\rho(y', e') = -\varepsilon$ in case $e' \in B$. In all cases (5.7) holds.

(ii) Let $y' = f_{\phi'}$ for $\phi' \in \mathcal{F}^+ \cup \mathcal{F}^{\rm new}$. Then $\rho(x', e') = \rho(y', e') = 0$ (as $x', y' \in N$). By (6.21)(ii), $\alpha^\varepsilon(\phi) - \alpha(\phi) = -(\alpha^\varepsilon(\phi') - \alpha(\phi'))$, whence $\widehat{\alpha}^\varepsilon(e') - \widehat{\alpha}(e') = 0$, and (5.7) holds.

(iii) Let $y' = f_{\phi'}$ for $\phi' \in \mathcal{F}^- - \{\phi\}$. Then $\rho(x', e') = \rho(y', e') = 0$, and (6.21)(iii) implies that $\widehat{\alpha}^\varepsilon(e') - \widehat{\alpha}(e') = 0$, whence (5.7) holds.

Return to $e = xf_\phi \in B^0$ for $\phi \in \mathcal{F}^-$. Taking into account that $\gamma^\varepsilon(e) = 0$ if e is labelled (by (5.3),(5.7)), it suffices to consider only cases in which e is unlabelled; then e belongs to U_ϕ.

(a) $x \in L$. Then $q := \widehat{\alpha}^\varepsilon(e) - \widehat{\alpha}(e) = -\varepsilon$, $\rho(e) = \rho(x, e) = \varepsilon$, whence, by (5.3), $\gamma^\varepsilon(e) = \gamma(e) + \rho(e) - q = 0 + \varepsilon - (-\varepsilon) \geq 0$.

(b) $y \in N \cup M$. Then $\rho^\varepsilon(e) = 0$. From (6.21)(ii),(iii), one can see that $q \leq 0$, whence, by (5.3), $\gamma^\varepsilon(e) \geq 0$.

To show the part of Statement 5.1 concerning λ^ε, we have to examine certain edges in J. For this and further purposes we need the following two statements.

Statement 6.3. *Let $e, e' \in J$ be a pair of mates in Γ^* connecting x and y.*

(i) *If $e, e' \in \zeta$ then either e, e' are unlabelled, or labelled in both directions, or labelled in the same direction, from x to y say; in the latter case, $y = f_\phi$ for some $\phi \in \mathcal{F}^-$.*

(ii) *If $e \in \beta$ (and hence $e' \in \zeta$) then either e, e' are unlabelled, or labelled in both directions, or labelled in opposite directions, say, e is labelled from y to x, while e' is labelled from x to y; in the latter case, $y = f_\phi$ for some $\phi \in \mathcal{F}^-$.*

Proof. (i) Let $e, e' \in \zeta$, and let some of e, e' be labelled; let for definiteness e be labelled from x to y. Clearly for $z = x, y$ and $u \in E(z)$, (u, z, e) is a fork if and only if (u, z, e') is a fork (since $l(z, e) = l(z, e') = 0$). Hence, e, e' are labelled in the same direction (or directions). Furthermore, if y is ordinary then (e, y, e') is a fork, whence e, e' are labelled in both directions. Now, if e is not labelled from y to x then $y = f_\phi$ for some $\phi \in \mathcal{F}^{\max}$, and we conclude that $\phi \in \mathcal{F}^-$ (since $e \in \zeta$ implies that e cannot be the root of ϕ, by (6.6)).

(ii) Let $e \in \beta$. We observe that for $z = x, y$, $\tau = (e, z, e')$ is a fork. Indeed, if z is a non-ordinary vertex f_ϕ then $e = e_\phi$ (by (6.6)), and τ is a fork, by (6.17). And if z

is ordinary then the facts that $e' \in Z$ and $l(z, e') = 0$ imply that (e', z, u) forms a fork for any feasible $u \in E(z)$. Thus, if some of e, e' is labelled from x to y say, then the other is labelled from y to x. Now let e be labelled from y to x but not from x to y. Suppose that y is ordinary. Then x is a non-ordinary vertex $f_{\phi'}$. Considering an active path P in Γ^* starting from T and passing e, and using the facts that $y \notin T$ (by (6.5)) and that P meets $f_{\phi'}$ at most once, we conclude that the edge u in P preceding e is different from e'. Then (u, y, e') is a fork, hence, e' must be labelled also from y to x; a contradiction. Thus, $y = f_\phi$ for some $\phi \in \mathcal{F}^{\max}$, whence $\phi \in \mathcal{F}^-$. •

Statement 6.4. *Let $e = xy \in J$ be an edge in Γ^* such that at least one end of e is labelled and e belongs to no blossom in $\mathcal{Q}$. Then e is labelled, from x to y say, and:*

(i) *if $e \in \beta$ then $x = f_\phi$ for some $\phi \in \mathcal{F}^-$ (whence $e = e_\phi$), and either $y \in L$, or $y = f_{\phi'} \in \mathcal{F}^+$ (whence $e_\phi = e_{\phi'}$), or y is in a blossom $F \in \mathcal{Q}$ (whence $e_\phi = e_F$);*

(ii) *if $e \in \zeta$ then $y = f_\phi$ for some $\phi \in \mathcal{F}^-$, and either $x \in L$, or $x = f_{\phi'}$ for some $\phi' \in \mathcal{F}^+$, or x is in a blossom $F \in \mathcal{Q}$.*

(Figure 6.2 illustrates possible cases for e.)

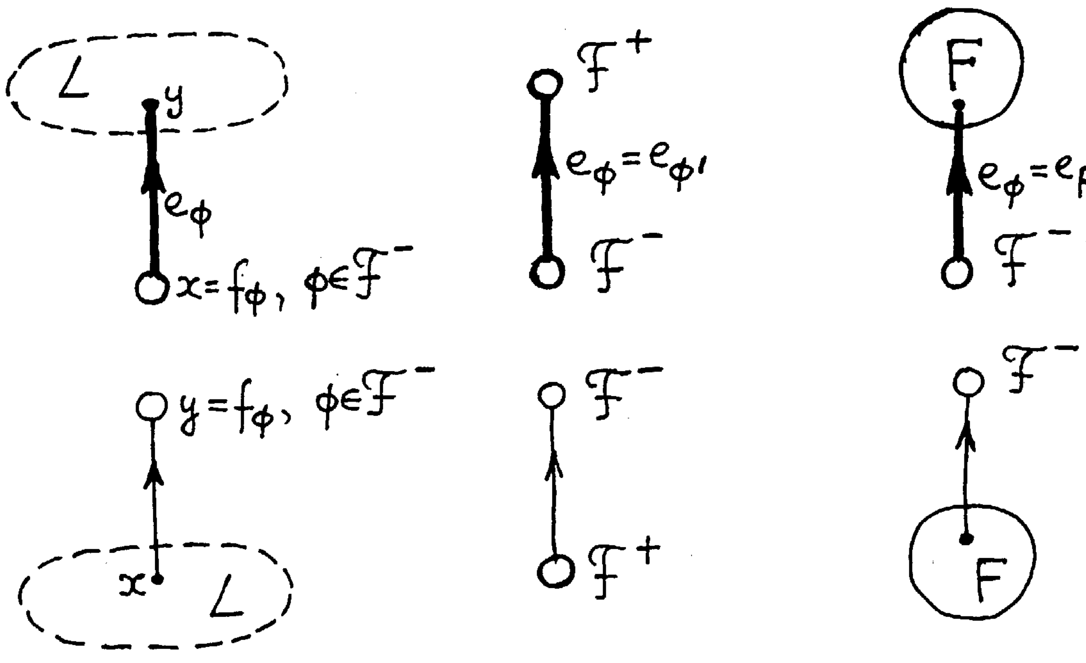


Fig. 6.2

Proof. Let e' be the mate of e. We know that at least one of $u \in \{e, e'\}$ is in ζ and that u has a labelled end; this implies that u is labelled itself. Next, since u belongs to no blossom, it is labelled in one direction only. Now the result easily follows from Statement 6.3. •

It should be noted that from Statement 6.4 it follows immediately that

(6.22) if $e \in J$ belongs to $\delta(VF)$ for a blossom $F \in \mathcal{Q}$, and ϕ' is the fragment in $\mathcal{F}^{\text{new}}$ corresponding to F then $e \notin U_{\phi'}$, i.e. (6.6) is true for ϕ'.

To see nonnegativity of λ^ε, we have to examine edges in J. Moreover, it suffices to examine edges $e = xy$ only as in Statement 6.4 (since if e is in a blossom or both x, y are unlabelled then $\lambda^\varepsilon(e) = \lambda(e) = 0$). A straightforward check-up shows that $\lambda^\varepsilon(e) = 0$ for each sort of edges e indicated in (i),(ii) of this statement, except the case when $e \in \zeta$, $x \in L$ and $y = f_\phi$ for $\phi \in \mathcal{F}^-$; in the latter case we have $\lambda^\varepsilon(e) = \lambda^\varepsilon(e) - \lambda(e) = \widehat{\alpha}^\varepsilon(e) - \widehat{\alpha}(e) = \varepsilon \geq 0$ (as $\alpha^\varepsilon(\phi) = \alpha(\phi) - \varepsilon$).

Thus, Statement 5.1 is valid. As to Statement 5.2, one can see that the only difference in comparison with the proof given in Section 5 is that in (5.11) one should replace $\mathcal{T}$ by $\mathcal{T} \cup \mathcal{T}'$, where

(6.23) $\mathcal{T}'$ is the set of unlabelled edges $e = xy \in Z$ such that $x \in X_\phi$ for $\phi \in \mathcal{F}^-$ and either $y \in M$ or $y \in L$ or $y \in X_{\phi'}$ for $\phi' \in \mathcal{F}^- - \{\phi\}$.

For e as in (6.23) we have (i) $\gamma^\varepsilon(e) = 0$ (as $e \in Z$); (ii) $\rho(e) = 0$ if $y \in M$ or $y \in X_{\phi'}$ (as $x, y \in M \cup N$); (iii) $\rho(e) = \rho(y, e) = -\varepsilon$ if $y \in L$ (by (5.5)); and (iv) $\widehat{\alpha}^\varepsilon(e) - \widehat{\alpha}(e)$ is equal to ε if $y \in M \cup L$, and 2ε if $y \in X_{\phi'}$ (as $e \notin U_\phi$ and $e \notin U_{\phi'}$). This yields $\lambda^\varepsilon(e) \geq \lambda(e) + \rho^\varepsilon(e)$ for $e \in \mathcal{T}'$, and now to prove Statement 5.2 we apply arguments similar to those given in Section 5. It should be noted that, by Statement 6.1, each element of B belongs to a strong line.

Thus, (ii) in Theorem 2 is valid for the general case (under assumptions (6.1)-(6.6),(6.8),(6.16),(6.18)). Note that maintenance of some of these properties after the transformation of (α, γ) has been already checked (e.g., validity of (6.6)), while that of the others will be shown in Section 8. In Section 7 we prove Statement 6.2 and examine some of above-mentioned properties after the transformation of B.

Let us emphasize once again one important property shown above:

(6.24) if $e \in E\Gamma$ is labelled or has both ends in X_ϕ for some $\phi \in \mathcal{F} \cup \mathcal{F}^{\text{new}}$ then $\lambda^\varepsilon(e) = \lambda(e) + \rho^\varepsilon(e)$ and $\gamma^\varepsilon(e) = \gamma(e)$.

Remark 6.5. The above transformation of (α, γ) may result in appearance of fragments $\phi \in \mathcal{F}^{\max}$ with $\alpha'(\phi) > 0$ for which (6.2) is violated for the new α', γ' and Γ' (as we explain later, this is possible only if $f_\phi \in M$). In Section 8 we show that α' and γ' can be "improved" in a certain way so that for the resulting functions (6.2) becomes true. Briefly speaking, instead of the transformation $(\alpha, \gamma) \to (\alpha', \gamma')$ as above, we will make a slightly different transformation $(\alpha, \gamma) \to (\alpha'', \gamma'')$, where α'' can differ from α' only on fragments ϕ with $X_\phi \subseteq M$, and γ'' can differ from γ' only on edges e with at least one end in M. Such a transformation will cause additional restrictions on the maximal appropriate ε (yet remaining ε positive). The exact upper bound for ε will be given in (8.26) [The choice of maximum possible ε is important for the algorithm to be polynomial.]

7. Proof of Statement 6.2

Let P be an augmenting path in Γ^*. Assume that the fragments in $\mathcal{F}$ are numbered as $\phi_1, \ldots, \phi_N$ so that $X_{\phi_i} \supset X_{\phi_j}$ implies $i < j$. Put $\mathcal{F}_j := \{\phi_j, \ldots, \phi_N\}$. Let Γ^j be obtained from Γ by shrinking X_ϕ for each $\phi \in \mathcal{F}_j^{\max}$ into a single vertex f_ϕ; thus, $\Gamma^1 = \Gamma^*$. Let P_1 and Γ^{N+1} stand for P and Γ, respectively. For $j = 2, \ldots, N+1$, we design an "augmenting path" P_j in Γ^j by extending the previous path P_{j-1} in Γ^{j-1}; the final path P_{N+1} will be just the path $\widetilde{P}$ as required in the statement. Let $B^{(j)}$ denote the set $B \triangle P_j$ in Γ. Let $Z^{(j)}, \beta^j, \zeta^j$ stand for $E\Gamma - B^{(j)}, J \cap B^{(j)}, J - B^{(j)}$, respectively. When it leads to no confuse, the image (projection) of $B^{(j)}$ in the graph Γ^j will be also denoted by $B^{(j)}$. It will be convenient to consider also the graph $\Gamma^0 := \Gamma^*$ and the null path P_0; then $B^{(0)}$ is the original set B.

Let us fix some j. In case $j > 0$, if the current P_j does not pass through f_{ϕ_j}, we obviously put $P_{j+1} := P_j$. Otherwise, assuming e_{ϕ_j} and u_j to be the edges incident to f_{ϕ_j} that are passed by P_j, the path P_{j+1} is designed from P_j by replacing the vertex f_{ϕ_j} in it by a rooted path $Q = Q^{(j)}$ (among those indicated in (6.18)) in $\Gamma^*_{\phi_j}$ whose last vertex is the end of u_j in $X^*_{\phi_j}$ (Q is considered without its first edge e_{ϕ_j}).

Consider a 0-component H of the subgraph in Γ induced by β^j. Let H^* be the projection of H to Γ^j; we call H^* a 0-component in Γ^j (for $B^{(j)}$). Let $\widehat{H}$ be the graph obtained from H^* by adding the edges $e \in B^{(j)} - \beta^j$ such that e has an end at an *ordinary* vertex of H^*. Using (6.7) and (6.18)(i), one can see that, independently of choice of $Q^{(i)}$ for $i = 1, \ldots, j-1$, the following is true:

(7.1) H^* is a tree, each non-ordinary vertex $f_{\phi'}$ in H^* is a *leaf* in it (i.e. is of valency at most one), and each ordinary vertex of H^* has an even valency in $\widehat{H}$.

A connected subgraph H' of H^* all vertices of which are ordinary is called a *proper subtree* of H^*; sometimes we will consider the maximal proper subtree of H^*, denoting it by $H^\sim$. [Figure 7.1 illustrates $H^*, H^\sim$ and $\widehat{H}$.] An edge $e \in EH^\sim$ is called *inner* for H^*; the edges in $E\widehat{H} - EH^\sim$ are called *outer* for H^*. [In particular, $e \in EH^*$ is outer if e is incident to a non-ordinary vertex.] For a proper subtree H' let $B^{(j)\text{in}}(H')$ (respectively, $B^{(j)\text{out}}(H')$) denote the set of inner (respectively, outer) edges for H^* with exactly one end in H'; for $s \in \langle -t, t\rangle$, $B_s^{(j)\text{out}}$ denotes the set of $e = xy \in B^{(j)\text{out}}$ with $x \in VH'$ and $l(x, e) = s$.

We proceed by induction on j, starting from $j = 0$. We assume by induction that for the current j the following hold:

(7.2) $B^{(j)}$ is non-excessive for each ordinary vertex $x \notin T$ of Γ^j contained in no 0-component;

(7.3) for each 0-component H^* in Γ^j, $B^{(j)}$ is non-excessive for any proper subtree H' for H^*; this means that for any $s \in \langle -t, t\rangle$,

$$|B_s^{(j)\text{out}}(H')| \le \frac{1}{2}|B^{(j)}(H')| \quad (= \frac{1}{2}|B^{(j)\text{in}}(H')| + \frac{1}{2}|B^{(j)\text{out}}(H')|).$$

As usual, we say that s is tight for H' if the inequality in (7.3) holds with equality; one can see that (6.11)-(6.12) are true (with $H^\sim$ instead of H). Note that for $j = 0$, (7.2)-(7.3) obviously follow from (6.14)(ii), and that for $j = N + 1$ (7.2)-(7.3) are equivalent to (6.8) for $B' = B \triangle \widetilde{P}$.

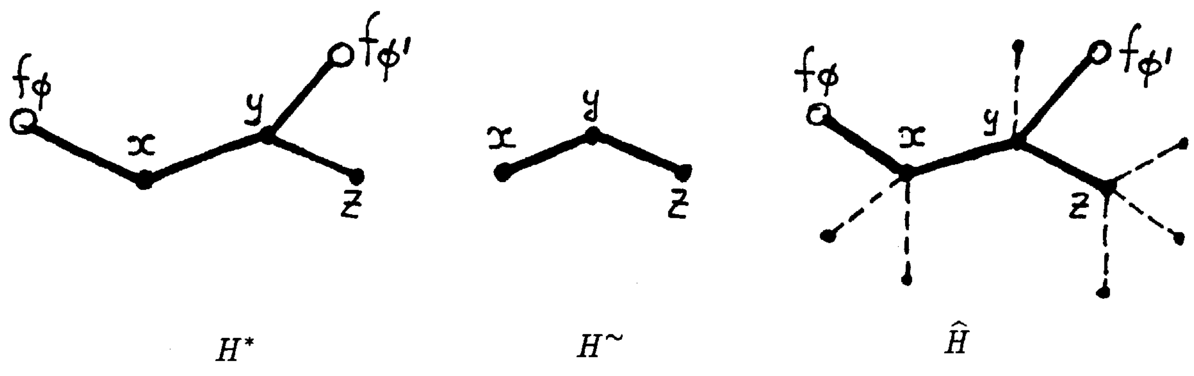


Fig. 7.1

First of all we examine the first, particular, step of the induction where the role of $Q^{(0)}$ plays the path P.

Claim 1. *(7.2)-(7.3) are valid for $j = 1$.*

Proof. Let $x \notin T$ be an ordinary vertex in $\Gamma^0 = \Gamma^*$. Consider possible cases.

(R1) P does not pass x. Then (7.2) for x or (7.3) for H^* containing x is obviously true (taking into account that H^* is a star as in (6.7)(iii), whence P meets no vertex in H^*).

(R2) P passes x. Let W be the set of edges in P incident to x; then $B^{(1)}(x) = B(x) \triangle W$. We observe that $\beta^{(1)}(x) = (\beta(x) - W) \cup (\zeta(x) \cap W)$. If $\beta^{(1)}(x) = \emptyset$ then x belongs to no 0-component for Γ^* and $B^{(1)}$. And if $\beta^{(1)}(x)$ is nonempty then, obviously, it is just the edge-set of a 0-component H^* for Γ^* and $B^{(1)}$, each edge of H^* is of the form $xf_{\phi'}$ for some $\phi' \in \mathcal{F}^{\max}$, and the maximal proper subtree $H^\sim$ of H^* is $(\{x\}, \emptyset)$. Now (7.2) for x and $B^{(1)}$ (in the former case) or (7.3) for H^* and $B^{(1)}$ (in the latter case) follow from the primitivity of P (by applying arguments as in the proof of Statement 4.1 and taking into account that for each $e \in \beta^{(1)}(x)$, if $l(x,e) = s$ then $e \in B_s^{(1)\mathrm{out}}(x)$ (in particular, $l(x,e) = 0$ and $e \in B_0^{(1)\mathrm{out}}(x)$ for $e \in \zeta(x) \cap W$)). •

Now we assume that $j \geq 1$. To prove (7.2)-(7.3) for $j+1$ we distinguish two cases, depending on whether or not P_j passes through f_{ϕ_j}.

Claim 2. *Let P_j do not meet f_{ϕ_j}. Then (7.2)-(7.3) hold for $j' := j+1$.*

Proof. Let $\phi := \phi_j$. It suffices to examine the case when $e_\phi \in \beta^j$ (and therefore, $e_\phi \in \beta$) and to show validity of (7.3) for j' and the 0-component H^* for $\Gamma^{j'}$ containing e_ϕ. Let x (y) be the end of e_ϕ in X^*_ϕ (respectively, in $V\Gamma^{j'} - X^*_\phi$). Clearly, H^* is the union of the 0-component h_1 for Γ^j and $B^{(j)}$ that contains e_ϕ and the 0-component h_2 for Γ^*_ϕ and B that contains e_ϕ (h_2 is the projection to Γ^*_ϕ of a 0-component for Γ and B). If $x = f_{\phi'}$ for some $\phi' \in \mathcal{F}_\phi$ (see Fig. 7.2(a)) then H^* coincides, in essense, with h_1, and (7.3) for j' and H^* immediately follows from that for j and h_1.

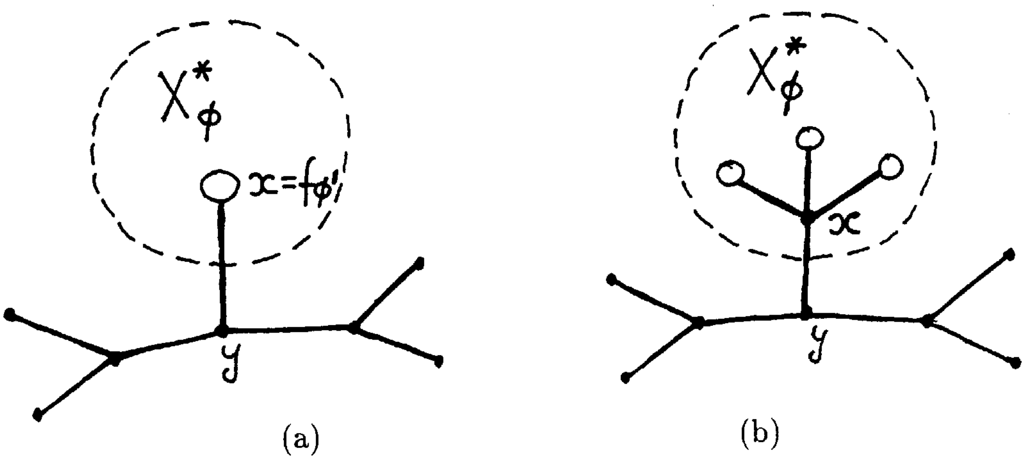


Fig. 7.2

Now suppose that x is an ordinary vertex (see Fig. 7.2(b)). Then h_2 is a star with the edge-set K that consists of e_ϕ and the edges in β of the form $xf_{\phi'}$ for $\phi' \in \mathcal{F}_\phi$ (if any). Clearly, $B^{(j')\mathrm{out}}(H^*) = (B^{(j)\mathrm{out}}(h_1) \cup Eh_2) - \{e_\phi\}$. Consider $s \in \langle -t, t \rangle$ and a proper subtree H' of H^*.

(i) If $H' \subseteq h_1$ then $B^{(j')}(H') = B^{(j)}(H')$. We observe also that $B^{(j')\mathrm{out}}(H') =$

$B^{(j)\text{out}}(H') - \{e_\phi\}$; whence $B_s^{(j')\text{out}} \subseteq B_s^{(j)\text{out}}$. Thus the inequality in (7.3) for H', s, j implies that for H', s, j'.

(ii) If $H' \subseteq h_2$ (i.e. $H' = (\{x\}, \emptyset)$) then the inequality in (7.3) for H', s, j' obviously follows from (6.14)(ii).

(iii) H' contains x and y. Then the graph $H' - \{e_\phi\}$ consists of two components, one of them, H_1 say, is a proper subtree of h_1, and the other, H_2, a proper subtree of h_2 (i.e. $H_2 = (\{x\}, \emptyset)$). Put ω_1 to be 1 if $l(y, e_\phi) = s$ and 0 otherwise; similarly, put ω_2 to be 1 if $l(x, e_\phi) = s$ and 0 otherwise. Obviously, $|B^{(j')}(H')| = |B^{(j)}(H_1)| + |B(x)| - 2$, and $|B_s^{(j')\text{out}}(H')| = |B_s^{(j)\text{out}}(H_1)| + |B_s(x)| - \omega_1 - \omega_2$. Let $q := 2|B_s^{(j')\text{out}}(H')| - |B^{(j')}(H')|$. Then

$$
\begin{aligned}
q &= (2|B_s^{(j)\text{out}}(H_1)| + 2|B_s(x)| - 2\omega_1 - 2\omega_2) - (|B^{(j)}(H_1)| + |B(x)| - 2) \\
&= (2|B_s^{(j)\text{out}}(H_1)| - |B^{(j)}(H_1)|) + (2|B_s(x)| - |B(x)|) - 2\omega_1 - 2\omega_2 + 2.
\end{aligned}
$$

Suppose that $q > 0$. We know that $q_1 := 2|B_s^{(j)\text{out}}(H_1)| - |B^{(j)}(H_1)|$ and $q_2 := 2|B_s(x)| - |B(x)|$ are non-positive (by (7.3) and (6.14)(ii)) and even. Hence, $q_1 = q_2 = \omega_1 = \omega_2 = 0$. In particular, s is tight for x and B. Then, according to rule (6.13), $l(y, e_\phi)$ must be equal to s, whence $\omega_1 = 1$; a contradiction. •

Now we assume that P_j passes through the vertex f_ϕ ($\phi = \phi_j$) using edges e_ϕ and u. Let x (y) be the end of u in X_ϕ^* (respectively, $V\Gamma^j - \{f_\phi\}$). If $u \in J$ then $u \in \zeta$, by (6.6) (as $u \neq e_\phi$), wence, $u \in \beta^j$, and u is an outer edge for a 0-component h in Γ^j. For such a u we define the number σ_u to be

(7.4) $\qquad \sigma_u := s$ if y is an ordinary vertex in Γ^j and $s \in \langle -t, t\rangle$ is tight for some proper subtree of h that contains y;

$\qquad := 0$ otherwise.

We choose the path $Q = Q^{(j)}$ according to the following rule (using notation as in (6.18)):

(7.5) (i) if x is a non-ordinary vertex $f_{\phi'}$, put $Q := Q_{\phi'}$;

(ii) if $u \in Z - \zeta$ (and therefore, $u \in B^{(j)} - \beta^j$), we put Q to be Q_x^s if $s := l(x, u)$ is tight for x and B; and to be Q_x otherwise;

(iii) if $u \in B - \beta$ (and therefore, $u \in Z^{(j)} - \zeta^j$), we put Q to be Q_x^s if there is $s \in \langle -t, t\rangle$ such that $s \neq l(x, u)$ and s is tight for x and B; and to be Q_x otherwise;

(iv) if $u \in \zeta$ (and therefore, $u \in \beta^j$), we put Q to be $Q_x^{\sigma_u}$ if $\sigma_u \neq 0$ and σ_u is tight for x and B; and to be Q_x otherwise;

here (ii)-(iv) concern an ordinary x in Γ^*_ϕ; as mentioned above, case $u \in \beta$ is impossible.

We have to prove (7.2)-(7.3) for $j' := j+1$. It suffices to examine certain vertices in X^*_ϕ and the set $\mathcal{H}$ of 0-components H^* for $\Gamma^{j'}$ and $B^{(j')}$ that meet X^*_ϕ. Let $\widetilde{\Gamma}$ denote the graph obtained from Γ^*_ϕ by adding the edge u (and the vertex y), and let P' be the part of $P_{j'}$ contained in $\widetilde{\Gamma}$ (i.e. P' is the extension of Q by the elements u, y added to its end). Note that if $e_\phi \in J$ then $e_\phi \in \zeta^i$ for $i = j, j+1$ (as $e_\phi \in \beta$, by (6.6)); hence e_ϕ belongs to no 0-component for Γ^j or $\Gamma^{j'}$.

Let $u \notin J$. Then each $H^* \in \mathcal{H}$ lies in $\Gamma^j - \{e_\phi\}$. Furthermore, (6.18) and (7.5)(i)-(iii) show that P' is a primitive path in $\widetilde{\Gamma}$ for B (assuming the corresponding "minimal self-intersecting" property for x to hold; here (6.18)(iv) is important). Now (7.2) (for j' and corresponding vertices in X^*_ϕ) and (7.3) (for j' and $H^* \in \mathcal{H}$) can be shown by use of arguments as in the proof of Claim 1.

Now let $u \in J$. Then $u \in \zeta$, whence $u \in \beta^j$ and u is contained in a 0-component $H^* \in \mathcal{H}$. By arguments above, the only point which needs to be explained is that (7.3) holds for j' and this H^*. One can see that if $x = f_{\phi'}$ for some $\phi' \in \mathcal{F}_\phi$ then H^* coincides, in essense, with the 0-component for Γ^j and $B^{(j)}$ that contains u, and (7.3) for j' and H^* is trivial.

Suppose that x is ordinary. Let h_1 (h_2) be the subtree of H^* induced by the edges outside Γ^*_ϕ (respectively, in $\widetilde{\Gamma}$). Then $h_1 \cup h_2 = H^*$ and $h_1 \cap h_2 = (\{x,y\}, u)$. Furthermore, one can see that h_2 is a star induced by u and edges $xf_{\phi'}$ for some $\phi' \in \mathcal{F}_\phi$. To check (7.3) for j' and H^*, we apply arguments similar to those in the proof of Claim 2.

More precisely, first of all let us define attachments $l'(x,e)$ by

(7.6)
$$\begin{aligned} l'(x,e) &:= l(x,e) \quad \text{if } e \in E(x) - \{u\}, \\ &:= \sigma_u \quad \text{if } e = u. \end{aligned}$$

Using (7.6) together with properties (6.18)(ii)-(iv) and (7.5)(iv) for Q, one can check that

(7.7) x is non-excessive for $B' := B \triangle P'$ and l'.

Suppose that there is a proper subtree H' of H^* and $s \in \langle -t, t\rangle$ such that

$$q := 2|B_s^{(j')\mathrm{out}}(H')| - |B^{(j')}(H')| > 0.$$

This is impossible for $H' \subset h_1$ (because in this case, $B^{(j')}(H') = B^{(j)}(H')$ and $B^{(j')\mathrm{out}}(H') \subseteq B^{(j)}(H')$) and for $H' = (\{x\}, \emptyset)$ (since (7.7),(7.6) and the fact that $l(x,u) = 0$ imply that x is non-excessive for B' and l). Hence, H' contains both x and y. Let H_1 (H_2) be the component of $H' - \{u\}$ that contains y (respectively, x). Put ω_1 to be 1 if $l(y,u) = s$ and 0 otherwise, and similarly, put ω_2 to be 1 if $l'(x,u) = s$ and 0 otherwise. By arguments as in the part (iii) of the proof of Claim 2,

$$\begin{aligned} 0 < q &= (2|B_s^{(j)\mathrm{out}}(H_1)| + 2|B_s'(x)| - 2\omega_1 - 2\omega_2) - (|B^{(j)}(H_1)| + |B'(x)| - 2) \\ &= (2|B_s^{(j)\mathrm{out}}(H_1)| - |B^{(j)}(H_1)|) + (2|B_s'(x)| - |B'(x)|) - 2\omega_1 - 2\omega_2 + 2 \end{aligned}$$

(where $B_s'(x)$ is defined with respect to l'). This implies $q_1 = q_2 = \omega_1 = \omega_2 = 0$, where $q_1 := 2|B_s^{(j)\mathrm{out}}(H_1)| - |B^{(j)}(H_1)|$ and $q_2 := 2|B_s'(x)| - |B'(x)|$. Since $q_1 = 0$, s is tight for H_1, whence $\sigma_u = s$ (by (7.4)) and $l'(x,u) = s$. Hence, $\omega_2 = 1$; a contradiction.

This completes the proof of Statement 6.2. • •

Let B' be the set obtained from B by the alteration along the "augmenting path" $\widetilde{P}$ as above, i.e. $B' := B \triangle \widetilde{P}$. We have to check that the conditions imposed in Section 6 on B, α, γ are satisfied for B', α, γ. (6.1)-(6.5) are trivial, (6.6) and (6.16) were shown in Section 6. (6.8) is provided by Statement 6.2. It remains to check (6.18).

Put $Z' := E\Gamma - B'$, $\beta' := B' \cap J$ and $\zeta' := Z' \cap J$. Let l' denote the attachment function for Γ and B' (recall that such a function is assigned uniquely; in particular, $l'(x,e)$ for $e \in \beta'$ is defined by rule (6.13) applied to 0-components H for β'). We know that $l'(x,e) = l(x,e)$ if $e \notin J$, and $l'(x,e) = 0$ if $e \in \zeta'$. Also the following is true:

(7.8) let $e = xy \in E\Gamma^*$ ($e = xy \in E\Gamma_\phi^* - \{e_\phi\}$ for $\phi \in \mathcal{F}$) belong to both β and β', and let x be ordinary in Γ^* (respectively, Γ_ϕ^*); then $l'(x,e) = l(x,e)$.

Indeed, observe that y must be a non-ordinary vertex $f_{\phi'}$ in Γ^* (Γ_ϕ^*), and $\widetilde{P}$ does not meet $X_{\phi'}$ (since $e \in \beta$ implies that $e = e_\phi$ and $e \in \beta \cap \beta'$ implies that $\widetilde{P}$ does not pass e). Thus, the 0-components for B and for B' that contain e coincide within $\Gamma_{\phi'}$. By (6.13), the attachments for (x,e) depend only on certain subtrees lying in these subgraphs, whence $l'(x,e) = l(x,e)$.

Statement 7.1. *Let $\phi \in \mathcal{F}$. Then (6.18) is valid for B' and l'.*

Proof. It falls into several parts. In what follows by $B_s'(x)$ and $Z_s'(x)$ we mean the sets

$\{e \in B'(x) : l'(x,e) = s\}$ and $\{e \in Z'(x) : l'(x,e) = s\}$, respectively, in contrast to the notations $B_s(x)$ and $Z_s(x)$ concerning B, Z and l. For $B'' \in \{B, B'\}$ and $l'' \in \{l, l'\}$ we say that a path is B'', l''-*active* (or B'', l''-*primitive*) if it is active (or primitive) with respect to B'' and l''; we may omit B'' or l'' in these terms if it leads to no confuse. Also we use the terms B'', l''-fork (B''-fork, l''-fork) and B'', l''-tight (B''-tight, l''-tight) for the corresponding objects.

We first consider the case when $\widetilde{P}$ does not meet X_ϕ. Then B' and B coincide within Γ_ϕ. In particular, each 0-component H for B with all vertices in X_ϕ is a 0-component for B', whence $l'(x,e) = l(x,e)$ for any $e = xy \in EH$. However, if $e_\phi \in \beta$ then the 0-components H and H' containing e_ϕ for B and B', respectively, may differ outside Γ_ϕ. Let h be the subgraph of H in Γ_ϕ (recall that Γ_ϕ contains e_ϕ), and H^* be the projection of h to Γ^*_ϕ.

Let v be the end of e_ϕ in X^*_ϕ. If v is non-ordinary then EH^* consists of the only edge e_ϕ. Moreover, for each (x,e) with an ordinary $x \in X^*_\phi$ the attachments l and l' are the same, whence (6.18) for B' and ϕ immediately follows.

If v is ordinary then H^* is a star induced by e_ϕ and edges $vf_{\phi'}$ for some $\phi' \in \mathcal{F}_\phi$. In view of (7.8), the only pair (x,e) ($x \in X^*_\phi$) for which $l'(x,e) \neq l(x,e)$ is possible is (v, e_ϕ). Let $q := l(v, e_\phi)$ and $q' := l'(v, e_\phi)$. If $q = q'$, we are done. So assume that $q \neq q'$. We observe that

(7.9) if $s \in \langle -t, t\rangle - \{q'\}$ is l'-tight for v then s is l-tight for v

(since $|B'(v)| = |B(v)|$ and $B'_s(v) \subseteq B_s(v)$ for $s \neq q'$). (7.9) implies that if $\tau = (e_\phi, v, e)$ is an l-fork then τ is an l'-fork as well.

Now we consider $x \in X^*_\phi$ and design a path coming x as required in (6.18) for l'. We first assume that $x \neq v$. Consider a rooted l-primitive path $Q = (x_0, e_1, x_1, \ldots, e_k, x_k)$ in Γ^*_ϕ with $x_k = x$; then $e_1 = e_\phi$ and $x_1 = v$. If Q meets v exactly once then Q is l'-active (as (e_1, x_1, e_2) is an l'-fork by the above argument). Suppose that $v = x_i$ for some $1 < i < k$. If $\tau = (e_i, x_i, e_{i+1})$ is an l'-fork then Q is l'-active. Suppose that τ is not an l'-fork. Then $e_i, e_{i+1} \in Z'_{s'}(v) \cup (B(v) - B'_{s'}(v))$, where $s' \in \langle -t, t\rangle$ is l'-tight for v. Since τ is an l-fork, (7.9) implies that $s' = q'$. Hence, (e_ϕ, v, e_{i+1}) is an l'-fork, and $Q' = (x_0, e_1, x_1, e_{i+1}, x_{i+1}, \ldots, e_k, x_k)$ is an l'-active path. These arguments easily imply the existence of a rooted primitive path as required in (6.18) for ϕ, l' and x.

Let $x = v$. If no $s' \in \langle -t, t\rangle - \{q'\}$ is l'-tight for v then, obviously, the path $Q = (x_0, e_\phi, v)$ (where $e_\phi = x_0 v$) is as required in (6.18)(ii) or (6.18)(iii) for l' and x. Finally, if such an s' exists then the path required in (6.18)(iii) for x, l', s' is just the rooted l-primitive path $Q^{s'}_x$.

We now begin to study the case when $\widetilde{P}$ meets X_ϕ. Let u be the edge in $\delta X_\phi \cap \widetilde{P}$

different from e_ϕ, and let v_0 be the end of u not in Γ^*_ϕ. Form $\widetilde{\Gamma}$ by adding to Γ^*_ϕ the edge u and the vertex v_0. Denote by P the projection of $\widetilde{P}$ to $\widetilde{\Gamma}$; we may assume that e_ϕ (u) is the first (last) edge of P. We know that P is B,l-primitive, and that u becomes the new root of ϕ when B turns into B'. Thus, a rooted path Q for ϕ, B', l' means a corresponding path whose first edge is u; it is convenient to call such a Q a *u-rooted* path, whereas a rooted path for B will be called, if needed, an *e_ϕ-rooted* path.

Consider an ordinary vertex x in X^*_ϕ. Taking into account (7.8), we have

(7.10) for $e \in E(x)$, $l'(x,e)$ can differ from $l(x,e)$ only if e is a 0-edge in P; and:

(i) if $e \in \zeta \cap P$ then $e \in \beta'$ and $l(x,e) = 0$;

(ii) if $e \in \beta \cap P$ then $e \in \zeta'$ and $l'(x,e) = 0$.

For $s \in \langle -t,t\rangle$ define $\widetilde{B}_s(x) := \{e \in B(x) \; : \; l'(x,e) = s\}$ and $\widetilde{Z}_s(x) := \{e \in Z(x) \; : \; l'(x,e) = s\}$. By (7.10),

$$\widetilde{B}_s(x) \subseteq B_s(x) \qquad \text{for any } s \in \langle -t,t\rangle, \tag{7.11}$$

which obviously implies that

(7.12) B is non-excessive for x with respect to l'.

We need two auxiliary assertions.

Claim 1. *Let $\eta = (z,g,x,e,f_{\phi'},e',x,g',z')$ be a part of P such that x is ordinary, $\phi' \in \mathcal{F}_\phi$, $e = e_\phi$, $e \in \beta$, and e' is the mate of e. Then:*

(i) *there is $s \in \langle -t,t\rangle$ such that s is B,l-tight for x, $l(x,e) = s$, and $g,g' \in Z_s(x) \cup (B(x) - B_s(x))$;*

(ii) *s as in (i) is B',l'-tight for x, and $l'(x,e') \neq s$.*

(See Fig. 7.3 for an illustration.)

Proof. (i) Since P is B,l-primitive and meets x twice, (g,x,g') is not a B,l-fork, whence, by (4.1), there exists $s \in \langle -t,t\rangle$ such that s is tight for x, and $g,g' \in Z_s(x) \cup (B(x) - B_s(x))$. [Note that if, first, case $g = u \in \zeta$ occurs, second, in the process described in the proof of Statement 6.2 the fragment ϕ was treated when the end of g different from x had become an ordinary vertex, and, third, $\sigma_u \neq 0$ (see (7.4)), then we assume by definition that $l(x,g) := \sigma_u$; and similarly for g'. Properties (iii) and (iv) ensure the above inclusion for g,g'.] Since (e,x,g) is a B,l-fork, we have $e \in (Z(x) - Z_s(x)) \cup B_s(x)$. Now $e \in B$ implies $e \in B_s(x)$.

(ii) Using (i), one can see that $|D| = |B'(x)|/2$, where $D := \{e'' \in B'(x) : l(x, e'') = s\}$. Note that if $g \in Z_s(x)$ then $g \in B'_s(x)$, and similarly for g'. Furthermore, for each $e'' \in E(x) - \{g, e, e', g'\}$, we have $l'(x, e'') = l(x, e'')$, by (7.10). Thus, $D \subseteq B'_s(x)$, whence $D = B'_s(x)$ (since B' is non-excessive for x and l'). Now $e' \notin D$ (as $l(x, e') = 0$) implies that $l'(x, e') \neq s$. •

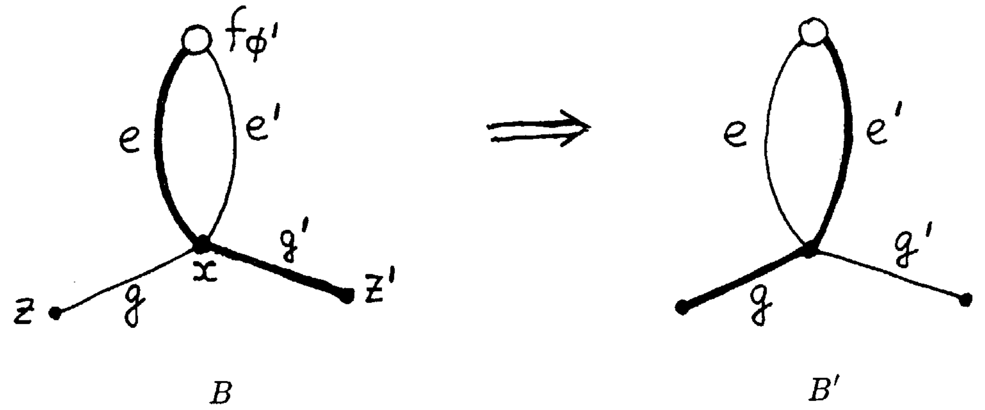


Fig. 7.3

Claim 2. *Let $\psi = (w, d, x, e, f_{\phi'}, e', x, d', w')$ be an (e_ϕ-rooted) B, l-primitive path in Γ^*_ϕ, where $\phi' \in \mathcal{F}_\phi$, $e = e_\phi$, $e \in \beta$, and e' is the mate of e. Let e, e' belong to P. Then $\tau = (d, x, d')$ is a B, l'-fork.*

Proof. Let η and s be as in Claim 1. Then $l(x, e) = s$. Since ψ is primitive and meets x twice, (d, x, d') is not a B, l-fork. Hence, there is $s' \in \langle -t, t \rangle$ such that s' is B, l-tight for x, $d, d' \in Z_{s'}(x) \cup (B(x) - B_{s'}(x))$, and $e \in B_{s'}(x)$. On the other hand, $e \in B_s(x)$. Hence $s' = s$.

Suppose that τ is not a B, l'-fork. Then there is $q \in \langle -t, t \rangle$ such that q is B, l'tight for x, and $d, d' \in \widetilde{Z}_q(x) \cup (B(x) - \widetilde{B}_q(x))$. We know that $\widetilde{B}_q(x) \subseteq B_q(x)$ (by (7.11)). Hence, $B_q(x) = \widetilde{B}_q(x)$, and q is B, l-tight for x. Note that s is not B, l'-tight for x (as $e \in B_s(x)$ and $l'(x, e) = 0$). Thus, $q \neq s$. Finally, since (e, x, d) is a B, l-fork, we have $e \in B_q(x)$; a contradiction with the fact that $e \in B_s(x)$. •

Next, for $x \in X^*_\phi$ let $\Gamma(x)$ be the graph obtained from $\widetilde{\Gamma}$ by adding a new vertex $\overline{x}$ and a new edge $h = h_x$ connecting x and $\overline{x}$; we regard h to be a feasible thin edge. If x is an ordinary vertex in V_s for $s \in T$, put $S_x := \{s, -s\}$, and if x is an ordinary central vertex, put $S_x := T$. If $x = f_{\phi'}$ for $\phi' \in \mathcal{F}_\phi$, we assume that $h \notin U_{\phi'}$ and $S_x := T$. It is easy to see that (6.18) (for ϕ, B, l) is equivalent to the following property:

(7.13) for any $x \in X^*_\phi$ and $s \in S_x$ there is an e_ϕ-rooted B, l-primitive path $\overline{Q}_{x,s}$ in $\Gamma(x)$ with the last vertex $\overline{x}$, where the attachment $l(x, h)$ is defined to be s.

A similar equivalence takes place for ϕ, B', l'. Fix some $x \in X_\phi^*$ and $s \in S_x$. We have to show that there is a u-rooted B', l'-primitive path $\overline{Q}'_{x,s}$ in $\Gamma(x)$. To do this, take an e_ϕ-rooted B, l-primitive path $Q = \overline{Q}_{x,s}$ as in (7.13). Note that $l(x,h) = l'(x,h) \neq 0$. We first associate with Q the path $\omega'(Q)$ formed from Q by removing some its parts and replacing some edges, as follows:

(7.14) (i) for $1 < i < k$, if e_i, e_{i+1} are mates in J belonging to P and x_i is non-ordinary, we remove e_i, x_i, e_{i+1} from Q;

(ii) unless the case as in (i) occurs, if $e_i \in \zeta \cap P$, we replace e_i by its mate e'.

Clearly e_ϕ is the first edge of $\omega'(Q)$. Using (7.11) and Claim 2, one can check that $\omega'(Q)$ is an e_ϕ-rooted B, l'-active path in $\Gamma(x)$ with the last vertex $\overline{x}$.

Let $\omega(Q)$ be a B, l'-primitive path $\omega(Q)$ formed from $\omega'(Q)$ by removing "superfluous" circuits (if any). The required path $\overline{Q}'_{x,s}$ will be obtained by combining certain parts of P and $\omega(Q)$.

Put $D := \omega(Q) \triangle P$ (considering D as an edge-set). Clearly D has even valency in all vertices except $\overline{x}$ and v_0, where v_0 is the end of u not in X_ϕ. We observe that

$$B' \triangle D = (B \triangle P) \triangle (\omega(Q) \triangle P) = B \triangle \omega(Q) =: \mathcal{B}. \qquad (7.15)$$

(here B, B' are considered to be restricted to $\widetilde{\Gamma}$). Since $\omega(Q)$ is B, l'-primitive,

(7.16) for any ordinary $y \in X_\phi^* - \{x\}$, $\mathcal{B}$ is non-excessive for y and l'.

We also know that

(7.17) for any ordinary $y \in X_\phi^*$, B' is non-excessive for y and l'.

Let $D(y)$ denote the set of edges in D incident to y. We need the following.

Claim 3. *Let $y = f_{\phi'}$ for $\phi' \in \mathcal{F}_\phi$, and let $e'_{\phi'}$ be the root of ϕ' with respect to B'. If $D(y) \neq \emptyset$ then $D(y)$ contains $e'_{\phi'}$, and $|D(y)| = 2$.*

Proof. The claim is obvious if y is not in P or of y is not in $\omega(Q)$. So we may assume that y is a common vertex of P and $\omega(Q)$. Let e, e' be the edges in P incident to y, and g, g' be the edges in $\omega(Q)$ incident to y. From the definition of $\omega(Q)$ it follows that g and g' are different and one of them, g say, is $e_{\phi'}$. Also one of e, e', e say, is $e_{\phi'}$. Then $e' = e'_{\phi'}$, and $\{e, e'\} \triangle \{g, g'\}$ is either $\{e', g'\}$ (if $e' \neq g'$) or $\emptyset$ (if $e' = g'$), as required. •

Now we design a path $R = (v_0, u_1, v_1, \dots)$ whose edges are in D, starting from v_0

as above and $u_1 = u$. Suppose that a B', l'-primitive path $R = (v_0, u_1, v_1, \ldots, u_i, v_i)$ has been already designed. Let $y := v_i$.

(i) If $y = f_{\phi'}$ for $\phi' \in \mathcal{F}_\phi$ then, by Claim 3, there is an edge $e = yz$ in $D(y)$ that is not in R. Extend R by adding e (as u_{i+1}) and z (as v_{i+1}).

(ii) Let y be ordinary. One can see from (7.16),(7.17), the primitivity of R and the fact that $|D(y)|$ is even that there is an edge $e = yz$ in $D(y)$ such that e is not in R and (u_i, y, e) is a B', l'-fork. Extend P by adding $e =: u_{i+1}$ and $z =: v_{i+1}$. If there is $j < i$ such that $v_j = y$ and (u_j, y, e) is a B', l'-fork, we delete from R (and simultaneously from D) the circuit formed by $u_{j+1}, \ldots, u_i$, thus maintaining the primitivity of R.

Clearly that eventually we reach the vertex $\overline{x}$. Then R is the required $\overline{Q}'_{x,s}$.

This completes the proof of Statement 7.1. • •

Thus, all conditions imposed in Section 6 on B, α, γ remain valid for B', α, γ. In the rest of this section we present two more statements.

Recall that each fragment $\phi \in \mathcal{F}$ was created, at some moment, from a blossom F appeared in the labelling process as described in Section 4 and specified in Section 6. We have to sheck correctness of (6.18) for such an initial ϕ.

Statement 7.2. *(6.18) is valid for a fragment ϕ at the moment when it is created from a blossom F.*

Proof. If F is an elementary blossom $(\{x\}, \emptyset)$ and $e_F = xy$ is its root then the path $Q = (y, e_F, x)$ satisfies (ii) and (iii) in (6.18) ((iii) obviously follows from (4.4)(iv).

Let F be a non-elementary blossom. Then for each $x \in X^*_\phi$ there is an edge $e_1 = xy \in EF$ labelled in both directions. Since e is labelled from x to y, and $e_1 \neq e_F$, there is an edge $e_2 = xy \in EF$ labelled from z to x and such that $\tau = (e_1, x, e_2)$ is a fork. Let Q_1 (Q_2) be a primitive rooted path in Γ^*_ϕ with the last vertex x and the last edge e_1 (respectively, e_2).

If $x = f_{\phi'}$ for some $\phi' \in \mathcal{F}_\phi$ then $e_\phi = e_i$ for some $i \in \{1, 2\}$ (as τ is a fork); so Q_i is $Q_{\phi'}$ as required in (6.18)(i). Let x be ordinary. It is easy to see that Q_1, Q_2 can be chosen so that at least one of them, Q_1 say, does not meet x twice. Then Q_1 satisfies (6.18)(ii),(iv). Finally, suppose that $s \in \langle -t, t \rangle$ is tight for x. Since τ is a fork, $e_i \in B_s(x) \cup (Z(x) - Z_s(x))$ for some $i \in \{1, 2\}$. If Q_i has no inner vertex v such that $v = x$ then Q_i is as required in (6.18)(iii),(iv). And if such a v exists, it is easy to see that either Q_i or its part from the beginning to the first meeting v satisfies (6.18)(iii),(iv). •

Finally, we demonstrate one more application of Statements 6.2 and 7.1; it was mentioned in Remark 1.2 in the Introduction.

Statement 7.3. $|U_\phi| \geq 3$ *for any* $\phi \in \mathcal{F}$.

Proof. By induction we may assume that if $\mathcal{F}_\phi \neq \emptyset$ then $|U_{\phi'}| \geq 3$ for each $\phi' \in \mathcal{F}_\phi$. To the contrary, suppose that $|U_\phi| = 1$. Consider two cases.

(i) $e_\phi \in Z$. Then $U_\phi = \{e_\phi\}$ and $\delta X_\phi \cap B = \emptyset$, whence $B(x) = \emptyset$ for each $x \in X_\phi$. Furthermore, $\mathcal{F}_\phi = \emptyset$. For otherwise the fact that $|U_{\phi'}| \geq 3$ for $\phi' \in \mathcal{F}_\phi$ would imply that $\delta X_{\phi'} \cap B \neq \emptyset$, whence $B(x) \neq \emptyset$ for some $x \in X_\phi$. Let x be the end of e_ϕ in X_ϕ, and let $s := l(x, e_\phi)$; then $s \neq 0$ (since $e_\phi \notin J$). Moreover, s is tight for x (as $B(x) = \emptyset$). Consider the path $Q_x^s = (x_0, e_1, x_2, \ldots, e_k, x_k)$ as in (6.18) (for B, l); here $e_1 = e_\phi$ and $x_k = x$. Then (e_k, x, e_ϕ) is in a line (since $l(x, e_k) \neq s$), and $e_k \notin J$ (since both ends of e_k are in X_ϕ, and $\mathcal{F}_\phi = \emptyset$). Hence, the path $(x_0, e_1 = e_\phi, x_1, \ldots, e_k, x_k, e_\phi, x_0)$ is a line, which is impossible.

(ii) $e_\phi \in B$. Since $|U_\phi| = 1$, $|\delta X_\phi \cap B| = 2$. Therefore, the edges in B having at least one end in X_ϕ form a simple path $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ in Γ with $x_1, \ldots, x_{k-1} \in X_\phi$ and $e_1 = e_\phi$. Then $\delta X_\phi \cap P = \{e_\phi, e_k\}$; let $x := x_{k-1}$. Since $e_k \in U_\phi$, we have $e_k \notin J$, whence $s := l(x, e_k) \neq 0$.

Let $\phi' \in \mathcal{F}^{\max}$ be such that $X_\phi \subseteq X_{\phi'}$ (possibly $\phi' = \phi$). Let $u := e_{\phi'}$ (possibly $u = e_\phi$), and let y (z) be the end (in Γ) of u that is not in (respectively, in) $X_{\phi'}$. Let us form a new graph $\widehat{\Gamma}$ from Γ, as follows.

(a) Add a new *thin* edge h connecting x as above and the terminal s_0 such that $s_0 = s$ if $s > 0$ (i.e. $s \in T$) and $s_0 = -s$ if $s < 0$.

(b) Partition u into two edges $u_1 = yv$ and $u_2 = vz$ in series, considering them as bold (thin) if u is bold (thin) and considering v as an ordinary vertex in $\widehat{\Gamma}^*$. We assume u_2 to be the new root of ϕ'. For $i = 1, 2$ assign lengths $\lambda(u_i)$ in such a way that $\lambda(u_1) + \lambda(u_2) = \lambda(u)$, and assign appropriate attachments for (v, u_i).

(c) Add a new *thin* edge h_1 connecting v and some terminal $s_1 \in T$. We choose s_1 in such a way that (h_1, v, u_2) forms a fork (this, obviously, can be done).

The lengths $\lambda(h)$ and $\lambda(h_1)$ should be assigned so that h and h_1 belong to T-lines of $\widehat{\Gamma}$ (and every T-line of Γ should remain a T-line in $\widehat{\Gamma}$).

We observe that $P = (s_1, h_1, v, u_2, f_{\phi'}, h, s_0)$ is an augmenting path in the corresponding graph $\widehat{\Gamma}^*$. Applying Statement 6.2 to $\widehat{\Gamma}$ and P, we obtain an "augmenting" path $\widetilde{P}$ in $\widehat{\Gamma}$ such that $B' := B \triangle \widetilde{P}$ is regular. Now return to ϕ, and consider the set B'' of elements of B' having at least one end in X_ϕ. Clearly e_k and h are the only edges of δX_ϕ belonging to B'', and h is just the root of ϕ with respect to B'. Moreover, since

e_k and h have a vertex in common, there are no other edges in B''. This implies that $\mathcal{F}_\phi$ is empty (for if $\phi'' \in \mathcal{F}_\phi$ then $B'' \cap \delta X_{\phi''} \neq \emptyset$, whence $v \in X_{\phi''}$; so h is the root of ϕ'', and we have $|U_{\phi''}| = 1$; a contradiction.)

If $s > 0$ then $l(x, e_k) = l(x, h) = s$, by the choice of s_0, whence $l'(x, e_k) = l'(x, h) = s$. So s is excessive for x and B' (as $|B'_s(x)| = 2 > 1 = |B'(x)|/2$); a contradiction. Let $s < 0$. Then e_k, h belong to a T-line, and s is tight for v and B'. Take the h-rooted B'-primitive path $Q_v^s = (y_0, d_1 = h, y_1 = v, d_2, y_2, \ldots, d_q, y_q = v)$ for ϕ (existing by Statement 7.1); let Q be its part from y_1 to y_q. Clearly $q > 1$. Furthermore, $\lambda(d_i) > 0$ for $i = 2, \ldots, q$ (since both ends of d_i are in X_ϕ, and $\mathcal{F}_\phi = \emptyset$). Hence, Q is a line of a non-zero λ-length; this is impossible since $y_1 = y_q$. •

8. Correctness of the α, γ-transformation

We use terminology and notation as in Sections 5 and 6; in particular, $\alpha', \gamma', \lambda', p'$, Γ' denote $\alpha^\varepsilon, \gamma^\varepsilon, c_{\alpha', \gamma'}, p_{\lambda'} = p + 2\varepsilon, \Gamma^{\lambda'}$, respectively, for a (sufficiently small) $\varepsilon > 0$ (see (6.20)).

A T-line P in Γ (or T, λ-line P) is called *non-broken* if $\lambda^\varepsilon(P) = p + 2\varepsilon$ for $\varepsilon > 0$ (in other words, P remains a T-line in Γ'). In particular, every strong T-line is non-broken (see (5.12)), whence each edge in B belongs to a non-broken T-line. Moreover, it was shown that

(8.1) A T-line $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ in Γ is non-broken if and only if P contains neither an edge in $\mathcal{T} \cup \mathcal{T}'$ nor a triple (e_i, x_i, e_{i+1}) as in (5.9), where $\mathcal{T}$ and $\mathcal{T}'$ are defined in (5.10) and (6.23).

Statements 8.1 and 8.3 below show the existence of non-broken lines of a special kind; they will play an important role as being key tools in the proofs that some conditions imposed in Section 6 are preserved under the ε-transformation of (α, γ). Let us say that $Q = (v_0, u_1, v_1, \ldots, u_m, v_m)$ is a *rooted line* for a fragment ϕ if Q is a line in Γ, $u_1 = e_\phi$, and $v_1, \ldots, v_m \in X_\phi$. Let $\mathcal{F}' := \mathcal{F} \cup \mathcal{F}^{\text{new}}$.

Statement 8.1. *Let $\phi \in \mathcal{F}'$, and $x \in X_\phi$. Then at least one of the following is true:*

(i) *there is a strong T-line P containing x;*

(ii) *there are two strong T-lines P, P' and a line Q such that: all vertices of Q are in X_ϕ, Q contains x, one end of Q is in P and the other in P';*

(iii) *there are a strong T-line P and a rooted line Q for ϕ such that: Q contains x, and the last vertex of Q is in P.*

Proof. If $B(x) \neq \emptyset$ then (i) holds. In particular, (i) is true if $|X_\phi| = 1$ (then (6.18) easily implies that $B(x) \neq \emptyset$). So we may assume that $B(x) = \emptyset$.

Observe that if there is a path η in Γ from x to x' in which all vertices are in X_ϕ and all edges are 0-edges, then statement is valid for x if and only if it is valid for x'. Note also that if all edges in Γ_ϕ are 0-edges then $e_\phi \in B$ (by (6.6)), and the result follows (taking into account that Γ_ϕ is connected, by (6.18)).

Thus, w.l.o.g. we may assume that there is an edge $e = xy \in E\Gamma_\phi$ with $\lambda(e) > 0$. Consider a line $R = (z_0, d_1, z_1, \ldots, d_q, z_q)$ in Γ_ϕ such that: (a) $y = z_{i-1}$, $e = d_i$, $x = z_i$ for some i; (b) $d_1, d_q \notin J$; (c) the number $\alpha(R)$ of edges of R that are not in J is as large as possible subject to (a),(b). Let W be the set of vertices connected with z_q by paths in Γ_ϕ having zero λ-length. We assert that

(8.2) (i) $B(z) \neq \emptyset$ for some $z \in W$; or

(ii) $q > i$ and $d_q = e_\phi$.

Indeed, suppose that neither (i) nor (ii) in (8.2) is true. Then each $z \in W$ is in X_ϕ (since if z is the end of e_ϕ not in X_ϕ then (i) or (ii) in (8.2) must be true). Let $s := l(z_q, d_q)$; then $s \neq 0$ (since $d_q \notin J$), and s is tight for every $z \in W$ (since $B(z) = \emptyset$). Choose $\phi' \in \mathcal{F} \cup \{\phi\}$ such that $X_{\phi'} \cap W \neq \emptyset$ and $X_{\phi''} \cap W = \emptyset$ for each ϕ'' preceding ϕ'. Let $z \in X_{\phi'} \cap W$, and let η be a path in Γ_ϕ of zero λ-length connecting z_q and z.

By the choice of ϕ', no edge $u = zz' \in J$ with $z' \in X_{\phi'}$ exists (as $u \in J$ would imply that $z' \in X_{\phi''}$ for some ϕ'' preceding ϕ'). Take the path Q_z^s as in (6.18)(iii) for ϕ', z, s; let u be the last edge in Q_z^s. Since $B(z) = \emptyset$, $u \in Z(z) - Z_s(v)$, whence $s' := l(z, u) \neq s$. Moreover, $\lambda(u) \neq 0$ (for if $u \in J$, the only possible case is when $u = e_{\phi'}$; then $u \in B$, contrary to (8.2)(i)). Hence, we can extend R by adding, after z_q, the path η and the edge u, forming a line R' with $\alpha(R') > \alpha(R)$; a contradiction.

Applying similar arguments to z_0 and d_1, we conclude that at least one of the following is true: (*) $B(z') \neq \emptyset$ for some $z' \in X_\phi$ such that there is a zero λ-length path η' in Γ_ϕ from z' to z_0, or (**) $d_1 = e_\phi$. Note that the case $d_1 = d_q = e_\phi$ is possible only if $e_\phi \in B$ (for $e_\phi \in Z$ would imply $\lambda(e_\phi) > 0$, whence R cannot be a line). Now the result obviously follows. •

Arguments as in Section 3 imply the following two elementary facts.

(8.3) If $\eta = \eta' \cdot \eta''$ and $\mu = \mu' \cdot \mu''$ are two T, λ-lines having a common vertex that is the last vertex of both η' and μ' then at least one of the following is true: (i) $\eta' \cdot \mu''$ and $\mu' \cdot \eta''$ are T, λ-lines, or (ii) $\eta' \cdot (\mu')^{-1}$ and $(\eta'')^{-1} \cdot \mu''$ are T, λ-lines.

(8.4) Let $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ and $Q = (v_0, u_1, v_1, \dots, u_m, v_m)$ be two lines in Γ such that $x_0, v_0 \in T$, $e_k = u_m$ and $\lambda(e_k) > 0$. Then:

(i) if $x_{k-1} = v_m$ then $P(x_0, x_{k-1}) \cdot Q^{-1}$ is a T, λ-line;

(ii) if $x_k = v_k$ then $\lambda(P) = \lambda(Q)$; if, in addition, $\lambda(P(x_0, x_{k-1})) < p/2$ then $x_0 = v_0$.

Let us say that a line is *strong* if it is a part of a strong T-line. Statement 8.1 has the following consequence.

Statement 8.2. *Let $\phi \in \mathcal{F}'$. For any vertex $x \in X_\phi$ (any edge $e \in E\Gamma_\phi$) there exists a T, λ-line $\psi = \psi_1 \cdot \psi_2 \cdot \psi_3$ in such that: either*

(i) *ψ_1 and ψ_3 are strong, ψ_2 contains x (respectively, e), and all vertices of ψ_2 are in X_ϕ; or*

(ii) *ψ_3 is strong, and ψ_2 is a rooted line for ϕ containing x (respectively, e).*

Proof. For each $x \in X_\phi$ there is an edge e in Γ_ϕ incident to x, therefore, it suffices to prove the statement for an edge $e \in E\Gamma_\phi$. If $e \in B$, we take as ψ a strong T-line containing e. Let $e \in Z$, and let x, y be the ends (in Γ_ϕ) of e; we may assume that $x \in X_\phi$. Choose a T-line η passing y, e, x in this order. From (8.3) and Statement 8.1 one can deduce that there is a T-line $\mu = \mu_1 \cdot \mu_2 \cdot \mu_3$ such that: μ_2 contains x, and either $(*)$ μ_1, μ_3 are strong, and all vertices of μ_2 are in X_ϕ; or $(**)$ μ_3 is strong, and μ_2 is a rooted line for ϕ. Represent η and μ in the forms $\eta = \eta' \cdot \eta''$ and $\mu = \mu' \cdot \mu''$, where x is the last vertex in both η' and μ'. By (8.3), there is a T-line $\omega = \eta' \cdot \omega_1 \cdot \omega_2$ such that: either ω_2 is strong and all vertices of ω_1 are in X_ϕ; or ω_1^{-1} is a rooted line for ϕ.

Represent ω in the form $\omega' \cdot \omega''$, where the first vertex and edge of ω'' are y and e, respectively. Applying Statement 8.1 to y and using (8.3), we observe that there is a T-line $\xi = \xi_1 \cdot \xi_2 \cdot \xi_3$ such that ξ_2 contains y, and either $(*')$ ξ_1, ξ_3 are strong, and all vertices of ξ_2 are in X_ϕ; or $(**')$ ξ_3 is strong, and ξ_2 is a rooted line for ϕ.

Represent ξ in the form $\xi = \xi' \cdot \xi''$, where y is the last vertex of ξ', and apply (8.3) to $\omega^{-1} = (\omega'')^{-1} \cdot (\omega')^{-1}$ and $\xi = \xi' \cdot \xi''$. As a result, we get a T-line $\psi = \psi_1 \cdot \psi_2 \cdot \psi_3$ such that either ψ is as required in (i) or (ii), or $\psi_2 = (v_0, u_1, v_1, \dots, u_k, v_k)$ lies in Γ_ϕ, contains e, and satisfies $v_0 = v_k$ and $u_1 = u_k = e_\phi$. The latter is possible only if $\lambda(e_\phi) = 0$, whence $e_\phi \in \beta$, and the result easily follows. •

Statement 8.3. *Each 1-labelled edge e belongs to a non-broken T-line.*

Proof. We may assume that $e \in Z$. It is convenient to consider the graph Γ^0 obtained from Γ^* by shrinking every blossom $F \in \mathcal{Q}$ (or the set X_ϕ for the fragment $\phi \in \mathcal{F}^{\text{new}}$ corresponding to F) into a vertex f_ϕ.

Consider an active path $R = (x_0, e_1, x_1, \dots, e_k, x_k)$ in Γ^0 beginning at T and containing e; let $e = e_i$. Let $R' = (x_{j-1}, e_j, x_j, \dots, e_q, x_q)$ be the maximal part of R such that $j \le i \le q$, $e_j, \dots, e_q \in Z$ and the vertices $x_{j+1}, \dots, x_{q-1}$ are ordinary in Γ^0. One can see that R' is a line, $e_{j+1}, \dots, e_{q-1} \notin J$, and the triples $(e_j, x_j, e_{j+1}), \dots, (e_{q-1}, x_{q-1}, e_q)$ are not as in (5.9). In addition, we may assume that R is chosen so that $q = k$ (i.e. R is of the form $R'' \cdot R'$) and $q - j$ as large as possible.

We first design a path R^0 as follows. Let $v = x_{j-1}$. If $j = 1$ (i.e. $x_0 \in T$), put $R^0 := R$ ($= R'$). Otherwise consider possible cases.

(C1) v is ordinary in Γ^0. Then $u := e_{j-1}$ is in B (by the maximality of R' in R). Let P be a strong T-line containing u, and let $P = P_1 \cdot P_2$, where v is the last vertex of P_1. It is easy to see that at least one path Q among $P_1 \cdot R'$ and $P_2^{-1} \cdot R'$ is a line, and the corresponding triple in Q containing v is not as in (5.9). Then we put $R^0 := Q$.

(C2) $v = f_\phi$ for $\phi \in \mathcal{F}^-$. Then $e_j = e_\phi$ (as e_j is labelled from v to x_j). Let x be the end of e_j in X_ϕ. Apply Statement 8.2 to ϕ, e_ϕ. Then there exists a T-line $\eta = \eta_1 \cdot \eta_2 \cdot \eta_3$ such that η_1 is strong, and η_2^{-1} is a rooted line for ϕ. Applying (8.4)(ii) to $\eta_1 \cdot \eta_2$ and R' (for which e_ϕ is a common edge with $\lambda(e_\phi) > 0$) shows that $R^0 := \eta' \cdot R'$ is a line, where η' is $\eta_1 \cdot \eta_2$ without the last edge and vertex.

(C3) $v = f_\phi$ for $\phi \in \mathcal{F}^+ \cup \mathcal{F}^{\text{new}}$. Let x be the end of e_j in X_ϕ. We again apply Statement 8.2 (to x) and arguments as in (C1) or (C2). As a result, there is a line $R'' = Q \cdot R'$ such that R'' has neither edges in $\mathcal{T} \cup \mathcal{T}'$ nor triples as in (5.9), and

(8.6) (i) either R'' has the first vertex in T;

(ii) or the first edge of R'' is e_ϕ and the first vertex of R'' is not in X_ϕ.

If (i) occurs, we put R^0 to be R''. And if (ii) occurs, we may assume that $e_\phi \in Z$ (otherwise we proceed as in (C1)). Then we apply arguments as above to R'' (rather than R'). [Note that R'' is a part of an active path $\widetilde{R}$ starting from T since e_ϕ is labelled as entering X_ϕ.] Since $l(e_\phi) > 0$ (as $e_\phi \in Z$), the λ-length of R'' is greater than that of R'. Thus, using a simple induction, we can conclude that there is R^0 such that

(8.7) R^0 is a line of the form $Q \cdot R'$ beginning at T and containing neither edges in $\mathcal{T} \cup \mathcal{T}'$ nor triples as in (5.9).

Now we extend R^0 to get a T-line R^1 of the form $R^0 \cdot Q'$. By the maximality of $q-j$ and the fact that $v' := x_q \notin T$ (otherwise R would be an augmenting path), one of the following holds: (a) v' is ordinary and $B(v') \neq \emptyset$; (b) $v' = f_\phi$ for $\phi \in \mathcal{F}'$. Using arguments similar to given in (C1)-(C3) above (exchanging the roles of $\phi \in \mathcal{F}^-$ and $\phi \in \mathcal{F}^+ \cup \mathcal{F}^{\mathrm{new}}$), we finally construct a T-line containing e and having neither edges in $\mathcal{T} \cup \mathcal{T}'$ nor triples as in (5.9). Then this line is non-broken, as required. •

Summing up Statements 8.2 and 8.3 and using (8.3), we get the following important properties:

(8.8) each edge $e \in E\Gamma$ that is labelled or is in Γ_ϕ for some $\phi \in \mathcal{F}^+ \cup \mathcal{F}^- \cup \mathcal{F}^{\mathrm{new}}$ belongs to a non-broken T-line, and therefore, e is contained in the new graph Γ'.

Let $\overline{\mathcal{F}}$ be the set of $\phi' \in \mathcal{F}'$ such that $X_{\phi'} \subseteq X_\phi$ for some $\phi \in \mathcal{F}^+ \cup \mathcal{F}^- \cup \mathcal{F}^{\mathrm{new}}$. (8.8) shows validity of (6.2)(i) for Γ' and every $\phi \in \overline{\mathcal{F}}$. Furthermore, by (6.24), $\gamma'(e_\phi) = \gamma(e_\phi) = 0$ for such a ϕ, i.e. (6.2)(ii) is also true for ϕ and γ'. However, (i) or (ii) in (6.2) may not be true, with respect to γ' and Γ', for ϕ in $\mathcal{H} := \mathcal{F}' - \overline{\mathcal{F}}$, as mentioned in Remark 6.5. We call such a ϕ *bad*.

We develop an improvement procedure for α', γ' the aim of which is to make $\mathcal{H}$ free of bad fragments. First of all we need some preliminary observations. Let L, N, M be the sets defined in Section 6. Then $X_\phi \subseteq M$ for any $\phi \in \mathcal{H}$. Define Z_M to be the set of edges in Z connecting M and $L \cup N$. Analysis in Sections 5,6 shows that $Z_M \subseteq \mathcal{T} \cup \mathcal{T}'$; more precisely,

(8.10) if $e = xy \in Z_M$ and $x \in M$ then:

(i) either $y \in L$, or $y \in X_{\phi'}$ for $\phi' \in \mathcal{F}^-$ and $e \notin U_{\phi'}$ (in particular, $e \neq e_{\phi'}$);

(ii) $\lambda'(e) = \lambda(e) + \rho^\varepsilon(e) + \varepsilon$.

The equality (ii) can be interpreted as follows:

(8.11) let P be a T-line in Γ containing an edge $e \in Z_M$; then $\lambda'(P) \geq p' + \varepsilon$ $(= p + 3\varepsilon)$, and this inequality holds with equality if and only if P contains no edges in $\mathcal{T} \cup \mathcal{T}'$ except e and no triples as in (5.9).

Another fact coming from analysis in Section 6 is that

(8.12) if $\gamma'(e_\phi) > 0$ for some $\phi \in \mathcal{H}$ with $e_\phi \in B$ then $e_\phi = xf_\phi$ is an (unlabelled) edge with either $x \in L$ or $x = f_{\phi'}$ for some $\phi' \in \mathcal{F}^-$; this implies that $\gamma'(e_\phi) = \varepsilon$.

Let us say that $\phi \in \mathcal{H}$ with $e_\phi \in B$ (respectively, $e_\phi \in Z$) is a fragment of *B-type* (respectively, Z-*type*). *Improvement procedure* consists of the operations (i)-(iv):

(i) choose a certain subset $\mathcal{H}' \subseteq \mathcal{H}^{\max}$ (of not necessarily bad fragments), where $\mathcal{H}^{\max} := \mathcal{H} \cap \mathcal{F}^{\max}$;

(ii) for each fragment $\phi \in \mathcal{H}'$ of B-type, put

$$
\begin{aligned}
\alpha''(\phi) &:= \alpha'(\phi) - \varepsilon; \\
\gamma''(\phi, e_\phi) &:= \gamma'(e_\phi) - \varepsilon; \\
\gamma''(\phi, e) &:= \gamma'(e) + \varepsilon \qquad \text{for each } e \in U_\phi;
\end{aligned}
\tag{8.13}
$$

(iii) for each fragment $\phi \in \mathcal{H}'$ of Z-type, put

$$
\begin{aligned}
\alpha''(\phi) &:= \alpha'(\phi) - \varepsilon; \\
\gamma''(\phi, e) &:= \gamma'(e) + \varepsilon \qquad \text{for each } e \in U_\phi - \{e_\phi\};
\end{aligned}
\tag{8.14}
$$

(iv) put $\alpha''(\phi')$ to be $\alpha'(\phi')$ for the remaining ϕ''s in $\mathcal{F}'$, and for $e \in E\Gamma$, put

$$
\begin{aligned}
\gamma''(e) &:= (\gamma''(\phi, e) + \gamma''(\phi', e))/2 \qquad \text{if } e \in \delta X_\phi, \delta X_{\phi'} \text{ for two } \phi, \phi' \in \mathcal{H}', \\
&:= \gamma''(\phi, e) \qquad \text{if } e \in \delta X_\phi \text{ for exactly one } \phi \in \mathcal{H}', \\
&:= \gamma'(e) \qquad \text{otherwise.}
\end{aligned}
\tag{8.15}
$$

Put $\lambda'' := c_{\alpha'', \gamma''}$. From (8.13)-(8.15) one can see that

(8.16) $\lambda''(e) = \lambda'(e)$ for any $e \in B$; in other words, the procedure does not change the length of any bold edge;

(8.17) if $\phi, \phi' \in \mathcal{H}'$ are such that $e_\phi \in \delta X_{\phi'}$ and $e_\phi \neq e_{\phi'}$ then $\gamma''(e_\phi) = \gamma'(e_\phi) = \gamma(e_\phi)$ and $\lambda''(e_\phi) = \lambda'(e_\phi) = \lambda(e_\phi)$;

(8.18) if $\phi \in \mathcal{H}'$ and $e_\phi \notin \delta X_{\phi'}$ for any $\phi' \in \mathcal{H}' - \{\phi\}$ then:

(i) $\gamma''(e_\phi) = \gamma'(e_\phi) - \varepsilon$ whenever $e_\phi \in B$;

(ii) $\lambda''(e_\phi) = \lambda'(e_\phi) - \varepsilon$ whenever $e_\phi \in Z$.

In particular, (8.16) shows that each strong T-line remains non-broken (i.e. its λ''-length is p') when we come from α', γ' to α'', γ''.

The core of improvement procedure is the rule for choosing $\mathcal{H}'$ described in (8.22) below. First of all we introduce a relation $\prec$ on pairs in $\mathcal{H}^{\max}$. We set $\phi \prec \phi'$ if there is a sequence $\phi = \phi_1, \phi_2, \dots, \phi_k = \phi'$ of distinct elements of $\mathcal{H}^{\max}$ with $k \geq 2$ such that $e_{\phi_i} \in \delta X_{\phi_{i-1}}$ for $i = 2, \dots, k$. This gives a quasi-order on $\mathcal{H}^{\max}$ in which for distinct $\phi, \phi' \in \mathcal{H}^{\max}$ both $\phi \prec \phi'$ and $\phi' \prec \phi$ take place if and only if $e_\phi = e_{\phi'}$, in view of (6.4)(ii). Thus, each week component $K = (K, \prec)$ of the digraph on $\mathcal{H}^{\max}$ defined by the relation $\prec$ satisfies the following:

(8.19) there is one $\phi \in K$ (called the *source* of K) with $e_\phi \notin \delta X_{\phi'}$ for any $\phi' \in \mathcal{H}^{\max} - \{\phi\}$, or there is a pair $\{\phi, \phi'\} \subseteq K$ (of *sources* of K) with $e_\phi = e_{\phi'}$ such that: for each $\phi'' \in \mathcal{H}^{\max}$, $\phi'' \in K$ if and only if $\phi \preceq \phi''$ (respectively, $\phi \preceq \phi''$ and $\phi' \preceq \phi''$).

We say that $K = (K, \prec)$ is a *one-*, or *bi-source fragment-tree*; for $\phi'' \in K$ denote by $\sigma(\phi'')$ the sequence $\phi = \phi_1, \phi_2, \dots, \phi_k = \phi''$, where ϕ is a source of K, $e_{\phi_i} \in \delta X_{\phi_{i-1}}$ for $i = 2, \dots, k$, and $e_\phi \notin \delta X_{\phi_2}$ (in bi-source case). We need two statements. Let $\Gamma(K)$ denote the union of graphs Γ_ϕ for $\phi \in K$.

Statement 8.4. *Let K be a bi-source fragment-tree with the sources ϕ, ϕ'. Let $\phi'' \in K$, and e be an edge in $\Gamma_{\phi''}$. Then there is a T-line $\omega = \omega_1 \cdot \omega_2 \cdot \omega_3$ in Γ such that ω_1 and ω_3 are strong, and ω_2 lies in $\Gamma(K)$ and contains e. In particular, ω is non-broken.*

Proof. Let $\sigma(\phi'') = (\phi_1, \dots, \phi_k = \phi'')$, and let for definiteness $\phi_1 = \phi$. Let $\psi = \psi_1 \cdot \psi_2 \cdot \psi_3$ be as in Statement 8.2 applied to ϕ'' and e. If ψ is as in (i) of that statement, we are done (with $\omega := \psi$). So we may assume that ψ is as in (ii) of that statement, and that $e_{\phi''} \in Z$ (for if $e_{\phi''} \in B$ then we can replace in ψ the part ψ_1 together with the first edge $e_{\phi''}$ of ψ_2 by the corresponding part of a strong T-line containing $e_{\phi''}$, whence the result follows). Two cases are possible.

(i) $k = 1$, i.e. $\phi'' = \phi$. Take $\psi' = \psi'_1 \cdot \psi'_2 \cdot \psi'_3$ as in Statement 8.2 for ϕ' and $e_{\phi'}$. Then $e_\phi = e_{\phi'}$ is the first edge of ψ_2 and the first edge of ψ'_2. Note that $\lambda(e_\phi) > 0$ (as $e_\phi \in Z$). Now applying (8.4)(i) to $(\psi_2 \cdot \psi_3)^{-1}$ and $(\psi'_2 \cdot \psi'_3)^{-1}$, we get ω as required.

(ii) $k > 1$. Let x be the end of $e_{\phi''}$ in $X_{\phi_{k-1}}$. By Statement 8.2 for ϕ_{k-1} and x, there is a T-line $\mu = \mu_1 \cdot \mu_2 \cdot \mu_3$ such that μ_3 is strong, μ_2 contains x, and either μ_1 is strong and all vertices of μ_2 are in $X_{\phi_{k-1}}$; or μ_2 is a rooted line for ϕ_{k-1}. Apply (8.3) to ψ and μ, which have the vertex x in common. We obtain a T-line $\psi' = \psi'_1 \cdot \psi'_2 \cdot \psi'_3$ such that ψ'_3 is strong, ψ'_2 contains e and lies in $\Gamma_{\phi_k} \cup \Gamma_{\phi_{k-1}}$, and either $(*)$ ψ'_1 is strong, or $(**)$ the first edge of ψ'_2 is $e_{\phi_{k-1}}$. In case $(*)$, we finish with ψ' to be ω as required. In case $(**)$, we continue the process by considering ψ' and the fragment ϕ_{k-2}, and so on. Clearly, eventually we either construct the required $\omega = \omega_1 \cdot \omega_2 \cdot \omega_3$ (with ω_2 lying in $\Gamma_{\phi_k} \cup \ldots \cup \Gamma_{\phi_i}$ for some $i \geq 1$) or obtain a T-line $\eta = \eta_1 \cdot \eta_2 \cdot \eta_3$, where η_3 is strong, while η_2 contains e, is contained in $\Gamma_{\phi_k} \cup \ldots \cup \Gamma_{\phi_1}$, and has e_ϕ as the first edge. Then we apply arguments as in case (i) above. •

Using similar arguments and (8.4)(ii) it is easy to show the following (it is left to the reader).

Statement 8.5. *Let K be a one-source fragment-tree with the source ϕ. Then:*

(i) for any edge e in $\Gamma(K)$, there is a T-line $\omega = \omega_1 \cdot \omega_2 \cdot \omega_3$ in Γ such that ω_3 is strong, ω_2 contains e, and either $(*)$ ω_1 is strong, ω_2 lies in $\Gamma(K)$ and does not use e_ϕ; or $(**)$ ω_2 lies in $\Gamma(K)$, the last edge of ω_1 is e_ϕ, and the last vertex of ω_1 is in X_ϕ; in the latter case, $e_\phi \in Z$;

(ii) for any two edges e, e' in $\Gamma(K)$, if $\omega = \omega_1 \cdot \omega_2 \cdot \omega_3$ and $\omega' = \omega_1' \cdot \omega_2' \cdot \omega_3'$ are T-lines as in (i) for e and e', respectively, and if case $(**)$ occurs for both ω and ω', then $\omega_1 = \omega_1'$. •

We say that the path ω_1 as in $(**)$ above (which is common for all corresponding e, by (ii)) is a *tail* for a one-source K with the source ϕ such that $e_\phi \in Z$. By (8.4), we can take as a tail any line $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ in Γ such that $x_0 \in T$, $e_k = e_\phi$ and x_k is in X_ϕ.

Now we wish to classify the fragment-trees. Let $\mathcal{K}^1$ ($\mathcal{K}^2$) be the set of one-source (respectively, bi-source) fragment-trees, and let $\mathcal{K}_B$ ($\mathcal{K}_Z$) be the set of $K \in \mathcal{K}^1$ whose source ϕ is of B-type (respectively, Z-type). We saw above that if $e_\phi \in B$ then case $(*)$ in Statement 8.5 occurs for any $e \in E\Gamma(K)$. This and Statement 8.4 show that

(8.20) for any $\phi \in \mathcal{H}$ such that $X_\phi \subseteq X_{\phi'}$ and ϕ' belongs to a fragment-tree $K \in \mathcal{K}^2 \cup \mathcal{K}_B$, each $e \in \Gamma_\phi$ belongs to a non-broken T-line ω; moreover, the edges of ω that are not in $\Gamma(K)$ belong to B.

For $\phi \in \mathcal{F}$ denote by v_ϕ (z_ϕ) the end (in Γ) of e_ϕ that is in (respectively, not in) X_ϕ. Let $\phi \in \mathcal{H}^{\max}$ be a fragment of B-type with $\gamma'(e_\phi) > 0$. Since $z_\phi \notin M$ (by (8.12)), ϕ is the source of a one-source fragment-tree K in $\mathcal{K}_B$. The set of such K's is denoted by $\mathcal{K}_B^+$. Let $\mathcal{K}_Z^0$ be the set of $K \in \mathcal{K}_Z$ with the source ϕ such that $e_\phi \in Z_M$ (where Z_M is defined in (8.10)).

Consider $K \in \mathcal{K}_Z$ with the source ϕ. For a tail $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ of K,

(8.21) put $m(P)$ to be the maximal index $i < k$ such that $x_i \notin M$ or $B(x) \neq \emptyset$ or x_i belongs to $X_{\phi'}$ for some $\phi \in \mathcal{H}$; and let $u(P)$ denote the edge $e_{m(P)+1}$ in P (then $u(P) \in Z$).

Now we recursively design a subset $\mathcal{K}_Z^+$ of $\mathcal{K}_Z$, starting from $\mathcal{K}_Z^+ := \mathcal{K}_Z^0$, as follows:

The core of improvement procedure is the rule for choosing $\mathcal{H}'$ described in (8.22) below. First of all we introduce a relation $\prec$ on pairs in $\mathcal{H}^{\max}$. We set $\phi \prec \phi'$ if there is a sequence $\phi = \phi_1, \phi_2, \dots, \phi_k = \phi'$ of distinct elements of $\mathcal{H}^{\max}$ with $k \geq 2$ such that $e_{\phi_i} \in \delta X_{\phi_{i-1}}$ for $i = 2, \dots, k$. This gives a quasi-order on $\mathcal{H}^{\max}$ in which for distinct $\phi, \phi' \in \mathcal{H}^{\max}$ both $\phi \prec \phi'$ and $\phi' \prec \phi$ take place if and only if $e_\phi = e_{\phi'}$, in view of (6.4)(ii). Thus, each week component $K = (K, \prec)$ of the digraph on $\mathcal{H}^{\max}$ defined by the relation $\prec$ satisfies the following:

(8.19) there is one $\phi \in K$ (called the *source* of K) with $e_\phi \notin \delta X_{\phi'}$ for any $\phi' \in \mathcal{H}^{\max} - \{\phi\}$, or there is a pair $\{\phi, \phi'\} \subseteq K$ (of *sources* of K) with $e_\phi = e_{\phi'}$ such that: for each $\phi'' \in \mathcal{H}^{\max}$, $\phi'' \in K$ if and only if $\phi \preceq \phi''$ (respectively, $\phi \preceq \phi''$ and $\phi' \preceq \phi''$).

We say that $K = (K, \prec)$ is a *one-*, or *bi-source fragment-tree*; for $\phi'' \in K$ denote by $\sigma(\phi'')$ the sequence $\phi = \phi_1, \phi_2, \dots, \phi_k = \phi''$, where ϕ is a source of K, $e_{\phi_i} \in \delta X_{\phi_{i-1}}$ for $i = 2, \dots, k$, and $e_\phi \notin \delta X_{\phi_2}$ (in bi-source case). We need two statements. Let $\Gamma(K)$ denote the union of graphs Γ_ϕ for $\phi \in K$.

Statement 8.4. *Let K be a bi-source fragment-tree with the sources ϕ, ϕ'. Let $\phi'' \in K$, and e be an edge in $\Gamma_{\phi''}$. Then there is a T-line $\omega = \omega_1 \cdot \omega_2 \cdot \omega_3$ in Γ such that ω_1 and ω_3 are strong, and ω_2 lies in $\Gamma(K)$ and contains e. In particular, ω is non-broken.*

Proof. Let $\sigma(\phi'') = (\phi_1, \dots, \phi_k = \phi'')$, and let for definiteness $\phi_1 = \phi$. Let $\psi = \psi_1 \cdot \psi_2 \cdot \psi_3$ be as in Statement 8.2 applied to ϕ'' and e. If ψ is as in (i) of that statement, we are done (with $\omega := \psi$). So we may assume that ψ is as in (ii) of that statement, and that $e_{\phi''} \in Z$ (for if $e_{\phi''} \in B$ then we can replace in ψ the part ψ_1 together with the first edge $e_{\phi''}$ of ψ_2 by the corresponding part of a strong T-line containing $e_{\phi''}$, whence the result follows). Two cases are possible.

(i) $k = 1$, i.e. $\phi'' = \phi$. Take $\psi' = \psi'_1 \cdot \psi'_2 \cdot \psi'_3$ as in Statement 8.2 for ϕ' and $e_{\phi'}$. Then $e_\phi = e_{\phi'}$ is the first edge of ψ_2 and the first edge of ψ'_2. Note that $\lambda(e_\phi) > 0$ (as $e_\phi \in Z$). Now applying (8.4)(i) to $(\psi_2 \cdot \psi_3)^{-1}$ and $(\psi'_2 \cdot \psi'_3)^{-1}$, we get ω as required.

(ii) $k > 1$. Let x be the end of $e_{\phi''}$ in $X_{\phi_{k-1}}$. By Statement 8.2 for ϕ_{k-1} and x, there is a T-line $\mu = \mu_1 \cdot \mu_2 \cdot \mu_3$ such that μ_3 is strong, μ_2 contains x, and either μ_1 is strong and all vertices of μ_2 are in $X_{\phi_{k-1}}$; or μ_2 is a rooted line for ϕ_{k-1}. Apply (8.3) to ψ and μ, which have the vertex x in common. We obtain a T-line $\psi' = \psi'_1 \cdot \psi'_2 \cdot \psi'_3$ such that ψ'_3 is strong, ψ'_2 contains e and lies in $\Gamma_{\phi_k} \cup \Gamma_{\phi_{k-1}}$, and either $(*)$ ψ'_1 is strong, or $(**)$ the first edge of ψ'_2 is $e_{\phi_{k-1}}$. In case $(*)$, we finish with ψ' to be ω as required. In case $(**)$, we continue the process by considering ψ' and the fragment ϕ_{k-2}, and so on. Clearly, eventually we either construct the required $\omega = \omega_1 \cdot \omega_2 \cdot \omega_3$ (with ω_2 lying in $\Gamma_{\phi_k} \cup \ldots \cup \Gamma_{\phi_i}$ for some $i \geq 1$) or obtain a T-line $\eta = \eta_1 \cdot \eta_2 \cdot \eta_3$, where η_3 is strong, while η_2 contains e, is contained in $\Gamma_{\phi_k} \cup \ldots \cup \Gamma_{\phi_1}$, and has e_ϕ as the first edge. Then we apply arguments as in case (i) above. $\bullet$

Using similar arguments and (8.4)(ii) it is easy to show the following (it is left to the reader).

Statement 8.5. *Let K be a one-source fragment-tree with the source ϕ. Then:*

(i) for any edge e in $\Gamma(K)$, there is a T-line $\omega = \omega_1 \cdot \omega_2 \cdot \omega_3$ in Γ such that ω_3 is strong, ω_2 contains e, and either $(*)$ ω_1 is strong, ω_2 lies in $\Gamma(K)$ and does not use e_ϕ; or $(**)$ ω_2 lies in $\Gamma(K)$, the last edge of ω_1 is e_ϕ, and the last vertex of ω_1 is in X_ϕ; in the latter case, $e_\phi \in Z$;

(ii) for any two edges e, e' in $\Gamma(K)$, if $\omega = \omega_1 \cdot \omega_2 \cdot \omega_3$ and $\omega' = \omega'_1 \cdot \omega'_2 \cdot \omega'_3$ are T-lines as in (i) for e and e', respectively, and if case $(**)$ occurs for both ω and ω', then $\omega_1 = \omega'_1$. $\bullet$

We say that the path ω_1 as in $(**)$ above (which is common for all corresponding e, by (ii)) is a *tail* for a one-source K with the source ϕ such that $e_\phi \in Z$. By (8.4), we can take as a tail any line $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ in Γ such that $x_0 \in T$, $e_k = e_\phi$ and x_k is in X_ϕ.

Now we wish to classify the fragment-trees. Let $\mathcal{K}^1$ ($\mathcal{K}^2$) be the set of one-source (respectively, bi-source) fragment-trees, and let $\mathcal{K}_B$ ($\mathcal{K}_Z$) be the set of $K \in \mathcal{K}^1$ whose source ϕ is of B-type (respectively, Z-type). We saw above that if $e_\phi \in B$ then case $(*)$ in Statement 8.5 occurs for any $e \in E\Gamma(K)$. This and Statement 8.4 show that

(8.20) for any $\phi \in \mathcal{H}$ such that $X_\phi \subseteq X_{\phi'}$ and ϕ' belongs to a fragment-tree $K \in \mathcal{K}^2 \cup \mathcal{K}_B$, each $e \in \Gamma_\phi$ belongs to a non-broken T-line ω; moreover, the edges of ω that are not in $\Gamma(K)$ belong to B.

For $\phi \in \mathcal{F}$ denote by v_ϕ (z_ϕ) the end (in Γ) of e_ϕ that is in (respectively, not in) X_ϕ. Let $\phi \in \mathcal{H}^{\max}$ be a fragment of B-type with $\gamma'(e_\phi) > 0$. Since $z_\phi \notin M$ (by (8.12)), ϕ is the source of a one-source fragment-tree K in $\mathcal{K}_B$. The set of such K's is denoted by $\mathcal{K}_B^+$. Let $\mathcal{K}_Z^0$ be the set of $K \in \mathcal{K}_Z$ with the source ϕ such that $e_\phi \in Z_M$ (where Z_M is defined in (8.10)).

Consider $K \in \mathcal{K}_Z$ with the source ϕ. For a tail $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ of K,

(8.21) put $m(P)$ to be the maximal index $i < k$ such that $x_i \notin M$ or $B(x) \neq \emptyset$ or x_i belongs to $X_{\phi'}$ for some $\phi \in \mathcal{H}$; and let $u(P)$ denote the edge $e_{m(P)+1}$ in P (then $u(P) \in Z$).

Now we recursively design a subset $\mathcal{K}_Z^+$ of $\mathcal{K}_Z$, starting from $\mathcal{K}_Z^+ := \mathcal{K}_Z^0$, as follows:

(8.22) if $K \in \mathcal{K}_Z - \mathcal{K}_Z^+$ is such that for *every* its tail P the edge $u(P)$ belongs to either Z_M or $\delta X_{\phi'}$ for $\phi' \in K' \in \mathcal{K}_B^+ \cup \mathcal{K}_Z^+$, then we add K to $\mathcal{K}_Z^+$; repeat until such a K does not exist.

The set $\mathcal{K}_B^+ \cup \mathcal{K}_Z^+$ is just the above mentioned $\mathcal{H}'$ to which we apply the improvement procedure (see (8.13)-(8.15)).

For a fragment-tree K let $Z(K)$ denote the set $\cup(z_\phi : \phi \in K)$, where $z_\phi := Z \cap (\delta X_\phi - \{e_\phi\})$. First of all it is easy to check that

(8.23) (i) $\gamma''(e_\phi) = \gamma'(e_\phi) - \varepsilon = 0$ for all bad fragment ϕ of type B, and $\gamma''(e_\phi) = \gamma'(e_\phi) = 0$ for the remaining fragments of type B in $\mathcal{H}$ (cf. (8.17),(8.18)(i));

(ii) $\lambda''(e_\phi) = \lambda'(e_\phi) - \varepsilon$ for the source ϕ of each $K \in \mathcal{K}_Z^+$; if $K \in \mathcal{K}_Z^0$ then $\lambda''(e_\phi) = \lambda(e_\phi) + \rho^\varepsilon(e_\phi)$ (cf. (8.10)(ii));

(iii) among the edges in Z (for Γ) with at least one end in M, the set of edges e for which $\lambda''(e) > \lambda'(e)$ is just $\cup(Z(K) : K \in \mathcal{K}_B^+ \cup \mathcal{K}_Z^+$; moreover, for such an e, $\lambda''(e) = \lambda'(e) + \varepsilon$ (respectively, $\lambda''(e) = \lambda'(e) + 2\varepsilon$) if e belongs to z_ϕ for exactly one (respectively, two) $\phi \in \cup(K : K \in \mathcal{K}_B^+ \cup \mathcal{K}_Z^+)$;

(iv) for the remaining edges $e \in E\Gamma$, $\lambda''(e) = \lambda'(e)$.

Define $\mathcal{K}'_Z := \mathcal{K}_Z - \mathcal{K}_Z^+$ and $\mathcal{K}'_B := \mathcal{K}_B - \mathcal{K}_B^+$.

Statement 8.6. *For any $K \in \mathcal{H} - (\mathcal{K}_B^+ \cup \mathcal{K}_Z^+)$ and $e \in \Gamma(K)$, there is a non-broken T-line Q containing e, i.e. $\lambda''(Q) = \lambda'(Q) = p'$.*

Proof. In view of (8.20), it suffices to examine $K \in K'_Z$ only. Let ϕ be the source of K. By (8.21)-(8.22), there exists a tail $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ for K such that for $x := x_{m(P)}$, either (i) x is an ordinary vertex in M with $B(x) \neq \emptyset$, or (ii) $x \in X_{\phi'}$ for some $\phi' \in K' \in \mathcal{K}'_B \cup \mathcal{K}'_Z \cup \mathcal{K}^2$ (note that $K' = K$ is possible). Let P' be the part of P from x to x_k. Consider $e \in \Gamma(K)$. Let $\omega = \omega_1 \cdot \omega_2 \cdot \omega_3$ be as in Statement 8.5. If $(*)$ takes place, we are done. Otherwise we may assume that $\omega_1 = P$.

In case (i), take a strong T-line $\mu = \mu_1 \cdot \mu_2$, where x is the last vertex of μ_1. Then at least one path Q among $\mu_1 \cdot P' \cdot \omega_2 \cdot \omega_3$ and $\mu_2^{-1} \cdot P' \cdot \omega_2 \cdot \omega_3$ is a T-line (by (8.3)). Now the obvious facts that $x \in M$ and $\lambda''(e') = \lambda(e')$ for all $e' \in P'$ imply that Q is non-broken.

In case (ii), there are two possibilities for $u := u(P)$, namely, $u = e_{\phi'}$ or $u \in Z(\phi')$. Suppose that $u = e_{\phi'}$. Then ϕ' is the source of K', and $K' \in \mathcal{K}'_Z$ (since $u \in Z$ and $x_{m(P)+1}$ is not in $\cup(X_{\phi''} : \phi'' \in K')$). Take a T-line $\omega' = \omega'_1 \cdot \omega'_2 \cdot \omega'_3$ as in Statement 8.5 for K' and $e_{\phi'}$. Since u is a common edge of T-lines ω and ω', and $\lambda(u) > 0$, the path $Q := (\omega_3')^{-1} \cdot (\omega_2')^{-1} \cdot P' \cdot \omega_2 \cdot \omega_3$ is a T-line (by (8.4)). Moreover, $\lambda''(e') = \lambda(e')$ for all $e' \in P'$. Hence, Q is non-broken.

Now suppose that $u \in Z(\phi')$ (recall that $Z(\phi')$ does not contain $e_{\phi'}$). If $K' \in \mathcal{K}_B \cup \mathcal{K}^2$, the result is easy. Let $K' \in \mathcal{K}_Z'$. Apply Statement 8.2 to ϕ' and x, and then apply (8.3) and, possibly, Statement 8.5. The only nontrivial case arises when we get a T-line ψ of the form $\psi_1 \cdot \psi_2 \cdot \psi_3$, where ψ_1 is a tail for K', ψ_2 lies in $\Gamma(K')$, and $\psi_3 = P' \cdot \omega_2 \cdot \omega_3$. On the other hand, by Statement 8.5 for K' and $e_{\phi''}$, where ϕ'' is the source of K', there is a T-line $\omega' = \omega_1' \cdot \omega_2' \cdot \omega_3'$ as in (**). Since both ψ_1 and ω_1' are tails for K', we may assume that $\psi_1 = \omega_1'$. Moreover, since $K' \notin \mathcal{K}_Z^+$, we may assume that ψ_1 satisfies the property similar to that described above for P. So we can apply to K', $e_{\phi''}$ and ψ arguments similar to those for K, e, ω; and so on. One can see that eventually we find either a T-line Q as required, or a non-simple T-line Q' having a circuit C which contains e_{ϕ^*}, where ϕ^* is the source of some $K^* \in \mathcal{K}_Z'$. The latter is impossible since $\lambda(e_{\phi^*}) > 0$. •

One observation will be useful in what follows:

(8.24) let $K_0, K_0' \in \mathcal{K}_Z^+$, let ϕ_0' be the source of K_0', and let P be a tail of K_0 such that $u(P) \in \delta X_{\phi''}$ for some $\phi'' \in K_0'$; then $u(P) \neq e_{\phi_0'}$.

Indeed, suppose that $u(P) = e_{\phi_0'}$. Let $\omega = \omega_1 \cdot \omega_2 \cdot \omega_3$ be a T-line as in case (*) in Statement 8.5 applied to K_0 and e_{ϕ_0}, where ϕ_0 is the source of K_0. Assuming that $\omega_1 = P$ and that P' is the part of P with the first edge $e_{\phi_0'}$ and the last edge e_{ϕ_0}, we observe that by the rule (8.22) neither K_0 nor K_0' can be added to the current $\mathcal{K}_Z^+$; a contradiction.

Statement 8.7. *Let $K \in \mathcal{K}_Z^+$, and $e \in E\Gamma(K)$. Then there exists a T-line Q in Γ such that Q contains e, and $\lambda''(Q) = p'$.*

Proof. In view of Statement 8.5, it suffices to consider the case when $e = e_\phi$ for the source ϕ of K. Rule (8.22), property (8.24) and standard arguments as above show that there is a T-line Q in Γ of the form

$$Q = \eta_1 \cdot \mu_1 \cdot \eta_2 \cdot \mu_2 \cdot \ldots \cdot \eta_r \cdot \mu_r,$$

and there is a sequence $K_1, K_2, \ldots, K_r = K$ of fragment-trees with the sources $\phi_1, \phi_2, \ldots, \phi_r = \phi$ such that:

(i) $K_2, \ldots, K_r \in \mathcal{K}_Z^+$ and $K_1 \in \mathcal{K}_Z^+ \cup \mathcal{K}_B^+$;

(ii) for $i = 1, \ldots, r-1$, μ_1 lies in $\Gamma(K_i)$; while $\mu_r = \mu \cdot \mu'$, where μ lies in $\Gamma(K)$ and

μ' is strong;

(iii) for $i = 2, \ldots, r$, the path $P_i := \eta_1 \cdot \mu_1 \cdot \eta_2 \cdot \mu_2 \cdot \ldots \cdot \eta_i$ is a tail of K_i; the first edge of η_i, u_i say, is $u(P_i)$ (see (8.21)), and $u_i \in Z(K_{i-1})$;

(iv) either (a) η_1 is strong, or (b) the last edge of η_1 is e_{ϕ_1}, $K_1 \in \mathcal{K}_Z^+$, and η_1 is a tail of K_1;

(v) if case (b) in (iv) occurs then the edge $u_1 := u(\eta_1)$ is in Z_M.

In case (iv)(b), let $\eta_1 = \eta \cdot \eta'$, where the first edge of η' is u_1. Then the last vertex x of η is not in M, whence, by (8.8), we may assume that η belongs to a non-broken line; furthermore, we may assume that the corresponding triple in η_1 containing x is not as in (5.9). In view of (8.11), we can see that

(8.25)
$$\begin{aligned} \lambda'(Q) &= p' \qquad \text{if case (a) in (iv) occurs,} \\ &= p' + \varepsilon \quad \text{if case (b) in (iv) occurs.} \end{aligned}$$

For $i = 2, \ldots, r$, we have $\lambda''(u_i) = \lambda'(u_i) + \varepsilon$ and $\lambda''(e_{\phi_i}) = \lambda'(e_{\phi_i}) - \varepsilon$, by (8.23). Hence, $\lambda''(\eta_i) = \lambda'(\eta_i)$ for $i = 2, \ldots, r$. Furthermore, $\lambda''(\mu_i) = \lambda'(\mu_i)$ for $i = 1, \ldots, r$, and $\lambda''(\eta_1) = \lambda'(\eta_1)$ in case (a) in (iv). Finally, in case (b) in (iv), we have $\lambda''(e_{\phi_1}) = \lambda'(e_{\phi_1}) - \varepsilon$ (by (8.23)(ii)), whence $\lambda''(\eta_1) = \lambda'(\eta_1) - \varepsilon$. We now conclude from (8.25) that $\lambda''(Q) = p'$. •

Statement 8.8. *For any T-line $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ in Γ, $\lambda''(P) \geq p'$.*

Proof. Let U be the set of elements e_i such that $e_i = e_\phi$, where ϕ is the source of some $K \in \mathcal{K}_Z^+$. Let W be the set of pairs (x_i, e_i) or (x_i, e_{i+1}) such that $x_i \in X_\phi$ and $e_i \in Z(K)$ (respectively, $e_{i+1} \in Z(K)$) for some $\phi \in K \in \mathcal{K}_Z^+ \cup \mathcal{K}_B^+$. Let Y be the set of elements e_i such that $e_i \in Z_M$. By (8.23) and (8.10)(ii),

$$\lambda''(P) = \lambda'(P) + \varepsilon|W| - \varepsilon|U| \geq p' + \varepsilon|Y| + \varepsilon|W| - \varepsilon|U|.$$

We show that $|Y| + |W| - |U| \geq 0$, whence the result will follow. To see this, consider $e_i \in U$. Let $e_i = e_\phi$ for the source ϕ of $K \in \mathcal{K}_Z^+$, and let for definiteness $v_\phi = x_i$ (where $v(\phi)$ is the end of e_ϕ in X_ϕ). The part P' of P from x_0 to x_i is a tail of K; let $x := x_{m(P')}$ (see (8.21)). Since $K \in \mathcal{K}_Z^+$, for $u := u(P')$ one of the following is true: (i) $u \in Z_M$, or (ii) $u \in Z(K')$ for some $K' \in \mathcal{K}_Z^+ \cup \mathcal{K}_B^+$ (taking into account (8.24)).

In case (i), we bring e_i to the element u of Z_M, and in case (ii), we bring e_i to the pair (x_{j-1}, e_j), where $e_j = u$. This gives a mapping τ of U into $Y \cup W$. Moreover, it is easy to see that $\tau(e) \neq \tau(e')$ for distinct $e, e' \in U$. Hence, $|Y| + |W| - |U| \geq 0$. •

This statement shows that for a sufficiently small $\varepsilon > 0$ the corresponding λ'' satisfies the inequality $\mathrm{dist}_{\lambda''}(s, s') \geq p'$ $(= p + 2\varepsilon)$ for all distinct $s, s' \in T$. Thus, we make the transformation $(\alpha, \gamma) \to (\alpha'', \gamma'')$ correct by taking ε to be

(8.26) $$\varepsilon := \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\},$$

where $\varepsilon_0, \ldots, \varepsilon_3$ were defined in Sections 5 and 6, and:

(8.27) $$\varepsilon_4 := \min\{\alpha(\phi) \; : \; \phi \in \mathcal{H}' = \mathcal{K}_B^+ \cup \mathcal{K}_Z^+\};$$

(8.28) ε_5 is the supremum of $\varepsilon' < \varepsilon_1$ such that (8.22) defines the same set $\mathcal{K}_Z^+$.

Remark. In fact, imposing (8.28) to restrict ε is redundant. More precisely, using arguments as in the proofs of Statements 8.6-8.8, one can deduce that if $\mathcal{K}_Z^+$ is changed when we reach some ε' then the further growth of ε should cause violation of the above distance inequalities (because a new T-line necessarily arises when ε' is reached); this means that $\varepsilon_5 \geq \varepsilon_3$. We leave it to the reader to check this phenomenon in details.

In what follows the corresponding objects arising when we come from α, γ to α'', γ'' will be denoted with two primes, e.g., $\mathcal{F}''$ denotes the support of α'', Γ'' denotes $\Gamma_{\lambda''}$, and etc. Let $\mathcal{H}''$ be the set of $\phi \in \mathcal{F}''$ with $X_\phi \subseteq M$, and let $\widetilde{N}$ be the union of N and the set of vertices $x \in M$ such that $B(x) \neq \emptyset$ or $x \in X_\phi$ for some $\phi \in \mathcal{H}''$. We say that a T-line P in Γ is *λ'-non-broken* (*λ''-non-broken*) if $\lambda'(P) = p'$ (respectively, $\lambda''(P) = p'$).

The immediate corollary from the above considerations is that (6.2) is true for $\mathcal{F}'', \Gamma'', \gamma''$. Also from the proofs of Statements 8.6-8.8 one can conclude that

(8.29) for any $\phi \in \mathcal{H}''$,

(i) each edge e in Γ_ϕ belongs to a λ''-non-broken T-line (and hence, e is in Γ'');

(ii) each $x \in X_\phi$ belongs to a λ''-non-broken T-line $P = P_1 \cdot P_2$ such that x is the last vertex of P_1, and for at least one $i \in \{1, 2\}$, $\lambda''(e) = \lambda'(e)$ for all edges e in P_i.

In order to examine some conditions from Section 6 we need one auxilliary assertion (Statement 8.9). First of all we observe that (8.8) can be strengthened as

(8.30) each edge e in Γ that is labelled or is in Γ_ϕ for some $\phi \in \mathcal{F}^+ \cup \mathcal{F}^- \cup \mathcal{F}^{\text{new}}$ (respectively, each vertex $x \in L \cup N$) belongs to a λ'-non-broken line $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ such that $\lambda''(e_i) = \lambda'(e_i)$ for $i = 1, \ldots, k$; in particular, P is λ''-non-broken.

Indeed, let $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ be a λ'-non-broken line with $e_j = e$. If P does not meet M then $\lambda''(e_i) = \lambda'(e_i)$ for all i. Otherwise there is an edge e_i connecting $L \cup N$ and M; one may assume that $i > j$ and $x_i \in M$. Since P has no edge in $\mathcal{T} \cup \mathcal{T}'$, $e_i \notin Z_M$, whence $e_i \in B$. Then, in view of (8.3), we can replace in P the part $P(x_i, x_k)$ by a corresponding part of a strong T-line passing x_i so that the resulting T-line P' is again λ'-non-broken. This argument shows that a λ'-non-broken P containing e can be chosen so that each edge of P with at least one end in M belongs to B. Then $\lambda''(e') = \lambda'(e')$ for all edges e' of P.

Statement 8.9. *Let $x \in \widetilde{N}$.*

(i) *If $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ is a λ''-non-broken T-line passing x, and Q is its part from x_0 to x, then $\lambda''(Q) = \lambda(Q) + \varepsilon$;*

(ii) *$\pi''(x) = \pi(x) + \varepsilon$.*

Proof. (ii) immediately follows from (i). It suffices to check (i) for some particular λ''-non-broken P passing x; then (i) would easily follow for any λ''-non-broken P passing x, in view of (8.3). We first consider case $x \in N$. Let $x := x_i$ for P as in (8.30). We know (cf. (5.14)) that $\lambda'(e_j) - \lambda(e_j) = \rho^\varepsilon(e_j)$ for $j = 1, \ldots, k$, and $\widehat{\rho}^\varepsilon(x_j) = 0$ for $j = 1, \ldots, k-1$. Hence,

$$\lambda''(Q) - \lambda(Q) = \lambda'(Q) - \lambda(Q) = \rho^\varepsilon(Q) = \rho^\varepsilon(x_0, e_1) + \sum_{j=1}^{i-1} \widehat{\rho}^\varepsilon(x_j) + \rho^\varepsilon(x, e_i)$$
$$= \rho^\varepsilon(x_0, e_1) + \rho^\varepsilon(x, e_i) = \varepsilon + \rho^\varepsilon(x, e_i) = \varepsilon.$$

(since $\rho^\varepsilon(x', e') = 0$ for any $x' \in N$).

Now let $x \in \widetilde{N} - N$. If $B(x) \neq \emptyset$ then we may assume that P is a strong T-line passing x and use arguments as above (taking into account that $\lambda''(e) = \lambda'(e)$ for all edges e in P, and that $\rho^\varepsilon(x, e') = 0$ (since $x \in M$)). Finally, if $x \in X_\phi$ for some $\phi \in \mathcal{H}''$, consider $P = P_1 \cdot P_2$ as in (8.29)(ii). Let for definiteness $\lambda''(e) = \lambda'(e)$ for all edges in $P_1 =: Q$. By arguments as above, $\lambda''(Q) - \lambda(Q) = \varepsilon$. •

This statement has an analogue for $x \in L$; it will be used in the next section.

Statement 8.10. *Let $x \in L$. If $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ is a λ'-non-broken T-line* *passing x, and Q is its part from x_0 to $x = x_i$, then $\lambda'(Q)-\lambda(Q) = \varepsilon+\rho^\varepsilon(x, e_i) \in \{0, 2\varepsilon\}$.*

Proof. We have $\lambda'(Q) - \lambda(Q) = \rho^\varepsilon(x_0, e_1) + \rho^\varepsilon(x, e_i) = \varepsilon + \rho^\varepsilon(x, e_i)$, and $\rho^\varepsilon(x, e_i) \in \{-\varepsilon, \varepsilon\}$ (by (5.5)). •

Now we check one key condition from Section 6.

Statement 8.11. *(6.6) is true for $\mathcal{F}''$ and J'' (where $J'' := \{e \in EG : \lambda''(e) = 0\}$).*

Proof. Let $e = vz \in J''$. If both v and z are in X_ϕ for some $\phi \in \mathcal{F}''$ then $e \in J$, and the result immediately follows from (6.6) for $\mathcal{F}$ and J. Hence, we may assume that $e \in \delta X_\phi$ for some $\phi \in \mathcal{F}''^{\max}$; let for definiteness $v \in X_\phi$. Since $c(e) > 0$ and $\lambda''(e) = 0$, we may assume that $e \notin U_\phi$ for this ϕ. We have seen in Section 6 that (6.4) is valid for $\mathcal{F}'$, so it is valid for $\mathcal{F}''$ too (since $\mathcal{F}'' \subseteq \mathcal{F}'$). Thus, $e \notin U_{\phi'}$ for all $\phi' \in \mathcal{F}''$ such that $X_{\phi'} \subset X_\phi$. If there is no $\phi' \in \mathcal{F}''$ such that $z \in X_{\phi'}$, we are done. So we may assume that such a ϕ' exists; then $X'_\phi \cap X_\phi = \emptyset$. We have to show that $e \notin U_{\phi'}$. By (6.4), it suffices to consider ϕ' with maximal $X_{\phi'}$ (i.e. $\phi' \in \mathcal{F}''^{\max}$).

(i) Let $e \in B$. Since $v, z \in N \cup M$, we have $\lambda''(e) = \lambda(e)$ (by Statement 8.9 and the fact that e is in a strong T-line). Hence, $e \in J$. If both ϕ, ϕ' are in $\mathcal{F}$ then the result follows from (6.6) for $\mathcal{F}$ and J. Suppose that some of ϕ, ϕ' is in $\mathcal{F}^{\text{new}}$. Then at least one end (in Γ^*) of e is labelled. Now by Statement 6.4(i), $e = e_\phi = e_{\phi'}$, whence $e \notin U_{\phi'}$.

(ii) Let $e \in Z$. Suppose, for a contradiction, that $e \in U_{\phi'}$. Then $e = e_{\phi'}$. Hence, e is a feasible edge in Γ (by (6.2) if $\phi' \in \mathcal{F}$, and by forming a blossom if $\phi' \in \mathcal{F}^{\text{new}}$). Also we must have $\lambda(e) > 0$ (by (6.6) for $\mathcal{F}$ and J). Consider possible cases.

(a) $v, z \in M$. Then there are a fragment-tree K and $\phi_1, \phi_2 \in K$ such that $X_\phi \subseteq X_{\phi_1}$ and $X_{\phi'} \subseteq X_{\phi_2}$. The fact that $e_{\phi'} = e \neq e_\phi$ implies that either $\phi_1 = \phi_2$, or $e_{\phi_2} = e \neq e_{\phi_1}$. In both cases, $\lambda(e) = \lambda'(e) = \lambda''(e)$, whence $\lambda(e) = 0$; a contradiction.

(b) $|\{z, v\} \cap M| = 1$. Since e is feasible, the only possible case is $\phi \in \mathcal{F}^-$ and $\phi' \in \mathcal{H}$. Then $X_{\phi'} \subseteq X_{\phi''}$, where ϕ'' is the source of some $K \in \mathcal{K}^0_Z$. We have $\lambda'(e) = \lambda(e) + \varepsilon$ (see (8.10)(ii); here $\rho^\varepsilon = 0$) and $\lambda''(e) = \lambda'(e) - \varepsilon$, whence $\lambda(e) = 0$; a contradiction.

(c) $z, v \in N$. Since e is feasible and $U_\phi \not\ni e \in U_{\phi'}$, there are $\phi_1 \in \mathcal{F}^+ \cup \mathcal{F}^{\text{new}}$ and $\phi_2 \in \mathcal{F}^-$ such that $X_\phi \subseteq X_{\phi_i}$ and $X_{\phi'} \subseteq X_{\phi_{3-i}}$ for some $i \in \{1, 2\}$. This implies that $\lambda(e) = \lambda'(e) = \lambda''(e)$, whence $e \in J$. Then $e \notin U_{\phi_{3-i}}$ (by (6.6) for $\mathcal{F}$ and J), whence $e \notin U_{\phi'}$ (by (6.4)); a contradiction. •

It remains to check (6.5),(6.8),(6.16) and (6.18). (6.16) for J'' is trivial. To see (6.5), consider an edge $e = xy \in EG$ with $\lambda''(e) = 0$ (i.e. $e \in J''$). Since $c(e) > 0$ and

$\gamma''(e) \geq 0$, e must belong to $\delta X_\phi - U_\phi$ for some $\phi \in \mathcal{F}''$; let for definiteness $x \in X_\phi$. By (8.29) or (8.30), there is a T-line P in Γ'' passing x. Replacing in P the vertex x by the sequence x, e, y, e, x we get a T-path P' for which $\lambda''(P') = \lambda''(P)$, whence e is in Γ''. Next, obviously, $\gamma''(e) = 0$ if $e \in Z$. And if $e \in B$ then $\gamma''(e) = 0$ is implied by (6.6) and (6.2)(ii). Finally, suppose that one end of e, x say, is in T. We know that no terminal belong to a blossom in $\mathcal{Q}$ (for otherwise an augmenting path in Γ would exist). Hence, $x \in L$. Since $\lambda''(e) = 0$ and $x \in T$, there is $\phi \in \mathcal{F}''$ such that $y \in X_\phi$. By Statement 8.9 (applied to y), $\pi''(y) = \pi(y) + \varepsilon > 0$. On the other hand, $\pi''(x) = 0$ (as $x \in T$) and $\lambda''(e) = 0$ show that $\pi''(y) = 0$; a contradiction. Thus, (6.5) for α'', γ'' is valid.

To show (6.8), consider a representation $B = \cup(P \in \mathcal{P})$ as in Statement 6.1 (for Γ, λ). It is easy to see that no $P \in \mathcal{P}$ can contain a triple as in (5.9) (otherwise some $P' \in \mathcal{P}$ passing x would have both ends at the same terminal). Hence, each $P \in \mathcal{P}$ is strong, so $\mathcal{P}$ consists of T-lines for Γ'', λ''. Now part "if" in Statement 6.1 applied to $B, \mathcal{P}, \lambda''$ says that B is regular with respect to λ'' (and the corresponding attachments l''). Thus, (6.8) is valid.

The final stage in the proof of Theorem 2 is the following.

Statement 8.12. *Each $\phi \in \mathcal{F}''$ satisfies (6.18) (for B, α'', γ'').*

Proof. We know that the graphs Γ_ϕ and Γ''_ϕ are the same (and similarly for Γ^*_ϕ and Γ''^*_ϕ); moreover, the sets of feasible edges in Γ_ϕ and Γ''_ϕ are the same. Consider $x \in X^*_\phi$. If $x = f_{\phi'}$ for $\phi' \in \mathcal{F}''_\phi = \mathcal{F}_\phi$ then, obviously,

(8.31) for edges e, e' in Γ^*_ϕ incident to x, if $\tau = (e, x, e')$ is an l-fork then τ is an l''-fork

(since the root $e_{\phi'}$ is not changed, and the attachments are not important in this case). Let x be ordinary in X^*_ϕ. We assert that

(8.32) if $e = xy$ is an edge in Γ such that $(*)$ $y \in X^*_\phi$, or $(**)$ $e \in B$ and $e \neq e_\phi$, then $l''(x, e) = l(x, e)$,

assuming by induction that a similar property is valid for any $\phi' \in \mathcal{F}$ with $X_{\phi'} \subset X_\phi$.

(i) If $e \in \zeta$ then $e \in \zeta''$, whence $l''(x, e) = l(x, e) = 0$.

(ii) Let either $y \in X^*_\phi$ and $\lambda(e) > 0$, or $y \notin X^*_\phi$ and $\lambda(e) > 0$, or $y \notin X^*_\phi$, $e \in B$ and $e \neq e_\phi$. Then $\lambda(e) > 0$ and $\lambda''(e) > 0$. Choose a λ''-non-broken T-line $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ containing e; let for definiteness $x = x_i$ and $y = x_{i+1}$. Put $s := x_0$, $s' := x_k$ and $Q := P(s, x)$. Then x is in one of $V_s, V_{s'}, V^\bullet$. By Statement

8.9, $\lambda''(Q) = \lambda(Q) + \varepsilon$. This implies that if $\lambda(Q)$ is equal to (less than, greater than) $p/2$, then $\lambda''(Q)$ is equal to (respectively, less than, greater than) $p'/2$. Hence, x is in V_s (respectively, $V_{s'}, V^\bullet$) if and only if x is in V''_s (respectively, $V''_{s'}, V''^\bullet$). Next, since $\lambda(e) > 0$ and $y = x_{i+1}$, $\mathrm{dist}_\lambda(s,x) < \mathrm{dist}_\lambda(s,y)$. This implies that if x is in V_s ($V_{s'}, V^\bullet$) then $l(x,e)$ is $-s$ (respectively, s', s'). Similarly, if x is in V''_s ($V''_{s'}, V''^\bullet$) then $l''(x,e)$ is $-s$ (respectively, s', s'). Hence, $l''(x,e) = l(x,e)$.

(iii) Let $e \in \beta$. Then $y \in X_{\phi'}$ and $e = e_{\phi'}$ for some $\phi' \in \mathcal{F}_\phi$. Let H and H'' be the 0-components for β and β'', respectively, that contain e. Let H_1 (H_2) be the component of $H - \{e\}$ (respectively, $H'' - \{e\}$) that contains y. Then H_1 lies in $\Gamma_{\phi'}$, and therefore, H_1 coincides with H''_1. Moreover, from (8.32) for ϕ' it follows that for any $z \in VH_1$ and $q \in \langle -t, t\rangle$ the sets $B^+_q(s)(z)$ and $B''^+_q(z)$ are the same. Since $l(x,e)$ is determined by the subtrees H' of H_1 together with the sets $B^+_q(z)$ for $z \in VH'$, and similarly for $l''(x,e)$, we conclude that $l''(x,e) = l(x,e)$.

Thus, (8.32) is true. Let v be the end of e_ϕ in X^*_ϕ. Clearly, (8.32) implies (8.31) for an ordinary x different from v. Hence, (8.31) holds for all $x \in X^*_\phi - \{v\}$. If (8.31) is true for $x := v$ (e.g., if v is non-ordinary) then the statement obviously follows. So assume that $x := v$ is ordinary, and $e = e_\phi$. If $\lambda(e) > 0$ and $\lambda''(e) > 0$, one can show that $l''(x,e) = l(x,e)$ by applying arguments as in (ii) above. Then (8.31) holds for v, and we are done.

Suppose that $s := l(x, e_\phi)$ and $s' := l''(x, e_\phi)$ are different, and that at least one of $\lambda(e_\phi), \lambda''(e_\phi)$ is zero. The latter implies that $e_\phi \in B$. By (8.32), we have,

$$B''(x) = B(x); \quad B''_s(x) = B_s(x) - \{e_\phi\}; \quad B''_{s'}(x) = B_{s'}(x) \cup \{e_\phi\}; \tag{8.33}$$
$$\text{and } B''_q(x) = B_q(x) \text{ for any } q \in \langle -t, t\rangle - \{s, s'\}.$$

This shows that if s' is not l''-tight for x then (8.31) holds. So we may assume that s' is l''-tight for x. (8.32)-(8.33) imply that, for an edge e in Γ^*_ϕ incident to x, if $\tau = (e_\phi, x, e)$ is an l-fork then τ is an l''-fork. Now the existence of paths as in (6.18) (for l'') is shown by a similar method as in the first part of the proof of Statement 7.1. •

This completes the proof of Theorems 2 and 1. • • •

9. Complexity of the algorithm

The above proof provides an algorithm for solving problem (1.2) (as well as (1.1)). It starts with $B=0$, $\alpha=0$, $\gamma=0$ and $p=p_\lambda$ (for $\lambda:=c_{\alpha,\gamma}$) and, on an iteration, either (i) it transforms B into B' with $\mu(B')=\mu(B)+2$, or (ii) it transforms the current α,γ into α'',γ'' so that $p':=p_{\lambda''}>p_\lambda$ (where $\lambda'':=c_{\alpha'',\gamma''}$, and (α'',γ'') is obtained from (α',γ') as described in the previous section). Let us say that the iteration is *positive* (*negative*) if (i) (respectively, (ii)) happens. Since $\mu(B')\le 2|EG|$, the number of positive iterations is $O(|EG|)$. Thus, to show that the algorithm is finite, and moreover, polynomial, we have to estimate the number of negative iterations going in succession (without interruption by a positive iteration). We prove that this amount is bounded by a polynomial in $|VG|+|EG|$. Hence, for an arbitrary p the algorithm finishs after executing a polynomial number of iterations (in fact, we assume that p is large; then the algorithm stops, on a negative iteration, whenever ε as in (8.26) becomes infinite). We also shall explain that an iteration can be executed in a strongly polynomial time. So it will provide that the whole algorithm is strongly polynomial.

Consider a negative iteration, using terminology and notation as in the previous section. The core of the proof that the number of consecutive negative iterations is bounded by a polynomial is the following.

Statement 9.1. *Let u be a 1-labelled edge in Γ^*. Then u is labelled in Γ''^* either in both directions or in the same direction as in Γ^*.*

Proof. Since u is labelled in Γ^*, there is an active path R in Γ^* containing u. We show that there is an active path R' in Γ''^* that passes u in the same direction as R does. We know that if an edge e is labelled in Γ^* then e is feasible with respect to α'',γ'' (i.e. e is in Γ''^* and $\gamma''(e)=0$), and that $\lambda''(e)=\lambda'(e)$. The folloving is obvious:

(9.1) if $\tau=(e,f_\phi,e')$ is a fork in Γ^* such that e,e' are labelled, and either $\phi\in\mathcal{F}^+$, or $\phi\in\mathcal{F}^-$ and $\alpha(\phi)>\varepsilon$ (i.e. $\phi\in\mathcal{F}''$) then τ is a fork in Γ''^*;

(9.2) if $\phi\in\mathcal{F}^{\text{new}}$, and $e\in\delta X_\phi-\{e_\phi\}$ is a labelled edge in Γ^*, then (e_ϕ,f_ϕ,e) is a fork in Γ''^*;

(9.3) if $\phi\in\mathcal{F}^-$, $\alpha(\phi)=\varepsilon$ (i.e. $\phi\notin\mathcal{F}''$), and $e\in\delta X_\phi-\{e_\phi\}$ is a labelled edge in Γ^*, then there is a sequence $(e_1,x_1,e_2\dots,x_k,e_k)$ such that $e_1=e_\phi$, $e_k=e$, $x_1,\dots,x_k\in X^*_\phi$, and each (e_i,x_i,e_{i+1}) is a fork in Γ''^*.

[The latter follows from (6.18) for α'',γ''. For brevity we say that a triple (e,x,e') is a fork in Γ^* (Γ''^*) if it is a fork for B,α,γ,l (respectively, for B,α'',γ'',l'').] Therefore, it suffices to prove the following.

Claim. *Let $x \in L$. Let W (W') be the set of edges in Γ incident to x and labelled as entering (respectively, leaving) x. Let $e = xy \in W$ and $e' = xz \in W'$. Then $\tau = (e, x, e')$ is a fork in Γ''^*.*

Proof. Since x does not form an elementary blossom (see (4.4)(iv)),

(9.4) there is $s \in \langle -t, t\rangle$ such that s is B, l-tight for x, $W \subseteq Z_s(x) \cup (B(x) - B_s(x))$ and $W' \subseteq (Z(x) - Z_s(x)) \cup B_s(x)$.

Consider a representation $B = \cup(P \in \mathcal{P})$ as in Statement 6.1 (for Γ). Then each $P \in \mathcal{P}$ is λ'-non-broken, and $\lambda''(h) = \lambda'(h)$ for any edge h in P. Let $\mathcal{P}'$ be the set of $P \in \mathcal{P}$ passing x. From (6.15) it follows that each $P \in \mathcal{P}'$ meets both $B^1 := B_s(x)$ and $B^2 := B(x) - B_s(x)$.

For $P \in \mathcal{P}'$ let s_P (t_P) denote the first (last) vertex in P, and let P_1 (P_2) denote the part of P from s_P to x (respectively, from x to t_P). We may assume that P meets B^1 earlier than B^2, i.e. $P_1 \cap B^1 \neq \emptyset$. Since s is tight for x, the only possible cases are (cf. (6.15)):

(9.5) (i) $s > 0$, $x \in V_s \cup V^\bullet$, and for any $P \in \mathcal{P}'$, $s_P = s$ and $t_P \neq s$;

(ii) $s' := -s > 0$, $x \in V_{s'}$, and for any $P \in \mathcal{P}'$, $t_P = s'$ and $s_P \neq s'$.

This implies that

(9.6) for any P, P', both $P_1 \cdot P_2'$ and $P_1' \cdot P_2$ are λ'-non-broken T-lines in Γ, and hence, they are λ''-non-broken T-lines.

Note also that

(9.7) for any λ''-non-broken T-line $P' = (x_0, e_1, x_1, \dots, e_k, x_k)$ with $x = x_i$, $l''(x_i, e_i) \neq l''(x_i, e_{i+1})$ unless $l''(x_i, e_i) = l''(x_i, e_{i+1}) = 0$; in particular, this is true for any $P' \in \mathcal{P}'$.

Return to e, e' as in the hypotheses of the claim.

Case 1. $e, e' \in B$. By (9.6), we may assume that some $P \in \mathcal{P}$ contains both e, e'. Suppose that τ is not a fork in Γ''^*. Then there is $q \in \langle -t, t\rangle$ such that

$$(9.8) \qquad |\{h \in B(x) \,:\, l''(x,h) = q\}| = |B(x)|/2, \quad l''(x,e) \neq q \text{ and } l''(x,e') \neq q.$$

This implies that there exists $P' \in \mathcal{P}' - \{P\}$ containing two edges $h, h' \in B(x)$ such that $l''(x,h) = l''(x,h') = q$; a contradiction with (9.7).

Case 2. $e, e' \in Z$. If $\lambda''(e) = 0$ then $l''(e) = 0$, whence τ is a fork in Γ''; and similarly for e'. So we may assume that $e, e' \notin \zeta''$. We assert that

(9.9) there is a λ''-non-broken T-line $R = (v_0, u_1, v_1, \dots, u_m, v_m)$ with $u_i = e$, $v_i = x$, $u_{i+1} = e'$.

Indeed, by (8.30), e (e') belongs to a λ'-non-broken T-line Q (respectively, D) in Γ with $\lambda''(h) = \lambda'(h)$ for all edges h in Q (respectively, D). Let $Q = Q_1 \cdot Q_2$ and $D = D_1 \cdot D_2$, where e and x (e' and x) are the last edge and vertex in Q_1(respectively, D_1). By (8.3), at least one of the following is true: (i) $R := Q_1 \cdot D_1^{-1}$ and $R' := Q_2^{-1} \cdot D_2$ are T-lines in Γ; (ii) $R := Q_1 \cdot D_2$ and $R' := D_1 \cdot Q_2$ are T-lines in Γ. Note that in both cases $\lambda''(R) + \lambda''(R') = \lambda''(Q) + \lambda''(D) = 2p'$ implies that $\lambda''(R) = \lambda''(R') = p'$. Thus, if (i) occurs, R is as required in (9.9). Suppose that (ii) occurs. Then $\lambda(Q_1) + \lambda(D_2) = \lambda(R) = p = \lambda(D) = \lambda(D_1) + \lambda(D_2)$ implies that $\lambda(Q_1) = \lambda(D_1)$; and similarly $\lambda''(Q_1) = \lambda''(D_1)$. On the other hand, $\lambda''(Q_1) = \lambda'(Q_1) = \lambda(Q_1)$ (in view of Statement 8.10 and the fact that $\rho^\varepsilon(x,e) = -\varepsilon$, by (5.5)), while $\lambda''(D_1) = \lambda'(D_1) = \lambda(D_1) + 2\varepsilon$ (since $\rho^\varepsilon(x,e') = \varepsilon$, by (5.5)); a contradiction. Hence, (ii) is impossible, and (9.9) is true.

Since $l''(x,e) \neq 0 \neq l''(x,e')$ (in view of $e, e' \notin \zeta''$), and e, e' belong to a T-line R in Γ'', we conclude that $l''(x,e) \neq l''(x,e')$ (by (9.7)). Then τ is a fork in Γ''^*.

Case 3. $e \in B$ and $e' \in Z$. Consider $P = P_1 \cdot P_2 \in \mathcal{P}'$ containing e (to be the last edge of P_1), and a path $D = D_1 \cdot D_2$ containing e', where D is as in Case 2. Let $R_1 := D_1 \cdot P_1^{-1}$ and $R_2 := D_1 \cdot P_2$. By arguments as in Case 2, for at least one $i \in \{1,2\}$, R_i is a λ''-non-broken T-line. Applying Statement 8.10 and (5.5), we get $\lambda''(D_1) = \lambda(D_1) + 2\varepsilon$ and $\lambda''(P_1) = \lambda(P_1) + 2\varepsilon$ (as $\rho^\varepsilon(x,e) = \varepsilon = \rho^\varepsilon(x,e')$), whence $\lambda''(R_1) = \lambda(R_1) + 4\varepsilon$. This shows that R_1 cannot be a T-line simultaneously in Γ and Γ''. Therefore, $i = 2$. Suppose that τ is not a fork in Γ''^*. Then $e \in B(x) - B''_q(x)$ and $e' \in Z''_q(x)$ for some $q \in \langle -t, t\rangle$ tight for x, B, l''. Let e'' be the first edge in P_2. Then $e'' \in B''_q(x)$. Hence, $l''(x,e') = l''(x,e'') = q \neq 0$; this is impossible (by (9.7)) because R_2 is a T-line in Γ''^*.

Case 4. $e \in Z$ and $e' \in B$. Consider a path $Q = Q_1 \cdot Q_2$ as in Case 2 and the path $P = P_1 \cdot P_2 \in \mathcal{P}$ containing e' (to be the first edge of P_2). Let $R_1 := Q_1 \cdot P_2$ and $R_2 := Q_1 \cdot P_1^{-1}$. Then for at least one $i \in \{1,2\}$, R_i is a λ''-non-broken T-line. By Statement 8.10 and (5.5), $\lambda''(Q_1) = \lambda(Q_1)$ and $\lambda''(P_2) = \lambda(P_2)$ (as $\rho^\varepsilon(x,e) = -\varepsilon = \rho^\varepsilon(x,e')$), whence $\lambda''(R_1) = \lambda(R_1)$. Therefore, $i = 2$. Now using arguments as in Case 3, we show that if τ is not a fork in Γ''^* then $l''(x,e) = l''(x,e'') = q \neq 0$, whence e'' is the last edge in P_1. Hence, R_2 cannot be a line in Γ''^*. •

This completes the proof of Statement 9.1. • •

Consider two concecutive iterations, i-th and $i+1$-th ones, of the algorithm. Let both iterations be negative. We denote by $\alpha_j, \gamma_j, \lambda_j, \Gamma_j, \mathcal{F}_j, \mathcal{F}_j^{\max}, \mathcal{H}^{\max}$ the corresponding objects at the beginning of j-th iteration, and $L_j, N_j, M_j, \mathcal{F}_j^+, \mathcal{F}_j^-, \varepsilon^j, \varepsilon_r^j$ ($r = 0, \ldots, 5$) the corresponding objects found on this iteration. Define

$$
\begin{aligned}
n_1 &:= |L_i|;\\
n_2 &:= \sum(|X_\phi|^2 \;:\; \phi \in \mathcal{F}_i^+ \cup \mathcal{F}_i^{\text{new}});\\
n_3 &:= \sum(|X_\phi|^2 \;:\; \phi \in \mathcal{F}_i^- \cup \mathcal{H}_i^{\max}).
\end{aligned}
$$

and define n_1', n_2', n_3' to be the corresponding numbers for $i+1$. From Statement 9.1 one can deduce that:

(i) each $x \in L_i$ is contained either in L_{i+1} or in a blossom found on $i+1$-th iteration (i.e. $x \in X_\phi$ for some $\phi \in \mathcal{F}_{i+1}^{\text{new}}$);

(ii) for each $\phi \in \mathcal{F}_i^+ \cup \mathcal{F}_i^{\text{new}}$, either $\phi \in \mathcal{F}_{i+1}^+$, or f_ϕ is contained in a blossom found on $i+1$-th iteration (i.e. $X_\phi \subset X_{\phi'}$ for some $\phi \in \mathcal{F}_{i+1}^{\text{new}}$).

This implies that

(9.10) $n_1 + n_2 \le n_1' + n_2'$, and $n_1 + n_2 = n_1' + n_2'$ holds if and only if $L_i = L_{i+1}$, $\mathcal{F}_{i+1}^+ = \mathcal{F}_i^+ \cup \mathcal{F}_i^{\text{new}}$ and $\mathcal{F}_{i+1}^{\text{new}} = \emptyset$.

Next, since $\alpha_{i+1}(\phi) \le \alpha_i(\phi)$ for any $\phi \in \mathcal{F}^- \cup \mathcal{H}^{\max}$,

(9.11) $n_3' \le n_3$, and $n_3' = n_3$ holds if and only if $\mathcal{F}_{i+1}^- = \mathcal{F}_i^-$ and $\mathcal{H}_{i+1}^{\max} = \mathcal{H}_i^{\max}$.

Thus,

$$n_1 + n_2 - n_3 \le n_1' + n_2' - n_3'. \tag{9.12}$$

We assert that this inequality must be strict. Indeed, by definition of ε^i (see (8.26)) at least one of the following events happens on i-th iteration:

(E1) for some $\phi \in \mathcal{F}^- \cup \mathcal{H}^{\max}$, $\alpha(\phi)$ becomes zero (i.e. $\alpha_i(\phi) > \alpha_{i+1}(\phi) = 0$);

(E2) for some $e \in E\Gamma$, $\lambda(e)$ becomes zero (i.e. $\lambda_i(e) > \lambda_{i+1}(e) = 0$);

(E3) for some $e \in E\Gamma$, $\gamma(e)$ becomes zero (i.e. $\gamma_i(e) > \gamma_{i+1}(e) = 0$);

(E4) some new T-line appears in the current Γ

(taking into account the Remark concerning ε_5^i in Section 8). Each of events (E1)-(E4) makes it impossible to take ε greater than the corresponding ε_r ($r \in \{0, \dots, 5\}$) stipulating this event (this is trivial for (E1)-(E3) and can be easily shown for (E4)). On the other hand, the equality in (9.12) would mean that the "configuration" (defined by $\mathcal{F}^+ \cup \mathcal{F}^{\text{new}}$, $\mathcal{F}^-$, and the set of 1-labelled edges) remains the same. This implies that we could apply to (α_i, γ_i) the transformation with $\varepsilon^i + \varepsilon^{i+1}$, instead of ε^i; a contradiction with the maximality of ε^i.

Since $|n_1' + n_2' - n_3'| \le |VG| + |VG|^2$, the number of consecutive negative iterations is $(O(|VG|^2)$, as required. (in fact, one can show that this amount is $O(|VG|)$, however, this is not important for us here).

As to the running time required to execute one iteration, the only point which needs some explanations is how to compute the value $\min\{\varepsilon_3, \overline{\varepsilon}\}$, where $\overline{\varepsilon} := \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_4\}$. We refer to [Ka1,Ka2] where a similar task is solved by a simple method in strongly polynomial time. (Roughly speaking, we first try $\varepsilon := \overline{\varepsilon}$. If $\widetilde{p} := p_{\lambda^\varepsilon}$ turns out to be less than $p + 2\varepsilon$, we search for a T-path P in G for which $\lambda^\varepsilon(P) = \widetilde{p}$, and try again with ε' ($< \varepsilon$) such that $\lambda^{\varepsilon'}(P) = p + 2\varepsilon'$, and so on. One can show that the required ε_3 will be found in a polynomial (in $|VG|$) number of applications of such a procedure.)

10. Proof of Theorem 3

Consider $\alpha, \gamma, \lambda, \Gamma, \mathcal{F}, L, N, M$ on the final iteration, i.e. when $\varepsilon = \infty$ occurs. Then $\mathcal{F}^- = \emptyset$. Fix a large positive real ε, and consider objects (denoting them with primes) arising when we transform (α, γ) into ($\alpha' := \alpha^\varepsilon$, $\gamma' := \gamma^\varepsilon$). We observe that

$$\gamma'(e) \ge \gamma(e) \text{ and } \lambda'(e) \ge \lambda(e) \text{ for any } e \in EG \tag{10.1}$$

(for if, e.g., $\gamma'(e) < \gamma(e)$ then the choice of ε to be large enough would imply that $\gamma'(e) < 0$).

Next, obviously, every T, λ'-line $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ (in $\Gamma' = \Gamma^\varepsilon$) is a non-broken T, λ-line (in Γ), and vice versa. For such a P we have $\widehat{\rho}'(x_i) = 0$ for $i = 1, \dots, k-1$, and $\lambda'(e_j) = \lambda(e_j) + \rho'(e_j)$ for $j = 1, \dots, k$ (where $\widehat{\rho}'$ and ρ' stand for $\widehat{\rho}^\varepsilon$ and ρ^ε, respectively). In view of $\rho'(x_0, e_1) = \rho'(x_k, e_k) = \varepsilon$ and $\lambda'(P) = \lambda(P) + 2\varepsilon$, this implies that there are $0 \le i < j \le k$ such that:

$$(10.2) \qquad \begin{aligned} \rho'(x_q, e_{q+1}) = -\rho'(x_q, e_q) = \varepsilon \quad & \text{for } q = 0, \dots, i; \\ \rho'(x_q, e_q) = -\rho'(x_q, e_{q+1}) = \varepsilon \quad & \text{for } q = j, \dots, k; \\ \rho'(x_q, e_q) = \rho'(x_q, e_{q+1}) = 0 \quad & \text{for } q = i+1, \dots, j-1; \end{aligned}$$

or, equivalently, i (j) is the maximum (minimum) index such that $\lambda'(P(x_0, x_i)) = \lambda(P(x_0, x_i))$ (respectively, $\lambda'(P(x_j, x_k)) = \lambda(P(x_j, x_k))$). Note that

$$(10.3) \qquad \lambda'(e) = \lambda(e) \quad \text{for each } e \in EG - E\Gamma'$$

since $\lambda'(e) > \lambda(e)$ for $e \in EG - E\Gamma'$ would imply that $e \in U_\phi$ for some $\phi \in \widehat{\mathcal{F}} := \mathcal{F}^+ \cup \mathcal{F}^{\text{new}}$, which is impossible. For $s \in T$ define W_s to be the set of $x \in V\Gamma'$ such that there is a line P' from s to x for which $\lambda'(P') = \lambda(P')$. Let Y_s be the union of W_s and the set of vertices $x \in VG$ such that x is reachable from W_s by a path having all its edges in $EG - E\Gamma'$. We assert that $\{Y_s : s \in T\}$ achives the equality in Theorem 3.

First of all, from (10.2),(10.3) and the fact that ε is large it follows that

(10.4) (i) $W_s \subseteq V'_s$;

(ii) $W_s \subset L$;

(iii) $Y_s \cap V\Gamma' = W_s$;

(iv) the sets Y_s are pairwise-disjoint;

(v) each T-line P in Γ' meets δY_s at most once, and if P meets δY_s then s is an end of P.

Let $B = \cup(P \in \mathcal{P})$, where $\mathcal{P}$ consists of $\mu(B)/2$ edge-disjoint T-lines. (10.4)(v) shows that the number ξ_s of edges $e \in \delta Y_s$ such that $e \in B$ is equal to the number of paths in $\mathcal{P}$ with one end at s. Hence,

$$(10.5) \qquad |\mathcal{P}| = \frac{1}{2} \sum (\xi_s \, : \, s \in T).$$

Now consider an edge $e = xy \notin B$ with $x \notin Y_s \ni y$. By (10)(iii),(v), $e \in E\Gamma'$, and there is a T-line $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ with $x_0 = s$ passing y, e, x in this order; let $e = e_i$ (then $x = x_i$). By (10.4)(i),(ii), $l'(y, e) = -s$ and $y \in L$, therefore, e is labelled from y to x. The case $x \in L$ is impossible (for if $x \in L$, we would have $\rho'(x, e) = -\varepsilon$ (by (5.5)), whence $\lambda'(P(x_0, x)) = \lambda(P(x_0, x))$ and $x \in W_s$). Thus, (10.6) $$e = e_\phi \quad \text{for some } \phi \in \widehat{\mathcal{F}}.$$

Let G' be the subgraph of G induced by $\overline{Y} := VG - \cup(Y_s \ : \ s \in T)$, and let K be the component of G' containing the above vertex x. To complete the proof of Theorem 3, it suffices to show that

(10.7) e is the only edge in $\delta^G(VK)$ that is not in B.

To see (10.7), consider a fragment $\phi' \in \widehat{\mathcal{F}}$ and an edge $u = vz \in \delta^G X_{\phi'} - \{e_{\phi'}\}$ with $v \in X_{\phi'}$.

(i) Let $u \in Z$. Then $u \notin U_{\phi'}$, whence $\widehat{\alpha}'(u) \le \widehat{\alpha}(u)$ (where $\widehat{\alpha}'$ stands for $\widehat{\alpha}^\varepsilon$). On the other hand, we have $0 \le \lambda'(u) = \lambda(u) + \widehat{\alpha}'(u) - \widehat{\alpha}(u)$ (as $\gamma'(u) = \gamma(u) = 0$). Hence, $\widehat{\alpha}'(u) = \widehat{\alpha}(u)$. This is possible only if $u = e_{\phi''}$ for some $\phi'' \in \widehat{\mathcal{F}}$.

(ii) Let $u \in B$. Since $\rho'(v,u) = 0$ and $\rho'(u) \ge 0$, $\rho'(z,u) \ge 0$. If $\rho'(z,u) > 0$ then $z \in V_{s'}$ for some $s' \in T$, in view of (10.2). Suppose that $\rho'(z,u) = 0$. Then $\lambda'(u) = \lambda(u)$, whence, in view of (5.3) and the fact that $u \in U_\phi$, $\gamma'(u) \ge \gamma(u)$ is possible only if $u \in \delta X_{\phi''} - U_{\phi''}$ for some $\phi'' \in \widehat{\mathcal{F}} - \{\phi'\}$. So u is the root of ϕ''.

Now return to e and ϕ as above, and consider the set $\mathcal{H}$ of $\phi' \in \widehat{\mathcal{F}}$ such that either $\phi' = \phi$ or there is a sequence $\phi_1, \ldots, \phi_k$ such that $\phi_1 = \phi'$, $\phi_k = \phi$ and $e_{\phi_i} \in \delta X_{\phi_{i+1}}$ for $i = 1, \ldots, k-1$. Let $K' := \cup(X_{\phi'} \ : \ \phi' \in \mathcal{H})$. Arguments in (i) and (ii) above easily imply that K' coincides with the set VK, whence K satisfies (10.7). •

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