

**MULTIFLOWS AND DISJOINT PATHS OF
MINIMUM TOTAL COST**

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Multiflows and Disjoint Paths of Minimum Total Cost

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Abstract. In this paper we survey some earlier and recent results in the field of combinatorial optimization and network flow theory that concern problems on minimum cost maximum value multiflows (multicommodity flows) and minimum cost maximum packings of edge-disjoint paths.

We deal with an undirected network N consisting of a supply graph G , a commodity graph H and nonnegative integer-valued functions of capacities and costs of edges of G , and consider the problems of minimizing the total cost among (i) all maximum multiflows, and (ii) all maximum integer multiflows.

We discuss the denominators behavior in optimal solutions to problem (i), in terms of the commodity graph. The main result here is that if H is complete (i.e. partial flows between any two terminals are allowed) then (i) has a *half-integral* optimal solution. Moreover, there are polynomial algorithms to find such a solution.

The main theorem concerning (ii) gives an explicit combinatorial minimax relation in case of H complete. This is a far generalization of a minimax relation obtained by Mader and, independently, Lomonosov for maximum number of edge-disjoint paths connecting arbitrary pairs among prescribed vertices. Also there exists a polynomial algorithm when the capacities are all-unit.

The minimax relation for (ii) with a complete H enables us to describe the dominant polyhedra for the sets of so-called T, d -joins (extending the notion of a T -join) and multi-joins of a graph. Also other results are reviewed.

We finish the paper with considering an analog of (ii) for openly disjoint paths and posing open problems.

Keywords. Multicommodity Flow, Disjoint Paths, Minimum Cost, Dominant Polyhedron

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1. Definitions, problems, results

Suppose that we are given vertices $s_1, \dots, s_k, t_1, \dots, t_k$ in a graph G , and we wish to find pairwise edge-disjoint paths P_1, \dots, P_r such that: (i) each P_i connects s_j and t_j for some j ; (ii) the number r of paths is as large as possible; and (iii) the sum of length of these paths is as small as possible, subject to (i),(ii). When can this problem be efficiently solved? It is known, due to [10], that the problem is, in general, NP-hard for $k = 2$ even if we drop condition (iii). On the other hand, it turns out that the desired paths can be found in polynomial time if the pairs $\{s_1, t_1\}, \dots, \{s_k, t_k\}$ form (the edge-set of) a complete graph. The latter result follows from some of the theorems and algorithms on minimum cost multicommodity flows and edge-disjoint paths that we survey in this paper.

We start with some definitions and conventions. Throughout, unless otherwise is explicitly stated, by a *graph* we mean an undirected graph without multiple edges and loops; VG and EG denote the vertex-set and edge-set of a graph G . An edge with end vertices u and v is denoted by uv .

We deal with a network $N = (G, H, c, a)$ consisting of a *supply graph* G , a *commodity graph* H with $VH \subseteq VG$, a *capacity function* $c : EG \rightarrow \mathbb{Z}_+$ and a *cost function* $a : EG \rightarrow \mathbb{Z}_+$ (\mathbb{Z}_+ is the set of nonnegative integers). The edges of H indicate the pairs of vertices of G that are allowed to connect by flows.

From the combinatorial viewpoint, it is more convenient to think of multicommodity flows as functions on certain paths. Let $\mathcal{P} = \mathcal{P}(G, H)$ be the set of simple paths in G connecting vertices s and t with $st \in EH$. Then a (c -admissible) multicommodity flow, or, briefly, a *multiflow*, is a nonnegative rational-valued function $f : \mathcal{P} \rightarrow \mathbb{Q}_+$ satisfying the capacity constraint

$$(1.1) \quad \zeta^f(e) := \sum (f_P : e \in P \in \mathcal{P}) \leq c(e) \text{ for all } e \in EG$$

(hereinafter we often consider a path as an edge-set). The sum of f_P 's over all $P \in \mathcal{P}$ is called the *total value* of f and denoted by $\text{val}(f)$. We denote by $\nu^* = \nu^*(G, H, c)$ the maximum total value of a multiflow f , and f is called *maximum* if $\text{val}(f) = \nu^*$. Similarly, considering the set of (c -admissible) integer-valued multiflows $f : \mathcal{P} \rightarrow \mathbb{Z}_+$, we define a *maximum integer multiflow* and the number $\nu = \nu(G, H, c)$. Clearly $\nu \leq \nu^*$.

We also associate with a multiflow f its *total cost* a_f that is $\sum (a_e \zeta^f(e) : e \in EG)$, or $\sum (a(P) f_P : P \in \mathcal{P})$, where $a(P)$ is the cost of P . (For a function $g : S \rightarrow \mathbb{R}$ and a subset $S' \subseteq S$, $g(S')$ stands for $\sum (g_e : e \in S')$.)

In this paper we discuss two problems:

(1.2) Find a maximum multiflow f with a_f as small as possible;

(1.3) Find a maximum integer multiflow f with a_f as small as possible.

Thus, (1.3) is the integer strengthening of (1.2), while (1.2) is the fractional relaxation of (1.3). We will refer to (1.2) ((1.3)) as the *fractional* (resp. *integer*) problem. We may assume that H has no isolated (i.e., zero degree) vertices; VH is called the set of *terminals* of the network and denoted by T . A path in G connecting two distinct terminals is called a T -*path*.

When $a = \mathbf{0}$, we obtain the pure maximum and maximum integer multiflow problems. When $c = \mathbb{I}$ and $a = \mathbb{I}$, (1.3) turns into the problem on edge-disjoint paths of minimum total length that we mentioned above. The examples below recall well-known facts and reveal possible behavior of problems (1.2) and (1.3) with respect to commodity graph H . Here and later on for $X \subseteq VG$, $\delta(X) = \delta^G(X)$ denotes the set of edges of G with exactly one end in X , called the *cut* in G induced by X . We say that $\delta(X)$ *separates* vertices u and v (or sets $Y, Z \subset VG$) if one of them is (entirely) contained in X and the other in $VG - X$. If $c(\delta(\{v\}))$ is even for each $v \in VG - T$, c is called *inner Eulerian*.

Example 1. EH consists of a unique element st . Then (1.2) is the undirected minimum cost maximum (single-commodity) flow problem, and by classical theorems in network flow theory (see, e.g., [12]), ν^* equals the minimum capacity $c(\delta(X))$ of a cut $\delta(X)$ separating s and t , and (1.2) has an integer optimal solution (o.s.); in particular, $\nu = \nu^*$. Moreover, there are many polynomial and strongly polynomial algorithms to solve (1.3) (see [1,16] for a survey).

Example 2. $T = \{s, t, s', t'\}$ and $EH = \{st, s't'\}$. In case $a = \mathbf{0}$, (1.2) turns into the (undirected) maximum two-commodity flow problem, and it has a *half-integral* o.s. [18] (or an integer o.s. in the inner Eulerian case [33]). However, we shall see later that in the general case of a , one cannot guarantee that (1.2) has an o.s. with bounded denominators. In its turn, (1.3) is strongly NP-hard, as it is NP-hard for $a = \mathbf{0}$ and $c = \mathbb{I}$ [10].

Example 3. H is the complete graph K_T with vertex-set T , and $|T| \geq 3$. In other words, flows connecting any two distinct terminals are allowed. We refer to a multiflow for G, K_T, c as a T -*multiflow*. Lovász [28] and, independently, Cherkassky [4] established two results on T -multiflows. First, $2\nu^*$ is equal to the sum over $s \in T$ of the minimum capacities of cuts separating s and $T - \{s\}$ (this minimax relation was originally stated in [26]). Second, if c is inner Eulerian then there exists a maximum T -multiflow that is integer-valued (and therefore, a half-integer maximum T -multiflow

for arbitrary integral capacities). Also there are strongly polynomial algorithms to find such a multiflow [4,19,13] (in [19] this is reduced to solving $\log |T|$ maximum flow problems). The maximum integer T -multiflow problem turned out to be much more complicated. An outstanding result, due to Mader [30] and, independently, Lomonosov [27], is that there is a minimax relation involving ν , which can be written as

$$(1.4) \quad \nu = \frac{1}{2} \min \left\{ \sum_{s \in T} c(\delta(Y_s)) - \eta \right\},$$

where the minimum is taken over the collections $\{Y_s : s \in T\}$ of pairwise disjoint sets $Y_s \subset VG$ with $Y_s \cap T = \{s\}$, and η is the number of components K occurring when the Y_s 's are removed from G and such that $c(\delta^G(VK))$ is odd.

The case $H = K_T$ will be most important in this paper.

We now outline results on problems (1.2) and (1.3).

1. A natural question arises: what is the smallest natural number that is a multiple of all denominators in some optimal solution to (1.2)? It seems to be hopeless to attempt to determine such a number for every instance of problem (1.2). Nevertheless, it turned out that this can be done in terms of commodity graph H . For a fixed H , define $\varphi(H)$, the *fractionality of problem (1.2) with H* , or, briefly, the *fractionality of H* , to be the minimum natural number k such that for any network (G, H, c, a) problem (1.2) has an optimal solution f for which kf is integer-valued. If such a k does not exist, we say that H has *unbounded fractionality*, denoting this as $\varphi(H) = \infty$.

For example, $\varphi(H) = 1$ if $|EH| = 1$. More generally, $\varphi(H) = 1$ for any complete bipartite graph H , by the multi-terminal version of the min-cost max-flow problem [12]. On the other hand, it is easy to show that $\varphi(H) \geq 2$ for all other graphs H . The next result is less trivial: if $H = K_T$ then (1.2) has a *half-integral* o.s. [20]; hence, $\varphi(K_T) = 2$ if $|T| \geq 3$. This fact was proved by considering the following slightly more general *parameteric problem* which combines both objectives figured in (1.2):

$$(1.5) \quad \text{given } p \in \mathbb{Q}_+, \text{ maximize the linear objective function } p \text{val}(f) - a_f \text{ among all multiflows } f \text{ for } G, K_T, c.$$

Obviously, (1.5) becomes equivalent to (1.2) (with $T = K_T$) when p is large enough (one shows that $p = 2a(EG)c(EG) + 1$ is sufficient). The above-mentioned result is an immediate corollary from the following theorem.

Theorem 1 [20]. *If $H = K_T$ then for any $p \in \mathbb{Q}_+$ problem (1.5) has a half-integral optimal solution f .*

As a consequence, we also conclude that $\varphi(H) = 2$ for any complete multi-partite

graph H with $k \geq 3$ parts (recall that H is *complete multi-partite* if there is a partition $\{T_1, \dots, T_k\}$ of T such that $\{s, t\} \in EH$ if and only if $s \in T_i$ and $t \in T_j$ for $i \neq j$). For we can add to G new vertices t_1, \dots, t_k and edges $t_i s$ ($s \in T_i$) with rather large capacities and the same rather large costs; then any optimal solution for the resulting network with the complete graph on $\{t_1, \dots, t_k\}$ as commodity graph yields in a natural way an optimal solution for the original network.

The complete multi-partite H 's exhibit just all cases when the fractionality is bounded.

Theorem 2 [21]. *If H is not complete multi-partite then $\varphi(H) = \infty$.*

This theorem is reduced to examination of few instances of H , in view of the following fact.

Statement 1.1. *If H' is an induced subgraph of H then $\varphi(H') \leq \varphi(H)$.*

Proof. Given a network $N' = (G', H', c', a')$, form graph G by adding to G' the elements $s \in VH - VH'$ as isolated vertices, obtaining network $N = (G, H, c, a)$. Then N and N' have the same sets of optimal solutions, and the result follows. •

There are exactly three minimal, under taking induced subgraphs, graphs that are not complete multi-partite, namely, H_1, H_2, H_3 drawn in Fig. 1. By Statement 1.1, Theorem 2 follows from the fact that $\varphi(H_i) = \infty$, $i = 1, 2, 3$. We explain why the fractionality of these H_i 's is unbounded in Section 3.

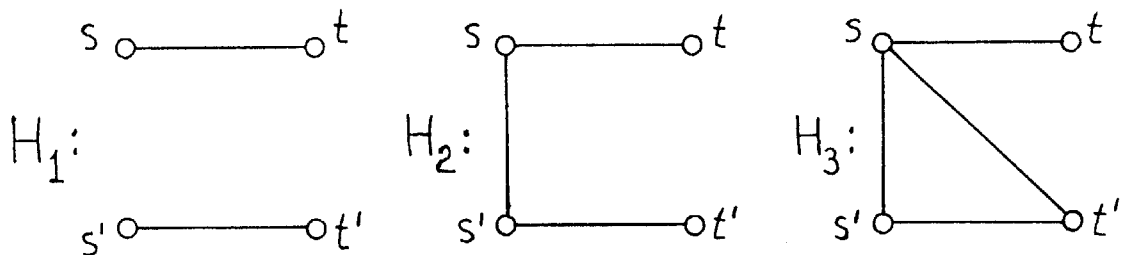


Fig.1

2. The program dual to (1.5) can be written as

$$(1.6) \quad \begin{array}{l} \text{minimize } c\gamma \quad \text{subject to} \\ \gamma \in \mathbb{Q}_+^{EG} \text{ and } \text{dist}_{a+\gamma}(s, t) \geq p \text{ for all } s, t \in T, s \neq t, \end{array}$$

where for $\ell : EG \rightarrow \mathbb{Q}_+$, $\text{dist}_\ell(u, v)$ denotes the ℓ -distance between vertices u and v , i.e., the minimum ℓ -length $\ell(P)$ of a path P in G that connects u and v .

Example 4. Let G be the graph shown in Fig. 2a, and let $T = \{s_1, \dots, s_6\}$, $c = \mathbb{I}$ and $a = \mathbb{I}$. There is a unique optimal T -multiflow, namely, that takes value $1/2$

on six paths of cost two as drawn in Fig. 2b, and zero on the other T -paths. Suppose $p = 7$. Then a (unique) optimal γ to (1.6) takes value zero on the edge uv and 2.5 on the other edges.

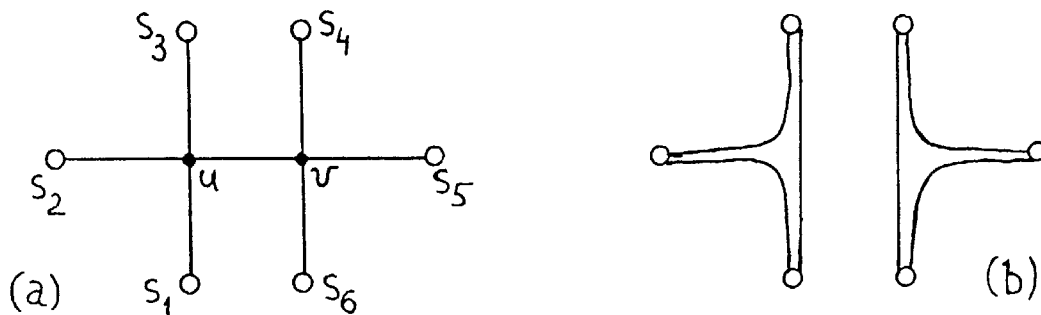


Fig. 2

The original proof of Theorem 1 given in [20] was constructive, provided by a pseudo-polynomial algorithm. Being within frameworks of the primal-dual l.p. method, this algorithm is based on the parametric approach, like that used in the classical algorithm of Ford and Fulkerson [12] for the min-cost max-flow problem, but now in more complicated context. In fact, it finds optimal primal and dual solutions simultaneously for all $p \in \mathbb{Q}_+$. More precisely, it constructs, step by step, a sequence $0 = p_0 \leq p_1 < p_2 < \dots < p_M$ of rationals, a sequence $\mathbf{0} = f_0, f_1, \dots, f_M$ of half-integral T -multiflows and a sequence $\mathbf{0} = \gamma_0, \mathbf{0} = \gamma_1, \gamma_2, \dots, \gamma_M, \gamma_{M+1}$ of functions on EG such that: (i) for $i = 0, \dots, M - 1$ and $0 \leq \varepsilon \leq 1$, f_i and $(1 - \varepsilon)\gamma_i + \varepsilon\gamma_{i+1}$ are o.s. to (1.5) and (1.6), respectively, with $p = (1 - \varepsilon)p_i + \varepsilon p_{i+1}$; and (ii) for $0 \leq \varepsilon < \infty$, f_M and $\gamma_M + \varepsilon\gamma_{M+1}$ are o.s. to these programs with $p = p_M + \varepsilon$. In particular, f_M is a maximum T -multiflow.

The crucial idea in [20] is that, at an iteration, the new f and γ can be obtained by solving the usual maximum flow problem in an auxiliary digraph, the so-called *double covering* over a certain subgraph of G . A shorter, non-algorithmic, proof of Theorem 1 was given in [22]; it is also based on double covering arguments. We outline this proof in Section 2.

Two more results were obtained in [22]. It was shown that the dual problem (1.6) has a half-integral o.s. whenever p is an integer. Also a strongly polynomial algorithm to find a half-integral o.s. to (1.2) with $H = K_T$ was developed there. However, this algorithm is not purely combinatorial as it uses (once) the ellipsoid method; we outline this in Section 2.

Recently Goldberg and the author [14] designed two polynomial algorithms for finding a half-integral o.s. to (1.2) with $H = K_T$. Both algorithms are purely combinatorial and they handle within the original graph G itself rather than the double covering. One of these applies scaling on capacities, while the other scaling on costs

(cf. [11,6] and [32,2] for the min-cost max-flow problem).

3. Apparently most significant results in the area we discuss were recently obtained for the integer problem (1.3) with $H = K_T$. W.l.o.g. we assume that the capacities are all-unit (since, for an arbitrary $c \in \mathbb{Z}_+^{EG}$, splitting each edge e into c_e parallel edges of the same cost a_e makes an equivalent problem). As before, it is convenient to deal with a parameteric problem, namely,

(1.7) given $p \in \mathbb{Q}_+$, maximize the objective function $\psi(p, \mathcal{D}) = p|\mathcal{D}| - a(\mathcal{D})$ among all sets \mathcal{D} of pairwise edge-disjoint T -paths in G ,

where $a(\mathcal{D})$ stands for $\sum(a(P) : P \in \mathcal{D})$. Note that the objective function in (1.6) gives an upper bound to $\psi(p, \mathcal{D})$, namely, $\psi(p, \mathcal{D}) \leq \gamma(EG)$ for any γ as in (1.6). Simple examples show that there can be a gap between $\max\{\psi(p, \mathcal{D})\}$ and $\min\{\gamma(EG)\}$. Nevertheless, one can modify γ in a certain way so that we get an exact upper bound. In other words, there is an explicit combinatorial minimax relation involving $\psi(p, \mathcal{D})$. To state it, we need some definitions. We refer to a set of pairwise edge-disjoint T -paths as a *packing*.

Definition. A pair $\phi = (X_\phi, U_\phi)$ is called an *inner fragment* if $X_\phi \subseteq VG - T$, $U_\phi \subseteq \delta(X_\phi)$, and $|U_\phi|$ is odd.

Let \mathcal{F}^0 denote the set of inner fragments. Define the *characteristic function* $\chi_\phi \in \mathbb{Z}^{EG}$ of ϕ by

$$(1.8) \quad \begin{aligned} \chi_\phi(e) &= 1 && \text{if } e \in U_\phi, \\ &= -1 && \text{if } e \in \delta(X_\phi) - U_\phi, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Given $\beta : \mathcal{F}^0 \rightarrow \mathbb{R}_+$ and $\gamma : EG \rightarrow \mathbb{R}_+$, define the function $\ell = \ell^{\beta, \gamma}$ on EG as

$$(1.9) \quad \ell = a + \gamma + \sum(\beta_\phi \chi_\phi : \phi \in \mathcal{F}^0)$$

We say that (β, γ) is *p-admissible* if:

$$(1.10) \quad \ell^{\beta, \gamma} \text{ is nonnegative;}$$

$$(1.11) \quad \text{dist}_{\ell^{\beta, \gamma}}(s, t) \geq p \text{ for all distinct } s, t \in T.$$

Theorem 3 [23]. For any $p \geq 0$,

$$(1.12) \quad \max\{\psi(p, \mathcal{D})\} = \min\{\gamma(EG) + \sum(\beta_\phi(|U_\phi| - 1) : \phi \in \mathcal{F}^0)\},$$

where \mathcal{D} ranges over all packings and (β, γ) ranges over all p -admissible pairs.

For instance, if G, T, c, a, p are as in Example 4 then any o.s. \mathcal{D} to (1.7) consists of three T -paths covering all edges of G . The equality in (1.12) is achieved by assigning $\beta_{\phi_1} = \beta_{\phi_2} = 1/2$, $\gamma_{uv} = 0$ and $\gamma_e = 2$ for the other six edges e of G , where ϕ_1, ϕ_2 are the inner fragments with $X_{\phi_1} = \{u\}$, $U_{\phi_1} = \{us_i : i = 1, 2, 3\}$, $X_{\phi_2} = \{v\}$ and $U_{\phi_2} = \{vs_i : i = 4, 5, 6\}$.

The inequality \leq in (1.12) is easy. Indeed, for a packing \mathcal{D} and a p -admissible (β, γ) we have:

$$\begin{aligned}
(1.13) \quad \psi(p, \mathcal{D}) &= \sum_{P \in \mathcal{D}} (p - a(P)) \\
&\leq \sum_{P \in \mathcal{D}} (\gamma(P) + \sum_{\phi \in \mathcal{F}^0} \beta_\phi \chi_\phi(P)) \quad (\text{by (1.11)}) \\
&\leq \gamma(EG) + \sum_{\phi \in \mathcal{F}^0} \beta_\phi \sum_{P \in \mathcal{D}} \chi_\phi(P) \quad (\text{as the paths in } \mathcal{D} \text{ are edge-disjoint}) \\
&\leq \gamma(EG) + \sum_{\phi \in \mathcal{F}^0} \beta_\phi (|U_\phi| - 1) \quad (\text{as } |U_\phi| \text{ is odd and } \chi_\phi(P) \text{ is an even } \leq |P \cap U_\phi|).
\end{aligned}$$

The proof of equality in (1.12) is more sophisticated and it uses numerous combinatorial arguments; a sketch of this proof is outlined in Section 4. Note that, being constructive, the proof gives rise to a strongly polynomial algorithm to solve (1.7), or a pseudo-polynomial algorithm for an arbitrary integral c . In particular, it gives a polynomial algorithm to compute $\nu(G, K_T, \Pi)$. The algorithm in [23] is also based on the parameteric approach, and it constructs a sequence $0 = p_0 \leq p_1 < p_2 < \dots < p_M$ of rationals, a sequence $\emptyset = \mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_M$ of packings and a sequence $\mathbf{0} = (\beta_0, \gamma_0), (\beta_1, \gamma_1), \dots, (\beta_{M+1}, \gamma_{M+1})$ of pairs so that M is a polynomial in $|VG|$ and: (i) for $i = 0, \dots, M-1$ and $0 \leq \varepsilon \leq 1$, \mathcal{D}_i and $((1-\varepsilon)\beta_i + \varepsilon\beta_{i+1}, (1-\varepsilon)\gamma_i + \varepsilon\gamma_{i+1})$ are optimal (i.e., they achieve the equality in (1.12)) for $p = (1-\varepsilon)p_i + \varepsilon p_{i+1}$; and (ii) for $0 \leq \varepsilon < \infty$, \mathcal{D}_M and $(\beta_M + \varepsilon\beta_{M+1}, \gamma_M + \varepsilon\gamma_{M+1})$ are optimal for $p = p_M + \varepsilon$. Furthermore, one shows that $(\beta_{M+1}, \gamma_{M+1})$ can be transformed into optimal cuts $\delta(Y_s)$ ($s \in T$) figured in (1.4).

Relation (1.12) and the algorithm can be obviously extended to a complete multi-partite H . On the other hand, the following is true.

Theorem 4. *If H is not complete multi-partite then (1.3) is NP-hard even for $c = \Pi$ and $a = \mathbf{0}$.*

To see this, it suffices to consider the minimal (under taking induced subgraphs) H 's that are not complete multi-partite, i.e., H_1, H_2, H_3 as in Fig. 1. The theorem for H_1 is just the corresponding result in [10]. This implies the theorem for H_2 and H_3 .

For, given a natural k , the problem to decide whether $k \leq \nu(G, H_1, \mathbb{I})$ is reduced to computing $\nu(G', H'_i, \mathbb{I})$ with $i = 2$ or $i = 3$, where G' is formed from G by adding new vertices t_1, \dots, t_4 and k parallel edges connecting s_j and t_j ($j = 1, \dots, 4$), and H'_i is the graph with $VH'_i = \{t_1, \dots, t_4\}$ and $EH'_i = \{t_j t_q : s_j s_q \in EH_i\}$.

4. Theorem 3 has interesting applications. Suppose we are given a graph G' , a set $T' \subseteq VG'$ and a function $d : T' \rightarrow \mathbb{Z}_+$ (of *demands*) such that $d(T')$ is even. We call a set $B \subseteq EG'$ a T', d -join if it is representable in the form $B = \cup(P \in \mathcal{D})$ for some set \mathcal{D} of mutually edge-disjoint T' -paths such that for each $s' \in T'$ there are exactly $d_{s'}$ members of \mathcal{D} beginning or ending at s' . We usually assume that B is minimal under this property and denote the set of T', d -joins by \mathcal{B}_d . We consider the *minimum weight T', d -join problem*:

$$(1.14) \quad \text{given } w : EG' \rightarrow \mathbb{Z}_+, \text{ find a } T', d\text{-join } B \text{ of weight } w(B) \text{ minimum.}$$

If $d = \mathbb{I}$ then $|T'|$ is even and we get the well-known notion of a T' -join; such an object originally appeared in connection with the Chinese postman problem [17,7]. Edmonds and Johnson [9] proved that the minimum weight $w(B)$ of a T' -join is equal to the maximum value of a (fractional) w -packing of T' -cuts ($\delta^{G'}(X)$ is called a T' -cut if $|X \cap T'|$ is odd). In polyhedral terms, this means that the *dominant polyhedron*

$$D(\mathcal{B}_1) = \text{conv}(\mathcal{B}_1) + \mathbb{R}_+^{EG'}$$

for \mathcal{B}_1 is formed by the vectors $x' \in \mathbb{R}_+^{EG'}$ satisfying $x'(\delta(X)) \geq 1$ for all T' -cuts $\delta(X)$. (Here for a family \mathcal{L} of subsets of a set E , $\text{conv}(\mathcal{L})$ is the convex hull of the incidence vectors $\xi_L \in \mathbb{R}^E$ of sets $L \in \mathcal{L}$, and for sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^E$, $\mathcal{X} + \mathcal{Y}$ denotes their Minkowsky sum $\{z : z = x + y \text{ some } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}$.) Also there are some other “nice” properties of T' -joins and T' -cuts and strongly polynomial algorithms to solve (1.14) with $d = \mathbb{I}$. For a survey, see, e.g. [29,15].

In case of arbitrary $d \in \mathbb{Z}_+^{T'}$, problem (1.14) becomes more involved. However, we can reduce it to (1.3) with $H = K_T$ and then apply results on the latter. More precisely, let G be obtained from G' by adding, for each $s' \in T'$, a new vertex s and $d_{s'}$ parallel edges ss' . Let $T = \{s : s' \in T'\}$ and $c_e = 1$ for all $e \in EG$. Assign the cost a_e to be w_e if $e \in EG'$, and 0 if $e \in EG - EG'$. Clearly, $\mathcal{B}_d \neq \emptyset$ if and only if $2\nu(G, K_T, \mathbb{I}) = d(T')$, in which case the algorithm for (1.7) yields an optimal solution to (1.14). Moreover, we explain in Section 5 that Theorem 3 enables us to derive a description of the dominant polyhedron for \mathcal{B}_d (Theorem 5 below).

Definition. A pair $\phi = (X_\phi, U_\phi)$ is called a *fragment* for G', T', d if $X_\phi \subseteq VG'$, $U_\phi \subseteq \delta^{G'}(X_\phi)$, and the numbers $|U_\phi|$ and $d(X_\phi \cap T')$ have different parity, that is,

$$(1.15) \quad |U_\phi| - \sum(d_{s'} : s' \in X_\phi \cap T') \equiv 1 \pmod{2}.$$

In particular, if $X_\phi \cap T' = \emptyset$, it turns into the above definition of an inner fragment. The characteristic function of a fragment is defined as in (1.8) (concerning G'). The set of fragments is denoted by \mathcal{F} .

Theorem 5 [3]. $\text{conv}(\mathcal{B}_d) \subseteq D' \subseteq \text{conv}(\mathcal{B}_d) + \mathbf{R}_+^{EG'}$, where D' is the set of vectors $x' \in \mathbf{R}^{EG'}$ satisfying

$$(1.16) \begin{aligned} \text{(i)} \quad & 0 \leq x'_e \leq 1 \quad \text{for } e \in EG'; \\ \text{(ii)} \quad & x'(\delta(Y)) \geq d_{s'} - d(Y \cap T' - \{s'\}) \quad \text{for } s' \in T' \text{ and } Y \subset VG' \text{ with } s' \in Y; \\ \text{(iii)} \quad & x' \chi_\phi \leq |U_\phi| - 1 \quad \text{for } \phi \in \mathcal{F}. \end{aligned}$$

In particular, $D' + \mathbf{R}_+^{EG'}$ is the dominant polyhedron for \mathcal{B}_d .

We are also able to describe, in a similar way, the dominant polyhedron of the set \mathcal{B}^{\max} of maximum multi-joins for G', T' . Here by a *maximum multi-join* we mean a minimal set $B \subseteq EG'$ such that the subgraph (VG', B) contains $\nu = \nu(G', K_{T'}, \mathbb{I})$ pairwise edge-disjoint T' -paths.

Definition. A collection $K = \{Y_{s'} : s' \in T'\}$ of pairwise disjoint sets $Y_{s'} \subset VG'$ with $Y_{s'} \cap T' = \{s'\}$ is called a T' -kernel family.

Let \mathcal{K} be the set of T' -kernel families. For $e \in EG'$ define $\zeta_K(e)$ to be the number of occurrences of e in the cuts $\delta(Y_{s'})$, $s' \in T'$ (thus $\zeta_K(e)$ is 0, 1 or 2).

Theorem 6 [3]. $\text{conv}(\mathcal{B}^{\max}) \subseteq Q' \subseteq \text{conv}(\mathcal{B}^{\max}) + \mathbf{R}_+^{EG'}$, where Q' is the set of vectors $x' \in \mathbf{R}^{EG'}$ satisfying

$$(1.17) \begin{aligned} \text{(i)} \quad & 0 \leq x'_e \leq 1 \quad \text{for } e \in EG'; \\ \text{(ii)} \quad & x' \zeta_K \geq 2\nu \quad \text{for } K \in \mathcal{K}; \\ \text{(iii)} \quad & x' \chi_\phi \leq |U_\phi| - 1 \quad \text{for each inner fragment } \phi. \end{aligned}$$

In particular, $Q' + \mathbf{R}_+^{EG'}$ is the dominant polyhedron for \mathcal{B}^{\max} .

This paper concludes with Section 6 where we describe an extension of Theorem 3 to openly disjoint T -paths and pose some open problems.

2. Proof of Theorem 1.

We now outline a proof of Theorem 1; for details, see [22]. Let f and γ be o.s. to (1.5) and (1.6), respectively. We have to show the existence of a half-integral o.s. f' to (1.5). W.l.o.g. we assume that $a_e > 0$ for all $e \in EG$ (as validity of the theorem for all positive a 's implies that for all nonnegative a 's, by obvious perturbation arguments).

Define the *length* function ℓ on EG to be $a + \gamma$; then ℓ is positive. Applying the l.p. duality theorem to (1.5)-(1.6), we observe that feasible f and γ are optimal if and only if they satisfy the (complementary slackness) conditions:

(2.1) $f_P > 0$ implies $\ell(P) = p$; in particular, P is an ℓ -shortest path;

(2.2) $\gamma_e > 0$ implies $\zeta^J(e) = c_e$ (i.e., e is *saturated* by f).

We may assume that $\min\{\text{dist}_\ell(s, t) : s, t \in T, s \neq t\}$ equals p (as if it exceeds p then $f = \mathbf{0}$ by (2.1), and we are done). Let $\mathcal{P}^0 = \mathcal{P}^0(\ell)$ be the set of T -paths of ℓ -length exactly p . Take the subgraph $G^0 = G^0(\ell)$ of G that is the union of T and all paths in \mathcal{P}^0 . Let $\text{dist}(\cdot, \cdot)$ stand for $\text{dist}_\ell(\cdot, \cdot)$.

Consider $v \in VG^0$. The *potential* $\pi(v)$ of v is the distance from v to T , i.e., $\text{dist}(v, T) = \min\{\text{dist}(v, s) : s \in T\}$. Denote by $T(v)$ the set of terminals s closest to v , i.e., $\text{dist}(v, s) = \pi(v)$. Obviously, $\pi(v) \leq p/2$ and: (i) if $\pi(v) < p/2$ then $T(v)$ consists of a unique terminal; (ii) if $\pi(v) = p/2$ then $|T(v)| \geq 2$. In case (ii), the vertex v is called *central*; let V^\bullet be the set of central vertices. Thus, VG^0 is partitioned into the sets V^\bullet and $V_s = \{v \in VG^0 : T(v) = \{s\}\}$, $s \in T$. The positivity of ℓ provides the following properties.

(2.3) Let $e = uv$ be an edge in G with $u, v \in VG^0$. Then e belongs to G^0 if and only if, up to the permutation of u and v , either (i) $u \in V_s$, $v \in V_s \cup V^\bullet$ and $\pi(v) - \pi(u) = \ell_e$ for some $s \in T$, or (ii) $u \in V_s$, $v \in V_t$ and $\pi(u) + \pi(v) + \ell_e = p$ for some distinct $s, t \in T$; in particular, no edge of G^0 connects two central vertices.

(2.4) Let $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ be a path in G^0 connecting distinct terminals $s = v_0$ and $t = v_k$. Then $P \in \mathcal{P}^0$ if and only if there is $0 \leq i < k$ such that $v_0, \dots, v_i \in V_s$; $v_{i+2}, \dots, v_k \in V_t$; $\pi(v_0) < \dots < \pi(v_i)$; $\pi(v_{i+2}) > \dots > \pi(v_k)$; and either $v_{i+1} \in V^\bullet$, or $v_{i+1} \in V_t$ and $\pi(v_{i+1}) > \pi(v_{i+2})$.

Property (2.3) enables us to construct $\Gamma = (V\Gamma, A\Gamma)$, called the *double covering digraph* over G^0 , as follows. Split each $v \in VG^0$ into $2|T(v)|$ nodes v_s^1 and v_s^2 ($s \in T(v)$). If $T(v) = \{s\}$, we may denote v_s^i as v^i . The arcs of Γ are designed as follows:

- (i) for $e = uv \in EG^0$ with $u \in V_s$, $v \in V_s \cup V^\bullet$ and $\pi(u) < \pi(v)$, split e into two arcs (u_s^1, v_s^1) and (v_s^2, u_s^2) ;
- (ii) for $e = uv \in EG^0$ with $u \in V_s$ and $v \in V_t$ ($s \neq t$), split e into two arcs (u_s^1, v_t^2) and (v_t^1, u_s^2) ;

(iii) for $v \in V^*$ make arcs (v_s^1, v_t^2) for all distinct $s, t \in T(v)$.

(See Fig. 3; here $T = \{s, t, q\}$, $p = 4$, and the number on the edges indicate values of ℓ .) Arcs in (i) and (ii) are provided with capacities c_e , and arcs in (iii) with capacity ∞ ; we use the same notation c for the capacities in Γ . We think of $T^1 = \{s^1 : s \in T\}$ as the set of *sources* and $T^2 = \{s^2 : s \in T\}$ as the set of *sinks* of Γ .

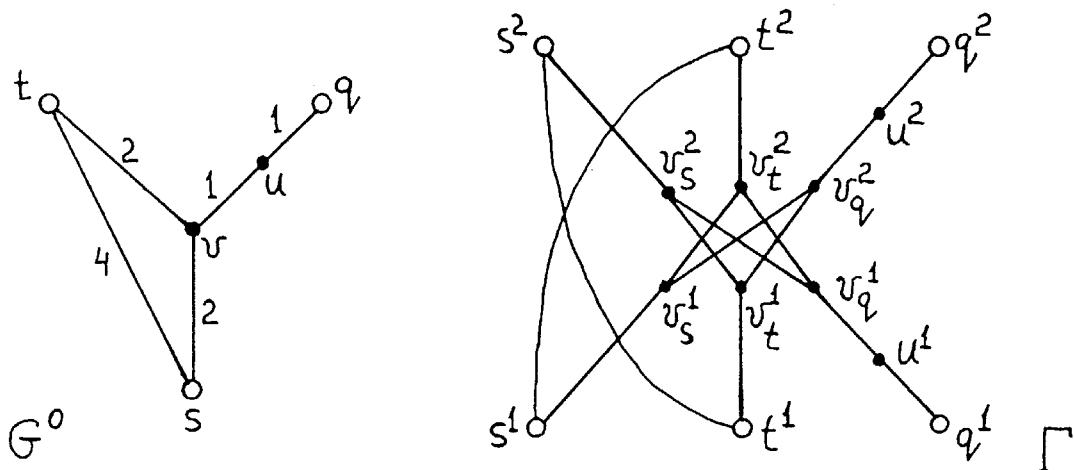


Fig. 3

Define $\sigma(v_s^i) = v_s^{3-i}$. Then for each arc $b = (u_s^i, v_t^j) \in A\Gamma$, (v_t^{3-j}, u_s^{3-i}) is also an arc in Γ , denoted as $\sigma(b)$; therefore, σ gives a (skew) *symmetry* of Γ . We extend σ to the dipaths of Γ in a natural way.

Next, the construction of Γ yields a natural mapping ω of $V\Gamma \cup A\Gamma$ to $VG^0 \cup EG^0$; it brings a node v_s^i to v , an arc (y_s^i, z_t^j) as in (i) or (ii) to the edge yz , and an arc (v_s^1, v_t^2) as in (iii) to the central vertex v . We naturally extend ω to a mapping of the dipaths of Γ into paths of G^0 . Then (2.4) and the construction of Γ imply the following key property:

- (2.5) (i) any dipath P in Γ is disjoint from $P' = \sigma(P)$, and the path $\omega(P')$ is reverse to $\omega(P)$;
- (ii) ω yields a one-to-one correspondence between \mathcal{P}^0 and the set of T^1 to T^2 dipaths in Γ .

Now we use this correspondence to get a relationship between the multiflows that are functions on \mathcal{P}^0 and flows in Γ . By a (c -admissible T^1 to T^2) *flow* in Γ we mean a function $h : A\Gamma \rightarrow \mathbb{Q}_+$ satisfying the conservation condition:

$$\operatorname{div}_h(y) := \sum_{z:(y,z) \in A\Gamma} h_{(y,z)} - \sum_{z:(z,y) \in A\Gamma} h_{(z,y)} \quad \text{for all } y \in V\Gamma - (T^1 \cup T^2),$$

and the capacity constraint $h_b \leq c_b$ for all $b \in A\Gamma$.

A routine fact is that a flow h as above can be represented as the sum of elementary flows along dipaths (note that Γ is acyclic). That is, there are T^1 to T^2 dipaths P_1, \dots, P_m and rationals $\alpha_1, \dots, \alpha_m \geq 0$ such that $\sum(\alpha_i : b \in P_i) = h_b$ for $b \in A\Gamma$; we call $\mathcal{D} = \{(P_i, \alpha_i)\}$ a *decomposition* of h . Such a \mathcal{D} induces the function $g = g^{\mathcal{D}}$ on \mathcal{P}^0 by setting $g_{\omega(P_i)} = \alpha_i/2$ for $i = 1, \dots, m$ and $g_P = 0$ for the remaining members P of \mathcal{P}^0 . We observe that for any $b \in A\Gamma$ with $e = \omega(b) \in EG^0$,

$$(2.6) \quad \zeta^g(e) = \frac{1}{2}(h_b + h_{\sigma(b)}) \leq \frac{1}{2}(c_b + c_{\sigma(b)}) = c_e,$$

hence, g is a c -admissible multiflow. Conversely, let $g : \mathcal{P}^0 \rightarrow \mathbb{Q}_+$ be c -admissible. Define the function $h = h^g$ on $A\Gamma$ so that for $b \in A\Gamma$, h_b is the sum of values $g_{\omega(P)}$ over all T^1 to T^2 dipaths P in Γ that contain b or $\sigma(b)$. Then h is a flow, and for $b \in A\Gamma$ with $e = \omega(b) \in EG^0$ we have

$$(2.7) \quad h_b = h_{\sigma(b)} = \zeta^g(e).$$

Now we are able to prove Theorem 1. Given f as above, form the flow $h = h^f$ in Γ . Let $E^+ = \{e \in EG^0 : \gamma_e > 0\}$. Then each $e \in E^+$ is saturated by f (by (2.2)), whence $h_b = c_b$ for $b \in A^+ := \omega^{-1}(E^+)$ (by (2.7)). Since all capacities in Γ are integral, there exists an *integer* c -admissible flow h' with $h'_b = h_b = c_b$ for all $b \in A^+$. Choose a decomposition $\mathcal{D} = \{(P_i, \alpha_i)\}$ of h' with all α_i 's integral. Let $f' = g^{\mathcal{D}}$. By (2.6), f' is a c -admissible function on \mathcal{P}^0 . Moreover, f' is half-integral and it saturates all edges in E^+ . Thus, f' and γ satisfy (2.1) and (2.2), so f' is the desired o.s. to (1.5).

In conclusion of this section we explain how to find a half-integral o.s. to (1.5) (and therefore, to (1.2) with $H = K_T$) in strongly polynomial time. Again, we may assume that a is positive. For if $Z = \{e : a_e = 0\} \neq \emptyset$, we can replace a by a' defined by $a'_e = 1$ for $e \in Z$ and $a'_e = (2c(Z) + 1)a_e$ otherwise; using the fact that there are half-integral o.s. for a and for a' , one can check that any half-integral o.s. for a' is an o.s. for a as well.

We first find an o.s. γ to (1.6) by use of the version [34] of the ellipsoid method [25] (it takes time polynomial in $n = |VG|$ since the size of the constraint matrix behind (1.6) is a polynomial in n). Second, we form G^0 for given p and $\ell = a + \gamma$ (by solving corresponding shortest paths problems) and construct Γ over G^0 . Third, we find an integer flow h in Γ with the restrictions $h_b = c_b$ for $b \in A^+$ (such an h must exist). Now an integer decomposition of h determines the desired half-integral multiflow for G .

3. Unbounded fractionality

As mentioned in the Introduction, to prove Theorem 2 it suffices to show that

$\varphi(H) = \infty$ for the commodity graphs $H = H_1, H_2, H_3$ as in Fig 1. Following [21], we design “bad networks” $N = (G, H, c, a)$ for these H 's.

Let k be an odd positive integer. Take k disjoint paths $(v_1^i, e_2^i, v_2^i, \dots, e_{2k}^i, v_{2k}^i)$, $i = 1, \dots, k$. Connect v_j^i and v_j^{i+1} by edge u_j^i for all i, j such that $i - j \equiv 1 \pmod{2}$. Add vertices $s, t, s', t', y, z, y', z'$ and edges

- (i) $sy, tz, s'y', t'z'$;
- (ii) yv_1^i and zv_{2k}^i for $i = 1, \dots, k$;
- (iii) $y'v_1^j$ for each odd j , and $z'v_{2k}^j$ for each even j ,

obtaining the graph G . Assign capacity $k - 1$ for edges $s'y', t'z'$, and 1 for the other edges of G . Assign the edge costs as follows:

- 0 for tz and e_{2j}^i , $i, j = 1, \dots, k$;
- 1 for all edges u_j^i and the remaining edges e_j^i ;
- k for $s'y', t'z'$ and the edges as in (ii) and (iii) ;
- $2k$ for sy .

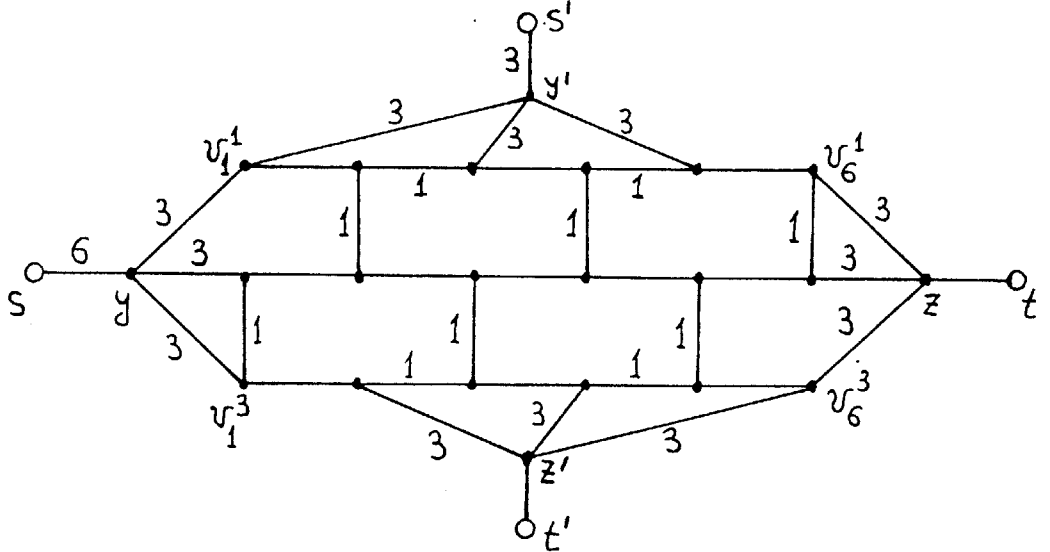


Fig. 4

(See Fig. 4 for $k = 3$; the numbers on edges indicate non-zero costs.) We identify s, t, s', t' with the corresponding vertices of the graph $H \in \{H_1, H_2, H_3\}$ in question; therefore $\{st, s't'\} \subseteq EH \subseteq \{st, s't', ss', st'\}$.

For $i = 1, \dots, k$, let $P_i (L_i)$ be the simple path going through the vertices $s, y, v_1^i, \dots, v_{2k}^i, z, t$ (respectively, $s', y', v_{2i-1}^1, v_{2i}^1, v_{2i}^2, v_{2i-1}^2, \dots, v_{2i-1}^{k-1}, v_{2i-1}^k, v_{2i}^k, z', t'$). We assign the multiflow f by $f_{P_i} = 1/k$, $f_{L_i} = (k - 1)/k$ ($i = 1, \dots, k$) and $f_Q = 0$ for the other paths Q in $\mathcal{P}(G, H)$. A straightforward examination shows that:

- (i) f satisfies the capacity constraints and $\text{val}(f) = k$, whence f is a maximum

multiflow in N (since ν^* cannot exceed the half-sum of the capacities of edges $sy, tz, s'y', t'z'$, that is k);

(ii) the cost of any path in $\mathcal{P}(G, H)$ is at least $5k - 1$, and it equals $5k - 1$ for the P_i 's and L_i 's and for only these paths in $\mathcal{P}(G, H)$;

(iii) f is a unique maximum multiflow which takes zero values outside $\{P_1, \dots, P_k, L_1, \dots, L_k\}$.

Thus, problem (1.2) for our network has a unique o.s. f , and this f has components with denominator k . Since k can be chosen arbitrarily large, $\varphi(H) = \infty$.

4. Sketch of the proof of Theorem 3

We outline the principal ideas of the proof, mostly attempting to give an impression of the approach rather than to go into particular details; for the complete proof, we refer the reader to [23]. As before, w.l.o.g. we assume that a is positive.

Given p , we say that a packing \mathcal{D} and a p -admissible (β, γ) are *good* if they achieve the equality in (1.12). Obviously, $\mathcal{D} = \emptyset$ and $(\beta, \gamma) = (\mathbf{0}, \mathbf{0})$ are good for $p = 0$. We grow p from 0 through ∞ and show the existence of good $\mathcal{D}, \beta, \gamma$ for every value of p . More precisely, by use of standard arguments, Theorem 3 is reduced to the following theorem.

Theorem 4.1. *Suppose that $\mathcal{D}, \beta, \gamma$ are good for some p . Then one of the following is true:*

- (i) *there exists a packing \mathcal{D}' with $|\mathcal{D}'| = |\mathcal{D}| + 1$ such that $\mathcal{D}', \beta, \gamma$ are good for p ;*
- (ii) *there exist $p' > p$ and (β', γ') such that for any $0 \leq \xi \leq 1$, the packing \mathcal{D} and functions $(1 - \xi)\beta + \xi\beta'$ and $(1 - \xi)\gamma + \xi\gamma'$ are good for $(1 - \xi)p + \xi p'$.*

The key idea in the proof of Theorem 4.1 is that in place of packings we can handle certain subsets of edges of G . Let $\ell = \ell^{\beta, \gamma}$ be as in (1.9).

Definition. Given a p -admissible (β, γ) , a set $B \subseteq EG$ is called *regular* if it is representable in the form $B = \cup(P : P \in \mathcal{D})$ with a packing \mathcal{D} consisting of simple T -paths P of ℓ -length $\ell(P)$ equal to p .

For $E' \subseteq EG$ and $v \in VG$ let $E'(v)$ denote the set of edges in E' incident to v . The *value* $\text{val}(B)$ of a regular B is defined as $\frac{1}{2} \sum(|B(s)| : s \in T)$; clearly, $\text{val}(B)$ equals the cardinality of a packing \mathcal{D} behind B , unless $p = 0$. We say that B, β, γ are good (for p) if $\mathcal{D}, \beta, \gamma$ are good. Considering the inequalities in (1.13), we observe that

B, β, γ are good if and only if they satisfy the (complementary slackness) conditions:

$$(4.1) \quad \gamma_e > 0 \text{ implies } e \in B;$$

$$(4.2) \quad \beta_\phi > 0 \text{ implies } \chi_\phi(B) = |U_\phi| - 1 \text{ (}\phi \text{ is saturated by } B\text{)}.$$

Let \bar{U}_ϕ denote $\delta(X_\phi) - U_\phi$. The equality in (4.2) is possible in two cases, namely,

$$(4.3) \quad \text{either (i) } B \text{ contains all but one element in } U_\phi \text{ and none in } \bar{U}_\phi, \text{ or (ii) } B \text{ contains } U_\phi \text{ and exactly one element in } \bar{U}_\phi.$$

This only element of $U_\phi - B$ in case (i), and of $B - U_\phi$ in case (ii) is called the *root* of (saturated) ϕ and denoted by r_ϕ . In the proof of Theorem 4.1 we impose some additional conditions (inductive assumptions) on B, β, γ we deal with. Let $\hat{\mathcal{F}} = \{\phi \in \mathcal{F}^0 : \beta_\phi > 0\}$. The first group consists of four conditions:

$$(A0) \quad \text{for } \phi \in \hat{\mathcal{F}}, r_\phi \text{ is in } G^0 \text{ and } \gamma_{r_\phi} = 0;$$

$$(A1) \quad \{X_\phi : \phi \in \hat{\mathcal{F}}\} \text{ is a nested family, i.e., for distinct } \phi, \phi' \in \hat{\mathcal{F}}, \text{ either } X_\phi \cap X_{\phi'} = \emptyset \text{ or } X_\phi \subset X_{\phi'} \text{ or } X_{\phi'} \subset X_\phi;$$

$$(A2) \quad \text{for } \phi_1, \phi_2 \in \hat{\mathcal{F}} \text{ with } X_{\phi_1} \subset X_{\phi_2}, \text{ the set } U_{\phi_1} \cap \delta(X_{\phi_2-i}) \text{ is contained in } U_{\phi_2-i}, \text{ } i = 1, 2 \text{ (or, equivalently, if } r_{\phi_1} \in \delta(X_{\phi_2-i}) \text{ for some } i \text{ then } r_{\phi_1} = r_{\phi_2}\text{)};$$

$$(A3) \quad \text{there are no } \phi_1, \dots, \phi_k \in \hat{\mathcal{F}} \text{ with } k > 1 \text{ such that the sets } X_{\phi_i} \text{ are pairwise disjoint, and } r_{\phi_i} \in \delta(X_{\phi_{i+1}}), i = 1, \dots, k \text{ (letting } \phi_{k+1} = \phi_1\text{)}.$$

Consider the subgraph G^0 of G that is the union of T and all (not necessarily simple) T -paths of ℓ -length p . The regularity of B implies $B \subseteq EG^0$. Let $J = \{e \in EG : \ell_e = 0\}$ (although a is positive, $\ell_e = 0$ is possible since χ_ϕ takes negative values on \bar{U}_ϕ). Define

$$B^0 = B \cap J \quad \text{and} \quad B^+ = B - J.$$

The second group of conditions concerns J , namely,

$$(A4) \quad \text{for } e = uv \in J, e \text{ is in } G^0, \gamma_e = 0, \text{ and both } u, v \text{ are not in } T;$$

$$(A5) \quad J \cap U_\phi = \emptyset \text{ for any } \phi \in \hat{\mathcal{F}}; \text{ in particular, } e \in B^0 \cap \delta(X_\phi) \text{ if and only if } e = r_\phi.$$

A component of the subgraph induced by B^0 is called a *0-component*. By (A4), the 0-components are disjoint from T . Also each 0-component is a tree. Indeed, if B^0 contains a circuit C then for any $\phi \in \mathcal{F}^0$, $|U_\phi \cap C| \geq |\bar{U}_\phi \cap C|$ (because of (4.3))

and the relations $|C \cap \delta(X_\phi)| \geq 2$ and $C \subseteq B$, whence $\chi_\phi(C) \geq 0$. Therefore, $\ell(C) - a(C) - \gamma(C) = \sum(\beta_\phi \chi_\phi(C) : \phi \in \mathcal{F}^0) \geq 0$. Since $\gamma(C) \geq 0$ and $a(C) > 0$, we have $\ell(C) > 0$; a contradiction.

The current B is transformed into a new regular set B' of bigger value by use of a certain augmenting path. To construct such a path, we first introduce the important notion of attachments and exhibit their properties. We identify T with the set of integers from 1 through $|T|$ and denote the set $\{-|T|, \dots, -1, 1, \dots, |T|\}$ by $\langle T \rangle$. Let

$$Z = EG^0 - B, \quad Z^0 = Z \cap J \quad \text{and} \quad Z^+ = Z - J.$$

For the vertices in G^0 define the potentials π and sets V^\bullet and V_s ($s \in T$) as in Section 2 with respect to our ℓ . For $v \in VG^0$ and $e = uv \in EG^0$ with $\ell_e > 0$ we assign the *attachment* $\alpha(v, e) \in \langle T \rangle$ by the following rule:

- (4.4) (i) if $v \in V_s \cup V^\bullet$, $u \in V_s$ and $\pi(u) < \pi(v)$, set $\alpha(v, e) = s$;
(ii) if $v \in V_s$ and either $u \notin V_s$, or $u \in V_s$ and $\pi(u) > \pi(v)$, set $\alpha(v, e) = -s$.

If $e = uv \in Z^0$, we assign for (v, e) the special attachment $\alpha(v, e) = 0$. To assign attachments for edges in B^0 is more sophisticated. Obviously, both ends of $e \in J$ (and therefore, the vertices of a 0-component) have the same potentials and belong to the same set among the V_s 's and V^\bullet . For a subgraph Q of G^0 let $B(Q)$ ($B^+(Q)$) denote the set of edges in B (resp. B^+) with exactly one end in Q . For $s \in \langle T \rangle$ define $B_s^+(Q)$ to be the set of edges $e = uv \in B^+(Q)$ with $v \in VQ$ and $\alpha(v, e) = s$. It will be convenient to think of a vertex $v \in VG^0 - T$ with $B^0(v) = \emptyset$ as a (trivial) 0-component. The next lemma easily follows from (2.4) if $B^0 = \emptyset$; in general case, the part "only if" is also easy (using the fact that when shrinking the edges in J we get the case as in Section 2), while the part "if" is slightly more involved and it is proved by induction on $|B|$. We say that $E' \subseteq EG$ is *inner Eulerian* if $|E'(v)|$ is even for each $v \in VG - T$.

Lemma 4.2. $B \subseteq EG^0$ is regular if and only if B is inner Eulerian and

$$(4.5) \quad |B_s^+(Q')| \leq |B(Q')|/2 \quad \text{for any 0-component } Q, \text{ subtree } Q' \subseteq Q \text{ and } s \in \langle T \rangle.$$

If the inequality in (4.5) holds with equality, we say that s is *tight* for Q' . E.g., if $v \in V_s$ is a trivial 0-component, then $\{B_s^+(v), B_{-s}^+(v)\}$ gives a partition of $B(v)$, and s and $-s$ are tight for v . Moreover, one can see that:

- (4.6) (i) if s is tight for Q' then for any $e \in EQ'$, s is tight for exactly one of two components of $Q' - \{e\}$;
(ii) for a 0-component Q and $e = uv \in EQ$, there is at most one $s \in \langle T \rangle$ such that s is tight for some subtree $Q' \subseteq Q$ containing u but not v .

Property (4.6)(ii) enables us to assign attachments to the edges in B^0 :

- (4.7) for a 0-component Q and $e = uv \in EQ$, set $\alpha(v, e) = s$ if $s \in \langle T \rangle$ is tight for some subtree $Q' \subseteq Q$ with $v \notin VQ' \ni u$, and $\alpha(v, e) = 0$ otherwise.

For $v \in VG^0$ let $E(v)$ stand for $EG^0(v)$. For $s \in \langle T \rangle \cup \{0\}$ define

$$E_s(v) = \{e \in E(v) : \alpha(v, e) = s\}, \quad B_s(v) = B \cap E_s(v) \quad \text{and} \quad Z_s(v) = Z \cap E_s(v).$$

Using (2.4) and (4.6)(i), one can check that the resulting attachments satisfy:

- (4.8) (i) for $e = uv \in VG^0$, $\alpha(v, e) \neq \alpha(u, e)$ unless $\alpha(v, e) = \alpha(u, e) = 0$;
(ii) $|B_s(v)| \leq |B(v)|/2$ for any $v \in VG^0 - T$ and $s \in \langle T \rangle$;
(iii) let $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ be a simple T -path with all edges in B ; then $B - P$ is a regular set of value $\text{val}(B) - 1$ if and only if for any consecutive $0 < i < j < k$ with non-zero $\alpha = \alpha(v_i, e_i)$ and $\alpha' = \alpha(v_{j-1}, e_j)$, one has $\alpha \neq \alpha'$.

In order to define an augmenting path we need to introduce the key notion of a fork, using the above attachments. Let \mathcal{F}^{\max} be the set of $\phi \in \widehat{\mathcal{F}}$ with X_ϕ maximal; by (A1), the sets X_ϕ , $\phi \in \mathcal{F}^{\max}$, are pairwise disjoint. For each $\phi \in \mathcal{F}^{\max}$ shrink in G^0 the subgraph $\langle X_\phi \rangle_{G^0}$ induced by X_ϕ into vertex v_ϕ , forming the graph G^* . These v_ϕ 's are called the *fragment-vertices*, whereas the other (non-shrunk) vertices in G^* are called *ordinary*; we keep the same notation for the corresponding edges in G^0 and G^* . Consider a vertex $v \in VG^* - T$ and distinct edges e, e' in G^* incident to v . We say that $\tau = (v, e, e')$ is a *fork* if

- (4.9) (i) v is ordinary and there is no $s \in \langle T \rangle$ tight for v with $e, e' \in Z_s(v) \cup (B(v) - B_s(v))$; or
(ii) v is a fragment-vertex v_ϕ and one of e, e' is r_ϕ .

We observe that (4.9)(i) means that making $B \Delta \{e, e'\}$ preserves the regularity (i.e., (4.8)(ii)) at v (Δ denotes the symmetric difference). Similarly, for $\phi \in \widehat{\mathcal{F}}$ make the graph G_ϕ from $\langle X_\phi \rangle_{G^0} \cup \{r_\phi\}$ by shrinking $\langle X_{\phi'} \rangle_{G^0}$ into vertex $v_{\phi'}$ for each $\phi' \in \mathcal{F}_\phi$, letting \mathcal{F}_ϕ be the set of $\phi' \in \widehat{\mathcal{F}}$ with maximal $X_{\phi'} \subset X_\phi$. Let X_ϕ^* denote the image of X_ϕ in G_ϕ . We define the forks in G_ϕ as in (4.9) (concerning triples in G_ϕ).

Next, one trick is used to make the desired augmenting paths non-self-intersecting on edges. Namely, we slightly modify G^0 by adding, for each $e \in J$, a parallel edge e' , called the *mate* of e ; we consider e' as an element of Z with $\ell_{e'} = 0$. Accordingly, we correct G^* and the G_ϕ 's. This slightly extends the set of forks in (4.9); e.g., for

$e = uv \in J$ and its mate e' , (v, e, e') is a fork. Note that if $r_\phi \in J$ then the mate of r_ϕ vanishes in G_ϕ . We keep notations $E(v), Z(v), E_s(v)$ and etc. for the corresponding sets in the new G^0 .

Let \tilde{E} denote $\{e \in GE^0 : \gamma_e = 0\}$, the set of *feasible* edges. A path $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ in G^* is called *active* if: (i) e_1, \dots, e_k are distinct and feasible; (ii) $v_0 \in T$, $e_1 \in Z$, $v_1, \dots, v_{k-1} \notin T$; and (iii) (v_i, e_i, e_{i+1}) is a fork, $i = 1, \dots, k-1$. Such an active P is called *minimal* if (v_i, e_i, e_{j+1}) is not a fork for any $1 \leq i < j < k$ with $v_i = v_j$ (otherwise we could cancel in P the part from v_i to v_j , getting again an active path). If, in addition, $v_k \in T$ and $e_k \in Z$, P is called *augmenting* in G^* . An active path in G_ϕ is defined by replacing (ii) by the conditions $e_1 = r_\phi$ and $v_1 \in X_\phi^*$. It is an easy exercise to show (using, e.g., (4.14) below) that if P is a minimal active path then

(4.10)(i) there are no $0 < i < j < q < k$ such that $v_i = v_j = v_q$;

(ii) each fragment-vertex can be passed by P at most once.

An augmenting path P' in G^0 is constructed from an augmenting path P in G^* as follows. If all vertices in P are ordinary then P is already the desired P' . Otherwise we repeatedly apply the *replacement procedure*, letting that the following “strong reachability” condition is imposed: for B and each G_ϕ ,

(A6) (i) each fragment-vertex $v_{\phi'}$ in X_ϕ^* is reachable by an active path with the last edge $r_{\phi'}$;

(ii) each ordinary $v \in X_\phi^*$ is reachable in G_ϕ by an active path L_v ; moreover, for each $s \in \langle T \rangle$ tight for v (if any), there is a minimal active path L_v^s to v such that $|B''_s(v)| = (|B''(v)| - 1)/2$, where $B'' = B \Delta L_v^s$ and B''_s is defined with respect to the old attachments.

(Moreover, the paths required in (A6) can be efficiently found.) We arbitrarily choose in P a fragment-vertex v_ϕ and consider the edges e, e' of P adjacent to v_ϕ (cf. (4.10)(ii)). By (4.9)(ii), one of these, e say, is r_ϕ ; let w be the end of e' in X_ϕ^* . We replace in P the vertex v_ϕ by a certain minimal active path L (without r_ϕ) in G_ϕ coming to w . If $w = v_{\phi'}$ for some $\phi' \in \mathcal{F}_\phi$, we take as L the path as in (A6)(i). If w is ordinary, we take as L the path L_w or L_w^s as in (A6)(ii) (we omit here the rule how to choose the “attachment” s). We repeat the procedure for the new path P (and a maximal fragment in $\mathcal{F}^0 - \{\phi\}$), and so on until no fragment-vertex in the current path exists. Then the resulting path is just the desired P' , and we transform B into

$$B' = B \Delta P'.$$

(If P' contains the mate e' of an original edge e but not e itself, then (A5) implies that

$e \in Z$, and we exchange the role of e' and e . And if P' contains both an original edge e and its mate e' , then e must be the root of some fragment ϕ with $e' \in \delta(X_\phi)$, whence $e \in B$; in this case e occurs in B' .)

We observe that, irrespectively of the choice of active paths as in (A6)(ii), for each $\phi \in \widehat{\mathcal{F}}$, either P' does not meet the cut $\delta(X_\phi)$ or P' intersects it twice, passing r_ϕ and another edge e . Obviously, in the latter case ϕ remains saturated ($\chi_\phi(B') = |U_\phi| - 1$), and e becomes the new root of ϕ if e is not the mate of r_ϕ , else the root preserves. Thus, (4.2) remains true. (4.1) for B' is also true since the edges of P' are feasible. It is easy to see that (A0)-(A5) continue to hold. Also B' is, obviously, inner Eulerian, and $\text{val}(B') = \text{val}(B) + 1$ (as both edges of P' meeting terminals are in Z , by the definition of an augmenting path in G^0). The next lemma is crucial to show the regularity of B' .

Lemma 4.3. *(4.5) holds for B' w.r.t. the 0-components and attachments induced by B' .*

In the simplest case when Q' is a trivial 0-component v in G^* and $\ell_e > 0$ for all $e \in B'(v)$, this obviously follows from the definitions of attachments (cf. (4.4)) and forks (cf. (4.9)(i)). If Q' is an ordinary vertex w with similar properties occurred in the above description of the replacement procedure, the statement is provided by an appropriate choice of an active path in (A6)(ii). The proof for remaining cases of Q' is more complicated. One should apply induction on the number of replacement and use (4.6)(ii) and the property (provided by (A5)) that, for $\phi \in \widehat{\mathcal{F}}$ and a 0-component Q for B , if Q meets X_ϕ^* then either Q entirely lies in $\langle X_\phi \rangle_{G^0}$ or it has a unique common edge (namely, r_ϕ) with $\delta(X_\phi)$.

Finally, one can prove (it is not straightforward) that (A6) preserves for B' and the corresponding roots and attachments. This completes the consideration of alternative (i) in Theorem 4.1.

To search an augmenting path in G^* (extended, as before, by the mates), one applies an efficient *labelling method* similar, in a sense, to that used for finding alternating paths in matching problems. During the labelling process, which grows in a certain way a digraph on feasible edges, a feasible edge $e = uv \in \widetilde{E}$ can be either *unlabelled*, or *labelled in one direction*, from u to v say, or *labelled in both directions*, from u to v and reversely; let E^{un}, E^1, E^2 denote their current sets, respectively. If $e = uv$ is labelled to v , we say that v is labelled.

The process is organized so that, at any moment, for an edge $e = uv$ labelled from u to v there is an active path containing u, e, v in this order and with all edges already labelled in forward direction. It terminates when (i) some edge $e = uv \in Z$ with $v \in T$ becomes labelled from u to v , or (ii) one can no longer label edges so that the set of labelled vertices enlarges or in the subgraph induced by E^2 the collection of vertex-sets

of components changes. In case (i), we get an augmenting path.

We now assume that the process terminates with case (ii) (but not (i)). A vertex $v \in VG^* - T$ is called *1-labelled* if it is incident to an edge in E^1 but none in E^2 ; for such a v denote by $E^{\text{in}}(v)$ ($E^{\text{out}}(v)$) the set of edges in $E^1(v)$ labelled to (resp. from) v . A component of the subgraph induced by E^2 is called a *pre-fragment*. We also introduce a special kind of pre-fragments. Namely, for an ordinary 1-labelled vertex v and an edge $e \in E^{\text{in}}(v)$, if $e \notin Z_s(v) \cup (B(v) - B_s(v))$ for each $s \in \langle T \rangle$ tight for v , then we consider the graph $(\{v\}, \emptyset)$ as an *elementary* pre-fragment (one can see that such a v is central). The following structural properties are analogs, in a sense, to ones of the so-called “Hungarian tree with blossoms” for matchings:

(4.11) each $e = uv \in Z$ with $u \in T$ is labelled from u to v but not from v to u ;

(4.12) if v is 1-labelled then each pair $\{e \in E^{\text{in}}(v), e' \in E^{\text{out}}(v)\}$ and none of the pairs in $E^{\text{in}}(v) \cup E^{\text{un}}(v)$ forms a fork;

(4.13) for each pre-fragment F all feasible edges in $\delta(VF)$ are labelled as leaving F except one, e_F say, labelled as entering F .

(To show (4.12) and (4.13), one can use the following easy corollary from (4.9):

(4.14) for $v \in VG^0$ and $e, e', e'' \in E(v)$, if neither (v, e, e') nor (v, e', e'') is a fork then (v, e, e'') is not a fork either.)

Our goal is to find β' and γ' as in alternative (ii) in Theorem 4.1. Each pre-fragment F yields the fragment $\phi = \phi(F)$ with X_ϕ that is the preimage of VF in G^0 , and:

(4.15) $U_\phi = (B \cap \delta(X_\phi)) \cup \{e_F\}$ if $e_F \in Z$, and $U_\phi = (B \cap \delta(X_\phi)) - \{e_F\}$ if $e_F \in B$

(for e_F as in (4.13)). Since B is inner Eulerian and $VF \cap T = \emptyset$, ϕ is well-defined; obviously, ϕ is saturated by B , and $e_F = r_\phi$.

Let \mathcal{F}^{new} be the set of fragments created from the pre-fragments (these fragments are to be added to $\widehat{\mathcal{F}}$ when β will change). Note that (A0), (A1) and (A3) are trivially true for $\widehat{\mathcal{F}} \cup \mathcal{F}^{\text{new}}$, while (A2) follows from the observation that if a (non-elementary) pre-fragment F would contain a fragment-vertex $v_{\phi'}$ so that e_F is incident to $v_{\phi'}$ but it is not the root of ϕ' , then (in view of (4.9)(ii)) none of the edges in $E(v_{\phi'})$ could be labelled in both directions, contradicting the definition of F .

Let \mathcal{F}^+ (\mathcal{F}^-) denote the set of $\phi \in \mathcal{F}^{\text{max}}$ such that v_ϕ is 1-labelled and $r_\phi \in E^{\text{in}}(v_\phi)$ (respectively, $r_\phi \in E^{\text{out}}(v_\phi)$). Let $\mathcal{F}' = \mathcal{F}^+ \cup \mathcal{F}^{\text{new}}$. We shall transform β into $\beta' = \beta^\varepsilon$ by increasing β by ε on \mathcal{F}' and decreasing by ε on \mathcal{F}^- , with some $\varepsilon \in \mathbb{Q}_+$ such that

