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RAPPORT DE RECHERCHE

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Abstract. A T, d-join arises as a natural generalization of the notion of a T-join. Given a graph $G = (V, E)$, a subset T of its vertices, and integers $d_s \geq 0$ for $s \in T$, a T, d-*join* is a set $B \subseteq E$ such that: (i) B is the union of (the edge sets of) some pairwise edge-disjoint paths $P_1, \ldots, P_k$ in G connecting pairs of distinct elements of T, and (ii) for each $s \in T$ exactly d_s of these paths have the beginning or end at s.

We introduce some polyhedron D', described by linear inequalities, and show that $D = D' + \mathbb{R}^E_+$ is the dominant polyhedron for the set of T, d-joins. To this purpose we consider the problem of minimizing over D' a nonnegative linear objective function and prove that it is, in fact, equivalent to the minimum weight T, d-join problem.

We also give a description, via linear inequalities, of a polyhedron Q' such that $Q' + \mathbb{R}^E_+$ is the dominant polyhedron for the set of maximum multi-joins for G, T. Here by a *multi-join* we mean a set $B \subseteq E$ satisfying (i) as above, and a multi-join B is called *maximum* if the number k of paths is as large as possible.

Both results are derived from a minimax relation obtained in [9] for the parameteric minimum cost edge-disjoint T-paths problem.

Key words. Dominant Polyhedron, T-join, Edge-disjoint Paths.

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1. Introduction

Throughout we deal with an undirected graph $G = (V, E)$, a subset T of its vertices, called *terminals* in G, and a nonnegative integer-valued function d (of *demands*) on T. A *T-path* is a path in G connecting two distinct terminals. A set $B \subseteq E$ is called a *T, d-join* if it is representable in the form $B = \cup(P \in \mathcal{P})$ for some set $\mathcal{P}$ of mutually edge-disjoint T-paths such that for each $s \in T$ exactly d_s paths in $\mathcal{P}$ begin or end at s (considering a path as an edge set). Unless otherwise explicitly stated, we also assume that such a B is minimal with respect to inclusion under the above property; in particular each path in $\mathcal{P}$ is simple. Let $\mathcal{B} = \mathcal{B}_d$ denote the set of T, d-joins for G, T, d. We assume that

$$\sum(d_s \ : \ s \in T) \text{ is even}$$

(otherwise $\mathcal{B}$ is obviously empty) and that some d_s is non-zero. If $d_s = 1$ for all $s \in T$, then $|T|$ is even and we get the notion of a *T-join* [13]; such an object originally appeared in connection with the so-called Chinese postman problem [8,3].

It is well-known that the set $\mathcal{B}_1$ of T-joins admits the "dual description" as being the set of all minimal $B \subseteq E$ that meet every odd-terminus cut δX; in other words, $\mathcal{B}_1$ and the set $\mathcal{C}$ of (minimal) T-cuts form a blocking pair [6]. [For $X \subset V$, $\delta X = \delta^G X$ denotes the set of edges of G with exactly one end in X (a *cut* in G), and for $|T|$ even, δX is called a *T-cut* if $|X \cap T|$ is odd.] Moreover, Edmonds and Johnson [5] proved the theorem that for any weighting $w : E \to \mathbb{R}_+$, the minimum weight $w(B)$ of a T-join is equal to the maximum value of a w-packing of T-cuts; in other words, $\mathcal{B}_1$ has the MFMC-property (see [12] for the definition). [For $f : S \to \mathbb{R}$ and $S' \subseteq S$, $f(S')$ denotes $\sum(f(e) \ : \ e \in S')$.] In polyhedral terms, this means that the *dominant polyhedron*

$$D(\mathcal{B}_1) = \text{conv}(\mathcal{B}_1) + \mathbb{R}^E_+$$

for $\mathcal{B}_1$ is formed by the vectors $x \in \mathbb{R}^E_+$ satisfying the system of inequalities

$$(1) \qquad x(\delta X) \geq 1 \quad \text{for } \delta X \in \mathcal{C}.$$

[Here for a family $\mathcal{F} \subseteq 2^E$ of subsets of E, $\text{conv}(\mathcal{F})$ is the convex hull of the incidence vectors $\xi_F \in \mathbb{R}^E$ of sets $F \in \mathcal{F}$, and for sets $X, Y \subseteq \mathbb{R}^E$, $X + Y$ denotes their Minkowsky sum, i.e., the set of $z \in \mathbb{R}^E$ such that $z = x + y$ for some $x \in X$ and $y \in Y$.] For a survey of the above-mentioned results see [11,7].

In the present paper we give a description of the dominant polyhedron $D = D(\mathcal{B}_d)$ for arbitrary demands d (Theorem 1). Such a description turns out to be somewhat more complicated than that for $\mathcal{B}_1$. It comes from consideration of the *minimum weight T, d-join problem*: given a weighting $w : E \to \mathbb{Z}_+$, find a T, d-join B of weight $w(B)$

minimum, and applying to the latter a minimax relation for the parameteric minimum cost edge-disjoint T-paths problem obtained in [9].

From the result in [9] we also derive a description of the dominant polyhedron Q for the set $\mathcal{B}^{\max}$ of maximum multi-joins for G, T (Theorem 2). Let $\nu = \nu(G,T)$ denote the maximum cardinality of a set of pairwise edge-disjoint T-paths in G. By a *maximum multi-join* we mean a minimal set $B \subseteq E$ such that the subgraph (V, B) contains ν pairwise edge-disjoint T-paths.

Note also that the above-mentioned parameteric problem can be solved in strongly polynomial time. This provides strongly polynomial algorithms to find optimal solutions to the minimum weight T,d-join problem and the minimum weight maximum multi-join problem (under nonnegative weights).

2. Theorems

We need some terminology and notation.

Definition. A pair $\phi = (X_\phi, U_\phi)$ is called a *fragment* if $X_\phi \subseteq V$, $U_\phi \subseteq \delta X_\phi$, and the numbers $|U_\phi|$ and $d(X_\phi \cap T)$ have different parity, that is,

$$|U_\phi| - \sum (d_s \, : \, s \in X_\phi \cap T) \equiv 1 \pmod 2. \tag{2}$$

In particular, U_ϕ has odd cardinality if $X_\phi \cap T = \emptyset$; such a fragment is called *inner*. Let $\mathcal{F}$ denote the set of all fragments for G, T, d. Define the *characteristic function* of $\phi \in \mathcal{F}$ by

$$\begin{aligned} \chi_\phi(e) &:= 1 \quad \text{if } e \in U_\phi, \\ &:= -1 \quad \text{if } e \in \delta X_\phi - U_\phi, \\ &:= 0 \quad \text{for the other edges in } G. \end{aligned}$$

We prove the following theorem.

Theorem 1. $\mathrm{conv}(\mathcal{B}_d) \subseteq D' \subseteq \mathrm{conv}(\mathcal{B}_d) + \mathbb{R}^E_+$, *where* $D' = D'(G,T,d)$ *is the set of vectors* $x \in \mathbb{R}^E$ *satisfying*

(3) (i) $x_e \geq 0$ *for* $e \in E$;

(ii) $x_e \leq 1$ *for* $e \in E$;

(iii) $x(\delta X) \geq d_s - d(X \cap T - \{s\})$ *for* $s \in T$ *and* $X \subset V$ *such that* $s \in X$;

(iv) $x\chi_\phi \leq |U_\phi| - 1$ *for* $\phi \in \mathcal{F}$.

In particular, $D = D' + \mathbb{R}^E_+$.

[For $a, b : S \to \mathbb{R}$, ab denotes the inner product $\sum(a_e b_e \ : \ e \in S)$.] Note that in case $d = \mathbb{1}$ system (3)(iv) implies (1). Indeed, for $\delta X \in \mathcal{C}$ the pair $\phi = (X, \emptyset)$ forms a fragment (since $d(X \cap T) = |X \cap T|$ is odd). Then $x\chi_\phi \leq |U_\phi| - 1 = -1$ shows that $x(\delta X) \geq 1$.

Let e_P denote the pair of end vertices of a path P.

To see the inclusion $\text{conv}(\mathcal{B}_d) \subseteq D'$, we observe that the incidence vector ξ_B of any T, d-join B belongs to D'. Indeed, (3)(i),(ii) are obvious, and (3)(iii) can be easily seen by considering a representation $B = \uplus(P \in \mathcal{P})$. Fix a fragment ϕ. For $P \in \mathcal{P}$, $|P \cap \delta X_\phi|$ is odd if $|e_P \cap X_\phi| = 1$, and even otherwise. Hence,

$$|B \cap \delta X_\phi| - d(X_\phi \cap T) \equiv 0 \pmod 2 \tag{4}$$

(taking into account that $|\{P \in \mathcal{P} : s \in e_P\}| = d_s$ for any $s \in T$). Obviously, $|B \cap \delta X_\phi|$ and $\xi_B \chi_\phi$ have the same parity. Thus, $\xi_B \chi_\phi - |U_\phi| \equiv 1 \pmod 2$, by (2) and (4). Now the evident fact that $\xi \chi_\phi \leq |U_\phi|$ for any 0,1-vector ξ in $\mathbb{R}^E$ implies

$$\xi_B \chi_\phi \leq |U_\phi| - 1, \tag{5}$$

that is, (3)(iv) holds for $x = \xi_B$.

We also show the following. For $W \subseteq E$ and $v \in V$ let $W_v = W^G_v$ denote the set of edges in W incident to v.

Statement 2.1. *Let x be an integer vector in D', and let $B = \{e \in E \ : \ x_e = 1\}$. Then B contains a T, d-join $\widetilde{B}$, and $B - \widetilde{B}$ is the union of pairwise edge-disjoint circuits (considered as edge-sets).*

Proof. By (3)(i),(ii), x is a 0,1-vector. We observe that $|B_v|$ is even for each $v \in V - T$. For if $|B_v|$ is odd for some $v \in V - T$ then for the inner fragment ϕ with $X_\phi = \{v\}$ and $U_\phi = B_v$ one has $x\chi_\phi = |U_\phi|$, contradicting (3)(iv). Also considering for $s \in T$ the fragment $\phi = \{\{s\}, B_s\}$ we conclude from (3)(iv) that $|B_s|$ and d_s have the same parity.

Next, form the graph $H = (V', B')$ by adding to the graph (V, B) a new vertex s' and d_s parallel edges connecting s and s', for each $s \in T$. Let $T' = \{s' \ : \ s \in T\}$ be the set of terminals in H. By the above argument, every vertex $v \in V' - T' = V$ has an even degree in H. Furthermore, (3)(iii) and the construction of H show that $|\delta^H X| \geq d_s$ for any $s \in T$ and $X \subset V'$ such that $X \cap T' = \{s'\}$, and that this inequality holds with equality for $X = \{s'\}$. Hence,

$$\min\{|\delta^H X| \ : \ X \subset V', \ X \cap T' = \{s'\}\} = d_s \ \text{ for any } \ s' \in T'.$$

Now the statement is implied by the following theorem due to Lovász [10] and, independently, Cherkassky [1]: if a graph $G'' = (V'', E'')$ and a set $T'' \subseteq V''$ are such that the degree of every vertex in $V'' - T''$ is even, then there exists a set $\mathcal{P}''$ of edge-disjoint T''-paths in G'' such that for each $t \in T''$ the number of paths $P \in \mathcal{P}''$ with $t \in e_P$ is exactly $\min\{|\delta^{G''}X| \ : \ X \subset V'',\ X \cap T'' = \{t\}\}$ paths in $\mathcal{P}''$. •

In view of Statement 2.1, in order to prove the second inclusion in Theorem 1 it suffices to show that (i) if $\mathcal{B}_d = \emptyset$ then $D' = \emptyset$, and (ii) if $\mathcal{B}_d \neq \emptyset$ then the problem:

(6) given weights $w_e \in \mathbb{Z}_+$ of edges $e \in E$, minimize wx over all $x \in D'$,

has an integer-valued optimal solution x. Indeed, in case (i) we have $D = \emptyset = \emptyset + \mathbb{R}_+^E = D' + E_+^E$. In case (ii), varying w, we conclude that all vertices of $D' + \mathbb{R}_+^E$ are integral. Then, by Statement 2.1, these vertices must be the incident vectors of T, d-joins, whence the result follows. In particular, (6) turns out to be equivalent, in essense, to the above-mentioned minimum weight T, d-join problem. We prove (i) and (ii) in the next section.

Now we state the theorem describing the dominant polyhedron Q for the set $\mathcal{B}^{\max}$ of maximum multi-joins for G, T. A set K of pairwise disjoint subsets $Y_s \subset V$, $s \in T$, is called a *T-kernel family* if $Y_s \cap T = \{s\}$ for all $s \in T$. Let $\mathcal{K} = \mathcal{K}(G, T)$ denote the set of T-kernel families for G, T. For $e \in E$ define $\zeta_K(e)$ to be the number of occurrencies of e in the cuts δY_s, $s \in T$, that is,

$$\zeta_K = \sum(\xi_{\delta Y_s} \ : \ Y_s \in K)$$

(thus ζ_K, the ***characteristic function*** of K, takes values only 0,1 or 2).

Theorem 2. *$\mathrm{conv}(\mathcal{B}^{\max}) \subseteq Q' \subseteq \mathrm{conv}(\mathcal{B}^{\max}) + \mathbb{R}_+^E$, where Q' is the set of vectors $x \in \mathbb{R}^E$ satisfying*

(7)
(i) $x_e \geq 0, \quad e \in E;$
(ii) $x_e \leq 1, \quad e \in E;$
(iii) $x\zeta_K \geq 2\nu$ *for any* $K \in \mathcal{K};$
(iv) $x\chi_\phi \leq |U_\phi| - 1$ *for each inner fragment* ϕ.

In particular, $Q = Q' + \mathbb{R}_+^E$.

Again, it is easy to show that the characteristic vector $x = \xi_B$ of every maximum multi-join B belongs to Q', thus proving the first inclusion in the theorem (the inequality in (7)(iii) follows from the fact that each T-path P meets at least two cuts δY_s for $Y_s \in K$). Next, arguing as in the proof of Statement 2.1 and using Lovász-Cherkassky' theorem, one can see that for every 0,1-vector $x \in Q'$ there is a maximum multi-join B with $\xi_B \leq x$.

To prove the remaining parts in Theorems 1 and 2, we utilize one general result on minimum cost edge-disjoint paths, as follows. Consider a graph $G' = (V', E')$ and a set $T' \subseteq V'$. For brevity, in the sequel we refer to a set of edge-disjoint T'-paths in G' as a *packing*. Let $w : E' \to \mathbb{Z}_+$ be a weighting. For a packing $\mathcal{P}$ let $w(\mathcal{P})$ denote the total weight (or *cost*) $\sum(w(P) : P \in \mathcal{P})$ of paths in $\mathcal{P}$. The *parameteric minimum cost problem* is:

(8) given $p \in \mathbb{R}_+$, find a packing $\mathcal{P}$ that maximizes the objective function $\psi(\mathcal{P}, p) = p|\mathcal{P}| - w(\mathcal{P})$.

Clearly, if p is large enough (e.g., $p = w(E') + 1$) then (8) becomes equivalent to the problem: among all packings $\mathcal{P}$ of maximum possible cardinality $|\mathcal{P}|$, find a packing $\mathcal{P}$ whose total cost $w(\mathcal{P})$ is as small as possible. [Therefore, (8) is a generalization of the minimum weight maximum multi-join problem and, in fact, of the minimum weight T, d-join problem, due to a simple reduction as explained in Section 3.]

Let $\mathcal{F}^0$ denote the set of inner fragments ϕ for G', T' (i.e., $X_\phi \cap T' = \emptyset$). For functions $\beta' : \mathcal{F}^0 \to \mathbb{R}_+$ and $\gamma' : E' \to \mathbb{R}_+$, define the *amortized cost function* $w^{\beta',\gamma'}$ on E to be

$$w^{\beta',\gamma'} = w + \gamma' + \sum(\beta'_\phi \chi_\phi \ : \ \phi \in \mathcal{F}^0) \tag{9}$$

(here χ_ϕ concerns E'). We say that (β', γ') is *p-admissible* if:

(10) $w^{\beta',\gamma'}$ is nonnegative;

(11) $\mathrm{dist}_{w^{\beta',\gamma'}}(s', t') \geq p$ for all distinct $s', t' \in T'$,

where $\mathrm{dist}_\ell(u, v)$ is the distance between vertices u and v in G' with length ℓ of edges, that is, the minimum length $\ell(P)$ of a path connecting u and v (the distances in (11) are well-defined because of (10)).

Theorem 3 [9]. *For any $p \geq 0$,*

$$\max\{\psi(\mathcal{P}, p)\} = \min\{\gamma'(E') + \sum(\beta'_\phi(|U_\phi| - 1) \ : \ \phi \in \mathcal{F}^0)\}, \tag{12}$$

where the maximum ranges over all packings $\mathcal{P}$ and the minimum ranges over all p-admissible (β', γ').

We shall also use the *optimality criterion* for problem (8): a packing $\mathcal{P}$ and p-admissible (β', γ') achieve the equality in (12) if and only if the following "complementary slackness" conditions hold:

(13) $w^{\beta',\gamma'}(P) = p$ for each $P \in \mathcal{P}$;

(14) for $e \in E'$, $\gamma'_e > 0$ implies that e is *covered* by $\mathcal{P}$, that is, e belongs to some $P \in \mathcal{P}$;

(15) for $\phi \in \mathcal{F}^0$, $\beta'_\phi > 0$ implies $\sum_{P\in\mathcal{P}} \chi_\phi(P) = |U_\phi| - 1$.

This criterion can be seen by considering, for arbitrary a packing $\mathcal{P}$ and a p-addmissible (β', γ'), the following expression:

$$\psi(\mathcal{P}, p) = \sum_{P\in\mathcal{P}} (p - w(P))$$

$$\leq \sum_{P\in\mathcal{P}} (\gamma'(P) + \xi_P \sum_{\phi\in\mathcal{F}^0} \beta'_\phi \chi_\phi) \qquad \text{(by (9))}$$

$$\leq \gamma'(E') + \sum_{\phi\in\mathcal{F}^0} \beta'_\phi \chi_\phi \sum_{P\in\mathcal{P}} \xi_P \qquad \text{(as the paths in } \mathcal{P} \text{ are edge-disjoint)}$$

$$\leq \gamma'(E') + \sum_{\phi\in\mathcal{F}^0} \beta'_\phi(|U_\phi| - 1) \qquad \text{(by (5))} \; .$$

3. Proof of Theorem 1

For $s \in T$ let $\mathcal{X}_s$ denote the collection of pairs (s, X) such that $X \subseteq V$ and $s \in X \cap T$, and let $\mathcal{X} = \cup(\mathcal{X}_s \, : \, s \in T)$. Assign a dual variable γ_e to $e \in E$ in (3)(ii), $\alpha_{s,X}$ to (s, X) in (3)(iii), and β_ϕ to ϕ in (3)(iv). Given $\alpha : \mathcal{X} \to \mathbb{R}$ and $\beta : \mathcal{F} \to \mathbb{R}$, for $e \in E$ define

(16) $$\widehat{\alpha}_e = \sum(\alpha_{s,X} \, : \, (s, X) \in \mathcal{X}, \, e \in \delta X), \qquad \widehat{\beta}_e = \sum(\beta_\phi \chi_\phi(e) \, : \, \phi \in \mathcal{F}),$$
$$\text{and} \qquad \ell_e = w_e + \gamma_e + \widehat{\beta}_e.$$

Then the linear program dual to (6) is:

(17) maximize

$$\Omega(\alpha, \beta, \gamma) = -\gamma(E) + \sum_{(s,X)\in\mathcal{X}} (d_s - d(X \cap T - \{s\}))\alpha_{s,X} - \sum_{\phi\in\mathcal{F}} (|U_\phi| - 1)\beta_\phi$$

subject to

$$\begin{aligned}&\text{(i)} \quad \alpha \geq 0, \ \beta \geq 0 \ \gamma \geq 0;\\&\text{(ii)} \quad -\gamma_e + \widehat{\alpha}_e - \widehat{\beta}_e \leq w_e, \quad e \in E.\end{aligned}$$

Suppose that the set $\mathcal{B}_d$ of T, d-joins is nonempty. Our goal is to find $B \in \mathcal{B}_d$ and α, β, γ satisfying (17)(i),(ii) so that the following relations hold:

(18)
$$\begin{aligned}&\text{(i)} \quad \alpha_{s,X} > 0 \text{ implies } |B \cap \delta X| = d_s - d(X \cap T - \{s\});\\&\text{(ii)} \quad \beta_\phi > 0 \text{ implies } \chi_\phi(B) = |U_\phi| - 1;\\&\text{(iii)} \quad \gamma_e > 0 \text{ implies } e \in B;\\&\text{(iv)} \quad \widehat{\alpha}_e < \ell_e \text{ implies } e \notin B.\end{aligned}$$

One can see that (18) gives the complementary slackness conditions for $x = \xi_B$ and (α, β, γ), whence x is an integer optimal solution to (6), and we are done.

To find the desired objects, we form the graph $G' = (V', E')$ by adding to G a new vertex s' and d_s parallel edges connecting s and s', for each $s \in T$. Let $T' = \{s' : s \in T\}$, and extend w by zero to the edges in $E' - E$. Clearly, each T', d-join in G' contains $E' - E$, and for $B \subseteq E$, the mapping $B \to B \cup (E' - E)$ yields a one-to-one correspondence between the set of T, d-joins in G and the set of T', d-joins in G'. By the above supposition, the set of T', d-joins in G' is nonempty.

Let $\mathcal{P}, \gamma', \beta'$ achieve the equality in (12) with a rather large p. Then $|\mathcal{P}|$ is as large as possible. Since $|\mathcal{P}|$ does not exceed $|E' - E|/2 = d(T)/2$, $B' = \cup(P \in \mathcal{P})$ is a T', d-join in G' and $B = B'|_E$ is a T, d-join in G. This B is just the desired T, d-join.

Next we explain how to obtain β and γ from β' and γ'. Consider a fragment $\phi \in \mathcal{F}^0$ with $\beta'_\phi > 0$. Note that (15) is equivalent to the fact that there is a unique element $u \in \delta^{G'} X_\phi$ such that

(19) either $u \in U_\phi$ and $\delta^{G'} X_\phi \cap B' = U_\phi - \{u\}$, or $u \notin U_\phi$ and $\delta^{G'} X_\phi \cap B' = U_\phi \cup \{u\}$.

Let $\mathcal{F}_1$ be the set of $\phi \in \mathcal{F}^0$ such that for each $s \in T \cap X_\phi$ (if any) all edges e connecting s and s' belong to U_ϕ (note that such an e is, obviously, in $\delta^{G'} X_\phi \cap B'$). From (19) one can see that for $\phi \in \mathcal{F}_1$, the pair $(X_\phi, U_\phi \cap E)$ forms a fragment, ϕ' say, for G, T, d; moreover, $\chi_{\phi'}(B) = |U_{\phi'}| - 1$. We denote ϕ' by $\sigma(\phi)$. The desired β is defined by

$$\begin{aligned}\beta_{\phi'} &= \beta'_\phi \quad \text{for } \phi' = \sigma(\phi), \ \phi \in \mathcal{F}_1,\\&= 0 \quad \text{for the other fragments in } \mathcal{F}.\end{aligned}$$

Then β and B satisfy (18)(ii). On the other hand, (19) shows that for each $\phi \in \mathcal{F}^0 - \mathcal{F}_1$, the set $U_\phi \cap E$ coincides with $B \cap \delta X_\phi$. Based on this property, we can "destroy" such fragments, accordingly increasing the function γ on E. More precisely, we define γ on E by

$$\begin{aligned}\gamma_e &= \gamma'_e + \sum(\beta'_\phi : \phi \in \mathcal{F}^0 - \mathcal{F}_1,\ e \in \delta X_\phi) \quad \text{for each } e \in B,\\ &= \gamma'_e \quad \text{otherwise.}\end{aligned}$$

By the above property, for $\phi \in \mathcal{F}^0 - \mathcal{F}_1$ and $e \in E$, $\chi_\phi(e) \geq 0$ if $e \in B$ and $\chi_\phi(e) \leq 0$ if $e \in E - B$. This and (14) imply that $\gamma_e > 0$ only if $e \in B$, and therefore (18)(iii) is true. Furthermore, we have

$$\begin{aligned}(20)\qquad \gamma'_e + \sum(\beta'_\phi \chi_\phi(e) : \phi \in \mathcal{F}^0) &= \widehat{\beta}_e + \gamma_e \quad \text{for } e \in B,\\ &\leq \widehat{\beta}_e + \gamma_e \quad \text{for } e \in E - B,\end{aligned}$$

defining $\widehat{\beta}$ as in (16) for our β. Let ℓ' stand for $w^{\beta',\gamma'}$ (see (9)). By (20),

$$(21)\qquad \ell_e = \ell'_e \ \text{ for } e \in B \quad \text{and} \quad \ell_e \geq \ell'_e \ \text{ for } e \in E - B.$$

It remains to define α. Consider $s' \in T'$. Introduce the set $Z_{s'}$ to be $\{v \in V' : \mathrm{dist}_{\ell'}(s', v) \leq p/2\}$ (recall that ℓ' is nonnegative, by (10)). From (11) it follows that if v is a common element for $Z_{s'}$ and $Z_{t'}$ with $s' \neq t'$ then $\mathrm{dist}_{\ell'}(s', v) = \mathrm{dist}_{\ell'}(t', v) = p/2$. Let $0 = \pi_0 < \pi_1 < \ldots < \pi_k = p/2$ be the sequence of all different values among $\mathrm{dist}_{\ell'}(s', v)$ ($v \in Z_{s'}$) and $p/2$. Define

$$(22)\qquad X^i = \{v \in Z_{s'} - \{s'\} : \mathrm{dist}_{\ell'}(s', v) < \pi_i\}, \quad i = 1, \ldots, k.$$

Now the desired α on $\mathcal{X}_s$ is defined by

$$\begin{aligned}(23)\qquad \alpha_{s,X} &= \pi_i - \pi_{i-1} \qquad \text{for } X = X^i \neq \emptyset,\ i = 1, \ldots, k,\\ &= 0 \qquad \text{for the other } (s, X)\text{'s in } \mathcal{X}_s.\end{aligned}$$

Note that the construction of G' shows that $\mathrm{dist}_{\ell'}(s', v) = \mathrm{dist}_{\ell'}(s', s) + \mathrm{dist}_{\ell'}(s, v)$ for any $v \in Z_{s'} - \{s'\}$, where s is the vertex in T corresponding to s'. This implies that if $X^i \neq \emptyset$ then $s \in X^i$, so α is well-defined.

Claim. *Let $P = (s' = v_0, e_1, v_1, \ldots, e_r, v_r = t')$ be a path in G' connecting distinct terminals $s', t' \in T'$ such that $\ell'(P) = p$. Then:*

(i) *$v_1 = s$ and $v_{r-1} = t$;*

(ii) *for every $(s, X) \in \mathcal{X}$ with $\alpha_{s,X} > 0$, P intersects $\delta^G X$ at most once;*

(iii) *for $i = 2, \ldots, r-1$, $\widehat{\alpha}_{e_i} = \ell'_{e_i}$.*

Proof. (i) is obvious. To show (ii) and (iii), we observe that P is a shortest path for ℓ' (by (11)). Hence,

(24) for $i = 1, \dots, r$,

$$\ell'_{e_i} = \mathrm{dist}_{\ell'}(s', v_i) - \mathrm{dist}_{\ell'}(s', v_{i-1}) = \mathrm{dist}_{\ell'}(t', v_{i-1}) - \mathrm{dist}_{\ell'}(t', v_i).$$

From (24) and the definition of $\alpha_{q,X}$ for $q = s, t$ (see (23)) one can conclude that: (a) for $(q, X) \in \mathcal{X}_s \cup \mathcal{X}_t$ with $\alpha_{q,X} > 0$, δX contains exactly one edge among $e_2, \dots, e_{r-1}$, and (b) for $i = 2, \dots, r-1$, $\ell'_{e_i} = \sum(\alpha_{q,X} \,:\, (q, X) \in \mathcal{X}_s \cup \mathcal{X}_t,\ e_i \in \delta X)$.

Next, for $q' \in T' - \{s', t'\}$ and $i = 0, \dots, r$, one has $\mathrm{dist}_{\ell'}(q', v_i) \ge p/2$. For otherwise for some $z \in \{s', t'\}$ we would have $\mathrm{dist}_{\ell'}(z, q') \le \mathrm{dist}_{\ell'}(z, v_i) + \mathrm{dist}_{\ell'}(q', v_i) < p$ because $\min\{\mathrm{dist}_{\ell'}(s', v_i), \mathrm{dist}_{\ell'}(t', v_i)\} \le p/2$). Hence, none of $X \subseteq V$ with $\alpha_{q,X} > 0$ meets P.

These arguments prove (ii) and (iii). •

In view of (13), part (ii) in Claim easily implies (18)(i), and part (iii) together with (21) shows that $\widehat{\alpha}_e = \ell_e$ holds for all $e \in B$. Next, if $e \in E - B$ belongs to a T'-path of l'-length p in G' then $\widehat{\alpha}_e \le \ell_e$, by (21) and (iii) in Claim. Finally, arguing as above, one can deduce that $\widehat{\alpha}_e \le \ell'_e$ for any $e \in E$ that belongs to no T'-path of l'-length p, and therefore $\widehat{\alpha}_e \le \ell_e$. Thus, (18)(iv) is true.

This completes the proof for case $\mathcal{B}_d \ne \emptyset$.

Now suppose that $\mathcal{B}_d = \emptyset$. We first show that the polyhedron $D'(G', T', d)$ is empty, where $G' = (V', E')$ and T' are formed as above from G, T, d, and then, using this property, we show that $D'(G, T, d)$ is empty as well. Let w be the all-unit function on E'. We shall show that the objective function $\Omega(\alpha', \beta', \gamma')$ in (17) is unbounded (considering G', T' instead of G, T). By the linear duality theorem, this fact will imply that $D'(G', T', d) = \emptyset$.

Let ν be the maximum cardinality of a packing for G', T'. Choose a rather large $p \in \mathbb{R}_+$ to ensure that $|\mathcal{P}| = \nu$ for an optimal solution to $\mathcal{P}$ to (8). For $s' \in T'$ let $\mu_{s'}$ be the number of paths $P \in \mathcal{P}$ with $s' \in e_P$; then $\mu_{s'} \le |E'_{s'}| = d_s$. Moreover, since $\mathcal{B}_d = \emptyset$, G' has no T', d-join, whence

$$2\nu = \sum(\mu_{s'} \,:\, s' \in T') \le d(T) - 2. \tag{25}$$

Let (β', γ') be p-admissible and achieve the minimum in (12). Then

$$p\nu \ge \psi(\mathcal{P}, p) = \gamma'(E') + \sum(\beta'_\phi(|U_\phi| - 1) \,:\, \phi \in \mathcal{F}^0). \tag{26}$$

The desired function α' on $\mathcal{X}'$ is defined similarly to α in the proof for case $\mathcal{B}_d \ne \emptyset$ (here $\mathcal{X}' = \cup(\mathcal{X}'_{s'} \,:\, s' \in T')$, and $\mathcal{X}'_{s'}$ is the set of pairs (s', X) with $X \subseteq V'$ and

$s' \in X \cap T'$). Namely, for $s' \in T'$ define X^i to be $\{v \in Z_{s'} : \mathrm{dist}_{\ell'}(s', v) < \pi_i\}$ (cf. (22)) and then define α' on $\mathcal{X}'_{s'}$ as in (23) (with s' instead of s). Repeating arguments as in the proof of the Claim, we observe that

$$(27) \qquad \widehat{\alpha}'_e \leq \ell'_e \quad (= w_e + \gamma'_e + \widehat{\beta}'_e) \qquad \text{for each } e \in E'.$$

Thus, $(\alpha', \beta', \gamma')$ is a feasible solution to (17) for G', T', d (assuming that β' is extended by zero on the fragments not in $\mathcal{F}^0$). Next, (23) (for α') shows that for each $s' \in T'$,

$$(28) \quad \sum(\alpha'_{s',X} : (s', X) \in \mathcal{X}'_{s'}) = p/2, \qquad \text{and } \alpha'_{s',X} > 0 \text{ implies } X \cap T' = \{s'\}.$$

Now, putting (25),(26) and (28) together, we have

$$\Omega(\alpha', \beta', \gamma') = \sum_{(s',X)\in\mathcal{X}'} (d_s - d(X \cap T' - \{s'\})\alpha'_{s',X} - \Big(\gamma'(E') + \sum_{\phi\in\mathcal{F}^0} \beta'_\phi(|U_\phi| - 1)\Big)$$
$$\geq \sum_{s\in T} d_s p/2 - p\nu = p(d(T)/2 - \nu) \geq p.$$

Since p can be chosen arbitrarily large, Ω is unbounded. Thus, $D'(G', T', d)$ is empty.

It remains to prove that $D'(G, T, d)$ is empty. Suppose that this is not so. Let x be a vector in $D'(G, T, d)$. Define $x' \in \mathbb{R}^{E'}$ by

$$x'_e = x_e \quad \text{for } e \in E, \qquad = 1 \quad \text{for } e \in E' - E.$$

We show that $x' \in D'(G', T', d)$, thus coming to a contradiction with the fact that $D'(G', T', d)$ is empty. The inequalities in (3)(i),(ii) (for E') are obvious. Consider $(s', X) \in \mathcal{X}'$. If $s \notin X$ then $\delta^{G'}X$ contains all d_s edges connecting s' and s, whence $x'(\delta^{G'}X) \geq d_s \geq d_s - \sum(d_t : t' \in X \cap T' - \{s'\})$. And if $s \in X$, consider $Y = X - T'$. Let Z be the set of $t \in T - \{s\}$ such that $t \in X \not\ni t'$. Then:

$$x'(\delta^{G'}X) \geq x(\delta^G Y) + \sum(|E'_{t'}| : t \in Z) = x(\delta^G Y) + d(Z)$$
$$\geq x(\delta^G Y) + d(Y \cap T - \{s\}) - \sum(d_t : t' \in X \cap T' - \{s'\}).$$

Now the inequality as in (3)(iii) for x' and (s', X) follows from (3)(iii) for x and (s, Y).

Finally, consider a fragment ϕ' for G', T', d. Let Z be the set of $s \in T$ such that $|X_{\phi'} \cap \{s, s'\}| = 1$; then $E'_{s'} \subseteq \delta^{G'}X_{\phi'}$ for all $s \in Z$. Suppose that there is $s \in Z$ such that some $e \in E'_{s'}$ is not in $U_{\phi'}$. Then $x'\chi_{\phi'} \leq x'(U_{\phi'}) - x'_e \leq |U_{\phi'}| - 1$. Thus, we may assume that $E'_{s'} \subseteq U_{\phi'}$ for all $s \in Z$. Consider the pair $\phi = (Y, W)$, where $Y = X_{\phi'} - T'$ and $W = U_{\phi'} - \cup(E'_{s'} \ : \ s \in Z)$. Then W is a subset of $\delta^G Y$. Moreover, obviously, $|U_{\phi'}| - \sum(d_s \ : \ s' \in X_{\phi'} \cap T')$ has the same parity as that of $|W| - d(Y \cap T)$. Hence, ϕ is a fragment for G, T, d. We have

$$x'\chi_{\phi'} = x\chi_\phi + x'(\cup(E'_{s'} \ : \ s \in Z)) \leq (|W| - 1) + |U_{\phi'} - W| = |U'_\phi| - 1.$$

Thus, (3)(iv) is true for x' and any fragment for G', T', d.

This completes the proof of Theorem 1.

4. Proof of Theorem 2

We show that the problem:

(29) given $w : E \to \mathbb{Z}_+$, minimize wx subject to $x \in Q'$,

has an integer optimal solution x; in other words, (29) is, in fact, equivalent to the minimum weight maximum multi-join problem:

(30) minimize $w(B)$ over all $B \in \mathcal{B}^{\max}$.

Assign a dual variable γ_e to $e \in E$ in (7)(ii), τ_K to $K \in \mathcal{K}$ in (7)(iii), and β_ϕ to $\phi \in \mathcal{F}^0$ (where $\mathcal{F}^0$ is the set of inner fragments for G, T). Then the program dual to (29) is

(31) maximize $\displaystyle -\gamma(E) + 2\nu \sum_{K \in \mathcal{K}} \tau_K - \sum_{\phi \in \mathcal{F}^0} (|U_\phi| - 1)\beta_\phi$ subject to

(i) $\beta \geq 0, \ \ \gamma \geq 0, \ \ \tau \geq 0;$

(ii) $-\gamma_e + \widehat{\tau}_e - \widehat{\beta}_e \leq w_e, \quad e \in E,$

where $\widehat{\beta}$ is defined in (16), and

$$\widehat{\tau} = \sum(\tau_K \zeta_K \ : \ K \in \mathcal{K}).$$

We may assume that $\nu > 0$; else $Q' \subseteq \mathrm{conv}(\mathcal{B}^{\max}) + \mathbb{R}^E_+$ is obviously true since $\emptyset$ is a maximum multi-join (whence $\mathrm{conv}(\mathcal{B}^{\max}) = \{0\}$). We have to show that there exist $B \in \mathcal{B}^{\max}$ and β, γ, τ satisfying (31)(i),(ii) so that the following (complementary slackness) conditions hold:

(32) (i) $\beta_\phi > 0$ implies $\chi_\phi(B) = |U_\phi| - 1$;
(ii) $\gamma_e > 0$ implies $e \in B$;
(iii) $\tau_K > 0$ implies $2\nu = \zeta_K(B)$ $(= \sum(|B \cap \delta Y_s| \,:\, Y_s \in K)$;
(iv) $\widehat{\tau}_e < \ell_e$ implies $e \notin B$;

where $\ell_e = w_e + \gamma_e + \widehat{\beta}_e$ (see (16). Consider $\mathcal{P}, \beta, \gamma$ that achieve the equality in (12) for a rather large p (using notation without primes). Then $|\mathcal{P}| = \nu$, $B = \cup(P \in \mathcal{P})$ is a maximum multi-join, and B, β, γ satisfy (32)(i),(ii) (by (14),(15)).

Now the desired τ is determined in a way close to that of determining α in Section 3. More precisely, letting $Z_s = \{v \in V \,:\, \mathrm{dist}_\ell(s,v) \leq p/2\}$ for $s \in T$, form the sequence $0 = \pi_0 < \pi_1 < \ldots < \pi_k = p/2$ of all different values among $\mathrm{dist}_\ell(s,v)$ ($s \in T$, $v \in Z_s$) and $p/2$. For $i = 1, \ldots, k$ define $K^i = \{Y^i_s \,:\, s \in T\}$ by

$$Y^i_s = \{v \in V \,:\, \mathrm{dist}_\ell(s,v) < \pi_i\}.$$

Obviously, $s \in Y^i_s$, and for any distinct $s,t \in T$ the sets Y^i_s and Y^i_t are disjoint; so K^i is a T-kernel family. Now putting $\tau_{K^i} = \pi_i - \pi_{i-1}$ for $i = 1, \ldots, k$, and $\tau_K = 0$ for the other T-kernel families, we get τ that satisfies (31)(ii) and (32)(iii),(iv). Indeed, arguing as in the proof of the Claim from the previous section, we observe that, for a fixed i, every path $P \in \mathcal{P}$ from s to t traverses only cuts δY^i_s and δY^i_t, each being traversed at exactly one edge. Hence, $|B \cap Y^i_s| = |\{P \in \mathcal{P} \,:\, s \in e_P\}|$ for each $s \in T$, whence (32)(iii) follows. Also we observe that $\widehat{\tau}_e = \ell_e$ if e belongs to a T-path P with $\ell(P) = p$, and $\widehat{\tau}_e \leq \ell_e$ otherwise; whence (31)(ii) and (32)(iv) follow.

5. Open problems

Theorem 1 shows the integrality of every vertex of the polyhedron $D' = D'(G,T,d)$ that remains a vertex in $D' + \mathbb{R}^E_+$. These vertices are exactly the incidence vectors of T,d-joins. An open question: is it true that *all* vertices of D' are integral? Clearly the answer is affirmative if and only if problem (6) has an integer optimal solution for any objective function $w : E \to \mathbb{R}$ (then the set of vertices of D' is exactly the set of incidence vectors of $B \subseteq E$ such that B contains a T,d-join B' and the graph $(V, B - B')$ is eulerian, by Statement 2.1).

Note that Theorem 3 (as well as the algorithm in [9]) concerns only a nonnegative w, and nothing is at present known for the case of an arbitrary w, which is crucial for studying the vertices of D'. The minimum (arbitrary) weight T, d-join problem with $d \neq \mathbb{1}$ seems to be more sophisticated than the same problem for T-joins. It is well-known (see [11]) that if B is a T-join and B' is a T'-join then $B \triangle B'$ contains a $T \triangle T'$-join, and that this property provides a simple reduction of the minimum (arbitrary) weight T-join problem to its nonnegative version ($S \triangle S'$ denotes the symmetric difference $(S - S') \cup (S' - S)$). A similar property does not remain, in general, true for an arbitrary d; there is a simple example with $T = \{s_1, s_2\}$, $d_{s_1} = d_{s_2} = 2$ and $T' = \{s_3, s_4\}$ that shows that for some T, d-join B and T'-join B', the set $B \triangle B'$ contains no T'', d''-join, where $T'' = \{s_1, \ldots, s_4\}$, $d_{s_i} = 2$ for $i = 1, 2$ and $d_{s_i} = 1$ for $i = 3, 4$.

A similar question concerning the integrality of the vertices of the polyhedron Q' (related to the maximum multi-joins) is also open.

Another interesting open problem is to describe the dominant polyhedra D and Q via systems of linear inequalities rather than the Minkowsky sums as above. Can it be done explicitly? (To compare: the perfect matching polytop of a graph has a "good" description via inequalities [4], but arguments in [2] make it unlikely that such a description exists for the corresponding dominant polyhedron.)

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