HOW TO TIDY UP A SYMMETRIC SET-SYSTEM

KARZANOV A V

Unité Associée au CNRS URA 410 : AL KHOWARIZMI

02/ 1995

Rapport de Recherche n° 948

How to tidy up a symmetric set-system

Alexander V. Karzanov

Institute for System Analysis of Russian Acad. Sci.
9, Prospect 60 Let Oktyabrya, 117312 Moscow, Russia *

Abstract. Let V be a finite set, and let $\mathcal{S} \subseteq 2^V$ be a collection of sets such that: (i) $\mathcal{S}$ is symmetric ($X \in \mathcal{S}$ implies $= V - X \in \mathcal{S}$), and (ii) for any two crossing $X, Y \in \mathcal{S}$, at least one of the pairs $\{X \cap Y, X \cup Y\}$ and $\{X - Y, Y - X\}$ is in $\mathcal{S}$. Red and Blue play the following game, starting with a symmetric family $\mathcal{F} \subseteq \mathcal{S}$. Red chooses two crossing sets X, Y in the current $\mathcal{F}$ and replace them by sets $X', Y' \in \mathcal{S}$ such that either $X = X \cap Y, Y' = X \cup Y$ or $X' = X - Y, Y' = Y - X$. Then Blue returns one of X, Y to $\mathcal{F}$. We assume that if a set Z is removed from (added to) $\mathcal{F}$ then the same is done for $V - Z$. The game terminates (and Red wins) when $\mathcal{F}$ no longer contains crossing sets.

[Hurkens at al. 1987] considers the special case when every member of $\mathcal{S}$ contains exactly one of two prescribed elements of V (in our terms). It was shown there that Red can win in time polynomial in $|V|$ and $|\mathcal{F}|$.

Extending this result, we develop an algorithm for Red to win, in polynomial time, in the above general case. Also a polynomial algorithm for a certain weighted version of the game is given. The key idea of both methods is that the whole problem can be split into a polynomial number of problems, each dealing with a cyclic family – a family of which members correspond to partitions of a cycle into two connected parts.

The results have applications in combinatorial optimization, e.g., when we deal with packing problems on certain cuts of a graph, such as T-cuts, directed cuts and etc., and we desire to transform a given optimal packing into another one which is free of crossing cuts.

Key words: Cut Packing Problem, Crossing Sets, Uncrossing.

* This research was partially supported by INTAS-93-2530 grant from European Union INTAS while the author was visiting LRI, University Paris-XI.

1. Introduction

Let V be a finite set. Two sets $X, Y \subseteq V$ are called *crossing* if none of $X - Y$, $Y - X$, $X \cap Y$ and $V - (X \cup Y)$ is empty; otherwise they are called *laminar*. If X, Y are crossing (laminar), we write $X \nparallel Y$ (respectively, $X \parallel Y$).

Suppose we are given a set-system $\mathcal{S} \subseteq 2^V$ consisting of subsets of V. If $\mathcal{S}$ has no crossing pairs, it is called *laminar*. To our purposes it does not matter whether $\mathcal{S}$ contains a set X or $\overline{X} = V - X$ or both $X, \overline{X}$. To this reason, we throughout assume that $\mathcal{S}$ (as well as any of its subsets under consideration later on) is *symmetric*, i.e., $X \in \mathcal{S}$ implies $\overline{X} \in \mathcal{S}$. Next, we assume that $\mathcal{S}$ is *cross-closed*. This means that for any crossing $X, Y \in \mathcal{S}$ there are $X', Y' \in \mathcal{S}$ such that either $X' = X - Y$ and $Y' = Y - X$, or $X' = X \cap Y$ and $Y' = X \cup Y$; we say that X', Y' are obtained by *uncrossing* X and Y, denoted as $X, Y \to X', Y'$. For example, the following four set-systems $\mathcal{S}$ are cross-closed.

(E1): $\mathcal{S} = \mathcal{S}' \cup \{\overline{X} : X \in \mathcal{S}'\}$, where $\mathcal{S}'$ is a *crossing family*, i.e., for any crossing $X, Y \in \mathcal{S}'$, the sets $X \cap Y, X \cup Y$ are also members of $\mathcal{S}'$ (cf. [Edmonds, Giles 1977]).

(E2): Given $T \subseteq V$ with $|T|$ even, $\mathcal{S}$ consists of the sets $X \subset V$ such that $|X \cap T|$ is odd. Such an $\mathcal{S}$ corresponds to the set of T-cuts of a graph $G = (V, E)$, which originally came up in connection with the Chinese postman problem [Guan 1962; Edmonds, Jonhson 1973].

(E3): Given a graph $G = (V, E)$ and a set $U \subseteq E$, $\mathcal{S}$ consists of all $X \subset V$ such that $|\delta(X) \cap U| = 1$ (cf. [Seymour 1981]).

(E4): Given a graph $G = (V, E)$ and a mapping $a : E \to \mathbb{Z}$, $\mathcal{S}$ consists of all $X \subseteq V$ such that $\sum(a(e) : e \in \delta^V(X))$ is odd (cf. [Karzanov 1984]).

[Here for a graph $G = (V, E)$ and a set $X \subseteq V$, $\delta(X) = \delta^G(X)$ is the set of edges of G with one end in X and the other in $\overline{X}$, a *cut* in G.]

We consider a game of two players, *Red* and *Blue*, as follows. It starts with a (symmetric) family $\mathcal{F} \subseteq \mathcal{S}$. At each step (move),

(1) (i) Red chooses crossing sets X, Y in the current $\mathcal{F}$ and makes uncrossing $X, Y \to X', Y'$, i.e., he replaces X, Y by X', Y' in $\mathcal{F}$; then

(ii) Blue returns one of X and Y to $\mathcal{F}$.

Multiple sets in $\mathcal{F}$ are ignored; if, say, X' has been a member of $\mathcal{F}$ before the move, it simply remains a member. We also assume that if a set Z is removed from (added to) $\mathcal{F}$ then its symmetric set $\overline{Z}$ is removed from (added to) $\mathcal{F}$ too. Note that if all $X \cap Y, X \cup Y, X - Y, Y - X$ occur in $\mathcal{S}$, Red has two possibilities to choose X' and Y' at his move, namely, $X', Y' = X \cap Y, X \cup Y$ or $X', Y' = X - Y, Y - X$. The game terminates (and Red wins) when the current $\mathcal{F}$ becomes laminar.

In fact, the goal of Red is to win as soon as possible. A priori, it is unclear whether Red can win at all. E.g., an unhappy choice of X, Y to uncross at each step may result in cycling, as this can be shown by simple examples. If the game still terminates and even if the initial $\mathcal{F}$ is small, the size of intermediate families may grow significantly during the game; so the number of steps may be large. We denote $n = |V|$ and $m = |\mathcal{F}|$ (for the initial $\mathcal{F}$) and take as a measure of *time* of the game the number of steps occurred in it, thus ignoring the real complexity of performing (i)-(ii) in (1).

An interesting special case was studied in [Hurkens, Lovász, Schrijver, Tardos 1987]. In our terms, two elements $s, t \in V$ are fixed and $\mathcal{S}$ consists of all $X \subset V$ that contain exactly one of s, t. Then for any two crossing $X, Y \in \mathcal{S}$ there is a unique choice of $X'Y'$ for uncrossing. [In fact, [Hurkens at al. 1987] deals with an arbitrary (not necessarily symmetric) family $\mathcal{F} \subseteq 2^{V'}$, Red can choose arbitrary (not necessarily crossing) $X, Y \in \mathcal{F}$ but replace them only by $X \cap Y$ and $X \cup Y$. To obtain the case as above, we add new elements s and t to V', add s to each member of $\mathcal{F}$ and then make symmetrization of $\mathcal{F}$.] It was shown that, following a simple rule, Red wins in time polynomial in n and m.

We prove the following theorem.

Theorem 1. *In general case, Red has a strategy to make $\mathcal{F}$ laminar in time polynomial in n, m.*

Moreover, the number of steps he uses is $O(n^4 m)$, as shown in Sections 2,3 where Theorem 1 is proved.

The algorithm developed for the game is applied to solve the following uncrossing problem. Suppose we are given a nonnegative integer-valued function $f : \mathcal{S} \to \mathbb{Z}_+$ that is *symmetric*, i.e., $f(X) = f(\overline{X})$. Let $\mathcal{F} = \mathcal{F}(f)$ denote the *support* $\{X \in \mathcal{S} : f(X) \neq 0\}$ of f. By the *uncrossing operation* we mean the following transformation of f (and $\mathcal{F}$):

(2) (i) choose some crossing $X, Y \in \mathcal{F}$; then

(ii) choose $X', Y' \in \mathcal{S}$ which are obtained by uncrossing X and Y; then

(iii) for $a = \min\{f(X), f(Y)\}$, decrease f by a on the sets $X, \overline{X}, Y, \overline{Y}$ and increase f by a on the sets $X', \overline{X}', Y', \overline{Y}'$.

The *uncrossing problem* is to arrange a sequence of uncrossing operations which results in a function f^* with $\mathcal{F}(f^*)$ laminar. Such a problem looks trivial as, independently of the choice of X, Y, X', Y', the process terminates in finite time (in the sense of the number of uncrossing operations we apply). Indeed, let us associate with V the complete graph $G = (V, E_V)$ with vertex-set V, and for $e \in E_V$, let $\widehat{f}(e)$ denote $\sum(f(X) : X \in \mathcal{S}, e \in \delta^G(X))$. If f' is obtained from f by the uncrossing operation (2), then it is easy to see that

(3) $\widehat{f}(e) - \widehat{f}'(e)$ is nonnegative for all $e \in E_V$, and equals $4a$ for some e.

Therefore, to solve the uncrossing problem takes time $\binom{n}{2}\|f\|/4$ for the initial f, where $\|f\| = 1 + \max\{\widehat{f}(e) : e \in E_V\}$. However, a stronger result is true.

Theorem 2. *The uncrossing problem defined by uncrossing operation (2) can be solved in time polynomial in n.m.*

Indeed, we can think of the uncrossing problem as the above game, interpreting (iii) in (2) as the move of Blue who always returns the set $Z \in \{X, Y\}$ for which $f(Z)$ remains positive, and then apply Theorem 1. Note that, to be consistent, we have to extend slightly our game by allowing Blue to return none of X, Y. Nevertheless, we shall see that such an extension does not affect our proof of Theorem 1.

The uncrossing problem comes up in combinatorial optimization when we deal with certain packing problems. E.g., the well-known T-cut packing problem is: given G, T as in (E2) and a function $w : E \to \mathbb{Z}_+$, find a function $g : \mathcal{C} \to \mathbb{R}_+$ on the set $\mathcal{C}$ of T-cuts such that $\sum(g(C) : C \in \mathcal{C})$ is maximum provided that $\sum(g(C) : e \in C \in \mathcal{C}) \leq w(e)$ for any $e \in E$. [A cut $\delta(X)$ in G is called a *T-cut* if $|T \cap X|$ is odd.] The set $\mathcal{S} = \{X : \delta(X)$ is a T-cut$\}$ is, obviously, symmetric and cross-closed. If g is an optimal solution to this problem, described by its support and values of g on it, then g can be transformed, in strongly polynomial time, into another optimal solution g' whose support is laminar (in the sense that $\{X : g'(\delta(X)) \neq 0\}$ is laminar). To do this, we solve the uncrossing problem for $\mathcal{S}$ and f, defined by $f(X) = g(\delta(X))$, using the fact that uncrossing operation (2) preserves both the value of the objective function and satisfying the above packing condition (by (3)). Similarly, one can apply our uncrossing method to problems concerning the sets in (E1),(E3),(E4).

One sort of uncrossing techniques was elaborated in [Grötschel, Lovász, Schrijver 1988], Ch. 10.3 for the dual submodular flow problem [Edmonds, Giles 1977]. It transforms, in polynomial time, an optimal solution to one with laminar support but uses uncrossing operations different from (2).

Next we consider another, weaker, kind of uncrossing operations. It is described via a game. We assume that $\mathcal{S} = 2^V$. Given $f : 2^V \to \mathbb{Z}_+$,

(4) (i) Red chooses crossing X, Y in $\mathcal{F} = \mathcal{F}(f)$; then

(ii) Blue chooses an integer b between 0 and $a = \min\{f(X), f(Y)\}$; then

(iii) f is decreased by a on $X, \overline{X}, Y, \overline{Y}$, increased by b on $X \cap Y$ and $X \cup Y$ and their complements, and increased by $a - b$ on $X - Y$ and $Y - X$ and their complements.

Red tends to minimize the total time, while Blue tends to maximize it. Clearly for an arbitrary cross-closed $\mathcal{S}$, step (ii) in uncrossing operation (2) can be realized by choosing a proper b in (4)(ii). So, to solve the uncrossing problem with (4) (in the sense of choosing a strategy for Red) takes time at least as much as that with respect to (2). Similarly to the previous case, the game always terminates in finite time (namely, $O(n^2\|f\|)$) since we observe that

(5) $\widehat{f}(e) - \widehat{f}'(e)$ is nonnegative for all $e \in E_V$ and there are $e, e' \in E_V$ such that $\widehat{f}'(e) + \widehat{f}'(e') = \widehat{f}(e) + \widehat{f}(e') - 4a$.

Theorem 3. *For game (4), Red has a strategy to make $\mathcal{F}$ laminar in time polynomial in n, m and $\log \|f\|$, where f is the initial function.*

Theorem 3 is proved in Section 4. The key idea of the proofs of Theorems 1 and 3 is that both games can be reduced to a polynomial number of games with cyclic set-systems on $n' \leq n$ elements. We say that $\mathcal{S}' \subseteq 2^W$ is cyclic if the elements of W can be numbered by $1, 2, \ldots, r$ ($r = |W|$) so that each $X \in \mathcal{S}'$ is of the form $\overline{i,j} = \{i, i+1, \ldots, j\}$ for some $1 \leq i, j \leq r$ (taking indices modulo r). The advantage of dealing with a cyclic set-system is twofold. First, the sets X', Y' obtained by uncrossing $X, Y \in \mathcal{S}'$ are obviously of similar form, therefore, any current family $\mathcal{F} \subseteq \mathcal{S}'$ that we handle remains cyclic under uncrossing. Second, $\mathcal{S}'$ consists of at most n^2 members, therefore, the size of any intermediate family in the process is polynomially bounded. We explain how both games are reduced to those on cyclic families in Section 2.

It should be mentioned that Theorems 2 and 3 were originally established in the preprint [Karzanov 1990]. Moreover, the proving method developed there for Theorem 2 is, in fact, very close to that we give for Theorem 1 in the present paper.

The proofs of Theorems 1 and 3 will provide polynomial algorithms which arrange the desired efficient uncrossing processes. Regarding computational aspects, we need to specify the input of the problem. In game (1), we may assume that $\mathcal{S}$ is given implicitly via the *membership oracle* (MO) that, being asked of a set $X \subseteq V$, returns whether or not X belongs to $\mathcal{S}$. In reasonable applications, MO is realized by a procedure polynomial in n. MO enables us to recognize efficiently which of the pairs $\{X - Y, Y - X\}$ and $\{X \cap Y, X \cup Y\}$ (or both) is contained $\mathcal{S}$, thus providing the choice of X', Y' in (1)(i). In game (4), the initial f is assumed to be given explicitly, e.g., by listing all members X of $\mathcal{F} = \mathcal{F}(f)$ and indicating $f(X)$ for these X's. Since in this paper we care only for polynomiality and do not aim to precise running time bounds, we need not come into details of how procedure (1) or (4) is performed.

In conclusion of this section we give two statements which will be used later on.

Statement 1.1. *Let $X, Y, Z \subseteq V$ be such that $X \nparallel Y$, $X \parallel Z$ and $Y \parallel Z$. Then $X' \parallel Z$ for any $X' \in \{X - Y, Y - X, X \cap Y, X \cup Y\}$.*

Statement 1.2. *If $\mathcal{F}' \subseteq 2^{V'}$ is laminar and $|V'| = n'$ then $|\mathcal{F}'| < 4n'$.*

Stamement 1.1 is trivial. To prove Statement 1.2 (see, e.g., [Karzanov 1979]), denote by $\alpha(n)$ the maximum cardinality of a laminar set-system on n-element set. It is easy to show by induction on n that $\alpha(n) \leq \alpha(n-1) + 4$, whence the statement follows.

An important corollary of Statement 1.1 is that if, at some step, a set Z in the current $\mathcal{F}$ becomes laminar to all other members of $\mathcal{F}$ then, independently of further steps, Z continues to satisfy this property up to termination of the game. So we will throughout assume that after each step such Z's are automatically excluded from consideration; i.e., for every current family $\mathcal{F}$,

(6) each member of $\mathcal{F}$ is crossing some other of its members.

We will also assume that such a property holds when we deal with the game on a subfamily of $\mathcal{F}$ being under consideration.

2. Reduction to cyclic families

We prove the following lemma.

Lemma 2.1. *For rule (1), the game is reduced to at most nm games, each for a cyclic family $\mathcal{R}$ on a set W with $|W| \leq n$.*

Proof. The desired strategy for Red is as follows. He fixes a laminar family $\mathcal{L}_1 \subseteq \mathcal{F}$ and chooses a set $A_1 \in \mathcal{F} - \mathcal{L}_1$. If $A \parallel X$ for all $X \in \mathcal{L}_1$, he simply adds $X, \overline{X}$ to $\mathcal{L}_1$. Otherwise he plays within the family $\mathcal{L}_1 \cup \{A_1, \overline{A}_1\}$. As a result, a laminar family $\mathcal{L}_2$ will be constructed. Then he chooses a set A_2 in the new current $\mathcal{F}$ which is not in $\mathcal{L}_2$ and plays within $\mathcal{L}_2 \cup \{A_2, \overline{A}_2\}$, and so on. Eventually, after $k \leq m/2$ iterations the obtained laminar family $\mathcal{L}_{k+1}$ will coincide with the current $\mathcal{F}$, and we are done.

We now explain how to play within a family $\mathcal{L} \cup \{A, \overline{A}\}$, where $\mathcal{L}$ is laminar and A is crossing some members of $\mathcal{L}$. For $X, Q \subseteq V$, Q is called *separating* X if both $X \cap Q$ and $X - Q$ are nonempty. We say that $X \subseteq V$ is *2-partitioned* with respect to a laminar family $\mathcal{D} \subset 2^V$ if there exists $Z \subseteq X$ (possibly $Z = \emptyset$ or X) such that

$$(7) \qquad X \cap Q \in \{Z, X - Z, X, \emptyset\} \quad \text{for any } Q \in \mathcal{D}.$$

Red plays in such a way that, at each step, the current family has a partition into laminar families $\mathcal{P}$ and $\mathcal{D}$ such that

(8) for each $X \in \mathcal{P}$, at least one of $X, \overline{X}$ is 2-partitioned with respect to $\mathcal{D}$.

Obviously, (8) holds for the initial $\mathcal{P} = \mathcal{L}$ and $\mathcal{D} = \{A, \overline{A}\}$. To maintain this property, Red chooses a *maximal* set $X \in \mathcal{P}$ which is 2-partitioned with respect to $\mathcal{D}$ and plays within $\mathcal{D} \cup \{X, \overline{X}\}$. Then for the resulting laminar family $\mathcal{D}'$ the following is true.

Claim. *For each $Y \in \mathcal{P}$ at least one of $Y, \overline{Y}$ is 2-partitioned with respect to $\mathcal{D}'$.*

Proof. Since $\mathcal{P}$ is laminar and symmetric, we may assume that either $Y \subseteq X$, or $X \cap Y = \emptyset$ and $X \cup Y \neq V$. First we observe that Y is 2-partitioned with respect to $\mathcal{D}$. Indeed, if $Y \subseteq X$, this easily follows from the fact that X is 2-partitioned with respect to $\mathcal{D}$. Otherwise $\overline{Y}$ strictly includes X, therefore, $\overline{Y}$ cannot be 2-partitioned because of the maximality of X.

Hence, there is $Z' \subseteq Y$ such that any $Q \in \mathcal{D}$ separates neither Z' nor $Y - Z'$. Since X does not separate Y, neither Z' nor $Y - Z'$ is separated by any Y arising during the game for $\mathcal{D} \cup \{X, \overline{X}\}$ whatever moves are applied by Red and Blue. This proves the claim. •

In view of the Claim, the game for $\mathcal{L} \cup \{A, \overline{A}\}$ as above is reduced to at most $|\mathcal{L}|/2$ games, each starting with a family $\mathcal{R} = \mathcal{D} \cup \{X, \overline{X}\}$ such that $\mathcal{D}$ is laminar and X is 2-partitioned with respect to $\mathcal{D}$. Since $|\mathcal{L}| \leq 4n$ (by Statement 1.2), the total number of games arising for the initial $\mathcal{F}$ does not exceed nm, as required in the lemma.

It remains to show that the game within $\mathcal{R}$ is in fact the game on a cyclic family. Indeed, we may assume that each $Y \in \mathcal{R}$ is crossing some other of its members (otherwise Y is excluded from the consideration). Then each set in $\mathcal{D}$ is crossing X. Let Z be as in (7) for our X and $\mathcal{D}$. Let $\mathcal{D}' = \{Q \in \mathcal{D} : X \cap Q = Z\}$; then $\mathcal{D} = \mathcal{D}' \cup \{\overline{Q} : Q \in \mathcal{D}'\}$. For each $Q \in \mathcal{D}'$ we have $\emptyset \neq Z \subset Q$ and $\emptyset \neq X - Z \subset \overline{Q}$. Hence, the laminarity of $\mathcal{D}'$ implies that for any two $Q, Q' \in \mathcal{D}'$, either $Q \subset Q'$ or $Q' \subset Q$. This means that there is a partition $\{V_1, V_2, ..., V_r\}$ of V such that $V_1 = Z$, $V_r = X - Z$ and each $Q \in \mathcal{D}'$ is a set of the form $V_1 \cup V_2 \cup \ldots \cup V_i$ for some $1 < i < r-1$. Furthermore, $X = V_1 \cup V_r$. Hence, $\mathcal{D} \cup \{X, \overline{X}\}$ is a cyclic family. This completes the proof of the lemma. •

The above proof shows that every cyclic family appeared during the game has a stronger form; this will be important for the proof of Theorem 1 in the next section.

Corollary 2.2. *Every cyclic family occurring in the above process is equivalent to a cyclic family $\mathcal{R} = \mathcal{D} \cup \{X, \overline{X}\}$ on a set $W = \{1, \dots, r\}$ such that*

(9) $$X = \{1, r\},$$

and

(10) *$\mathcal{D}$ consists of the sets $\overline{1,i}$ for $i = 2, \dots, r-2$, and their complements to W.* •

Next, one can see that the above arguments remain valid if we consider the game with step rule (4) instead of (1). Furthermore, the function $\widehat{f}$ is monotone non-increasing during the game, by (5). Thus, the following is true.

Statement 2.3. *For rule (4), the game for f is reduced to at most nm games, each for a cyclic family $\mathcal{R}$ on W with $|W| \le n$ and a function $g : \mathcal{R} \to \mathbb{Z}_+$ with $\|g\| \le \|f\|$, where $m = |\mathcal{F}(f)|$.* •

3. Proof of Theorem 1

In view of Lemma 2.1 and Corollary 2.2, it suffices to consider the game that starts with a cyclic family $\mathcal{R} = \mathcal{D} \cup \{X, \overline{X}\}$ on $W = \{1, \dots, r\}$ satisfying (9) and (10), and show that Red can win in a polynomial number of moves. Note that W was formed by shrinking the subsets V_i in a partition $\{V_1, \dots, V_r\}$ of V. Let $\mathcal{S}^*$ be the collection of subsets of W that correspond to members $Y \in \mathcal{S}$ of the form $Y = V_i \cup V_{i+1} \cup \dots V_j$ (taking indices modulo r). Obviously, $\mathcal{S}^*$ is cross-closed and cyclic, and we may think of $\mathcal{S}^*$ as the set-system behind $\mathcal{R}$. Note that (10) trivially implies that

(11) for $i = 2, \dots, r-2$, the set $\overline{1,i}$ belongs to $\mathcal{S}^*$.

To prove the theorem, we are forced to include into consideration slightly more general cyclic families on W. Namely, we assume that $\mathcal{R} \subseteq \mathcal{S}^*$ is partitioned into laminar families $\mathcal{L}$ and $\mathcal{D}$ such that:

(12) each set in $\mathcal{D}$ separates 1 and r (i.e., it is of the form $\overline{1,i}$ or $W - \overline{1,i}$ for some $2 \le i \le r-2$);

(13) each set in $\mathcal{L}$ separates 1 and 2 (i.e., it is of the form $\overline{2,i}$ or $W - \overline{2,i}$ for some $3 \le i \le r-1$).

As before, we also assume that each member of $\mathcal{R}$ is crossing some other of its members. We show that for such an $\mathcal{R}$ Red can win in time $O(r^3)$, thus proving the theorem. Let $d = d(\mathcal{R})$ denote the minimum number i such that $\mathcal{L}$ contains $\overline{2,i}$; then $3 \le d \le r-1$. We use induction on

$$\omega = \omega(r, \mathcal{R}) := r^3 + r|\mathcal{L}|/2 + d,$$

considering $\mathcal{S}^*$ and $\mathcal{R} = \mathcal{L} \cup \mathcal{D}$ satisfying (11)-(13). We may assume that

(14) for $i = 1, \dots, r$, there is a member of $\mathcal{R}$ separating $i-1$ and i

(hereinafter indices are taken modulo r). For if (14) is violated for some i then shrinking $i-1$ and i into one element yields an equivalent problem with smaller ω. Properties (12)-(14) imply

(15) $$\overline{2, r-1} \in \mathcal{L} \quad \text{and} \quad \overline{1,i} \in \mathcal{D} \text{ for } i = 2, \dots, d-1.$$

In what follows X, Y denote the crossing sets in a current $\mathcal{R}$ that Red chooses to uncross; X', Y' denote the sets obtained by uncrossing X, Y; and $\mathcal{R}', \mathcal{L}', \mathcal{D}'$ denote the corresponding objects that arise after the answering move of Blue and then by deleting the sets of the resulting family which are laminar to all other sets. For such an $\mathcal{R}'$ we denote by $r(\mathcal{R}')$ the number of maximal subsets $\overline{i,j}$ which are separated by no member of $\mathcal{R}'$. If $r' = r(\mathcal{R}') < r$ then contracting corresponding subsets in W results in a family $\mathcal{R}''$ on an r'-element set. Obviously, $\omega(r', \mathcal{R}'') < \omega(r, \mathcal{R})$, and therefore, we can immediately apply induction.

First we suppose that $\{1\} \notin \mathcal{S}^*$. Then Red takes $X = \overline{2, r-1}$ and $Y = \{1, 2\}$ to uncross. Since $Y - X = \{1\} \notin \mathcal{S}^*$ and $\mathcal{S}^*$ is cross-closed, we have $X \cap Y = \{2\} \in \mathcal{S}^*$ and $X \cup Y = W - \{r\} \in \mathcal{S}^*$. Red takes just $X \cap Y$ and $X \cup Y$ as X' and Y', respectively. Since both X' and $W - Y'$ are singletons (so they are not in $\mathcal{R}'$) and one of X, Y vanishes after the move of Blue, we conclude that at least one of the pairs $\{2, 3\}$ and $\{r-1, r\}$ is separated by no member of $\mathcal{R}'$. Hence, $r(\mathcal{R}') < r$, and the result follows by induction.

Thus we may assume that $\{1\} \in \mathcal{S}^*$. We may also assume that

(16) if $d = r-1$ then $\{r\}$ (and therefore, $\overline{1, r-1}$) belongs to $\mathcal{S}^*$.

Indeed, suppose that $\{r\} \notin \mathcal{S}^*$. Since $d = r-1$, $\mathcal{L}$ consists of only $\{1, r\}$ and its complement. Therefore, $\mathcal{D}$ consists of $\overline{1,i}$ and its complements for $i = 2, \dots, r-2$. Then renumbering $1, 2, \dots, r$ as $r, r-1, \dots, 1$ yields an isomorphism of $\mathcal{R}$. Now we have $\{1\} \notin \mathcal{S}^*$ and obtain the case above.

Let k be the maximal number such that $1 \le k \le d$ and $\{k\} \in \mathcal{S}^*$.

Claim. (i) $k \geq 2$. (ii) *If* $k < d$ *then* $Z = \overline{k+1, d} \notin \mathcal{S}^*$.

Proof. From (14) and the fact that $\mathcal{S}^*$ is cross-closed we deduce that any minimal nonempty set in $\mathcal{S}^*$ is a singleton. Therefore, $\{i\} \in \mathcal{S}^*$ for some $2 \leq i \leq d$ (as $\overline{2,d} \in \mathcal{R}$), which implies (i). Next, if $k < d$ and $Z \in \mathcal{S}^*$ then there is a $j \in Z$ such that $\{j\} \in \mathcal{S}^*$. Then $k < j \leq d$, contrary to the maximality of k. •

Consider three possible cases. In each case Red takes as X the set $\overline{2,d}$.

Case 1. $k = d$. Let $Y = \overline{1, d-1}$, $X' = X - Y$ and $Y' = Y - X$. Then $Y \in \mathcal{D}$ (by (15)), $X' = \{d\} \in \mathcal{S}^*$ and $Y' = \{1\} \in \mathcal{S}^*$. Red makes uncrossing $X, Y \to X', Y'$. If Blue returns X then $Y \notin \mathcal{R}'$ and no set in $\mathcal{R}'$ separates $d-1$ and d, whence $r(\mathcal{R}') < r$. And if Blue returns Y then $X \notin \mathcal{R}'$, whence $\mathcal{L}' = \mathcal{L} - \{X, \overline{X}\}$ and $|\mathcal{L}'| < |\mathcal{L}|$. In both cases, we get $\omega(r(\mathcal{R}'), \mathcal{R}') < \omega(r, \mathcal{R})$ and apply induction.

Case 2. $k = 2$. Let $Y = \{1, 2\}$, $X' = X \cap Y$ and $Y' = X \cup Y$. Then $X' = \{2\} \in \mathcal{S}^*$ and $Y' = \overline{1,d} \in \mathcal{S}^*$ (by (11) and (16)). Red makes uncrossing $X, Y \to X', Y'$. Then, after the move of Blue, at least one of the following situations takes place: (i) no set in $\mathcal{R}'$ separates 2 and 3, or (ii) $\mathcal{L}' = \mathcal{L} - \{X, \overline{X}\}$. The result follows by induction.

Case 3. $2 < k < d$. Let $Y = \overline{1,k}$, $X' = X \cap Y$, $Y' = X \cup Y$ and $Z = \overline{k+1, d}$. By the Claim, $Z = X - Y$ is not in $\mathcal{S}^*$. Therefore, both X' $(= \overline{2,k})$ and $Y' = X \cup Y$ $(= \overline{1,d})$ are in $\mathcal{S}^*$. Make the uncrossing operation $X, Y \to X', Y'$. Suppose that Blue returns Y. Then $X \notin \mathcal{R}'$ and $X' \in \mathcal{R}$, whence $|\mathcal{L}'| = |\mathcal{L}|$. Since $d(\mathcal{R}') = k < d = d(\mathcal{R})$, the result follows by induction.

Now suppose that Blue returns X. Then $X, X' \in \mathcal{R}'$; therefore, $\mathcal{L}' = \mathcal{L} \cup \{X', \overline{X}'\}$, and we cannot apply induction immediately. Nevertheless, we can use the property that

(17) k and $k+1$ are separated in $\mathcal{R}'$ by only $\overline{2,k}$ and its complement

(since $\overline{1,k}$ vanishes by uncrossing). Consider the sets $\widetilde{X} = \overline{2,k}$ and $\widetilde{Y} = \overline{1, k-1}$ in $\mathcal{R}'$. Both sets $\widetilde{X}' = \widetilde{X} - \widetilde{Y}$ $(= \{k\})$ and $\widetilde{Y}' = \widetilde{Y} - \widetilde{X}$ $(= \{1\})$ are in $\mathcal{S}^*$, so Red can apply to $\mathcal{R}'$ the next uncrossing operation $\widetilde{X}, \widetilde{Y} \to \widetilde{X}', \widetilde{Y}'$. Let $\mathcal{R}''$ be the family obtained after the move of Blue. Two cases are possible.

(i) Blue returns $\widetilde{Y}$. Then $\widetilde{X} \notin \mathcal{R}''$, and now there is no set in $\mathcal{R}''$ separating k and $k+1$, in view of (17).

(ii) Blue returns $\widetilde{X}$. Then $\widetilde{Y} \notin \mathcal{R}''$, therefore, no set in $\mathcal{R}''$ separates $k-1$ and k.

In both cases, we have $r(\mathcal{R}'') < r$ and apply induction.

4. Proof of Theorem 3

In view of Statement 2.3, Theorem 3 is implied by the following lemma.

Lemma 4.1. *Let g be a nonnegative integer-valued function of which support forms a cyclic family on a set $W = \{1, \ldots, r\}$. For rule (4), Red has a strategy to win in time polynomial in r and* $\log \|g\|$.

Proof. Let h and $\mathcal{R}$ be the current function and its support before a move of Red. We know that $\mathcal{R}$ is cyclic and $\|h\| \le \|g\|$ (by (5)).

Red plays as follows. He fixes a set $X \in \mathcal{R}$ such that $h(X)$ is maximum provided that $2 \le |X| \le r-2$; if such an X does not exist, $\mathcal{R}$ is already laminar, and we are done. He takes this X and an arbitrary $Y \in \mathcal{R}$ ($Y \not\parallel X$) to uncross, then he takes X and another Y' ($Y' \not\parallel X$), and so on until a function h' is obtained such that $h'(X) = 0$ or all members of the support $\mathcal{R}'$ of h' are laminar to X. We call this sequence of moves a *big iteration*. Let $\mathcal{R}_X = \{Y \in \mathcal{R} : Y \not\parallel X\}$. Consider possible cases.

Case 1. $h(X) \ge \frac{1}{2} h(\mathcal{R}_X)$, where $h(\mathcal{R}_X)$ stands for $\sum(h(Y) : Y \in \mathcal{R}_X)$. Then $h'(Y) = 0$ for all $Y \in \mathcal{R}_X$ (and no new set Y' such that $Y' \not\parallel X$ and $h'(Y') > 0$ can appear, by Statement 1.1). Therefore, all members of $\mathcal{R}'$ are laminar to X. Thus, we can split the game for $\mathcal{R}'$ into two games, one with the family $\mathcal{R}_1 = \{Y \in \mathcal{R}' : Y \subset X$ or $W - Y \subset X\}$ and the other with $\mathcal{R}_2 = \{Y \in \mathcal{R}' : X \subset Y$ or $X \subset W - Y\}$. In fact, we may assume that the former (latter) game deals with a cyclic family on the set W_1 (respectively, W_2) obtained from W by contracting the elements of $W - X$ (respectively, X). Then $|W_i| < r$ ($i = 1, 2$) and $|W_1| + |W_2| = r + 2$. Furthermore, if $|W_i| \le 3$ then $\mathcal{R}_i$ is obviously laminar. Using these, it is easy to show by induction on $|W|$ that the total number of big iterations (for families on W and those occurred on reduced sets) at which Case 1 takes place is bounded by a polynomial in r.

Case 2. $h(X) < \frac{1}{2} h(\mathcal{R}_X)$. For $e \in E_W$, let $\gamma(e)$ be the sum of $h(Y)$'s among $Y \in \mathcal{R}$ with $2 \le |Y| \le r-2$ and $e \in \delta^G(Y)$, and let $\beta = \sum(\gamma(e) : e \in E_W)$, where $G = (W, E_W)$ is the complete graph on W. Let $\gamma'(e)$ and β' be the corresponding numbers for h'.

Obviously, $\beta < r^2 \|h\|$. Since the sets $X, \overline{X}$ vanish at the big iteration, we deduce from (5) (with $\gamma(e)$ and $\gamma'(e)$ instead of $\widehat{f}(e)$ and $\widehat{f}'(e)$) that β' is at most $\beta - 4h(X)$. Furthermore, the maximality of $h(X)$ implies that $\beta \le |E_W||\mathcal{R}|h(X) < r^4 h(X)$ (taking into account that $|\mathcal{R}| < r^2$, as $\mathcal{R}$ is cyclic). Therefore,

$$\beta' < \beta(1 - 4/r^4). \tag{18}$$

Suppose that Case 2 occurs in k consecutive big iterations, and let β_0 and β_1 be the values of β at the first and last of these iterations, respectively. We may assume that $\beta_1 \geq 1$. Then (18) together with $\beta_0 < r^2\|g\|$ (for the initial g) implies that k is bounded by a polynomial in r and log $\|g\|$, whence the lemma easily follows. •

An open question. Can Red win in polynomial time in non-symmetric analogs of games (1) and (4)?

Acknowledgement. I am thankful to Éva Tardos for correcting some errors in the original version of this paper and many useful suggestions.

REFERENCES

[Grötschel, Lovász, Schrijver 1988] M. Grötschel, L. Lovász and A. Schrijver: *Geometric Algorithms and Combinatorial Optimization* (Springer, 1988).

[Edmonds, Giles 1977] J. Edmonds and R. Giles: A min-max relation for submodular functions on graphs. In: P.L. Hammer, E.L. Johnson, B. Korte, G.L. Nemhauser, eds., *Studies in Integer Programming* (North-Holland, Amsterdam, 1977) 185-204 (*Annals of Discrete Math.* **1**).

[Edmonds, Jonhson 1973] J. Edmonds and E.L. Jonhson: Matching, Euler tours and Chinese postman. *Math. Programming* **5**(1973) 88-124.

[Guan 1962] M. Guan: Graphic programming using odd and even points, *Chinese Mathematics* **1**(1962) 273-277.

[Hurkens, Lovász, Schrijver, Tardos 1987] C.A.J. Hurkens, L. Lovász, A. Schrijver, and E. Tardos: How to tidy up your set-system? In: *Colloquia Mathematica Societies Janos Bolyai, 52. Combinatorics* (North Holland, Amsterdam, 1987), pp. 309-314.

[Karzanov 1979] A.V. Karzanov: Combinatorial methods to solve cut-determined multiflow problems. In: *Combinatorial Methods for Flow Problems* (Inst. for System Studies, Moscow, 1979, iss. 3), pp. 6-69, in Russian.

[Karzanov 1984] A.V. Karzanov: A generalized MFMC-prorerty and multicommodity cut problems. In: *Finite and Infinite Sets* (Proc. 6th Hungar. Comb. Coll. (Eger,1981)) (North-Holland, Amsterdam, 1984), v.2, pp. 443-486.

[Karzanov 1990] A.V. Karzanov: Polynomial uncrossing processes, *Research Report* 90647-OR (Institut für Diskrete Mathematik, Bonn, 1990) 11p.

[Seymour 1981] P.D. Seymour: On odd cuts and plane multicommodity flows, *Proc. London Math. Soc.*, Ser. 3, **42**(1981) 178-192.