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Plücker environments, wiring and tiling diagrams, and weakly separated set-systems

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Abstract

For the ordered set $[n]$ of n elements, we consider the class $\mathcal{B}_n$ of bases B of tropical Plücker functions on $2^{[n]}$ such that B can be obtained by a series of so-called weak flips (mutations) from the basis formed by the intervals in $[n]$. We show that these bases are representable by special wiring diagrams and by certain arrangements generalizing rhombus tilings on an n-zonogon. Based on the generalized tiling representation, we then prove that each weakly separated set-system in $2^{[n]}$ having maximum possible size belongs to $\mathcal{B}_n$, yielding the affirmative answer to one conjecture due to Leclerc and Zelevinsky. We also prove an analogous result for a hyper-simplex $\Delta_n^m = \{S \subseteq [n]\colon\ |S| = m\}$.

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1. Introduction

For a positive integer n, let $[n]$ denote the ordered set of elements $1, 2, \dots, n$. In this paper we consider a certain "class" $\mathcal{B}_n \subseteq 2^{2^{[n]}}$. The collections (set-systems) $B \subseteq 2^{[n]}$ constituting $\mathcal{B}_n$ have

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equal cardinalities $|B|$, and for some pairs of collections, one can be obtained from the other by a single "flip" (or "mutation") that consists in exchanging a pair of elements of a very special form in these collections. The class we deal with arises, in particular, in a study of bases of so-called tropical Plücker functions (this seems to be the simplest source; one more source will be indicated later). For this reason, we may liberally call $\mathcal{B}_n$ along with mutations on it a *Plücker environment*.

More precisely, let f be a real-valued function on the subsets of $[n]$, or on the Boolean cube $2^{[n]}$. Following [1], f is said to be a *tropical Plücker function*, or a *TP-function* for short, if it satisfies

$$f(Xik) + f(Xj) = \max\{f(Xij) + f(Xk), f(Xi) + f(Xjk)\} \tag{1.1}$$

for any triple $i < j < k$ in $[n]$ and any subset $X \subseteq [n] - \{i, j, k\}$. Throughout, for brevity we write $Xi' \dots j'$ for $X \cup \{i'\} \cup \dots \cup \{j'\}$. For sets A, B, $A - B$ stands for the set difference $\{e\colon A \ni e \notin B\}$.

The set of TP-functions on $2^{[n]}$ is denoted by $\mathcal{TP}_n$.

Definition. A collection $B \subseteq 2^{[n]}$ is called a *TP-basis*, or simply a *basis*, if the restriction map $res : \mathcal{TP}_n \to \mathbb{R}^B$ is a bijection. In other words, each TP-function is determined by its values on B, and moreover, values on B can be chosen arbitrarily.

Such a basis does exist and the simplest instance is the set $\mathcal{I}_n$ of all intervals $\{p, p+1, \dots, q\}$ in $[n]$ (including the empty set); see, e.g., [2]. In particular, the dimension of the polyhedral conic complex $\mathcal{TP}_n$ is equal to $|\mathcal{I}_n| = \binom{n+1}{2} + 1$. The basis $\mathcal{I}_n$ is called *standard*.

(Note that the notion of a TP-function is extended to other domains, of which most popular are an *integer box* $\mathbf{B}^{n,a} := \{x \in \mathbb{Z}^{[n]}\colon 0 \leqslant x \leqslant a\}$ for $a \in \mathbb{Z}^{[n]}$ and a *hyper-simplex* $\Delta_n^m := \{S \subseteq [n]\colon |S| = m\}$ for $m \in \mathbb{Z}$ (in the latter case, (1.1) should be replaced by a relation with quadruples $i < j < k < \ell$). Aspects involving TP-bases or related objects are encountered in [1,5,9,10,12–14] and some other works. Generalizing some earlier known examples, [2] constructs a TP-basis for a "truncated integer box" $\{x \in \mathbf{B}^{n,a}\colon m \leqslant x_1 + \dots + x_n \leqslant m'\}$, where $0 \leqslant m \leqslant m' \leqslant a_1 + \dots + a_n$. The domains different from Boolean cubes are beyond the main part of this paper; they will appear in Section 10 and Appendix A.)

One can see that for a basis B, the collection $\{[n] - X\colon X \in B\}$ forms a basis as well, called the *complementary basis* of B and denoted by co-B. An important instance is the collection co-$\mathcal{I}_n$ of *co-intervals* in $[n]$.

Once we are given a basis B (e.g., the standard one), we can produce more bases by making a series of elementary transformations relying on (1.1). More precisely, suppose there is a cortege (X, i, j, k) such that the four sets occurring in the right-hand side of (1.1) and one set $Y \in \{Xj, Xik\}$ in the left-hand side belong to B. Then the replacement in B of Y by the other set Y' in the left-hand side results in a basis B' as well (and we can further transform the latter basis in a similar way). The basis B' is said to be obtained from B by the *flip* (or *mutation*) with respect to X, i, j, k. When Xj is replaced by Xik (thus increasing the total size of sets in the basis by 1), the flip is called *raising*. When Xik is replaced by Xj, the flip is called *lowering*. We sometimes write $j \rightsquigarrow ik$ and $ik \rightsquigarrow j$ for such flips. The standard basis $\mathcal{I}_n$ does not admit lowering flips, whereas its complementary basis co-$\mathcal{I}_n$ does not admit raising flips.

We now distinguish between two sorts of flips in another way, which inspire consideration of two classes of bases.

Definitions. For a TP-basis B and a cortege (X, i, j, k) as above, we say that the flip $j \rightsquigarrow ik$ or $ik \rightsquigarrow j$ is *weak*. If, in addition, both sets X and $Xijk$ belong to B as well, the flip is called *strong*. (The former (latter) is also called a "flip in the presence of four (resp. six) witnesses," by terminology in [9].) A basis is called *normal* (by terminology in [2]) if it can be obtained by a series of *strong* flips starting from $\mathcal{I}_n$. A basis is called *semi-normal* if it can be obtained by a series of *weak* flips starting from $\mathcal{I}_n$.

Leclerc and Zelevinsky [9] showed that the normal bases (in our terminology) are exactly the collections $C \subseteq 2^{[n]}$ of maximum possible size $|C|$ that possess the strong separation property (defined later). Also the class of normal bases admits a nice "graphical" characterization, even for a natural generalization to the integer boxes (see [2,4]): such bases one-to-one correspond to the rhombus tilings on a related *zonogon*.

Let $\mathcal{B}_n$ denote the set of *semi-normal* TP-bases for the Boolean cube $2^{[n]}$; this set (together with weak flips on its members) is just the Plücker environment of our interest mentioned at the beginning. Note that it is still open at present whether there exists a non-semi-normal (or "wild") TP-basis; we conjecture that there is none.

The first goal of this paper is to characterize $\mathcal{B}_n$. We give two characterizations for semi-normal bases: via a bijection to special collections of curves, that we call *proper wirings*, and via a bijection to certain graphical arrangements, that we call *generalized tilings*, or *g-tilings* for short (in fact, these characterizations are interrelated via planar duality). We associate to a proper wiring W (a g-tiling T) a certain collection of subsets of $[n]$ called its *spectrum*. It turns out that proper wirings and g-tilings are rigid objects, in the sense that any of these is determined by its spectrum.

(By a *general* wiring we mean a set of n directed non-self-intersecting curves $w_1, \dots, w_n$ on a disc D in the plane such that: w_i begins at a point s_i and ends at a point s'_i, and the points $s_1, \dots, s_n, s'_1, \dots, s'_n$ are different and occur in this order in the boundary of D. A *special* wiring W that we deal with is, in fact, a certain generalization of the so-called "pseudo-line arrangement" associated with a reduced word of the longest permutation on $[n]$ (see [1] for definitions). Such a W is defined by three axioms (W1)–(W3). Axiom (W1) is standard, it says that W preserves (topologically) under small deformations, i.e., no three wires have a common point, any two wires meet at a finite number of points and they cross, not touch, at each of these points. (W2) says that the common points of w_i, w_j follow in the opposed orders along these wires (such an axiom is stated in [10] for a somewhat different sort of wirings, arising in connection with set-systems in a hyper-simplex Δ_n^m; see also [12]). The crucial axiom (W3) says that in the planar graph induced by W, there is a certain bijection between the faces ("chambers") whose boundary is a directed cycle and the regions ("lenses") surrounded by pieces of two wires between their consecutive common points. W is called *proper* if none of "cyclic" faces is a whole lens. The spectrum of W is the collection of subsets $X \subseteq [n]$ associated to the "non-cyclic" faces F, where X consists of the elements i such that F "lies on the left" from the wire w_i, by the direction of w_i.

When any two wires intersect exactly once, W is equivalent to a pseudo-line arrangement mentioned above (more precisely, to a "commutation class" of these objects). In this case the dual planar graph is representable by a rhombus tiling (for a more general result of this sort, see [6]). The construction of a g-tiling is more sophisticated.)

The characterization of semi-normal bases via generalized tilings helps us to answer one conjecture of Leclerc and Zelevinsky concerning weakly separated set-systems; this is the second goal of our work. Recall corresponding definitions from [9], using slightly different binary relations on sets. Let $X, Y \subseteq [n]$. We write $X \lessdot Y$ if $Y - X \neq \emptyset$ and $i < j$ holds for any $i \in X - Y$ and $j \in Y - X$. We write $X \rhd Y$ if $Y - X$ has a (unique) bipartition $\{Y_1, Y_2\}$ such that $Y_1, Y_2, X - Y \neq \emptyset$ and $Y_1 \lessdot X - Y \lessdot Y_2$. (These relations need not be transitive in general. For example, $13 \lessdot 23 \lessdot 24$ but $13 \not\lessdot 24$, where 13 stands for $\{1, 3\}$, and so on. Similarly, $346 \rhd 256 \rhd 157$ but $346 \not\rhd 157$.)

Definitions. Sets $X, Y \subseteq [n]$ are called *weakly separated* if either $X \lessdot Y$, or $Y \lessdot X$, or $X \rhd Y$ and $|X| \geqslant |Y|$, or $Y \rhd X$ and $|Y| \geqslant |X|$, or $X = Y$. If merely either $X \lessdot Y$ or $Y \lessdot X$ or $X = Y$ takes place, the sets X, Y are called *strongly separated*. Accordingly, a collection $C \subseteq 2^{[n]}$ is called weakly (strongly) separated if any two members of C are weakly (resp. strongly) separated.

We will abbreviate the term "weakly separated collection" to "*ws-collection*." (As is seen from a discussion in [9], an interest in studying ws-collections is inspired, in particular, by the problem of characterizing all families of quasicommuting quantum flag minors, which in turn comes from exploration of Lusztig's canonical bases for certain quantum groups. It is proved in [9] that, in an $n \times n$ generic q-matrix, the flag minors with column sets $I, J \subseteq [n]$ quasicommute if and only if the sets I, J are weakly separated. See also [8].)

Important properties shown in [9] are that any ws-collection $C \subseteq 2^{[n]}$ has cardinality at most $\binom{n+1}{2} + 1$ and that the set of such collections is closed under weak flips (which are defined as for TP-bases above). Let $\mathcal{C}_n$ denote the set of *largest* ws-collections in $[n]$, i.e., having size exactly $\binom{n+1}{2} + 1$. It turns into a poset by regarding C as being less than C' if C can be obtained from C' by a series of weak lowering flips. This poset contains $\mathcal{I}_n$ and co-$\mathcal{I}$ as minimal and maximal elements, respectively, and it is conjectured in [9, Conjecture 1.8] that there are no other minimal and maximal elements in it. This would imply that $\mathcal{C}_n$ coincides with $\mathcal{B}_n$. We prove this conjecture.

The main results in this paper are summarized as follows.

Theorem A *(Main). For $B \subseteq 2^{[n]}$, the following statements are equivalent:*

(i) *B is a semi-normal TP-basis;*
(ii) *B is the spectrum of a proper wiring;*
(iii) *B is the spectrum of a generalized tiling;*
(iv) *B is a largest weakly separated collection.*

The paper is organized as follows. Section 2 contains basic definitions and states two results involved in Theorem A. It introduces the notions of proper wirings and generalized tilings, claims the equivalence of (i) and (ii) in the above theorem (Theorem 2.1) and claims the equivalence of (i) and (iii) (Theorem 2.2). Section 3 describes some "elementary" properties of g-tilings that will be used later. The combined proof of Theorems 2.1 and 2.2 consists of four stages and is lasted throughout Sections 4–7. In fact, g-tilings are the central objects of treatment in the paper; we take advantages from their nice graphical visualization and structural features, and all implications that we explicitly prove involve just g-tilings. (Another preference of g-tilings is that they admit "local" defining axioms; see Remark 1 in Section 3.) Implication (i) → (iii) in Theorem A is proved in Section 4, (iii) → (i) in Section 5, (iii) → (ii) in Section 6, and (ii) → (iii) in Section 7. In fact, Section 5 shows that if the spectrum B of a g-tiling is different from $\mathcal{I}_n$, then B admits a lowering weak flip. Section 8 establishes important interrelations between g-tilings in dimensions n and $n - 1$ (giving, as a consequence, a relation between the classes $\mathcal{B}_n$

and $\mathcal{B}_{n-1}$). Here we describe two operations, called the *n-contraction* and *n-expansion*; the former canonically transforms a g-tiling for n into one for $n-1$, and the latter is applied to a pair consisting of a g-tiling for $n-1$ and a certain path in it and transforms this pair into a g-tiling for n. These operations are essentially used in Section 9 where we prove (iv) → (iii) by induction on n, thus answering Leclerc–Zelevinsky's conjecture mentioned above. This completes the proof of Theorem A, taking into account that (i) → (iv) was established in [9]. Section 10 discusses two generalizations of our theorems: to an integer box and to an arbitrary permutation on $[n]$. In Appendix A we show that the equivalence (i) ↔ (iv) as in Theorem A is valid when, instead of TP-bases and largest ws-collections in $2^{[n]}$, one considers a natural class of bases of tropical Plücker functions on a hyper-simplex Δ_n^m and the ws-collections of maximum possible cardinality in Δ_n^m.

It should be noted that some methods, constructions and results presented in this paper are essentially used in the sequel [3] where we prove that any inclusion-wise maximal ws-collection in $2^{[n]}$ is a largest one and generalize this to ws-collections concerning an arbitrary permutation on $[n]$, thus answering another conjecture raised in [9].

2. Wirings and tilings

Throughout the paper we assume that $n > 1$. This section gives precise definitions of the objects that we call proper wiring and generalized tiling diagrams. Such diagrams live within a zonogon, which is defined as follows.

In the upper half-plane $\mathbb{R} \times \mathbb{R}_+$, take n non-colinear vectors $\xi_1, \dots, \xi_n$ so that:

(2.1) (i) $\xi_1, \dots, \xi_n$ follow in this order clockwise around $(0,0)$, and
(ii) all integer combinations of these vectors are different.

Then the set

$$Z = Z_n := \{\lambda_1\xi_1 + \cdots + \lambda_n\xi_n \colon\ \lambda_i \in \mathbb{R},\ 0 \leqslant \lambda_i \leqslant 1,\ i = 1, \dots, n\}$$

is a $2n$-gone. Moreover, Z is a *zonogon*, as it is the sum of n line-segments $\{\lambda\xi_i \colon\ 1 \leqslant \lambda \leqslant 1\}$, $i = 1, \dots, n$. Also it is the image by the linear projection π of the solid cube $conv(2^{[n]})$ into the plane $\mathbb{R}^2$, defined by $\pi(x) = x_1\xi_1 + \cdots + x_n\xi_n$. The boundary $bd(Z)$ of Z consists of two parts: the *left boundary* $\ell bd(Z)$ formed by the points (vertices) $z_i^\ell := \xi_1 + \cdots + \xi_i$ ($i = 0, \dots, n$) connected by the line-segments $z_{i-1}^\ell z_i^\ell := z_{i-1}^\ell + \{\lambda\xi_i \colon\ 0 \leqslant \lambda \leqslant 1\}$, and the *right boundary* $rbd(Z)$ formed by the points $z_i^r := \xi_{i+1} + \cdots + \xi_n$ ($i = 0, \dots, n$) connected by the segments $z_i^r z_{i-1}^r$. So $z_0^\ell = z_n^r$ is the minimal vertex of Z, denoted as z_0, and $z_n^\ell = z_0^r$ is the maximal vertex, denoted as z_n. We direct each segment $z_{i-1}^\ell z_i^\ell$ from z_{i-1}^ℓ to z_i^ℓ and direct each segment $z_i^r z_{i-1}^r$ from z_i^r to z_{i-1}^r. Then $\ell bd(Z)$ and $rbd(Z)$ can be regarded as directed paths going from z_0 to z_n. Let s_i (resp. s_i') denote the median point of the segment $z_{i-1}^\ell z_i^\ell$ (resp. $z_i^r z_{i-1}^r$).

When it is not confusing, a subset $X \subseteq [n]$ is identified with the corresponding vertex of the n-cube and with the point $\sum_{i \in X} \xi_i$ in the zonogon Z (and we will usually use capital letters to emphasize that a vertex (or a point) is considered as a set). Due to (2.1)(ii), all such points in Z are different.

Although the generalized tiling model will be used much more extensively later on, we prefer to start with describing the special wiring model, which looks more transparent.

2.1. Wiring diagrams

A *special wiring diagram*, also called a *W-diagram* or a *wiring* for brevity, is an ordered collection W of n wires $w_1, \dots, w_n$ satisfying three axioms below. A *wire* w_i is a continuous injective map of the segment $[0, 1]$ into Z (or the curve in the plane represented by this map) such that $w_i(0) = s_i$, $w_i(1) = s'_i$, and $w_i(\lambda)$ lies in the interior of Z for $0 < \lambda < 1$. We say that w_i begins at s_i and ends at s'_i, and direct w_i from s_i to s'_i. The diagram W is considered up to a homeomorphism of Z stable on $bd(Z)$, and up to parameterizations of the wires. Axioms (W1)–(W3) specify W as follows.

(W1) No three different wires w_i, w_j, w_k have a common point, i.e., there are no $\lambda, \lambda', \lambda''$ such that $w_i(\lambda) = w_j(\lambda') = w_k(\lambda'')$. Any two different wires w_i, w_j intersect at a finite number of points, and at each of their common points v, the wires *cross*, not touch (i.e., when passing v, the wire w_i goes from one connected component of $Z - w_j$ to the other).

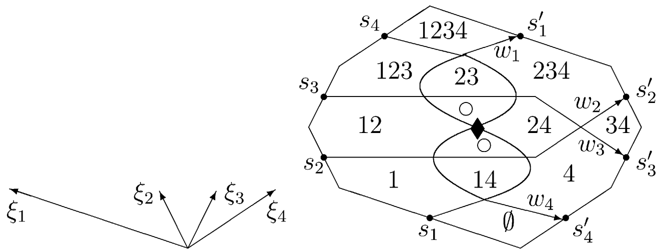
(W2) For $1 \leqslant i < j \leqslant n$, the common points of w_i, w_j follow in opposed orders along these wires, i.e., if $w_i(\lambda_q) = w_j(\lambda'_q)$ for $q = 1, \dots, r$ and if $\lambda_1 < \dots < \lambda_r$, then $\lambda'_1 > \dots > \lambda'_r$.

Since the order of s_i, s_j in $\ell bd(Z)$ is different from the order of s'_i, s'_j in $rbd(Z)$, wires w_i, w_j always intersect; moreover, the number $r = r_{ij}$ of their common points is odd. Assuming that $i < j$, we denote these points as $x_{ij}(1), \dots, x_{ij}(r)$ following the direction of w_i from $w_i(0)$ to $w_i(1)$. When $r > 1$, the (bounded) region in the plane surrounded by the pieces of w_i, w_j between $x_{ij}(q)$ and $x_{ij}(q+1)$ (where $q = 1, \dots, r-1$) is denoted by $L_{ij}(q)$ and called the q-th *lens* for w_i, w_j. The points $x_{ij}(q)$ and $x_{ij}(q+1)$ are regarded as the *lower* and *upper* points of $L_{ij}(q)$, respectively. When q is odd (even), we say that $L_{ij}(q)$ is an *odd* (resp. *even*) lens. Note that at each point $x_{ij}(q)$ with q odd the wire with the bigger number, namely, w_j, crosses the wire with the smaller number (w_i) *from left to right* w.r.t. the direction of the latter; we call such a point *white* (or *orientation-preserving*). In contrast, when q is even, w_j crosses w_i at $x_{ij}(q)$ *from right to left*. In this case, which will be of especial interest for us, we call $x_{ij}(q)$ *black* (or *orientation-reversing*) and say that this point is the *root* of the lenses $L_{ij}(q-1)$ and $L_{ij}(q)$. In the simplest case, when any two distinct wires intersect exactly once, there are no lenses at all and all intersection points for W are white. (The adjectives "white" and "black" for intersection points of wires match terminology that we will use for corresponding elements of tilings.)

The wiring W is associated, in a natural way, with a planar directed graph G_W embedded in Z. The vertices of G_W are the points $z_i^\ell, z_i^r, s_i, s'_i$ and the intersection points of wires. The edges of G_W are the corresponding directed line-segments in $bd(Z)$ and the pieces of wires between neighboring points of intersection with other wires or with the boundary, which are directed according to the direction of wires. We say that an edge contained in a wire w_i has *color* i, or is an i-*edge*. Let $\mathcal{F}_W$ be the set of inner (bounded) faces of G_W. Here each face F is considered as the closure of a maximal connected component in $Z - \bigcup(w \in W)$. We say that a face F is *cyclic* if its boundary $bd(F)$ is a directed cycle in G_W.

(W3) There is a bijection ϕ between the set $\mathcal{L}(W)$ of lenses in W and the set $\mathcal{F}_W^{cyc}$ of cyclic faces in G_W. Moreover, for each lens L, $\phi(L)$ is the (unique) face lying in L and containing its root.

We say that W is *proper* if none of cyclic faces is a whole lens, i.e., for each lens $L \in \mathcal{L}(W)$, there is at least one wire going across L. An instance of proper wirings for $n = 4$ is illustrated in the picture; here the cyclic faces are marked by circles and the unique black point is indicated by the black rhombus.



$$B_W = \{\emptyset, 1, 4, 12, 14, 23, 24, 34, 123, 234, 1234\}$$

Now we associate to W a set-system $B_W \subseteq 2^{[n]}$ as follows. For each face F, let $X(F)$ be the set of elements $i \in [n]$ such that F lies *on the left* from the wire w_i, i.e., F and the maximal point z_n^ℓ lie in the same of the two connected components of $Z - w_i$. We define

$$B_W := \{X \subseteq [n]:\ X = X(F) \text{ for some } F \in \mathcal{F}_W - \mathcal{F}_W^{cyc}\},$$

referring to it as the *effective spectrum*, or simply the *spectrum* of W. Sometimes it will also be useful to consider the *full spectrum* $\hat{B}_W$ consisting of all sets $X(F)$, $F \in \mathcal{F}_W$. (In fact, when W is proper, all sets in $\hat{B}_W$ are different; see Lemma 7.2. When W is not proper, different faces F, F' with $X(F) = X(F')$ always exist. We can turn such a W into a proper wiring W' by getting rid, step by step, of lenses forming faces (by making a series of Reidemeister moves of type II, namely, $() \to)($ operations). This preserves the effective spectrum: $B_{W'} = B_W$, whereas the full spectrum may decrease.)

Note that when any two wires in W intersect at exactly one point (i.e., when no black points exist), B_W is a normal basis, and conversely, any normal basis is obtained in this way. Such a W one-to-one corresponds to a commutation class of pseudo-line arrangements for the longest permutation ω_0 on $[n]$ and B_W is a largest strongly separated collection, as is shown in [9]; for definitions and a discussion, see also [1,7].

Our main result on wirings is the following

Theorem 2.1. *For any proper wiring W (obeying (W1)–(W3)), the spectrum B_W is a semi-normal TP-basis. Conversely, for any semi-normal TP-basis B, there exists a proper wiring W such that $B_W = B$.*

This theorem will be obtained in Sections 6–7.

2.2. Generalized tilings

Assuming that the vectors ξ_i have the same Euclidean norm, a *rhombus tiling diagram* is a subdivision T of Z into rhombi of the form $x + \{\lambda\xi_i + \lambda'\xi_j:\ 0 \leqslant \lambda, \lambda' \leqslant 1\}$ for some $i < j$ and a point x in Z, i.e., the rhombi are pairwise non-overlapping (have no common interior points) and their union is Z. From (2.1)(ii) it follows that for each rhombus in T determined by x, i, j as above, x represents a subset in $[n] - \{i, j\}$. We associate to T the directed planar graph G_T whose vertices and edges are, respectively, the points and line-segments occurring as vertices and sides in the rhombi in T (not counting multiplicities). An edge connecting X and Xi is directed from the former to the latter; such an edge (parallel to ξ_i) is called an edge of *color* i, or an i*-edge*. It is shown in [2,4] that the vertex set of T forms a normal basis and that each normal basis is obtained in this way.

In fact, it makes no difference whether we take vectors $\xi_1, \dots, \xi_n$ with equal or arbitrary norms (subject to (2.1)); to simplify technical details and visualization, we will assume throughout the paper that these vectors have *unit height*, i.e., each ξ_i is of the form $(a, 1)$. This leads to a subdivision T of Z into parallelograms of height 2, and for convenience we refer to T as a *(pure) tiling* and to its elements as *tiles*. A tile τ determined by X, i, j (with $i < j$) is called an ij*-tile* at X and denoted by $\tau(X; i, j)$. The edge from $b(\tau)$ to $\ell(\tau)$ is denoted by $b\ell(\tau)$, and the other three edges of τ are denoted as $br(\tau), \ell t(\tau), rt(\tau)$ in a similar way. According to a natural visualization of τ, its vertices X, Xi, Xj, Xij are called the *bottom, left, right, top* vertices of τ and denoted by $b(\tau)$, $\ell(\tau)$, $r(\tau)$, $t(\tau)$, respectively. Also we say that: a point (subset) $Y \subseteq [n]$ is of *height* $|Y|$; the set of vertices of height h in G_T forms h-th *level*; and a point Y *lies on the right* from a point Y' if Y, Y' have the same height and $\sum_{i \in Y} \xi_i \geqslant \sum_{i \in Y'} \xi_i$.

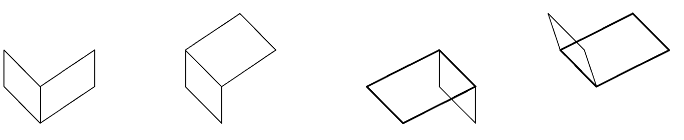
In a *generalized tiling*, or a *g-tiling*, some tiles may overlap. It is a collection T of tiles $\tau(X; i, j)$ which is partitioned into two subcollections T^w and T^b, of *white* and *black* tiles, respectively, obeying axioms (T1)–(T4) below. When $T^b = \emptyset$, we will obtain a pure tiling. As before, we associate to T the directed graph $G_T = (V_T, E_T)$, where V_T and E_T are the sets of vertices and edges, respectively, occurring in tiles of T.

For a vertex $v \in V_T$, the set of edges incident with v is denoted by $E_T(v)$, and the set of tiles having a vertex at v is denoted by $F_T(v)$.

(T1) Each boundary edge of Z belongs to exactly one tile. Each edge in E_T not contained in $bd(Z)$ belongs to exactly two tiles. All tiles in T are different, in the sense that no two coincide in the plane.

(T2) Any two white tiles having a common edge do not overlap, i.e., they have no common interior point. If a white tile and a black tile share an edge, then these tiles do overlap. No two black tiles share an edge.

See the picture; here all edges are directed up and the black tiles are drawn in bold.



(T3) Let τ be a black tile. None of $b(\tau), t(\tau)$ is a vertex of another black tile. All edges in $E_T(b(\tau))$ *leave* $b(\tau)$, i.e., they are directed from $b(\tau)$. All edges in $E_T(t(\tau))$ *enter* $t(\tau)$, i.e., they are directed to $t(\tau)$.

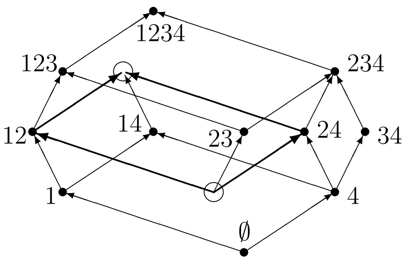
We refer to a vertex $v \in V_T$ as a *terminal* one if v is the bottom or top vertex of some black tile. A nonterminal vertex v is called *ordinary* if all tiles in $F_T(v)$ are white, and *mixed* otherwise (i.e. v is the left or right vertex of some black tile). Note that a mixed vertex may belong, as the left or right vertex, to several black tiles.

Each tile $\tau \in T$ corresponds to a square in the solid cube $conv(2^{[n]})$, denoted by $\sigma(\tau)$: if $\tau = \tau(X; i, j)$ then $\sigma(\tau)$ is the convex hull of the points X, Xi, Xj, Xij in the cube (so $\pi(\sigma(\tau)) = \tau$). Axiom (T1) implies that the interiors of these squares are pairwise disjoint, and that $\bigcup(\sigma(\tau)\colon\ \tau \in T)$ forms a 2-dimensional surface, denoted by D_T, whose boundary is the preimage by π of the boundary of Z. Note that the vertices in $bd(D_T)$ correspond to the *principal intervals* $\emptyset$, $[q]$ and $[q..n]$, $q = 1, \dots, n$, where for $1 \leqslant p \leqslant r \leqslant n$, we denote the interval $\{p, p+1, \dots, r\}$ by $[p..r]$. The last axiom is:

(T4) D_T is a disc, i.e., it is homeomorphic to $\{x \in \mathbb{R}^2\colon\ x_1^2 + x_2^2 \leqslant 1\}$.

The *reversed* g-tiling T^{rev} of a g-tiling T is formed by replacing each tile $\tau(X; i, j)$ of T by the tile $\tau([n] - Xij; i, j)$ (or, roughly speaking, by changing the orientation of all edges in E_T, in particular, in $bd(Z)$). Clearly (T1)–(T4) remain valid for T^{rev}.

The *effective spectrum*, or simply the *spectrum*, of a g-tiling T is the collection B_T of (the subsets of $[n]$ represented by) *nonterminal* vertices in G_T. The *full spectrum* $\hat{B}_T$ is formed by all vertices in G_T. An example of g-tilings for $n = 4$ is drawn in the picture, where the unique black tile is indicated by thick lines and the terminal vertices are surrounded by circles (this corresponds to the wiring shown on the previous picture).



$$B_T = \{\emptyset, 1, 4, 12, 14, 23, 24, 34, 123, 234, 1234\}$$

Our main result on g-tilings is the following

Theorem 2.2. *For any generalized tiling T (obeying (T1)–(T4)), the spectrum B_T is a semi-normal TP-basis. Conversely, for any semi-normal TP-basis B, there exists a generalized tiling T such that $B_T = B$.*

(In particular, the cardinalities of the spectra of all g-tilings on Z_n are the same and equal to $\binom{n+1}{2} + 1$.) The first part of this theorem will be proved in Section 5, and the second one in Section 4.

We will explain in Section 7 that for each semi-normal basis B, there are precisely one proper wiring W and precisely one g-tiling T such that $B_W = B_T = B$ (see Theorem 7.5); this is similar to the one-to-one correspondence between the normal bases and pure tilings.

In what follows, when it is not confusing, we may speak of a vertex or edge of G_T as a vertex or edge of T. The map σ of the tiles in T to squares in $conv(2^{[n]})$ is extended, in a natural way, to the vertices, edges, subgraphs or other objects in G_T. Note that the embedding of $\sigma(G_T)$ in the disc D_T is *planar* (unlike G_T and Z, in general), i.e., any two edges of $\sigma(G_T)$ can meet only at their end points. It is convenient to assume that the clockwise orientations on Z and D_T are agreeable, in the sense that the image by σ of the boundary cycle $(z_0, z_1^\ell, \dots, z_n^\ell, z_1^r, \dots, z_n^r = z_0)$ is oriented clockwise around the interior of D_T. Then the orientations on a tile $\tau \in T$ and on the square $\sigma(\tau)$ are consistent when τ is white, and different when τ is black.

3. Elementary properties of generalized tilings

In this section we give additional definitions and notation and demonstrate several consequences from axioms (T1)–(T4) which will be used later on. Let T be a g-tiling on $Z = Z_n$.

1. Let us say that the edges of T occurring in black tiles (as side edges) are *black*, and the other edges of T are *white*. For a vertex v and two edges $e, e' \in E_T(v)$, let $\Theta(e, e')$ denote the cone (with angle $< \pi$) in the plane pointed at v and generated by these edges (ignoring their directions). When another edge $e'' \in E_T(v)$ (a tile $\tau \in F_T(v)$) is contained in $\Theta(e, e')$, we say that e'' (resp. τ) lies *between* e and e'. When these e, e' are edges of a tile τ, we also write $\Theta(\tau; v)$ for $\Theta(e, e')$ (the conic hull of τ at v), and denote by $\theta(\tau, v)$ the angle of this cone taken with sign + if τ is white, and sign − if τ is black. The sum $\sum(\theta(\tau, v)\colon\ \tau \in F_T(v))$ is denoted by $\theta(v)$ and called the *full angle* at v. Axioms (T1)–(T3) show that terminal vertices behave as follows.

Corollary 3.1. *Let v be a terminal vertex belonging to a black ij-tile τ. Then:*

- (i) *v is not connected by edge with any other terminal vertex of T; therefore, $E_T(v)$ contains exactly two black edges, namely, those belonging to τ;*
- (ii) *$E_T(v)$ contains at least one white edge and all such edges e, as well as all tiles in $F_T(v)$, lie in the cone $\Theta(\tau; v)$; in particular, e is a q-edge with $i < q < j$;*
- (iii) *$\theta(v) = 0$;*
- (iv) *v does not belong to the boundary of Z; in particular, each boundary edge e of Z, as well as the tile containing e, is white.*

Indeed, since each edge of G_T belongs to some tile, at least one of its end vertices has both entering and leaving edges, and therefore (by (T3)), this vertex cannot be terminal (yielding (i)). Next, if $|E_T(v)| = 2$, then $F_T(v)$ would consist only of the tile τ and its white copy; this is not the case by (T1). Assume that $v = t(\tau)$. Then v is the top vertex of all tiles in $F_T(v)$ (by (T3)). This together with the facts that all tiles in $F_T(v) - \{\tau\}$ are white and that any two white tiles sharing an edge do not overlap (by (T2)) implies (ii) and (iii). When $v = b(\tau)$, the argument is similar. Finally, v cannot be a boundary vertex z^ℓ_k or z^r_k for $k \neq 0, n$ since the latter vertices have both entering and leaving edges. In case $v = z_0$, the tile τ would contain both boundary edges $z_0 z^\ell_1$ and $z_0 z^r_{n-1}$ (in view of (iii)). But then the white tile sharing with τ the edge $rt(\tau)$ would trespass the boundary of Z, which is impossible. The case $v = z^\ell_n$ is impossible for a similar reason. This yields (iv).

In view of (ii) in this corollary, we have:

(3.1) if a black ij-tile τ and a white tile τ' share an edge e, then: (a) either $b\ell(\tau) = b\ell(\tau')$ or $br(\tau) = br(\tau')$ or $\ell t(\tau) = \ell t(\tau')$ or $rt(\tau) = rt(\tau')$; (b) τ' is either an iq-tile or a qj-tile for some $i < q < j$; and (c) τ lies in the cone $\Theta(\tau'; w)$, where w is the nonterminal vertex of e.

2. The following lemma specifies the full angle at nonterminal vertices.

Lemma 3.2. *Let v be a nonterminal vertex of T.*

(i) *If v belongs to $bd(Z)$, then $\theta(v)$ is equal to the (positive) angle between the boundary edges incident to v.*
(ii) *If v is inner (i.e., not in $bd(Z)$), then $\theta(v) = 2\pi$.*

Proof. (i) For $v \in bd(Z)$, let e, e' be the boundary edges incident to v, where e, Z, e' follow clockwise around v. Consider the maximal sequence $S = (e = e_0, \tau_1, e_1, \dots, \tau_r, e_r)$ of edges in $E_T(v)$ and tiles in $F_T(v)$ such that for $q = 1, \dots, r$, e_{q-1} and e_q are distinct edges of the tile τ_q, and $\tau_q \neq \tau_{q+1}$ (when $q < r$). Using (3.1), one can see that all tiles in S are different and give the whole $F_T(v)$; also $e_r = e'$ and the tiles τ_1, τ_r are white. For each q, the ray at v containing e_q is obtained by rotating the ray at v containing e_{q-1} by the angle $\theta(\tau_q, v)$ (where the rotation is clockwise if the angle is positive). Summing the angles at v over the tiles in S, we obtain the angle of e, e'.

To show (ii), let $V := V_T$ and $E := E_T$. Also denote the set of terminal vertices by V^t, and the set of *inner* nonterminal vertices by $\hat{V}$. Since the boundary of Z contains $2n$ vertices and by (i),

$$|V| = |V^t| + |\hat{V}| + 2n \quad \text{and} \quad \sum_{v \in V \cap bd(Z)} \theta(v) = \pi \cdot 2n - 2\pi = 2\pi(n-1). \tag{3.2}$$

Let $\Sigma := \sum(\theta(v)\colon v \in V)$ and $\hat{\Sigma} := \sum(\theta(v)\colon v \in \hat{V})$. The contribution to Σ from each white (black) tile is 2π (resp. -2π). Therefore, $\Sigma = 2\pi(|T^w| - |T^b|)$. On the other hand, in view of Corollary 3.1(iii) and the second relation in (3.2), $\Sigma = \hat{\Sigma} + 2\pi(n-1)$. Then

$$\hat{\Sigma} = 2\pi\big(|T^w| - |T^b| - n + 1\big). \tag{3.3}$$

Considering G_T as a planar graph embedded (by σ) in the disc D_T and applying Euler formula to it, we have $|V| + |T| = |E| + 1$. Each tile has four edges, the number of boundary edges is $2n$, and each inner edge belongs to two tiles; therefore, $|E| = 2n + (4|T| - 2n)/2 = 2|T| + n$. Then $|V|$ is expressed as

$$|V| = |E| - |T| + 1 = 2|T| + n - |T| + 1 = |T| + n + 1. \tag{3.4}$$

Also $|V| = |\hat{V}| + 2|T^b| + 2n$ (using the first equality in (3.2) and the equality $|V^t| = 2|T^b|$). This and (3.4) give

$$|\hat{V}| = |V| - 2|T^b| - 2n = \big(|T| + n + 1\big) - 2|T^b| - 2n = |T^w| - |T^b| - n + 1.$$

Comparing this with (3.3), we obtain $\hat{\Sigma} = 2\pi|\hat{V}|$. Now the desired equality $\theta(v) = 2\pi$ for each vertex $v \in \hat{V}$ follows from the fact that $\theta(v)$ equals $2\pi \cdot d$ for some integer $d \geqslant 1$. The latter is shown as follows. Let us begin with a white tile $\tau_1 \in F_T(v)$ and its edges $e_0, e_1 \in E_T(v)$, in this order clockwise, and form a sequence $e_0, \tau_1, e_1, \dots, \tau_r, e_r, \dots$ similar to that in (i) above, until we return to the initial edge e_0. Let R_q be the ray at v containing e_q. Since $\theta(\tau_q, v) > 0$ when τ_q is white, and $\theta(\tau_q, v) + \theta(\tau_{q+1}, v) > 0$ when τ_q is white and τ_{q+1} is black (cf. (3.1)), the current ray $R_\bullet$ must make at least one turn clockwise before it returns to the initial ray R_0. If the sequence uses not all tiles in $F_T(v)$ (which is, in fact, impossible by (T4)), we can start with a new white tile to form a next sequence (for which the corresponding ray makes at least one turn clockwise as well), and so on. Thus, $d \geqslant 1$, as required (implying $d = 1$). □

Remark 1. If we postulate property (ii) in Lemma 3.2 as axiom (T4′) and add it to axioms (T1)–(T3), then we can eliminate axiom (T4); in other words, (T4′) and (T4) are equivalent subject to (T1)–(T3). Indeed, reversing reasonings in the above proof, one can conclude that $\hat{\Sigma} = 2\pi|\hat{V}|$ implies $|V| + |T| = |E| + 1$. The latter is possible only if D_T is a disc. (Indeed, if D_T forms a regular surface with g handles and c cross-caps, from which an open disc is removed, then Euler formula is modified as $|V| + |T| = |E| + 1 - 2g - c$. Also $|V|$ decreases when some vertices merge.) Note that each of the axioms (T1)–(T3), (T4′) is "local"; this gives rise to a local characterization for semi-normal bases.

3. Using (3.1) and Lemma 3.2, one can obtain the following useful description of the local structure of edges and tiles at nonterminal vertices.

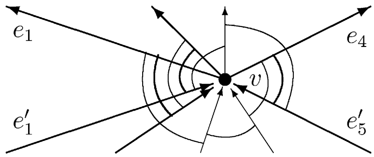
Corollary 3.3. *Let v be a nonterminal (ordinary or mixed) vertex of T different from z_0, z_n. Let $e_1, \dots, e_p$ be the sequence of edges leaving v and ordered clockwise around v (i.e., by increasing their colors), and $e'_1, \dots, e_{p'}$ the sequence of edges entering v and ordered counterclockwise around v (i.e., by decreasing their colors). Then there are integers $r, r' \geqslant 0$ such that:*

(i) *$r + r' < \min\{p, p'\}$, the edges $e_{r+1}, \dots, e_{p-r'}$ and $e'_{r+1}, \dots, e'_{p'-r'}$ are white, the other edges in $E_T(v)$ are black, $r = 0$ if $v \in \ell bd(Z)$, and $r' = 0$ if $v \in rbd(Z)$;*
(ii) *for $q = r + 1, \dots, p - r' - 1$, the edges e_q, e_{q+1} are spanned by a white tile (so such tiles have the bottom at v and lie between e_{r+1} and $e_{p-r'}$);*
(iii) *for $q = r + 1, \dots, p' - r' - 1$, the edges e'_q, e'_{q+1} are spanned by a white tile τ (so such tiles have the top at v and lie between e'_{r+1} and $e'_{p'-r'}$);*
(iv) *unless $v \in \ell bd(Z)$, each of the pairs $\{e_1, e'_{r+1}\}, \{e_2, e'_r\}, \dots, \{e_{r+1}, e'_1\}$ is spanned by a white tile, and each of the pairs $\{e_1, e'_r\}, \{e_2, e'_{r-1}\}, \dots, \{e_r, e'_1\}$ is spanned by a black tile (all tiles have the right vertex at v);*
(v) *unless $v \in rbd(Z)$, each of the pairs $\{e_p, e'_{p'-r'}\}, \{e_{p-1}, e'_{p'-r'+1}\}, \dots, \{e_{p-r'}, e'_{p'}\}$ is spanned by a white tile, and each of the pairs $\{e_p, e'_{p'-r'+1}\}$, $\{e_{p-1}, e'_{p'-r'+2}\}, \dots, \{e_{p-r'+1}, e'_{p'}\}$ is spanned by a black tile (all tiles have the left vertex at v).*

In particular, (a) there is at least one white edge leaving v and at least one white edge entering v; (b) the tiles in (ii)–(v) give a full list of tiles in $F_T(v)$; and (c) any two tiles $\tau, \tau' \in F_T(v)$ with $r(\tau) = \ell(\tau') = v$ do not overlap (have no common interior point).

Also: for $v = z_0, z_n$, all edges in $E_T(v)$ are white and consecutive pairs of these edges are spanned by white tiles.

(When v is ordinary, we have $r = r' = 0$.) The case with $p = 4$, $p' = 5$, $r = 2$, $r' = 1$ is illustrated in the picture; here the black edges are drawn in bold and the thin (bold) arcs indicate the pairs of edges spanned by white (resp. black) tiles.



Note that Corollary 3.3 implies the following property (which will be used, in particular, in Section 4.3):

(3.5) for a tile $\tau \in T$ and a vertex $v \in \{\ell(\tau), r(\tau)\}$, let e, e' be the edges of τ entering and leaving v, respectively, and suppose that there is an edge $\tilde{e} \neq e, e'$ incident to v and lying between e and e'; then $\tilde{e}$ is black; furthermore: (a) e' is black if $\tilde{e}$ enters v; (b) e is black if $\tilde{e}$ leaves v.

4. We will often use the fact (implied by (2.1)(ii)) that for any g-tiling T,

(3.6) the graph $G_T = (V_T, E_T)$ is *graded* for each color $i \in [n]$, which means that for any closed path P in G_T, the numbers of forward i-edges and backward i-edges in P are equal.

Hereinafter, a path in a directed graph is meant to be a sequence $P = (\tilde{v}_0, \tilde{e}_1, \tilde{v}_1, \dots, \tilde{e}_r, \tilde{v}_r)$ in which each $\tilde{e}_p$ is an edge connecting vertices $\tilde{v}_{p-1}, \tilde{v}_p$; an edge $\tilde{e}_p$ is called *forward* if it is directed from $\tilde{v}_{p-1}$ to $\tilde{v}_p$ (denoted as $\tilde{e}_p = (\tilde{v}_{p-1}, \tilde{v}_p)$), and *backward* otherwise (when $\tilde{e}_p = (\tilde{v}_p, \tilde{v}_{p-1})$). When $v_0 = v_r$ and $r > 0$, P is a *closed* path, or a *cycle*. The path P is called *directed* if all its edges are forward, and *simple* if all vertices $v_0, \dots, v_r$ are different. P^{rev} denotes the reversed path $(\tilde{v}_r, \tilde{e}_r, \tilde{v}_{r-1}, \dots, \tilde{e}_1, \tilde{v}_0)$.

4. From semi-normal bases to generalized tilings

In this section we prove the second assertion in Theorem 2.2, namely, the inclusion

$$\mathcal{B}_n \subseteq \mathcal{BT}_n, \tag{4.1}$$

where $\mathcal{B}_n$ is the set of semi-normal bases in $2^{[n]}$ and $\mathcal{BT}_n$ denotes the collection of the spectra of g-tilings on Z_n. The proof falls into three parts, given in Sections 4.1–4.3.

4.1. Flips in g-tilings

Let T be a g-tiling. By an *M-configuration* in T we mean a quintuple of vertices of the form Xi, Xj, Xk, Xij, Xjk with $i < j < k$ (as it resembles the letter "M"), which is denoted as $CM(X; i, j, k)$. By a *W-configuration* in T we mean a quintuple of vertices Xi, Xk, Xij, Xik, Xjk with $i < j < k$ (as resembling "W"), denoted as $CW(X; i, j, k)$. A configuration is called *feasible* if all five vertices are nonterminal, i.e., belong to B_T.

We know that any normal basis B (in particular, $B = \mathcal{I}_n$) is expressed as B_T for some pure tiling T, and therefore, $B \in \mathcal{BT}_n$. Thus, to conclude with (4.1), it suffices to prove the following assertion, which says that the set of g-tilings is closed under transformations analogous to weak flips for semi-normal bases.

Proposition 4.1. *Let a g-tiling T contain five nonterminal vertices Xi, Xk, Xij, Xjk, Y, where $i<j<k$ and $Y\in\{Xik, Xj\}$. Then there exists a g-tiling T' such that $B_{T'}$ is obtained from B_T by replacing Y by the other member of $\{Xik, Xj\}$.*

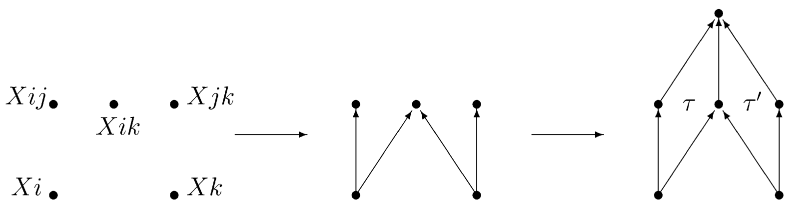
Proof. We may assume that $Y=Xik$, forming a feasible W-configuration $CW(X;i,j,k)$ (since an M-configuration in T turns into a W-configurations in the reversed g-tiling T^{rev}). We rely on the following two facts which will be proved in Sections 4.2 and 4.3.

(4.2) Any pair of nonterminal vertices of the form X', $X'i'$ in T is connected by edge.

(Therefore, T as above contains the edges (Xi, Xij), (Xi, Xik), (Xk, Xik) and (Xk, Xjk). Note that vertices X', $X'i'$ need not be connected by edge if some of them is terminal; e.g., in the picture before the statement of Theorem 2.2, the vertices with $X'=\emptyset$ and $i'=2$ are not connected.)

(4.3) T contains the jk-tile τ with $b(\tau)=Xi$ and the ij-tile τ' with $b(\tau')=Xk$.

Then $\ell(\tau)=Xij$, $r(\tau)=\ell(\tau')=Xik$, $r(\tau')=Xjk$, and $t(\tau)=t(\tau')=Xijk$. Since the vertices Xi, Xk are nonterminal, both tiles τ, τ' are white. See the picture.



Assuming that (4.2) and (4.3) are valid, we argue as follows. First of all we observe that

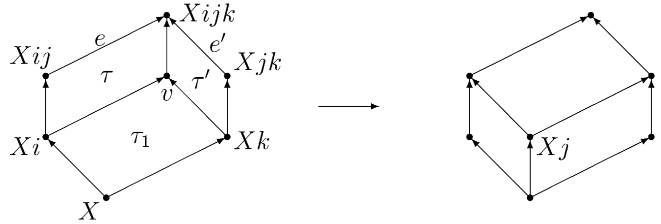
(4.4) the vertex $v:=Xik$ is ordinary.

Indeed, since both vertices Xi, Xik are nonterminal, the edge (Xi, Xik) cannot belong to a black tile. So this edge (which belongs to the white tile τ and enters v) is white. Also the edge $(Xik, Xijk)$ of τ that leaves v is white (for if it belongs to a black tile $\bar\tau$, then $\bar\tau$ should have $v':=Xijk$ as its top vertex, but then the cone of $\bar\tau$ at v' cannot simultaneously contain both edges $(Xij, Xijk)$ and $(Xjk, Xijk)$, contrary to Corollary 3.1(ii)). Now one can conclude from Corollary 3.3(b) that there is no black tile having its left or right vertex at v. So v is ordinary.

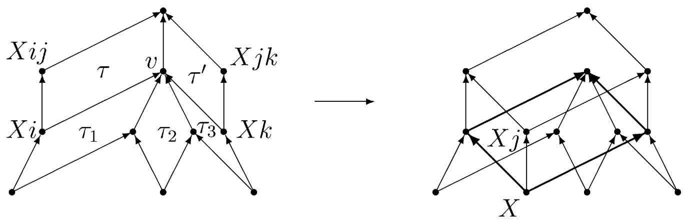
Let $e_0,\dots,e_q$ be the sequence of edges entering v in the counterclockwise order; then $e_0=(Xi, Xik)$ and $e_q=(Xk, Xik)$. Since v is ordinary, each pair e_{p-1}, e_p $(p=1,\dots,q)$ belongs to a white tile τ_p. Two cases are possible.

Case 1. The edges $e:=(Xij, Xijk)$ and $e':=(Xjk, Xijk)$ do not belong to the same black tile. Consider two subcases.

(a) Let $q=1$. Then the tiles τ, τ', τ_1 give a subdivision of a hexagon, and we replace in T these tiles by three new white tiles: $\tau(X;i,j)$, $\tau(X;j,k)$ and $\tau(Xj;i,k)$. So the vertex $v=Xik$ is replaced by Xj. See the picture.

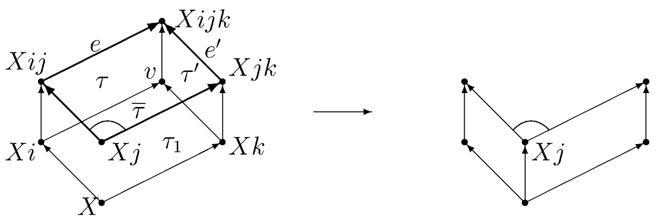


(b) Let $q > 1$. We remove the tiles τ, τ' and add four new tiles: the white tiles $\tau(X; i, j)$, $\tau(X; j, k)$, $\tau(Xj; i, k)$ (as before) and the black tile $\tau(X; i, k)$ (so v becomes terminal). See the picture with $q = 3$; here the added black tile is indicated in bold.

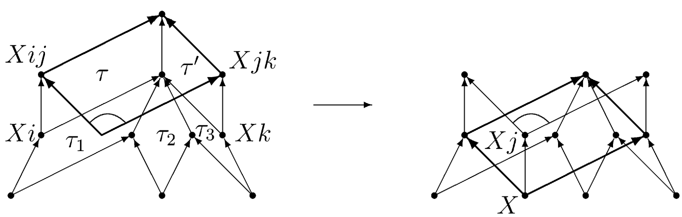


Case 2. Both edges e and e' belong to a black tile $\bar{\tau}$ (which is nothing else than $\tau(Xj; i, k)$). We act as in Case 1 with the only differences that $\bar{\tau}$ is removed from T and that the white ik-tile at Xj (which is a copy of $\bar{\tau}$) is not added. Then the vertex $Xijk$ vanishes, v either vanishes or becomes terminal, and Xj becomes nonterminal. See the picture; here (a') and (b') concern the subcases $q = 1$ and $q > 1$, respectively, and the arc above the vertex Xj indicates the bottom cone of $\bar{\tau}$ in which some white edges (not indicated) are located.

(a')



(b')



Let T' be the resulting collection of tiles. It is routine to check that in all cases the transformation of T into T' maintains the conditions on tiles and edges involved in axioms (T1)–(T3) at the vertices Xi, Xk, Xij, Xjk, as well as at the vertices Xik and $Xijk$ when the last ones do not vanish. Also the conditions continue to hold at the vertex X in Cases 1(a) and 2(a') (with $q = 1$), and at the vertex Xj in Case 2 (when the terminal vertex Xj becomes nonterminal). A less trivial task is to verify for T' the correctness at Xj in Case 1 and at X in Cases 1(b) and 2(b'). We assert that

(4.5) (i) V_T does not contain Xj in Case 1; and
(ii) V_T does not contain X in Cases 1(b) and 2(b').

Then these vertices (in the corresponding cases) are indeed new in the arising T', and now the required properties for them become evident by the construction. Note that this implies (T4) as well. We will prove (4.5) in Section 4.3.

Thus, assuming validity of (4.2), (4.3), (4.5), we can conclude that T' is a g-tiling and that $B_{T'} = (B_T - \{Xik\}) \cup \{Xj\}$, as required. □

Remark 2. Adopting terminology used for set-systems, we say that for the g-tilings T, T' as in the proof of Proposition 4.1, T' is obtained from T by the (weak) *lowering flip* w.r.t. the feasible W-configuration $CW(X; i, j, k)$. One can see that Xi, Xj, Xk, Xij, Xjk are nonterminal vertices of T'; so they form a feasible M-configuration for it. Moreover, one can check that the corresponding lowering flip applied to the reverse of T' results in the g-tiling T^{rev}. Equivalently: the *raising flip* for T' w.r.t. the configuration $CM(X; i, j, k)$ returns the initial T. An important consequence of this fact will be demonstrated in Section 7 (see Theorem 7.5).

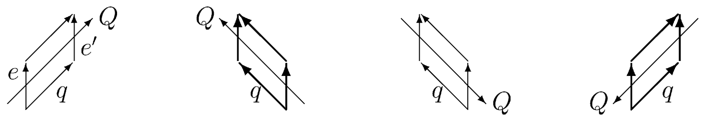
4.2. Strips in a g-tiling

In this subsection we show property (4.3), subject to (4.2). For this purpose, we introduce the following notion (which will be extensively used subsequently as well).

Definition. For $i \in [n]$, an *i-strip* (or a *dual i-path*) in a g-tiling T is a maximal sequence $Q = (e_0, \tau_1, e_1, \ldots, \tau_r, e_r)$ of edges and tiles in it such that: (a) $\tau_1, \ldots, \tau_r$ are different tiles, each being an iq- or qi-tile for some q, and (b) for $p = 1, \ldots, r$, e_{p-1} and e_p are the opposite i-edges of τ_p.

(Recall that speaking of an $i'j'$-tile, we assume that $i' < j'$.) Clearly Q is determined uniquely, up to reversing it and up to shifting cyclically when $e_0 = e_r$, by any of its edges or tiles. Also, unless $e_0 = e_r$, one of e_0, e_r lies on the left boundary, and the other on the right boundary of Z; we default assume that Q is directed so that $e_0 \in \ell bd(Z)$. In this case, going along Q, step by step, and using (T2), one can see that

(4.6) for consecutive elements e, τ, e' in an i-strip Q: (a) if τ is either a white iq-tile or a black qi-tile (for some q), then e leaves $b(\tau)$ and e' enters $t(\tau)$; and (b) if τ is either a white qi-tile or a black iq-tile, then e enters $t(\tau)$ and e' leaves $b(\tau)$ (see the picture where the i-edges e, e' are drawn vertically and q stands for the color of corresponding edges).



Let v_p (resp. v'_p) be the beginning (resp. end) vertex of an edge e_p in Q. Define the *right boundary* of Q to be the path $R_Q = (v_0, a_1, v_1, \ldots, a_r, v_r)$, where a_p is the edge of τ_p connecting

v_{p-1}, v_p. The *left boundary* L_Q of Q is defined in a similar way, regarding the vertices v'_p. From (4.6) it follows that

(4.7) for an i-strip Q, the forward edges of R_Q are exactly those edges in it that belong to either a white iq-tile or a black qi-tile in Q, and similarly for the forward edges of L_Q.

For $I \subseteq [n]$, we call a *maximal alternating I-subpath* in R_Q a maximal subsequence P of *consecutive* elements in R_Q such that each edge $a_p \in P$ is a q-edge with $q \in I$, and in each pair a_p, a_{p+1}, one edge is forward and the other is backward in R_Q (i.e., one of the tiles τ_p, τ_{p+1} is black). A maximal alternating I-subpath in L_Q is defined in a similar way. The following fact is of importance.

Lemma 4.2. *A strip Q cannot be cyclic, i.e., its first and last edges are different.*

Proof. For a contradiction, suppose that some i-strip $Q = (e_0, \tau_1, e_1, \dots, \tau_r, e_r)$ is cyclic ($e_0 = e_r$). One may assume that (4.6) holds for Q (otherwise reverse Q). Let $a_1, \dots, a_r = a_0$ be the sequence of edges of the right boundary R_Q of Q. For $q \in [n]$, let α_q (β_q) denote the number of forward (resp. backward) q-edges in R_Q. Since G_T is graded, $\alpha_q = \beta_q$ (cf. (3.6)).

Assume that R_T contains a q-edge with $q > i$. Put $I^> := [i+1..n]$ and consider a maximal alternating $I^>$-subpath P in R_Q (regarding Q up to shifting cyclically and taking indices modulo r). Using (3.1), we observe that if a_p is an edge in P such that τ_p is black, then the edges a_{p-1}, a_{p+1} are contained in P as well; also both tiles τ_{p-1}, τ_{p+1} are white. This together with (4.7) implies that the difference Δ_P between the number of forward edges and the number of backward edges in P is equal to 0 or 1, and that $\Delta_P = 0$ is possible only if P coincides with the whole R_Q (having equal numbers of forward and backward edges). On the other hand, the sum of numbers Δ_P over all maximal alternating $I^>$-subpaths P must be equal to $\sum_{q>i}(\alpha_q - \beta_q) = 0$.

So R_Q is an alternating $I^>$-cycle. To see that this is impossible, notice that if a_{p-1}, a_p, a_{p+1} are q'-, q-, and q''-edges, respectively, and if the tile τ_p is black, then (3.1) implies that $q', q'' < q$. Therefore, taking the maximum edge color q in R_Q, we obtain $\alpha_q = 0$ and $\beta_q > 0$; a contradiction. Thus, R_Q has no q-edges with $q > i$ at all.

Similarly, considering maximal alternating $[i-1]$-subpaths in R_Q and using (3.1) and (4.7), we conclude that R_Q has no q-edge with $q < i$. Thus, a cyclic i-strip is impossible. □

Corollary 4.3. *For a g-tiling T and each $i \in [n]$, there is a unique i-strip Q_i. It contains all i-edges of T, begins at the edge $z^\ell_{i-1} z^\ell_i$ and ends at the edge $z^r_i z^r_{i-1}$ of $bd(Z)$.*

Based on strip techniques, we now prove property (4.3) in the assumption that (4.2) is valid (the latter will be discussed in the next subsection).

Proof of (4.3). Let X, i, j, k be as in the hypotheses of Proposition 4.1 (with $Y = Xik$). We consider the part Q of the j-strip between the j-edges $e := (Xi, Xij)$ and $e' := (Xk, Xjk)$ (these edges exist by (4.2) and Q exists by Corollary 4.3). Suppose that Q begins at e and ends at e' and consider the right boundary R_Q of Q. This is a path from Xi to Xk; let $a_1, \dots, a_r$ be the sequence of edges in it. Comparing R_Q with the path $\tilde{P}$ from Xi to Xk formed by the forward k-edge (Xi, Xik) and the backward i-edge (Xik, Xk), we have (since G_T is graded):

$$\alpha_q - \beta_q = \begin{cases} -1 & \text{for } q = i, \\ 1 & \text{for } q = k, \\ 0 & \text{otherwise}, \end{cases} \tag{4.8}$$

where α_q (β_q) is the number of forward (resp. backward) q-edges in R_Q. We show that $\alpha_i = 0$, $\beta_i = 1$, $\alpha_k = 1$, $\beta_k = 0$ and $\alpha_q = \beta_q = 0$ for $q \neq i, k$, by arguing as in the proof of Lemma 4.2.

Let $P_1, \ldots, P_d$ be the sequence of maximal alternating $J^>$-subpaths in R_Q, where $J^> := [j+1..n]$. Each path P_h begins and ends with forward edges (taking into account (3.1), (4.7) and the fact that the edges e, e' are white). Therefore, $\Delta_{P_h} = 1$. Then $\Delta_{P_1} + \cdots + \Delta_{P_d} = \sum_{q>j}(\alpha_q - \beta_q) = \alpha_k - \beta_k = 1$ (cf. (4.8)) implies $d = 1$. Moreover, $|P_1| = 1$. For if $|P_1| > 1$, then P_1 contains a backward edge (belonging to a black tile in Q), and taking the maximum edge color q in P_1, we obtain $\alpha_q = 0$ and $\beta_q > 0$, which is impossible. Hence P_1 consists of a unique forward edge, and now (4.8) implies that it is a k-edge.

By similar reasonings, there is only one maximal alternating $[j-1]$-subpath P' in R_Q, and P' consists of a unique backward i-edge.

Thus, $R_Q = (a_1, a_2)$, and one of a_1, a_2 is a forward k-edge, while the other is a backward i-edge in R_Q. If $R_Q = \tilde{P}$ (i.e., a_1 is a k-edge), then the tiles in Q are as required in (4.3). The case when a_1 is an i-edge is impossible. Indeed, this would imply that the first tile τ in Q is generated by the edges $a_1 = (X, Xi)$ and $e = (Xi, Xij)$; but then the cone of τ at Xi contains the white edge (Xi, Xik), contrary to (3.5).

Now suppose that Q goes from e' to e. Then L_Q begins at Xk and ends at Xi. Define the numbers α_q, β_q as before. Then the sum $\sum_{q>j}(\alpha_q - \beta_q)$ (equal to the number of maximal alternating $J^>$-subpaths in L_Q) is nonnegative. But a similar value for the path reverse to $\tilde{P}$ (going from Xk to Xi as well) is equal to -1, due to the k-edge (Xi, Xik) which is backward in this path; a contradiction. □

4.3. Strip contractions

The remaining properties (4.2) and (4.5) are proved by induction on n, relying on a natural contracting operation on g-tilings (also important for purposes of Sections 8, 9).

Let T be a g-tiling on Z_n and $i \in [n]$. Partition T into three subsets T_i^0, T_i^-, T_i^+ consisting, respectively, of all $i*$- and $*i$-tiles, of the tiles $\tau(X; i', j')$ with $i', j' \neq i$ and $i \notin X$, and of the tiles $\tau(X; i', j')$ with $i', j' \neq i$ and $i \in X$. Then T_i^0 is the set of tiles in the i-strip Q_i, and the tiles in T_i^- are vertex disjoint from those in T_i^+.

Definition. The *i-contraction* of T is the collection T/i of tiles obtained from T by removing the members of T_i^0, keeping the members of T_i^-, and replacing each $\tau(X; i', j') \in T_i^+$ by $\tau(X - \{i\}; i', j')$. The black/white coloring of tiles in T/i is inherited from T.

The tiles of T/i live within the zonogon generated by the vectors ξ_q, $q \in [n] - \{i\}$ (and cover this zonogon). The regions $D_{T_i^-}$ and $D_{T_i^+}$ in the disc D_T are simply connected, as they arise when the interior of (the image by σ of) the strip Q_i is removed from D_T. The shape $D_{T/i}$ is obtained as the union of $D_{T_i^-}$ and $D_{T_i^+} - \epsilon_i$, where ϵ_i is the i-th unit base vector in $\mathbb{R}^{[n]}$. In other words, $D_{T_i^+}$ is shifted by $-\epsilon_i$ and the (image by σ of) the left boundary L_{Q_i} of Q_i in it merges with (the image of) R_{Q_i} in $D_{T_i^-}$. In general, $D_{T_i^-}$ and $D_{T_i^+} - \epsilon_i$ may intersect at some other points, and therefore, $D_{T/i}$ need not be a disc (this happens when G_T contains two vertices X, Xi not connected by edge, or equivalently, such that $X \notin R_{Q_i}$ and $Xi \notin L_{Q_i}$).

For our purposes, it suffices to deal with the case $i = n$. We take advantages from the important property that

(4.9) T/n is a feasible g-tiling, i.e., it obeys (T1)–(T4).

Instead of a direct proof of this property (which is rather tiresome technically), we prefer to appeal to explanations in Section 7 where a similar property is obtained on the language of wirings; see Corollary 7.4 in Remark 4. (More precisely, by results in Sections 6, 7, there is a bijection β of the g-tilings to the proper wirings. Furthermore, one shows that removing w_n from a proper n-wiring $W = (w_1, \dots, w_n)$ results in a proper $(n-1)$-wiring W'. It turns out that the g-tiling $\beta^{-1}(W')$ is just $(\beta^{-1}(W))/n$, yielding the desired property.)

Assuming that (4.9) is valid, we make two observations, exposed in (i), (ii) below.

(i) Validity of (T4) for T/n implies that

(4.10) any two vertices of the form $X', X'n$ in G_T are connected by an (n-)edge in Q_n (in particular, $X' \in R_{Q_n}$ and $X'n \in L_{Q_n}$).

(ii) For a black $i'j'$-tile $\tau \in T$ with $i', j' \neq n$, none of the terminal vertices $t(\tau), b(\tau)$ can occur in (the boundary of) Q_n. Indeed, all edges incident to such a vertex are q-edges with $q \leqslant j' < n$, whereas each vertex occurring in Q_n is incident to an n-edge. Also for a vertex X' not in Q_n, the local tile structure of T/n at $X' - \{n\}$ (including the white/black coloring of tiles) is inherited by that of T at X'. It follows that

(4.11) if X' is a nonterminal vertex for T, then $X' - \{n\}$ is such for T/n.

Now we are ready to prove (4.2) and (4.5).

Proof of (4.2). Let $X', X'i'$ be nonterminal vertices for T. If $i' = n$ then these vertices are connected by edge in G_T, by (4.10). Now let $i' \neq n$. Then $G_{T/n}$ contains the vertices $\tilde{X}, \tilde{X}i'$, where $\tilde{X} := X' - \{n\}$, and these vertices are nonterminal, by (4.11). We assume by induction that these vertices are connected by an edge e in $G_{T/n}$. Let τ' be a tile in T/n containing e. Then the tile $\tau \in T_n^- \cup T_n^+$ generating τ' has an edge connecting X' and $X'i'$, as required. □

Proof of (4.5). We use notation as in the proof of Proposition 4.1 and consider three possible cases.

(A) Let $k < n$ and $n \notin X$. Then all tiles in T containing the vertex $v = Xik$ are tiles in T/n, and Xi, Xk, Xik, Xij, Xjk are vertices for T/n forming a feasible W-configuration in it (as they are nonterminal, by (4.11)). By induction $G_{T/n}$ contains as a vertex neither Xj in Case 1, nor X in Cases 1(b) and 2(b’). Then the same is true for G_T, as required.

(B) Let $k < n$ and $n \in X$. The argument is similar to that in (A) (taking into account that all vertices X' for T that we deal with contain the element n, and the corresponding vertices for T/n are obtained by removing this element).

(C) Let $k = n$. First we consider Case 1 and show (i) in (4.5). Suppose G_T contains the vertex Xj. Then G_T contains the n-edge $\tilde{e} = (Xj, Xjn)$, by (4.10). This edge lies between the edges

$br(\tau') = (Xn, Xjn)$ and $e' = rt(\tau') = (Xjn, Xijn)$ of the white tile τ'. By (3.5), the presence of the edge $\tilde{e}$ (entering Xjn) in the cone $\Theta(\tau'; Xjn)$ implies that the edge e' of τ' is black. Let $\tilde{\tau}$ be the black tile containing e'; then $\tilde{\tau}$ has the top vertex at $Xijn$ (since Xjn is nonterminal). Obviously, the n-edge $e = (Xij, Xijn)$ coincides with $\ell t(\tilde{\tau})$. So both e, e' are edges of the same black tile $\tilde{\tau}$, which is not the case.

Now we consider Cases 1(b) and 2(b') and show (ii) by arguing in a similar way. Suppose G_T contains the vertex X. Then G_T contains the n-edge $\tilde{e} = (X, Xn)$, by (4.10). This edge lies in the cone of τ_q at Xn (where, according to notation in the proof of Proposition 4.1, τ_q is the white tile in T with $rt(\tau_q) = (Xn, Xin)$). By (3.5), the edge $rt(\tau_q)$ is black (since $\tilde{e}$ enters Xn). But each of the end vertices Xn, Xin of $rt(\tau_q)$ has both entering and leaving edges, and therefore, it cannot be terminal; a contradiction. □

This completes the proof of inclusion (4.1) (provided validity of (4.9)).

5. From generalized tilings to semi-normal bases

In this section we complete the proof of Theorem 2.2 by proving the first assertion in it, namely, we show the inclusion

$$\mathcal{BT}_n \subseteq \mathcal{B}_n. \tag{5.1}$$

This together with the reverse inclusion (4.1) will give $\mathcal{BT}_n = \mathcal{B}_n$, as required.

Let T be a g-tiling on Z_n. We have to prove that B_T is a semi-normal basis.

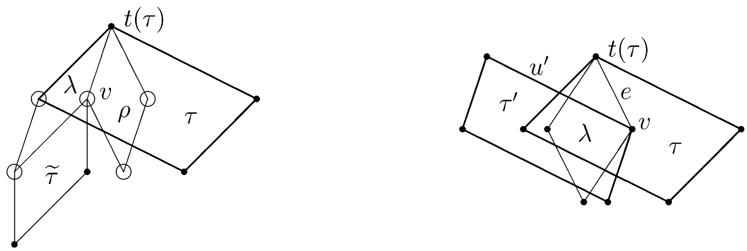
If T has no black tile, then B_T is a normal basis, and we are done. So assume $T^b \neq \emptyset$. Our aim is to show the existence of a feasible W-configuration $CW(X; i, j, k)$ for T (formed by nonterminal vertices Xi, Xk, Xij, Xik, Xjk, where $i < j < k$). Then we can transform T into a g-tiling T' as in Proposition 4.1, i.e., with $B_{T'} = (B_T - \{Xik\}) \cup \{Xj\}$. Under such a *lowering flip* (concerning g-tilings), the sum of sizes of the sets involved in $B_\bullet$ decreases. Then the required relation $B_T \in \mathcal{B}_n$ follows by induction on $\sum(|X'|\colon X' \in B_T)$ (this sort of induction is typical when one deals with tilings or related objects, cf. [2,4,9]).

In what follows by the *height* $h(v)$ of a vertex $v \in V_T$ we mean the size of the corresponding subset of $[n]$. The height $h(\tau)$ of a tile $\tau \in T$ is defined to be the height of its left vertex; then $h(\tau) = h(r(\tau)) = h(b(\tau)) + 1 = h(t(\tau)) - 1$. The height of a W-configuration $CW(X; i, j, k)$ is defined to be $|X| + 2$.

In fact, we are able to show the following sharper version of the desired property.

Proposition 5.1. *Let $h \in [n]$. If a g-tiling T has a black tile of height h, then there exists a feasible W-configuration $CW(X; i, j, k)$ of the same height h. Moreover, such a $CW(X; i, j, k)$ can be chosen so that $Xijk$ is the top vertex of some black tile (of height h).*

Proof. Let τ be a black tile of height h. Denote by $M(\tau)$ the set of vertices v such that there is a *white* edge from v to $t(\tau)$. This set is nonempty (by Corollary 3.1(ii)) and each vertex in it is nonterminal. Suppose that some $v \in M(\tau)$ is ordinary, and let λ and ρ be the (white) tiles sharing the edge $(v, t(\tau))$ and such that $v = r(\lambda) = \ell(\rho)$. Then the five vertices $b(\lambda), b(\rho), \ell(\lambda), v, r(\rho)$ form a W-configuration of height h (since $h(v) = h(\tau) = h$). Moreover, this configuration is feasible. Indeed, the vertices $\ell(\lambda), v, r(\rho)$ are nonterminal (since each has an entering edge and a leaving edge). And the tile $\tilde{\tau}$ that shares the edge $(b(\lambda), v)$ with λ has v as its top vertex (taking into account that $\tilde{\tau}$ is white and overlaps neither λ nor ρ since v is ordinary); then $b(\lambda)$ is the left vertex of $\tilde{\tau}$, and therefore $b(\lambda)$ is nonterminal. The vertex $b(\rho)$ is nonterminal for a similar reason. For an illustration, see the left fragment on the picture.



We assert that a black tile τ of height h whose set $M(\tau)$ contains an ordinary vertex does exist (yielding the result).

Suppose this is not so. Let us construct an alternating sequence of white and black edges as follows. Choose a black tile τ of height h and a vertex $v \in M(\tau)$. Let e be the white edge $(v, t(\tau))$. Since v is mixed (by the supposition), there is a black tile τ' (of height h) such that either (a) $v = r(\tau')$ or (b) $v = \ell(\tau')$. We say that τ' *lies on the left from* τ in the former case, and *lies on the right from* τ in the latter case. Let u' be the edge $rt(\tau')$ of τ' in case (a), and $\ell t(\tau')$ in case (b). Case (a) is illustrated on the right fragment of the above picture.

Repeat the procedure for τ': choose $v' \in M(\tau')$ (which is mixed again by the supposition); put $e' := (v', t(\tau'))$; choose a black tile τ'' such that either (a) $v' = r(\tau'')$ or (b) $v' = \ell(\tau'')$; and define u' to be the edge $rt(\tau'')$ in case (a), and $\ell t(\tau'')$ in case (b). Repeat the procedure for τ'', and so on. Sooner or later we must return to a black tile that has occurred earlier in the process. Then we obtain an alternating cycle of white and black edges.

More precisely, there appear a cyclic sequence of different black tiles $\tau_1, \dots, \tau_{r-1}, \tau_r = \tau_0$ of height h and an alternating sequence of white and black edges $C = (e_0 = e_r, u_1, e_1, \dots, u_r = u_0)$ (forming a cycle in G_T) with the following properties:

- for $q = 1, \dots, r$: (a) e_q is the edge $(v_q, t(\tau_q))$ for some $v_q \in M(\tau_q)$; (b) τ_{q+1} is a black tile whose right of left vertex is v_q; and (c) $u_{q+1} = rt(\tau_{q+1})$ when $r(\tau_{q+1}) = v_q$, and $u_{q+1} = \ell t(\tau_{q+1})$ when $\ell(\tau_{q+1}) = v_q$,

where the indices are taken modulo r. We consider C up to renumbering the indices cyclically and assume that τ_q is an $i_q k_q$-tile, that e_q is a j_q-edge, and that u_q is a p_q-edge. Then $i_q < j_q < k_q$, $p_q = i_q$ if τ_q lies on the left from τ_{q-1}, and $p_q = k_q$ if τ_q lies on the right from τ_{q-1}. Note that in the former (latter) case the vertex v_q lies on the left (resp. right) from v_{q-1} in the horizontal line at height h in Z_n. This implies that there exists a q such that τ_q lies on the left from τ_{q-1}, and there exists a q' such that $\tau_{q'}$ lies on the right from $\tau_{q'-1}$.

To come to a contradiction, consider a maximal subsequence Q of consecutive tiles in the above sequence in which each but first tile lies on the left from the previous one; one may assume that $Q = (\tau_1, \tau_2, \dots, \tau_d)$. Then τ_1 lies on the right from τ_0; so $u_1 = \ell t(\tau_1)$, whence $p_1 = k_1$. We observe that

$$k_1 \geqslant k_2 \geqslant \cdots \geqslant k_d. \tag{5.2}$$

Indeed, for $1 \leqslant q < d$, let λ be the (white) tile containing the edge e_q and such that $r(\lambda) = v_q$ (then $e_q = rt(\lambda)$). This tile lies in the cone of τ_q at $t(\tau_q)$. So λ is an $i'k'$-tile with $i_q \leqslant i' < k' \leqslant k_q$, and therefore the edge $\tilde{e} := br(\lambda)$ entering v_q has color $k' \leqslant k_q$. Since the edge e_q of λ is white, we observe from Corollary 3.3 that the edge $\tilde{e}$ of λ lies in the cone of the black tile τ_{q+1} at $v_q = r(\tau_{q+1})$. This implies that the edge $e' := br(\tau_{q+1})$ (entering v_q) has color at most k'. Therefore, we obtain $k_{q+1} \leqslant k' \leqslant k_q$, as required.

By (5.2), we have $j_q < k_q \leqslant k_1 = p_1$ for all $q = 1, \dots, d$. Also if a tile $\tau_{q'}$ lies on the right from the previous tile $\tau_{q'-1}$, then $u_{q'} = \ell t(\tau_{q'})$, whence $j_{q'} < k_{q'} = p_{q'}$. Thus, the maximum of $p_1, \dots, p_r$ is strictly greater than the maximum of $j_1, \dots, j_r$. This is impossible since all $u_1, \dots, u_r$ are forward edges, all $e_1, \dots, e_r$ are backward edges in C, and the graph G_T is graded. □

This completes the proof of Theorem 2.2.

Remark 3. For black tiles $\tau, \tau' \in T^b$, let us denote $\tau' \prec^\diamond \tau$ if there is a white edge $(v, t(\tau))$ such that v is the right or left vertex of τ'. The proof of Proposition 5.1 gives the following additional result.

Corollary 5.2. *The relation $\prec^\diamond$ determines a partial order on T^b.*

Similarly, the relation $\prec_\diamond$ determines a partial order on T^b, where for $\tau, \tau' \in T^b$, we write $\tau' \prec_\diamond \tau$ if there is a white edge $(b(\tau), v)$ such that v is $r(\tau')$ or $\ell(\tau')$.

We conclude this section with one more result which follows from Proposition 5.1.

Proposition 5.3. *Let a g-tiling T be such that, for some $h < n$, all nonterminal vertices of height $h+1$ are intervals in $[n]$ and there is no feasible W-configuration of height h. Then all nonterminal vertices of height h are intervals as well. Symmetrically, if for some $h > 0$, all nonterminal vertices of height $h-1$ are co-intervals and there is no feasible M-configuration of height h, then all nonterminal vertices of height h are co-intervals.*

Proof. If T has a black tile of height h, then there exists a feasible W-configuration of height h, by Proposition 5.1. So this is not the case.

Let u be a nonterminal vertex of height h and take an edge (u, v). Then the vertex v (of height $h+1$) is nonterminal (for otherwise v would be the top vertex of a black tile of height h). Let τ, τ' be the tiles sharing the edge (u, v); then both tiles are white and non-overlapping. Suppose $b(\tau) = u$. Then both vertices $\ell(\tau), r(\tau)$ lie in level $h+1$, and therefore, they are intervals. This easily implies that u is an interval as well. Similarly, u is an interval if $b(\tau') = u$.

Now suppose that u is neither $b(\tau)$ nor $b(\tau')$. Then $t(\tau) = t(\tau') = v$. Letting for definiteness that $u = r(\tau) = \ell(\tau')$, we obtain that the vertices $u, b(\tau), b(\tau'), \ell(\tau), r(\tau')$ form a feasible W-configuration of height h; a contradiction (the case when $b(\tau)$ or $b(\tau')$ is terminal is impossible, otherwise it would belong to a black tile of height h).

The second assertion in the proposition follows from the first one applied to T^{rev}. □

(Note also that if all nonterminal vertices of height h in a g-tiling are intervals, then there is no feasible W-configuration of height h. Indeed, suppose such a configuration $CW(X; i, j, k)$ exists. Then Xik is a nonterminal vertex of height h. Since $i < j < k$ and $j \notin X$, the set Xik is not an interval.)

In view of the coincidence of the set of spectra of g-tilings on Z_n with the set of largest weakly separated collections in $2^{[n]}$ (proved in Section 9), Proposition 5.3 answers affirmatively Conjecture 5.5 in Leclerc and Zelevinsky [9].

Finally, analyzing the proof of Proposition 5.3, one can see that this proposition can be slightly strengthened as follows: if there is no feasible W-configuration of height h, then each nonterminal vertex $Y \subset [n]$ of height h not contained in the boundary of Z_n is representable as $Y' \cap Y''$ for some nonterminal vertices $Y', Y'' \subset [n]$ of height $h+1$. (Similarly, if there is no feasible M-configuration of height h, then each nonterminal vertex Y of height h not contained in the boundary of Z_n is representable as $Y' \cup Y''$ for some nonterminal vertices Y', Y'' of height $h-1$.)

6. From generalized tilings to proper wirings

In this section we show the following

Proposition 6.1. *For any g-tiling T on Z_n, there exists a proper wiring W on Z_n such that $B_W = B_T$.*

This and the converse assertion established in the next section will imply that the collection $\mathcal{BT}_n$ of the spectra B_T of g-tilings T on Z_n coincides with the collection $\mathcal{BW}_n$ of the spectra B_W of proper wirings W on Z_n, and then Theorem 2.1 will follow from Theorem 2.2.

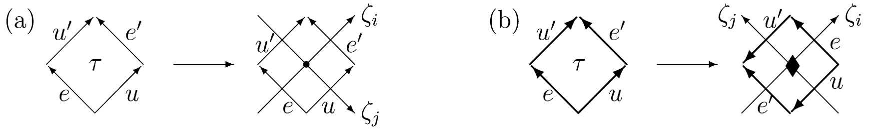
Proof of Proposition 6.1. For convenience we use the same notation for vertices, edges and tiles concerning a g-tiling T on $Z = Z_n$ and their corresponding points, line-segments and squares (respectively) in the disc D_T (every time it will be clear from the context or explicitly indicated which of Z and D_T we deal with). Accordingly, the planar graph $G_T = (V_T, E_T)$ is regarded as properly embedded in D_T.

In order to construct the desired wiring, we first draw curves on D_T corresponding to strips (dual paths) in G_T. More precisely, for each $i \in [n]$, take the i-strip $Q_i = (e_0, \tau_1, e_1, \dots, \tau_r, e_r)$ for T (defined in Section 4), considering it as the corresponding sequence of edges and squares in D_T. (Recall that Q_i contains all $i*$- and $*i$-tiles in T, e_0 is the edge $z^\ell_{i-1} z^\ell_i$ on the left boundary $\ell bd(Z)$, and e_r is the edge $z^r_i z^r_{i-1}$ on $rbd(Z)$; cf. Corollary 4.3.) For $q = 1, \dots, r$, draw the line-segment on the square τ_q connecting the median points of the edges e_{r-1} and e_r. This segment meets the central point of τ_q, denoted by $c(\tau_q)$. The concatenation of these segments gives the desired (piece-wise linear) curve ζ_i corresponding to Q_i; we direct ζ_i according to the direction of Q_i.

Fix a homeomorphic map $\gamma : D_T \to Z$ such that each boundary edge of D_T is linearly mapped onto the corresponding edge of $bd(Z)$. Then the curves (“wires”) ζ_i on D_T generate the wires $w_i := \gamma(\zeta_i)$ on Z (where w_i begins at the median point s_i of $z^\ell_{i-1} z^\ell_i$ on $\ell bd(Z)$ and ends at the medial point s'_i of $z^r_i z^r_{i-1}$ on $rbd(Z)$). We assert that the wiring $W = (w_1, \dots, w_n)$ is as required in the proposition.

Obviously, W satisfies axiom (W1). To verify the other axioms, we first explain how the planar graphs G_T and G_ζ on D_T are related to each other, where G_ζ is the “preimage” by γ of the graph G_W (defined as in Section 2.1 by considering $\zeta_1, \dots, \zeta_n$ in place of $w_1, \dots, w_n$). The vertices of G_ζ are the central points $c(\tau)$ of squares τ (where corresponding wires cross one another) and the points s_i, s'_i. (We identify corresponding points on the boundaries of D_T and Z, writing s_i for $\gamma^{-1}(s_i)$.) Each vertex v of G_T one-to-one corresponds to the face of G_ζ where v is located, denoted by v^*. The edges of color i in G_ζ (which are the pieces of ζ_i in its subdivision by the central points of squares lying on ζ_i) one-to-one correspond to the i-edges of G_T. More precisely, if an i-edge $e \in E_T$ belongs to squares τ, τ' and if τ, e, τ' occur in this order in the i-strip, then the i-edge of G_ζ corresponding to e, denoted by e^*, is the piece of ζ_i between $c(\tau)$ and $c(\tau')$, and this e^* is directed from $c(\tau)$ to $c(\tau')$. Observe that e crosses e^* *from right to left* on the disc. (We assume that the clockwise orientation on D_T is agreeable by γ with that of Z.) The first and last pieces of ζ_i correspond to the boundary i-edges $z^\ell_{i-1} z^\ell_i$ and $z^r_i z^r_{i-1}$ of G_T, respectively.

Consider an ij-tile $\tau \in T$, and let e, e' be its i-edges, and u, u' its j-edges, where e, u leave $b(\tau)$ and e', u' enter $t(\tau)$. We know that: (a) if τ is white, then e occurs in Q_i before e', while u occurs in Q_j after u', and (b) if τ is black, then e occurs in Q_i after e', while u occurs in Q_j before u'. In each case, in the disc D_T, both e, e' cross the wire ζ_i from right to left (w.r.t. the direction of ζ_i), and similarly both u, u' cross ζ_j from right to left. Also (by (T1), (T2)) when τ is white, the orientation of the tile τ in Z coincides with that of the square τ in D_T, whereas when τ is black, the clockwise orientation of τ in Z turns into the counterclockwise orientation of τ in D_T (causing the "orientation-reversing" behavior of wires at the vertex $c(\tau)$ of G_ζ). It follows that: in case (a), ζ_j crosses ζ_i at $c(\tau)$ from left to right, and therefore, the vertex $c(\tau)$ of G_ζ is white, and in case (b), ζ_j crosses ζ_i at $c(\tau)$ from right to left, and therefore, the vertex $c(\tau)$ is black. (Both cases are illustrated in the picture.) So the white (black) tiles of T generate the white (resp. black) vertices of G_ζ.



(in Z) (in D_T) (in Z) (in D_T)

Consider a vertex v of G_T and an edge $e \in E_T(v)$. Then the edge e^* belongs to the boundary of the face v^* of G_ζ. As mentioned above, e crosses e^* from right to left on D_T. This implies that e^* is directed clockwise around v^* if e leaves v, and counterclockwise if e enters v. In view of axiom (T3), we obtain that

(6.1) the terminal vertices of G_T and only these generate cyclic faces of $G_W \simeq G_\zeta$; moreover, for $\tau \in T^b$, the boundary cycle of $(t(\tau))^*$ is directed counterclockwise, while the boundary cycle of $(b(\tau))^*$ is directed clockwise.

(We use the fact that any nonterminal vertex $v \neq z_0, z_n$ has both entering and leaving edges, and therefore, the boundary of the face v^* has edges in both directions. When $v = z_0$ ($v = z_n$), a similar fact for v^* is valid as well, since v^* contains the edges (z_0, s_1) and (z_0, s'_n) (resp. (s_n, z_n) and (s'_1, z_n)) lying on the boundary of Z.)

Next, for each $i \in [n]$, removing from D_T the interior of the i-strip Q_i (i.e., the relative interiors of all edges and tiles in it) results in two closed regions Ω_1, Ω_2, the former containing the vertex $\emptyset$, and the latter containing the vertex $[n]$ (regarding the vertices as subsets of $[n]$). The fact that all edges in Q_i (which are the i-edges of G_T) go from Ω_1 to Ω_2 implies that each vertex v of G_T occurring in Ω_1 (in Ω_2) is a subset of $[n]$ not containing (resp. containing) the element i. So $i \notin X(v^*)$ if $v \in \Omega_1$, and $i \in X(v^*)$ if $v \in \Omega_2$. This implies the desired equality for spectra: $B_W = B_T$.

A less trivial task is to check validity of (W2) for W. One can see that axiom (W2) is equivalent to the condition that if wires w_i, w_j intersect at a white point x, then the parts of w_i, w_j after x do not meet. So it suffices to show the following

Claim. *Let wires ζ_i, ζ_j with $i < j$ intersect at a white point x. Then the part ζ of ζ_i from x to s'_i and the part ζ' of ζ_j from x to s'_j have no other common points.*

Proof. Suppose this is not so and let y be the common point of ζ, ζ' next to x in ζ. Since x is white, y is black. Therefore, the ij-tile τ such that $x = c(\tau)$ is white, and the ij-tile τ' such that $y = c(\tau')$ is black. Also in both strips Q_i and Q_j, the tile τ occurs earlier than τ'. One can see (cf. (4.6)) that in the strip Q_i, the edge succeeding τ is $rt(\tau)$ and the edge preceding τ' is $rt(\tau')$, whereas in the strip Q_j, the edge succeeding τ is $br(\tau)$ and the edge preceding τ' is $br(\tau')$. So the right boundary of Q_i passes the vertices $r(\tau)$ and $r(\tau')$, in this order, and similarly for the left boundary of Q_j.

Consider the part R of R_{Q_i} from $r(\tau)$ to $r(\tau')$ and the part L of L_{Q_j} from $r(\tau)$ to $r(\tau')$. For $q \in [n]$, let $\alpha_q, \alpha'_q, \beta_q, \beta'_q$ be the numbers of q-edges that are forward in R, forward in L, backward in R, and backward in L, respectively. Since G_T is graded, we have $(*)$ $\alpha_q - \beta_q = \alpha'_q - \beta'_q$.

Next we argue in a similar spirit as in the proof of Lemma 4.2. Define $I := [i+1..j-1]$, $\Delta := \sum_{q \in I}(\alpha_q - \beta_q)$, and $\Delta' := \sum_{q \in I}(\alpha'_q - \beta'_q)$. We assert that $\Delta > 0$ and $\Delta' < 0$, which leads to a contradiction with $(*)$ above.

To see $\Delta > 0$, consider a q-edge e in R with $q \in I$, and let τ^e denote the tile in Q_i containing e. Since $q > i$, τ^e is white if e is forward, and τ^e is black if e is backward in R (cf. (4.7)). Using this, one can see that:

(i) for a q-edge $e \in R$ such that $q \in I$ and τ^e is black, the next edge e' in R_{Q_i} is a forward q'-edge in R with $q' \in I$ (since the fact that τ^e is a black iq-tile implies that $\tau^{e'}$ is a white $q'q$-tile with $i < q' < q$, in view of (3.1)); a similar property holds for the edge in R_{Q_i} preceding e;
(ii) the last edge e of R is a forward q-edge with $q \in I$ (since the tile τ^e shares an edge with the black ij-tile τ');
(iii) if the first edge e of R is backward, then it is a q-edge with $q \notin I$ (since τ^e is black and shares an edge with the white ij-tile τ).

These observations show that the first and last edges of any maximal alternating I-subpath P in R are forward, and therefore, P contributes $+1$ to Δ. Also at least one such P exists, by (ii). So $\Delta > 0$, as required.

The inequality $\Delta' < 0$ is shown in a similar way, by considering L and swapping "forward" and "backward" in the above reasonings (due to replacing $q > i$ by $q < j$). More precisely, for a q-edge e in L with $q \in I$, the tile τ^e in Q_j containing e is black if e is forward, and white if e is backward (in view of $q < j$ and (4.7)). This implies that:

(i') for a q-edge $e \in L$ such that $q \in I$ and τ^e is black, the next edge e' in L_{Q_j} is a backward q'-edge in L with $q' \in I$; and similarly for the edge in L_{Q_j} preceding e;
(ii') the last edge e of L is a backward q-edge with $q \in I$;
(iii') if the first edge e of R is forward, then it is a q-edge with $q \notin I$.

Then the first and last edges of any maximal alternating I-subpath P in L are backward, and therefore, P contributes -1 to Δ'. Also at least one such P exists, by (ii'). Thus, $\Delta' < 0$, obtaining a contradiction with $\Delta = \Delta'$. $\square$

Thus, (W2) is valid. Considering lenses formed by a pair of wires and using (6.1) and (W2), one can easily obtain (W3). Finally, since $|E_T(v)| \geqslant 3$ for each terminal vertex v in G_T (by Corollary 3.1(ii)), each cyclic face in G_W is surrounded by at least three edges, and therefore, this face cannot be a lens. So the wiring W is proper.

This completes the proof of Proposition 6.1. $\square$

7. From proper wirings to generalized tilings

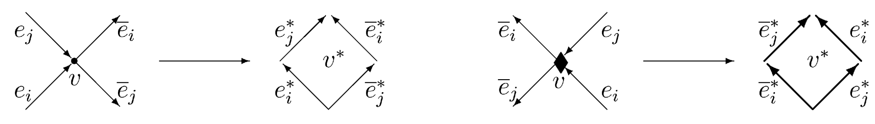
In this section we complete the proof of Theorem 2.1 by showing the converse to Proposition 6.1.

Proposition 7.1. *For a proper wiring W on $Z = Z_n$, there exists a g-tiling T on Z such that $B_T = B_W$.*

Proof. The construction of the desired T is converse, in a sense, to that described in the proof of Proposition 6.1; it combines planar duality techniques and geometric arrangements.

We associate to each (inner) face F of the graph G_W the point (viz. subset) $X(F)$ in the zonogon, also denoted as F^*. These points are just the vertices of tiles in T. The edges concerning T are defined as follows. Let faces $F, F' \in \mathcal{F}_W$ have a common edge e formed by a piece of a wire w_i, and let F lie on the right from w_i according to the direction of this wire (and F' lies on the left from w_i). Then the vertices F^*, F'^* are connected by edge e^* going from F^* to F'^*. Note that in view of the evident relation $X(F') = X(F) \cup \{i\}$, the direction of e^* matches the edge direction for g-tilings.

The tiles in T one-to-one correspond to the intersection points of wires in W. More precisely, let v be a common point of wires w_i, w_j with $i < j$. Then the vertex v of G_W has four incident edges $e_i, \bar{e}_i, e_j, \bar{e}_j$ such that: $e_i, \bar{e}_i \subset w_i$; $e_j, \bar{e}_j \subset w_j$; e_i, e_j enter v; and $\bar{e}_i, \bar{e}_j$ leave v. Also one can see that for the four faces F containing v, the subsets $X(F)$ are of the form X, Xi, Xj, Xij for some $X \subset [n]$. The tile surrounded by the edges $e_i^*, \bar{e}_i^*, e_j^*, \bar{e}_j^*$ connecting these subsets (regarded as points) is just the ij-tile in T corresponding to v, denoted as v^*. Observe that the edges $e_i, e_j, \bar{e}_i, \bar{e}_j$ follow in this order counterclockwise around v if v is black (orientation-reversing), and clockwise if v is white. The tile v^* is regarded as black in T if v is black, and white otherwise. Both cases are illustrated in the picture where the right fragment concerns the orientation-reversing case.



Next we examine properties of the obtained collection T of tiles. The first and second conditions in (T2) (concerning overlapping and non-overlapping tiles with a common edge) follow from the above construction and explanations.

1) Consider an i-edge $e = (u, v)$ in G_W (a piece of the wire w_i). If $u \neq s_i$ and $v \neq s'_i$, then the dual edge e^* belongs to exactly two tiles, namely, u^* and v^*. If $u = s_i$, then e^* belongs to

the unique tile v^*. Furthermore, for the faces $F, F' \in \mathcal{F}_W$ containing e, the sets $X(F)$, $X(F')$ are the principal intervals $[i-1]$ and $[i]$ (letting $[0] := \emptyset$). This implies that e^* is the boundary edge $z^\ell_{i-1}z^\ell_i$ of Z, and this edge belongs to a unique tile in T (which is, obviously, white). Considering $v = s'_i$, we obtain a similar property for the edges in $rbd(Z)$. This gives the first and second condition in (T1).

The proper wiring W possesses the following important property, which will be proved later (see Lemma 7.2): (∗) each face F in G_W has at most one i-edge for each i, and all sets $X(F)$ among $F \in \mathcal{F}_W$ are different. This implies that T has no tile copies, yielding the third condition in (T1). Also property (∗) and the planarity of G_W imply validity of axiom (T4).

2) For a face $F \in \mathcal{F}_W$, let $E(F)$ denote the set of its edges not contained in $bd(Z)$. By the construction and explanations above,

(7.1) the edges in $E(F)$ one-to-one correspond to the edges incident to the vertex $v = F^*$ of G_T; moreover, for $e \in E(F)$, the corresponding edge e^* enters v if e is directed counterclockwise around F, and leaves v otherwise.

This implies that v has both entering and leaving edges if and only if F is non-cyclic, unless $v = z_0$ or z_n. (Here we also use the observation that if F contains a vertex z^ℓ_i or z^r_i for some $1 \leqslant i < n$, then $E(F)$ has edges in both directions.)

Consider a cyclic face $F \in \mathcal{F}^{cyc}_W$, and let $C = (v_0, e_1, v_1, \dots, e_r, v_r = v_0)$ be its boundary cycle, where each edge e_p goes from v_{p-1} to v_p. Denote the color of e_p by i_p. Suppose C is directed clockwise. Then for each p, we have $i_p < i_{p+1}$ if v_p is white, and $i_p > i_{p+1}$ if v_p is black (taking the indices modulo r). Hence C contains at least one black point (for otherwise we would have $i_1 < \cdots < i_r < i_1$). Moreover,

(7.2) C contains exactly one black point, and if v_p is black, then the color of each edge of C is between i_{p+1} and i_p.

Indeed, let v_p be black. Then v_p is the root of the (even) lens L of wires $w_{i_{p+1}}$ and w_{i_p} such that $F \subseteq L$. By axiom (W3), L is bijective to F. The existence of another black vertex in C would cause the appearance of another lens bijective to F, which is impossible. This implies (7.2). As a consequence, the vertex F^* (which has leaving edges only, by (7.1)) is the bottom vertex of exactly one black tile. When C is directed counterclockwise, we have $i_p > i_{p+1}$ if v_p is white, and $i_p < i_{p+1}$ if v_p is black, and (7.2) is valid again. As a consequence, F^* is the top vertex of exactly one black tile.

Thus, T obeys (T3).

3) If a cyclic face F and another face F' in G_W have a common edge $e = (u, v)$, then F' is non-cyclic. Indeed, the edge e' preceding e in the boundary cycle of F enters the vertex u. The wire in W passing through e' leaves u by an edge e''. Obviously, e'' belongs to F'. Since the edges e, e'' of F' have the same beginning vertex, F' is non-cyclic. Hence the cyclic faces in G_W are pairwise edge-disjoint, implying that no pair of black tiles in T share an edge (the third condition in (T2)).

Thus, T is a g-tiling. If a face F of G_W lies on the left from a wire w_i, then the vertices F^* and $[n]$ occur in the same region when the interior of the i-strip is removed from the disc D_T. This implies that the sets $X(F)$, $F \in \mathcal{F}_W$, are just the vertices of T, i.e., the full spectra for T and W are the same. Now the correspondence between cyclic faces for W and terminal vertices for T yields $B_T = B_W$, as required. □

It remains to show the following (cf. $(*)$ in the above proof).

Lemma 7.2. *Let W be a proper wiring. Then:*

(i) *for each face F in G_W, all edges surrounding F belong to different wires;*
(ii) *there are no different faces $F, F' \in \mathcal{F}_W$ such that $X(F) = X(F')$.*

Proof. Suppose that a face F contains two i-edges e, e' for some i. One can see that:

(a) e, e' have the same direction in the boundary of F, and
(b) the face $F' \neq F$ containing e is different from the face $F'' \neq F$ containing e'.

Property (a) implies $X(F') = X(F'')$. Therefore, (i) follows from (ii).

To show (ii), we use induction on n (the assertion is obvious if $n = 2$). Let $W' := (w_1, \dots, w_{n-1})$. Clearly W' obeys axioms (W1), (W2). In its turn, (W3) for W' follows from the property that the cyclic faces for W' are exactly those cyclic faces for W that are bijective to the lenses formed by wires in W'. To see the latter, consider a lens L formed by wires w_i, w_j with $i, j < n$, the root v of L, and the face F in G_W such that $v \in F \subset L$. This face is cyclic (by (W3) for W), and the color of each edge in the boundary C of F is between i and j, by (7.2). So no edge of C belong to w_n, whence F is a (cyclic) face of $G_{W'}$ as well. Conversely, let F' be a cyclic face of $G_{W'}$. Then the boundary C' of F' must contain a black point v (by monotonicity reasonings as above). Let e, e' be the edges of C' incident to v; then their colors are strictly less than n. Take the face F of G_W such that $v \in F \subseteq F'$. The facts that v is black for W and that one of e, e' enters v and the other leaves v imply that the face F (which, obviously, contains e, e') is cyclic. By (7.2), the color of each edge in the boundary of F is less than n. This implies $F = F'$.

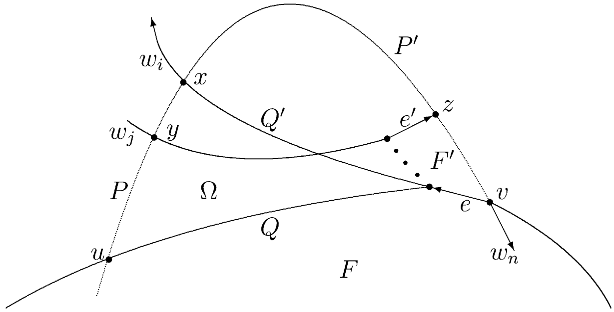
Thus, W' is a wiring (obeying (W1)–(W3)). Moreover, W' is proper. Indeed, suppose that some lens L formed by wires w_i, w_j $(i, j < n)$ is "empty," i.e., it is a face of $G_{W'}$. Let v be the root of L, and F the face of G_W such that $v \in F \subseteq L$. Then F is cyclic and the colors of all edges in its boundary are between i and j (by (7.2)). This implies $F = L$, contradicting the fact that W is proper.

Now we prove (ii) in the lemma as follows. By induction all sets $X(F)$, $F \in \mathcal{F}_{W'}$, are different. Suppose that there are different faces $\tilde{F}, \tilde{F}' \in \mathcal{F}_W$ such that $X(\tilde{F}) = X(\tilde{F}')$. Let F and F' be the faces for W' containing $\tilde{F}$ and $\tilde{F}'$, respectively. Then $X(F) = X(\tilde{F}) - \{n\}$ and $X(F') = X(\tilde{F}') - \{n\}$. This implies $X(F) = X(F')$, and therefore $F = F'$. Furthermore, w_n goes across F at least twice (for if it traversed F only once, we would have $F = \tilde{F} \cup \tilde{F}'$, which implies $X(\tilde{F}) \neq X(\tilde{F}')$).

It follows that w_n and the boundary of F have two common points u, v such that:

(a) u occurs in w_n earlier than v, and
(b) the piece P of w_n between u and v (not including u, v) lies outside F.

Let Q be the part of the boundary of F between u and v such that the simply connected region Ω surrounded by P and Q is disjoint from the interior of F. Consider the case when P goes clockwise around Ω; see the picture.



Let e be the edge of G_W contained in Q and incident to v; then e has color $i < n$. Take the maximal connected piece Q' of w_i lying in Ω and containing e. Since w_i does not meet the interior of F, the end x of Q' different from v lies on P. Then Q' and the piece P' of P from x to v form an in-lens L for W. Since P' is directed from x to v, Q' must be directed from v to x (by (W2)); in particular, e leaves v. So w_n crosses w_i at v from right to left, and therefore, the vertex v is black and is the root of L. Let F' be the cyclic face in G_W lying in L and containing v, and let C be its boundary cycle. Since W is proper, $F' \neq L$, whence $C \neq P' \cup Q'$. Then C contains an edge e' with color $j \neq i, n$ (one can take as e' the edge of C that either succeeds e or precedes the last edge on P). Take the maximal connected piece R of w_j, from a point y to a point z say, that lies in Ω and contains e'. It is not difficult to realize that y occurs in P earlier than z. This violates (W2) for w_j, w_n.

When P goes counterclockwise around Ω, a contradiction is shown in a similar way. (In this case, we take as e the edge on Q incident to u; one shows that e enters u, whence the vertex u is black.)

Thus, Lemma 7.2 is proven, and this completes the proof of Proposition 7.1. □

Propositions 6.1 and 7.1 imply the desired equality $\mathcal{BT}_n = \mathcal{BW}_n$, and now Theorem 2.1 follows from Theorem 2.2. Analyzing the transformation of a g-tiling into a proper wiring described in Section 6 and the converse transformation described above, one can conclude that their composition returns the initial g-tiling (or the initial proper wiring). This implies the following result (where, as before, B_T and $\hat{B}_T$ stand for the effective and full spectra of a g-tiling T, respectively, and similarly for wirings).

Theorem 7.3. *There is a bijection β of the set $\mathbf{T}_n$ of g-tilings to the set $\mathbf{W}_n$ of proper wirings on Z_n such that $B_T = B_{\beta(T)}$ holds for each $T \in \mathbf{T}_n$. Furthermore, for each proper wiring W, all subsets $X(F) \subseteq [n]$ determined by the faces F for W are different, and one holds $\hat{B}_W = \hat{B}_{\beta^{-1}(W)}$.*

We conclude this section with several remarks and additional results.

Remark 4. As is shown in the proof of Lemma 7.2, for any proper wiring $W = (w_1, \dots, w_n)$, the set $W' = (w_1, \dots, w_{n-1})$ forms a proper wiring as well (concerning the zonogon Z_{n-1}). Clearly a similar result takes place when we remove the wire w_1. As a generalization, we obtain that for any $1 \leqslant i < j \leqslant n$, the set $(w_i, \dots, w_j)$ forms a proper wiring on the corresponding subzonogon. One can see that removing w_n from W corresponds to the n-contraction operation applied to the g-tiling $\beta^{-1}(W)$ (described in Section 4.3), and that the resulting set T/n of tiles just corresponds to W'. This gives the following important result to which we have appealed in Section 4.

Corollary 7.4. *For a g-tiling T on Z_n, its n-contraction T/n is a g-tiling on Z_{n-1}.*

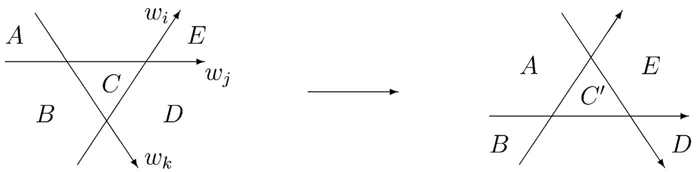
Remark 5. Properties of g-tilings and proper wirings established during the proofs of Theorems 2.1 and 2.2 enable us to obtain the following result saying that these objects are determined by their spectra.

Theorem 7.5. *For each semi-normal basis B, there are a unique g-tiling T and a unique proper wiring W such that $B = B_T = B_W$.*

Proof. Due to Theorem 7.3, it suffices to prove such a rigidity property for the set of g-tilings. We apply induction on $h(B) := \sum(|X|\colon X \in B)$. Suppose there are different g-tilings T, T' with $B_T = B_{T'} =: B$. This is impossible when none of T, T' has black tiles. Indeed, the vertices of G_T and $G_{T'}$ (which are the sets in B) are the same and they determine the edges of these graphs, by (4.2). So $G_T = G_{T'}$. This graph is planar and subdivides Z_n into little parallelograms, which are just the tiles in T and the tiles in T'. Then $T = T'$. Now let T (say) have a black tile. By Proposition 5.1, T has a feasible W-configuration $CW(X; i, j, k)$, and we can make the corresponding lowering flip for T, obtaining a g-tiling $\tilde{T}$ with $B_{\tilde{T}} = (B - \{Xik\}) \cup \{Xj\}$. Since $B_T = B_{T'}$, $CW(X; i, j, k)$ is a feasible W-configuration for T' as well, and making the corresponding lowering flip for T', we obtain a g-tiling $\tilde{T}'$ such that $B_{\tilde{T}'} = B_{\tilde{T}}$. We have $h(B_{\tilde{T}}) < h(B)$, whence, by induction, $\tilde{T} = \tilde{T}'$. But the raising flip in $\tilde{T}$ w.r.t. the (feasible) M-configuration $CM(X; i, j, k)$ returns T, as mentioned in Remark 2 in Section 4.1. Hence $T = T'$; a contradiction. □

One can develop an efficient procedure that, given the spectrum B_T of a g-tiling T, restores T itself (in essence, the procedure uses only "local" operations). This is provided by the possibility of efficiently constructing the graph G_T, as follows. We know that B_T is the set of nonterminal vertices of G_T, and the edges connecting these vertices are of the form (X, Xi) for all corresponding X, i; let G' be the graph formed by these vertices and edges. The goal is to construct the terminal vertices (if any) and the remaining edges of G_T (in particular, obtaining the full spectrum $\hat{B}_T$). This relies on the observation that each terminal vertex X one-to-one corresponds to a maximal collection $\mathcal{Y} \subset B_T$ such that $|\mathcal{Y}| \geqslant 3$ and: either (a) each $Y \in \mathcal{Y}$ is of the form $Y = Xi$ for some i and at least one member of $\mathcal{Y}$ has no entering edge in G'; or (b) each $Y \in \mathcal{Y}$ is of the form $Y = X - \{i\}$ for some i and at least one member of $\mathcal{Y}$ has no leaving edge in G'. In case (a), X is the bottom vertex of a black tile τ, and $E_T(X) = \{(X, Y)\colon Y \in \mathcal{Y}\}$. In case (b), X is the top vertex of a black tile τ, and $E_T(X) = \{(Y, X)\colon Y \in \mathcal{Y}\}$. So, by extracting all such collections $\mathcal{Y}$, we are able to obtain the whole G_T. Now to construct the tiles of T is easy (using Corollary 3.3). Note also that, in view of the equivalence (i) $\leftrightarrow$ (iv) in Theorem A, it is "easy" to decide, given a collection $B \subseteq 2^{[n]}$, whether or not B forms a semi-normal basis.

Remark 6. There is an alternative method of proving Theorems 2.1 and 2.2 in which the former theorem is proved directly and then the latter is obtained via the relationship of g-tilings and proper wirings established in this and previous sections. (Other possible methods: prove the first (second) assertion in Theorem 2.1 and the second (resp. first) assertion in Theorem 2.2.) This alternative method (which is beyond this paper) is based on ideas and techniques different from those applied in Sections 4, 5: the former extensively exploit Jordan curve theorem (a "topological" approach), while the latter extensively appeal to the fact that the graph of a g-tiling is graded for each color (a "geometrical" approach). For some illustration, let us briefly outline how the lowering flip is viewed on the language of wirings, for simplicity considering the situation corresponding to Case 1(a) in the proof of Proposition 4.1. Here we handle three wires w_i, w_j, w_k of a wiring W such that $i < j < k$ and there are five non-cyclic faces A, B, C, D, E whose local configuration is as illustrated on the left fragment of the picture below. The sets $X(A), X(B), X(C), X(D), X(E)$ are, respectively, Xij, Xi, Xik, Xk, Xjk, the face C looks like a triangle, and no other wire in W traverses some open neighborhood Ω of C.



The lowering flip replaces Xik by Xj. This corresponds to a deformation of the wire w_j within Ω which makes it pass below the intersection point of w_i and w_k, as illustrated on the right fragment of the picture. The triangle-shaped face C' arising instead of C satisfies $X(C') = Xj$ and is non-cyclic (as well as the modified A, B, D, E).

Note that if, in the initial wiring W, the face above C is also triangle-shaped (i.e., the wires w_i, w_k form a lens L containing C and such that w_j is the unique wire going across L), then the above flip turns L into a face and the wiring becomes non-proper. In this case the flip should be followed by the corresponding operation $() \to)($ which eliminates L and makes the wiring proper again. Such a transformation of W corresponds to that in Case 2(a') in the proof of Proposition 4.1.

8. n-Contraction and n-expansion

In Section 4.3 we introduced the n-contraction operation for a g-tiling on the zonogon Z_n. In this section we examine this operation more systematically. Then we introduce and study a converse operation that transforms a g-tiling on Z_{n-1} into a g-tiling on Z_n. The results obtained here (which are interesting by its own right) will be essentially used in Section 9.

Consider a g-tiling T on Z_n. Let P be the reversed path to the right boundary R_Q of the n-strip Q. It possesses a number of important features, as follows:

(8.1) For the path $P = (v_0, e_1, v_1, \dots, e_r, v_r)$ as above and the colors $i_1, \dots, i_r$ of its edges $e_1, \dots, e_r$, respectively, the following hold:
- (i) P begins at the minimal point z_0 of Z_n and ends at the point z^ℓ_{n-1};
- (ii) none of $v_0, \dots, v_r$ is the top or bottom vertex of a black ij-tile with $i, j < n$;
- (iii) P has no pair of consecutive backward edges;
- (iv) if $e_q = (v_{q-1}, v_q)$ and $e_{q+1} = (v_{q+1}, v_q)$ (i.e., e_q is forward and e_{q+1} is backward in P), then $i_q > i_{q+1}$;
- (v) if $e_q = (v_q, v_{q-1})$ and $e_{q+1} = (v_q, v_{q+1})$ (i.e., e_q is backward and e_{q+1} is forward in P), then $i_q < i_{q+1}$.

Indeed, the first and last edges of Q are $z^\ell_{n-1} z_n$ and $z_0 z^r_{n-1}$, yielding (i). Property (ii) follows from the facts that each vertex v_q has an incident n-edge (which belongs to Q) and that all edges incident to the top or bottom vertex of a black ij-tile have colors between i and j (cf. Corollary 3.1(ii)). The forward (backward) edges of P are the backward (resp. forward) edges of R_Q.

Therefore, each forward (backward) edge e_q of P belongs to a white (resp. black) $i_q n$-tile, taking into account the maximality of color n; cf. (4.7). Then for any two consecutive edges e_q, e_{q+1}, at least one of them is forward, yielding (iii). Next, let τ be the $i_q n$-tile (in Q) containing e_q, and τ' the $i_{q+1}n$-tile containing e_{q+1}. If e_q is forward and e_{q+1} is backward in P, then τ' is black, v_q is the left vertex of τ', and the i_q-edge e opposite to e_q in τ enters the top vertex of τ'. Since e lies in the cone of τ' at $t(\tau')$, we have $i_{q+1} < i_q < n$, as required is (iv). And if e_q is backward and e_{q+1} is forward, then τ is black and v_q is its bottom vertex. Since e_{q+1} lies in the cone of τ at $b(\tau)$, we have $i_q < i_{q+1} < n$, as required in (v).

Recall that the n-contraction operation applied to T shrinks the n-strip in such a way that R_Q merges with the left boundary L_Q of Q. From (8.1)(ii) it follows that in the resulting g-tiling T/n on Z_{n-1}, the path P as above no longer contains terminal vertices at all.

Next we describe a converse operation that transforms a pair consisting of an arbitrary g-tiling T' on Z_{n-1} and a certain path in $G_{T'}$ into a g-tiling on Z_n. To explain the construction, we first consider an arbitrary simple path P in $G_{T'}$ which begins at z_0, ends at the maximal point z^{ℓ}_{n-1} of Z_{n-1}, and may contain backward edges. Since the graph $G_{T'}$ has planar layout (by σ) in the disc $D_{T'}$, the path P subdivides $G_{T'}$ into two connected subgraphs $G' = G'_P$ and $G'' = G''_P$ such that: $G' \cup G'' = G_{T'}$, $G' \cap G'' = P$, G' contains $\ell bd(Z_{n-1})$, and G'' contains $rbd(Z_{n-1})$; we call G' (G'') the *left* (resp. *right*) *subgraph* w.r.t. P. Then each tile of T' becomes a face of exactly one of G', G'' (and all inner faces of G', G'' are such), and for an edge e of P not in $bd(Z_{n-1})$, the two tiles sharing e occur in different subgraphs. So T' is partitioned into two subsets, one being the set of faces of G', and the other of G''.

The *n-expansion operation* for (T', P) disconnects G', G'' by cutting $G_{T'}$ along P and then glue them by adding the corresponding n-strip. More precisely, we shift the vertices of G'' by the vector ξ_n, i.e., each vertex X in it changes to Xn; this induces the corresponding shift of edges and tiles in G''. The vertices of G' preserve. So each vertex X occurring in the path P produces two vertices, namely, X and Xn. As a result, for each edge $e = (X, Xi)$ of P, there appears its copy $\tilde{e} = (Xn, Xin)$ in the shifted G''; we connect e and $\tilde{e}$ by the corresponding (new) in-tile, namely, by $\tau(X; i, n)$. This added tile is colored white if e is a forward edge of P, and black if e is backward. The colors of all old tiles are inherited.

We refer to the resulting set T of tiles, with the partition into white and black ones, as the *n-expansion of* T' *along* P. Since the right boundary of the shifted G'' becomes the part of $rbd(Z_n)$ from the point z^r_{n-1} $(= \{n\})$ to z_n $(= [n])$, it follows that the union of the tiles in T is Z_n. Also it is easy to see that the shape D_T in $conv(2^{[n]})$ associated to T is again a disc (as required in axiom (T4)), and that T obeys axiom (T1). The path P generates the n-strip Q for T (consisting of the added $*n$-tiles and the edges of the form (X, Xn)), and we observe that $R_Q = P^{rev}$ and that L_Q is the shift of P^{rev} by ξ_n. Therefore, the n-contraction operation applied to T returns T'.

To ensure validity of the remaining axioms (T2) and (T3), we have to impose additional conditions on the path P. In fact, they are similar to those exposed in (8.1). Moreover, these conditions are necessary and sufficient.

Lemma 8.1. *Let* $P = (z_0 = v_0, e_1, v_1, \dots, e_r, v_r = z^{\ell}_{n-1})$ *be a simple path in* $G_{T'}$. *Then the following are equivalent:*

(i) *the n-expansion* T *of* T' *along* P *is a (feasible) g-tiling on* Z_n*;*
(ii) P *contains no terminal vertices for* T' *and satisfies (8.1)(iii), (iv), (v).*

Proof. Let P be as in (ii). We have to verify axioms (T2), (T3) for T. Let $P' = (v'_0, e'_1, v'_1, \dots, e'_r, v'_r)$ and $P'' = (v''_0, e''_1, v''_1, \dots, e''_r, v''_r)$ be the copies of P in the graphs G' and G'' (taken apart), respectively. It suffices to check conditions in (T2), (T3) for objects involving elements of P', P'' (since for any vertex of G' not in P', the structure of its incident edges and tiles, as well as the white/black coloring of tiles, is inherited from $G_{T'}$, and similarly for G'').

Consider a vertex v_q with $1 \leqslant q < r$. Let E^L_q (E^R_q) denote the set of edges in $E_{T'}(v_q)$ lying on the left (resp. right) in the zonogon when we move along P and pass through e_q, v_q, e_{q+1}; we include e_q, e_{q+1} in both E^L_q and E^R_q. Let F^L_q (F^R_q) denote the set of tiles in $F_{T'}(v_q)$ of which both edges incident to v_q belong to E^L_q (resp. E^R_q). Note that each tile $\tau \in F_{T'}(v_q)$ must occur in either F^L_q or F^R_q, i.e., τ is not separated (in Z_{n-1}) by e_q or e_{q+1} (taking into account that all vertices of P are nonterminal, and therefore the edges of P are white, and using Corollary 3.3). By the construction of G', G'', any two tiles of T' that share an edge not in P are faces of the same graph among G', G'', and if a tile $\tau \in T'$ has an edge contained in $\ell bd(Z_{n-1}) - P$ (resp. $rbd(Z_{n-1}) - P$), then τ is a face of G' (resp. G''). Using these observations, one can conclude that

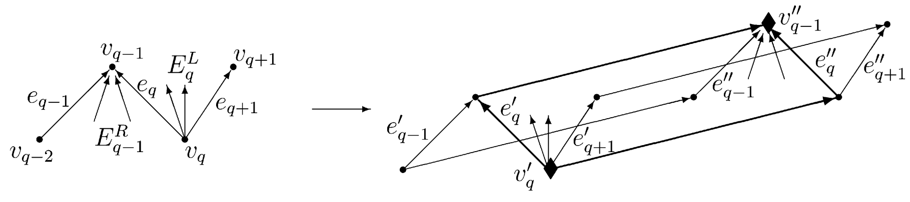
(8.2) for $1 \leqslant q < r$, E^L_q and F^L_q are entirely contained in G', while E^R_q and F^R_q are entirely contained in G''.

For $q = 1, \dots, r$, let τ^L_q (τ^R_q) denote the tile in T' (if exists) that contains the edge e_q and lies on the left (resp. right) when we traverse e_q from v_{q-1} to v_q. By (8.2), τ^L_q is in G' and τ^R_q is in G''. Also each of τ^L_q, τ^R_q is white. Let τ_q be the $i_q n$-tile in T that was added to connect the edges e'_q and e''_q. Then

$$e'_q = b\ell(\tau_q) \quad \text{and} \quad e''_q = rt(\tau_q). \tag{8.3}$$

Suppose that e_q is forward in P. Then τ_q is white. Since e_q is directed from v_{q-1} to v_q and τ^L_q lies on the left from e_q when moving from v_{q-1} to v_q, e_q belongs to the right boundary of τ^L_q. This and (8.3) imply that τ_q and τ^L_q do not overlap. In its turn, τ^R_q contains e_q in its left boundary; this together with (8.3) implies that τ_q and the shifted τ^R_q (sharing the edge e''_q) do not overlap as well. Now suppose that e_q is backward in P. Then τ_q is black. Since e_q is directed from v_q to v_{q-1} and τ^L_q lies on the left from e_q when moving from v_{q-1} to v_q, e_q belongs to the left boundary of τ^L_q. This implies that τ_q and τ^L_q overlap. Similarly, τ_q and τ^R_q overlap. Thus, (T2) holds for τ_q, τ^L_q and for τ_q, τ^R_q, as required. Also the non-existence of pairs of consecutive reverse edges in P implies that no two black tiles in T share an edge.

To verify (T3), consider a black tile τ_q. Then $1 < q < r$, the edges e_{q-1}, e_{q+1} are forward, and e_q is backward in P. Also $i_{q-1}, i_{q+1} > i_q$ (by (8.1)(iv), (v)). Observe that the set E^R_{q-1} consists of the edges in $E_{T'}(v_{q-1})$ that enter v_{q-1} and have color j such that $i_q \leqslant j \leqslant i_{q-1}$ (including e_{q-1}, e_q). All these edges are white (as is seen from Corollary 3.3). The second copies of these edges (shifted by ξ_n) plus the n-edge (v'_{q-1}, v''_{q-1}) are exactly those edges of G_T that are incident to the top vertex v''_{q-1} of τ_q. It its turn, the set E^L_q consists of the edges in $E_{T'}(v_q)$ that leave v_q and have color j such that $i_q \leqslant j \leqslant i_{q+1}$, and these edges are white. Exactly these edges plus the n-edge (v'_q, v''_q) form the set of edges of G_T incident to $b(\tau_q)$. (See the picture.) This gives (T3) for T.



Thus, (ii) implies (i) in the lemma. The converse implication (i) → (ii) follows from (8.1) and the fact (mentioned earlier) that for the n-expansion T of T' along P, the n-contraction operation applied to T produces T', and under this operation the n-strip for T shrinks into P^{rev}. This completes the proof of the lemma. □

Let us call a path P as in (ii) of Lemma 8.1 *legal*. It is the concatenation of $P_1, \dots, P_{n-1}$, where P_h is the maximal subpath of P whose edges connect levels $h-1$ and h, i.e., are of the form (X, Xi) with $|X| = h-1$. We refer to P_h as h-th *subpath* of P and say that this subpath is *ordinary* if it has only one edge, and *zigzag* otherwise. The beginning vertices of these subpaths together with z^{ℓ}_{n-1} are called *critical* in P (so there is exactly one critical vertex in each level); these vertices will play an important role in what follows. Note that the critical vertices of a legal path $P = (v_0, e_1, v_1, \dots, e_r, v_r)$ are $v_0 = z_0$, $v_r = z^{\ell}_{n-1}$ and the intermediate vertices v_q such that e_q enters and e_{q+1} leaves v_q. We distinguish between two sorts of non-critical vertices v_q by saying that v_q is a $\vee$-*vertex* if both e_q, e_{q+1} leave v_q, and a $\wedge$-*vertex* if both e_q, e_{q+1} enter v_q. Observe that

(8.4) regarding a vertex of P as a subset X of $[n-1]$, the following hold: (a) if X is critical, then both X, Xn are in B_T; (b) if X is a $\wedge$-vertex, then $X \in B_T$ and $Xn \notin B_T$; and (c) if X is a $\vee$-vertex, then $X \notin B_T$ and $Xn \in B_T$ (where T is the n-expansion of T' along P).

Indeed, from the proof of Lemma 8.1 one can see that: if X is critical, then both vertices X, Xn of G_T have entering and leaving edges, so they are nonterminal; if X is a $\wedge$-vertex, then Xn is terminal while X is not; and if X is a $\vee$-vertex, then X is terminal while Xn is not (see the above picture).

It follows that

(8.5) (i) $B_T = B' \cup B''$, where B' consists of all nonterminal vertices X in G'_P that are not $\vee$-vertices in P, and B'' consists of all Xn such that X is a nonterminal vertex in G''_P and is not a $\wedge$-vertex of P;

(ii) for each $h = 0, \dots, n-1$, there is exactly one set $X \subseteq [n-1]$ with $|X| = h$ such that both X and Xn belong to B_T; moreover, this X is just the unique critical vertex of P in level h.

Summing up the above observations and results, we can conclude with the following

Corollary 8.2. *The correspondence $(T', P) \mapsto T$, where T' is a g-tiling on Z_{n-1}, P is a legal path for T', and T is the n-expansion of T' along P, gives a bijection between the set of such pairs (T', P) and the set of g-tilings on Z_n. Moreover, under this correspondence, T' is the n-contraction T/n of T and P is the reverse to the right boundary of the n-strip in T.*

We conclude this section with an additional result which will be important for purposes of the next section. For a g-tiling T on Z_n and for $1 \leqslant h \leqslant n$, let H_h denote the subgraph of G_T induced by the set of *white* edges connecting levels $h-1$ and h.

Lemma 8.3. *For each $h = 1, \dots, n$, the graph H_h is a forest. Furthermore:*

- (i) *there exists a (connected) component K of H_h that contains both boundary edges $z^\ell_{h-1}z^\ell_h$ and $z^r_h z^r_{h-1}$ and such that all vertices of K are nonterminal; moreover, K has planar layout in the zonogon (i.e., non-adjacent edges in K do not intersect);*
- (ii) *any other component K' of H_h contains exactly one terminal vertex v and all edges of K' are incident to v (i.e., K' is a star).*

Proof. We observe that

(8.6) for any edge $e = (u, v)$ of H_h, if there are edges $e', e'' \neq e$ in H_h such that e' leaves u and e'' enters v, then one of e', e'' lies on the right from e, and the other on the left from e; equivalently: either $i', i'' < i$ or $i', i'' > i$, where i, i', i'' are the colors of e, e', e'', respectively.

Indeed, if $e \notin \ell bd(Z_n)$, then e belongs to the right boundary of some white tile τ, i.e., either $e = br(\tau)$ or $e = rt(\tau)$. By (3.5), any edge incident to $r(\tau)$ and lying strictly in the cone of τ at $r(\tau)$ is black. Similarly, if $e \notin rbd(Z_n)$, then e belongs to the left boundary of some white tile τ' and, by (3.5), any edge incident to $\ell(\tau')$ and lying strictly in the cone of τ' at $\ell(\tau')$ is black. This implies (8.6).

In view of (8.6), any edge-simple path P in H_h goes monotonically in one direction, either from left to right or from right to left; so P is not a cycle. Hence H_h is a forest in which any component K (a tree) has planar layout in Z_n. Suppose that H_h contains a terminal vertex u in level $h-1$, and consider an edge $e = (u, v)$ in H_h. Since u is terminal, each of the two white tiles τ', τ'' containing e has the bottom vertex at u and the right of left vertex at v. By (3.5), there is no white edge incident to v and lying strictly inside the cone of τ' at v, and similarly for τ''. This implies that e is the unique edge of H_h incident to v. Hence the component of H_h containing u is a star of which all edges are incident to u. A similar property holds for the components of H_h meeting a terminal vertex in level h.

Finally, consider a component K without terminal vertices (it exists since the boundary edge $z^\ell_{h-1}z^\ell_h$ is white and both of its ends are nonterminal). We assert that K contains $z^\ell_{h-1}z^\ell_h$. Indeed, take the leftmost edge $e = (u, v)$ in K and suppose that $e \neq z^\ell_{h-1}z^\ell_h$. Then there is a white tile τ containing e on its right boundary. Assume that $b(\tau) = u$; then $r(\tau) = v$. Then the edge $e' := b\ell(\tau)$ is black (as e' connects levels $h-1$ and h and lies on the left from e). Since $\ell(\tau)$ has both entering and leaving edges, it cannot be terminal. So $b(\tau)$ is terminal, contradicting the choice of K. The case $t(\tau) = v$ leads to a similar contradiction. Thus, K contains $z^\ell_{h-1}z^\ell_h$. Considering the rightmost edge of K and arguing similarly, we conclude that K contains the boundary edge $z^r_h z^r_{h-1}$ as well. □

We will refer to the component K as in (i) of this lemma as the *principal* one. Considering a legal path P for T and taking into account that all vertices of P are nonterminal and that h-th subpath in it is contained in H_h, for each h, we obtain the following property as a consequence of Lemma 8.3.

Corollary 8.4. *Any legal path for a g-tiling is determined by the set of its critical vertices.*

9. Weakly separated set-systems

The goal of this section is to prove the following theorem answering Leclerc–Zelevinsky's conjecture on weakly separated collections (or ws-collections) mentioned in Section 1.

Theorem 9.1. *Any largest ws-collection $C \subseteq 2^{[n]}$ is a semi-normal TP-basis.*

Recall that a ws-collection $C \subseteq 2^{[n]}$ is *largest* if its cardinality $|C|$ is maximum among all ws-collections in $2^{[n]}$; this maximum is equal to $\binom{n+1}{2} + 1$ [9]. An important example is the set $\mathcal{I}_n$ of intervals in $[n]$ (including the empty set). Also it was shown in [9] that a (lowering or raising) weak flip in a ws-collection produces again a ws-collection. Moreover, its cardinality preserves under a flip since it replaces one set in some pair $\{Xj, Xik\}$ (say) by the other and these sets are not weakly separated from each other. Due to these facts, the set $\mathcal{C}_n$ of largest ws-collections includes $\mathcal{B}_n$ (the set of semi-normal bases for $\mathcal{TP}_n$). Theorem 9.1 says that the converse inclusion takes place as well. As a result, we will conclude with the following

Corollary 9.2. $\mathcal{C}_n = \mathcal{B}_n$.

In view of Theorem 2.2, to obtain Theorem 9.1, it suffices to show the following

Theorem 9.3. *Any $C \in \mathcal{C}_n$ is the spectrum B_T of some g-tiling T on Z_n.*

This theorem is proved by combining additional facts established in [9] and results from the previous sections. Let $C \in \mathcal{C}_n$. To construct the desired tiling for C, we consider the projection C' of C into $2^{[n-1]}$, i.e., the collection of subsets $X \subseteq [n-1]$ such that either $X \in C$ or $Xn \in C$ or both. Partition C' into three subcollections M, N, S, where

$$M := \{X\colon\ X \in C \not\ni Xn\}, \qquad N := \{X\colon\ Xn \in C \not\ni X\}, \qquad S := \{X\colon\ X, Xn \in C\}.$$

Also for $h = 0, \dots, n-1$, define

$$C'_h := \{X \in C'\colon\ |X| = h\}, \qquad M_h := M \cap C'_h, \qquad N_h := N \cap C'_h.$$

It is shown in [9] that

(9.1) for each $h = 0, \dots, n-1$, $S \cap C'_h$ contains exactly one element.

We call S the *separator* of C' and denote its elements by $S_0, \dots, S_{n-1}$, where $|S_h| = h$. In view of (9.1), $|C'| = |C| - |S| = \binom{n+1}{2} + 1 - n = \binom{n}{2} + 1$, and as is shown in [9],

(9.2) C' is a ws-collection, and therefore it is a *largest* ws-collection in $2^{[n-1]}$.

Two more observations in [9] are:

(9.3) (i) $S_0 \lessdot S_1 \lessdot \dots \lessdot S_{n-1}$;
(ii) for each $h = 0, \dots, n-1$, any sets $Y \in M_h$ and $Y' \in N_h$ satisfy $Y \lessdot S_h$ and $S_h \lessdot Y'$.

An important consequence of (9.3) is that the collection C can be uniquely restored from the pair C', S. Indeed, C consists of the sets X, Xn such that $X \in S$, the sets $X \subseteq [n-1]$ such that $X \lessdot S_{|X|}$, and the sets Xn such that $X \subseteq [n-1]$ and $S_{|X|} \lessdot X$.

The proof of Theorem 9.3 is led by induction on n. The result is trivial for $n \leqslant 2$. Let $n > 2$ and assume by induction that there is a g-tiling T' on Z_{n-1} such that $B_{T'} = C'$. Our aim is to transform T' into a g-tiling on Z_n whose spectrum is C. The following assertion is crucial.

Lemma 9.4. *There exists a legal path P for T' whose set of critical vertices coincides with the separator S.*

Proof. It uses the following fact from [9]:

(9.4) for distinct $A, A', A'' \subseteq [n']$, if $|A| \leqslant |A'| \leqslant |A''|$, $A \lessdot A' \lessdot A''$, and A'' is weakly separated from A, then $A \lessdot A''$.

The desired path P is constructed by relying on Lemma 8.3. For $h = 1, \dots, n-1$, let H_h be the subgraph of $G_{T'}$ induced by the white edges connecting levels $h-1$ and h, and let K_h be the principal component of H_h. Since all vertices of K_h are nonterminal, they (regarded as sets) belong to C'.

Consider two vertices X, Y with $|X| \leqslant |Y|$ in K_h. If they are connected by edge, then $Y = Xi$ for some $i \in [n]$ and, obviously, $X \lessdot Y$. If the path P from X to Y in K_h has two or more edges and goes from left to right, then $X \lessdot Y$ as well. (When X, Y belong to different levels, we say that P goes from left to right if the vertex Y' preceding Y in P lies on the right from X in the level containing X, Y'.) Indeed, for any three consecutive vertices Z, Z', Z'' in P (occurring in this order) either $Z = Z'i$ and $Z'' = Z'j$, or $Z = Z' - \{j\}$ and $Z'' = Z - \{i\}$ for some $i < j$ (since Z'' lies on the right from Z), which implies $Z \lessdot Z''$. Then $X \lessdot Y$ follows by transitivity from (9.4).

We assert that the separating vertex S_{h-1} belongs to K_h. Indeed, suppose this is not so. Then S_{h-1} belongs to a star component K' of H_h, and therefore, the white edge e in H_h leaving S_{h-1} enters the top vertex of some black tile $\tau = \tau(X; p, q)$. We have $S_{h-1} = Xpq - \{i\}$ and $p < i < q$, where i is the color of e. On the other hand, the bottom vertex X of τ has a leaving j-edge (X, Xj) for some $p < j < q$ (cf. Corollary 3.1). The vertex Xj is nonterminal, and we have: $|Xj| = |S_{h-1}|$, $S_{h-1} - Xj = \{p, q\}$, and $Xj - S_{h-1} = \{i, j\}$. Since $p < i, j < q$, we come to a contradiction with the fact that S_{h-1} is comparable by $\lessdot$ with any nonterminal vertex in level $h-1$. Thus, S_{h-1} is in K_h. Arguing similarly, one shows that S_h is in K_h as well. Let P_h be the path from S_{h-1} to S_h in K_h. Since $S_{h-1} \lessdot S_h$ (by (9.3)), P_h goes from left to right (when it has two or more edges).

Concatenating $P_1, \dots, P_{n-1}$, we obtain a legal path P for T' in which h-th subpath is P_h and the critical vertices are exactly $S_0, \dots, S_{n-1}$, as required. $\square$

Now we finish the proof of Theorem 9.3 as follows. Let T be the n-expansion of T' along P. Then B_T is a ws-collection, moreover, it is a largest ws-collection since $|B_T| = \binom{n+1}{2} + 1$. By (8.5) and Corollary 8.2, the projection of B_T into $2^{[n-1]}$ (defined by $X \mapsto X - \{n\}$) is just $B_{T'} = C'$ and, moreover, the set of $X \subseteq [n-1]$ such that $X, Xn \in B_T$ is exactly the set of critical vertices in P, i.e., S. Since the pair C', S generates the corresponding largest ws-collection in $2^{[n]}$ in a unique way, we obtain $B_T = C$, and Theorem 9.3 follows.

This yields Theorem 9.1 and completes the proof of Theorem A.

10. Generalizations

In this section we outline two generalizations, omitting details and proofs.

A. The obtained relationships between semi-normal bases, proper wirings and generalized tilings are extendable to the case of an integer n-box $\mathbf{B}^{n,a} = \{x \in \mathbb{Z}^{[n]}:\ 0 \leqslant x \leqslant a\}$, where $a \in \mathbb{Z}^n_+$. A description in details will be given in a separate paper. Recall that a function f on $\mathbf{B}^{n,a}$ is a TP-function if it satisfies

$$\begin{aligned} &f(x+\epsilon_i+\epsilon_k)+f(x+\epsilon_j) \\ &\quad = \max\bigl\{f(x+\epsilon_i+\epsilon_j)+f(x+\epsilon_k),\ f(x+\epsilon_i)+f(x+\epsilon_j+\epsilon_k)\bigr\} \end{aligned} \tag{10.1}$$

for any x and $1 \leqslant i < j < k \leqslant n$, provided that all six vectors occurring as arguments in this relation belong to $\mathbf{B}^{n,a}$, where ϵ_q is q-th unit base vector in $\mathbb{R}^{[n]}$. In this case the standard basis of TP-functions consists of the vectors x such that $x_i, x_j > 0$ for $i < j$ implies $x_q = a_q$ for $q = i+1, \dots, j-1$ (see [2] where such vectors are called *fuzzy-intervals*). Normal and semi-normal bases are corresponding collections of integer vectors in $\mathbf{B}^{n,a}$, defined by a direct analogy with the Boolean case.

The semi-normal bases in the box case admit representations via natural generalizations of proper wiring and g-tiling diagrams for the Boolean case. They are viewed as follows.

The zonogon for a given a is the set $Z_{n,a} := \{\lambda_1\xi_1 + \dots + \lambda_n\xi_n:\ \lambda_i \in \mathbb{R},\ 0 \leqslant \lambda_i \leqslant a_i,\ i = 1, \dots, n\}$, where the vectors ξ_i are chosen as above. For each $i \in [n]$ and $q = 0, 1, \dots, a_i$, define the point $z^\ell_{i,q} := a_1\xi_1 + \dots + a_{i-1}\xi_{i-1} + q\xi_i$ (on the left boundary of $Z_{n,a}$) and the point $z^r_{i,q} := a_n\xi_n + \dots + a_{i+1}\xi_{i+1} + q\xi_i$ (on the right boundary). These points are regarded as the vertices on the boundary of $Z_{n,a}$, and the edges in it are the directed line-segments $z^\ell_{i,q-1}z^\ell_{i,q}$ and $z^r_{i,q}z^r_{i,q-1}$. When $q \geqslant 1$, we define $s_{i,q}$ ($s'_{i,q}$) to be the median point on the edge $z^\ell_{i,q-1}z^\ell_{i,q}$ (resp. $z^r_{i,q}z^r_{i,q-1}$).

A generalized tiling T on $Z = Z_{n,a}$ is defined by essentially the same axioms (T1)–(T4) from Section 2.2. A wiring W on Z consists of wires $w_{i,q}$ going from $s_{i,q}$ to $s'_{i,q}$, $i = 1, \dots, n$, $q = 1, \dots, a_i$. Again, it is defined by the same axioms (W1)–(W3) from Section 2.1.

Note that for any i and $1 \leqslant q < q' \leqslant a_i$, the point $s'_{i,q}$ occurs earlier than $s'_{i,q'}$ in the right boundary of Z (beginning at z_0), which corresponds to the order of $s_{i,q}, s_{i,q'}$ in the left boundary of Z. This and axiom (W2) imply that the wires $w := w_{i,q}$ and $w' := w_{i,q'}$ are always disjoint. Indeed, suppose that w and w' meet and take the first point x of w' that belongs to w. Let Ω_0, Ω_1 be the connected components of $Z - (P \cup P')$, where P is the part of w from x to $s'_{i,q}$, P' is the part of w' from $s_{i,q'}$ to x, and Ω_0 contains z_0. Then the end point $s'_{i,q'}$ of w' is in Ω_1. Furthermore, w' crosses w at x from left to right (since x is the first point of w' where it meets w); this implies that when passing x, the wire w' enters the region Ω_0. Therefore, the part of w' from x to $s'_{i,q'}$ must intersect $P \cup P'$ at some point $y \neq x$. But $y \in P$ is impossible by (W2) and $y \in P'$ is impossible because w' is not self-intersecting.

Like the Boolean case, for a g-tiling T, the spectrum B_T is defined to be the set of nonterminal vertices (viz. n-vectors) for T. For a wiring W and an (inner) face F of its associated planar graph, let $x(F)$ denote the n-vector whose i-th entry is the number of wires $w_{i,q}$ such that F lies on the left from $w_{i,q}$. Then B_W is defined to be the collection of vectors $x(F)$ over all non-cyclic faces F.

Theorems 2.1 and 2.2 remain valid for these extended settings (where Z_n is replaced by $Z_{n,a}$), and proving methods are essentially the same as those in Sections 4–7, with minor refinements on some steps. (E.g., instead of a unique dual i-path (i-strip) for each i, we now deal with a_i dual i-paths $Q_{i,1},\dots,Q_{i,a_i}$, each $Q_{i,q}$ connecting a boundary edge $z^\ell_{i,q-1}z^\ell_{i,q}$ to $z^r_{i,q-1}z^r_{i,q}$, which does not cause additional difficulty in the proof.)

B. The second generalization involves an arbitrary permutation ω on $[n]$. (In fact, so far we have dealt with the longest permutation ω_0, where $\omega_0(i) := n+1-i$.) For $i,j\in[n]$, we write $i\prec_\omega j$ if $i<j$ and $\omega(i)<\omega(j)$. This relation is transitive and gives a partial order on $[n]$. Let $\mathcal{X}_\omega\subseteq 2^{[n]}$ be the set (lattice) of ideals X of $([n],\prec_\omega)$, i.e., $i\prec_\omega j$ and $j\in X$ implies $i\in X$; such an X is called in [1,9] an *ω-chamber set*. In particular, $\mathcal{X}_\omega$ is closed under taking a union or intersection of its members. Below we specify settings and outline how results concerning ω_0 can be extended to ω.

(i) Speaking of a TP-function for ω, or an *ω-TP-function*, we mean a function f defined on the set $\mathcal{X}_\omega$ (rather than $2^{[n]}$) and satisfying (1.1) when all six sets in it belong to $\mathcal{X}_\omega$. Note that $Xi, Xk, Xij, Xjk\in\mathcal{X}_\omega$ implies that each of $X, Xj, Xik, Xijk$ is in $\mathcal{X}_\omega$ as well (since each of the latter sets is obtained as the intersection or union of a pair among the former ones). The notion of TP-basis is extended to the set $\mathcal{TP}_\omega$ of ω-TP-functions in a natural way. It turns out that the role of standard basis is now played by the set $\mathcal{I}_\omega$ of *ω-dense* sets $X\in\mathcal{X}_\omega$, which means that there are no triples $i<j<k$ such that $i,k\in X\not\ni j$ and each of the sets $X-\{i\}$, $X-\{k\}$ and $(X-\{i,k\})\cup\{j\}$ belongs to $\mathcal{X}_\omega$. In particular, $\mathcal{I}_\omega$ contains the sets $[i]$, $\{i'\colon i'\preccurlyeq_\omega i\}$ and $\{i'\colon \omega(i')\leqslant\omega(i)\}$ for each $i\in[n]$; when $\omega=\omega_0$, $\mathcal{I}_\omega$ turns into the set $\mathcal{I}_n$ of intervals in $[n]$. (It is rather easy to prove that any ω-TP-function is determined by its values on $\mathcal{I}_\omega$; this is done by exactly the same method as applied in [2] to show a similar fact for $\mathcal{TP}_n$ and $\mathcal{I}_n$. The fact that the restriction map $\mathcal{TP}_\omega\to\mathbb{R}^{\mathcal{I}_\omega}$ is surjective (which is more intricate) can be shown by extending a flow approach developed in [2] for the cases of TP-functions on Boolean cubes and integer boxes.) Normal and semi-normal bases for the ω-TP-functions are defined via flips from the standard basis $\mathcal{I}_\omega$, by analogy with those for ω_0.

(ii) Instead of the zonogon Z_n, we now should consider the region Z_ω in the plane bounded by two paths: the left boundary of Z_n and the path P_ω formed by the points $v^0_\omega := z_0$ and $v^i_\omega := \xi_{\omega^{-1}(1)}+\cdots+\xi_{\omega^{-1}(i)}$ for $i=1,\dots,n$, that are connected by the directed line-segments $e^1_\omega,\dots,e^n_\omega$, where e^j_ω begins at v^{j-1}_ω and ends at v^j_ω. (Then e^j_ω is a parallel translation of the vector $\xi_{\omega^{-1}(j)}$. Observe that $P_{id}=\ell bd(Z_n)$ and $P_{\omega_0}=rbd(Z_n)$, where id is the identical permutation.) A wiring for ω is a collection W of wires $w_1,\dots,w_n$ on Z_ω satisfying axioms (W1)–(W3) and such that each w_i begins at the median point s_i of the i-edge on $\ell bd(Z_n)$ (as before) and ends at the median point $s^{\omega(i)}_\omega$ of the i-edge $e^{\omega(i)}_\omega$ (a wire w_i degenerates into a single point if these edges coincide). Note that if $i\prec_\omega j$ then $s^{\omega(i)}_\omega$ occurs earlier than $s^{\omega(j)}_\omega$ in the right boundary P_ω of Z_ω, and therefore, the wires w_i and w_j does not meet (as explained in part A above). This implies that all sets in the full spectrum of W belong to $\mathcal{X}_\omega$.

In its turn, a generalized tiling T for ω is defined by axioms (T2), (T3) as before and slightly modified axioms (T1), (T4), where: in (T1), the first condition is replaced by the requirement that each edge in $(\ell bd(Z_n)\cup P_\omega)-(\ell bd(Z_n)\cap P_\omega)$ belong to exactly one tile; and in (T4), it is now required that the region $D_T\cup\sigma(\ell bd(Z_n)\cap P_\omega)$ be simply connected. Also we should include in the graph G_T all common vertices and edges of $\ell bd(Z_n)$ and P_ω. One easily shows that the union of tiles in T and edges in $\ell bd(Z_n)\cap P_\omega$ is exactly Z_ω, and that all vertices in the boundary of Z_ω are nonterminal.

The constructions and arguments in Sections 6, 7, based on planar duality, can be transferred without essential changes to the ω case, giving a natural one-to-one correspondence between the g-tilings and proper wirings for ω. (In particular, i-th wire w_i in a proper wiring W for ω turns into the i-strip Q_i in the corresponding g-tiling T, which begins with the i-edge $z^\ell_{i-1}z^\ell_i$ in $\ell bd(Z_n)$ and ends with the i-edge $e^{\omega(i)}_\omega$ in P_ω.) The arguments in Sections 4, 5 continue to work in the ω case as well. As a result, one can obtain direct generalizations of Theorems 2.1 and 2.2 to an arbitrary permutation ω.

(iii) It is shown in [9] that the maximum possible cardinality of a weakly separated collection $\mathcal{C}$ in $\mathcal{X}_\omega$ is equal to $\ell(\omega)+n+1$, where $\ell(\omega)$ is the *length* of ω (i.e., the number of inversions in it). It turns out (see [3]) that such *largest ω-chamber ws-collections* $\mathcal{C}$ are precisely the spectra of g-tilings on Z_ω, similar to the equivalence of (iii) and (iv) in Theorem A.

Summing up the above explanations, one can conclude with a corresponding generalization of Theorem A to the case of an arbitrary permutation ω on $[n]$.

Remark 7. In fact, the generalization in part A is a special case of the one in part B. More precisely, given $a \in \mathbb{Z}^n_+$, define $\bar a_i := a_1 + \cdots + a_i$, $i = 0, \ldots, n$ (letting $\bar a_0 := 0$). Let us form a permutation ω' on $[\bar a_n]$ as follows: for $i = 1, \ldots, n$ and $q = 1, \ldots, a_i$,

$$\omega'(\bar a_{i-1} + q) := \bar a_n - \bar a_i + q,$$

i.e., ω' permutes the blocks $B_1, \ldots, B_n$, where $B_i := \{\bar a_{i-1}+1, \ldots, \bar a_i\}$, according to the permutation ω_0 on $[n]$, and preserves the order of elements within each block. Then there is a one-to-one correspondence between the vectors $x \in \mathbf{B}^{n,a}$ and the ideals X of $([\bar a_n], \prec_{\omega'})$, namely: $X \cap B_i$ consists of the first x_i elements of B_i, for each i. Under this correspondence, (10.1) is equivalent to (1.1). Although the shape of the zonogon $Z_{n,a}$ looks somewhat different compared with $Z_{\omega'}$ (since the generating vectors $\xi_\bullet$ for different elements in a block are non-colinear), it is easy to see that the wirings for the former and the latter are, in fact, the same. (This implies an equivalence of the g-tilings for these two cases, which is not seen immediately.) So the integer box case is reduced, in all aspects we deal with, to the permutation one.

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Appendix A. TP-bases and weakly separated set-systems on a hyper-simplex

Our results on TP-bases, generalized tilings and weakly separated set-systems and techniques elaborated in previous sections enable us to obtain an analog of the equivalence (i) $\leftrightarrow$ (iv) in Theorem A to hyper-simplexes. Let us start with basic definitions and backgrounds.

When dealing with a hyper-simplex $\Delta^m_n = \{S \subseteq [n]\colon\ |S| = m\}$ rather than the Boolean cube $2^{[n]}$, the notion of TP-functions and TP-bases are modified as follows. Let $f : \Delta^m_n \to \mathbb{R}$. Instead of relation (1.1) involving triples $i < j < k$, one considers relation

$$f(Xik) + f(Xj\ell) = \max\bigl\{f(Xij) + f(Xk\ell),\ f(Xi\ell) + f(Xjk)\bigr\} \qquad \text{(A.1)}$$

for a quadruple $i < j < k < \ell$ in $[n]$ and a subset $X \subseteq [n] - \{i, j, k, \ell\}$ of size $m-2$. When this holds for all such X, i, j, k, ℓ, we refer to f as a *TP-function* on Δ^m_n. Let $\mathcal{TP}^m_n$ denote the set of such functions f. By an analogy with the Boolean cube case, a subset $B \subseteq \Delta_n^m$ is called a *TP-basis* if the restriction map $\mathcal{TP}_n^m \to \mathbb{R}^B$ is bijective.

An important instance of TP-bases for Δ_n^m is the collection $\mathcal{IS}_n^m = \mathcal{I}^m \cup \mathcal{S}^m$, where $\mathcal{I}^m = \mathcal{I}_n^m$ consists of the intervals of size m and $\mathcal{S}^m = \mathcal{S}_n^m$ consists of the sets of size m representable as the union of two nonempty intervals $[1..p]$ and $[q..r]$ with $q > p + 1$ (see [2] where the elements of $\mathcal{S}^m$ are called *sesquialteral intervals*).

When a TP-basis contains four sets $Xij, Xk\ell, Xi\ell, Xjk$ as above and one set $Y \in \{Xik, Xj\ell\}$, the replacement of Y by the other set Y' in $\{Xik, Xj\ell\}$ gives another TP-basis B'. We call such a transformation $Y \rightsquigarrow Y'$ (or $B \mapsto B'$) a *raising (lowering) 4-flip* if $Y = Xik$ (resp. $Y = Xj\ell$). One can see that $\mathcal{IS}_n^m$ does not admit lowering flips and we call this TP-basis *standard for* Δ_n^m (analogously to the basis $\mathcal{I}_n$ for $2^{[n]}$ where weak lowering flips are absent as well).

The object of our interest is the class $\mathcal{B}_n^m$ of TP-bases that can be obtained by making a series of 4-flips starting from $\mathcal{IS}_n^m$. It is analogous to the class $\mathcal{B}_n$ of semi-normal bases for the Boolean cube case (and $\mathcal{B}_n^m$ along with the 4-flips on its members represents another interesting instance of Plücker environments).

A direct calculation shows that $|\mathcal{IS}_n^m| = m(n-m) + 1$; so all TP-bases for Δ_n^m have this cardinality. Besides, one can associate to $B \in \mathcal{B}_n^m$ the number $\eta(B) := \sum_{X \in B} \sum_{i \in X} i$. Clearly any lowering 4-flip decreases η; we shall see later that any $B \in \mathcal{B}_n^m$ is reachable from $\mathcal{IS}_n^m$ by a series of merely raising 4-flips, and therefore, $\eta(B) > \eta(\mathcal{IS}_n^m)$ unless $B = \mathcal{IS}_n^m$.

Our goal is to show that $\mathcal{B}_n^m$ coincides with the set $\mathcal{C}_n^m$ of *largest* weakly separated collections $C \subseteq \Delta_n^m$, i.e., having maximum possible cardinality $|C|$. We rely on two known facts.

First, Scott [11] showed that if a ws-collection $C \subseteq \Delta_n^m$ contains four sets $Xij, Xk\ell, Xi\ell, Xjk$ (with X, i, j, k, ℓ as above), then each of $Xik, Xj\ell$ is weakly separated from any member of C. (Note that $Xik, Xj\ell$ are not weakly separated from each other.)

This implies that any TP-basis in $\mathcal{B}_n^m$ is weakly separated, since the standard basis is such (which is easy to check; cf. [9]).

Second, a simple, but important, fact noticed in [9] is that: for $0 \leqslant m \leqslant m' \leqslant n$ and a ws-collection $C \subseteq 2^{[n]}$ whose members have size at least m and at most m', if we add to C all intervals of size $> m'$ and all co-intervals of size $< m$, then the resulting collection is again weakly separated. Let us call the latter collection the *straight extension* of C and denote it by C^*. Note that the numbers of added intervals and co-intervals are $\binom{n-m'+1}{2}$ and $\binom{m+1}{2}$, respectively. When $m = m'$, the fact that the maximum cardinality of a ws-collection in $2^{[n]}$ amounts to $\binom{n+1}{2} + 1$ and the identity $\binom{n+1}{2} = \binom{n-m+1}{2} + \binom{m+1}{2} + m(n-m) + 1$ imply $|C| = |C^*| - \binom{n-m+1}{2} - \binom{m+1}{2} \leqslant m(n-m) + 1 = |\mathcal{IS}_n^m|$.

Thus, $\mathcal{IS}_n^m$ is a largest ws-collection in Δ_n^m, implying that all members of $\mathcal{B}_n^m$ are such, i.e., $\mathcal{B}_n^m \subseteq \mathcal{C}_n^m$. We show the following analog of a result from Section 9 to hyper-simplexes.

Theorem A.1. *Let $C \in \mathcal{C}_n^m$ and $C \neq \mathcal{IS}_n^m$. Then C admits a lowering 4-flip (defined in the same way as for TP-bases). In particular, all members of $\mathcal{C}_n^m$ belong to one and the same orbit w.r.t. 4-flips.*

In light of the above discussion, this gives the desired result:

Corollary A.2. $\mathcal{B}_n^m = \mathcal{C}_n^m$.

Proof of Theorem A.1. Note that C contains all intervals and co-intervals of size m (since they are weakly separated from any member of Δ_n^m).

We use induction on n, assuming w.l.o.g. that $0 < m < n$. The ws-collection C^* (defined as above when $m = m'$) is largest in $2^{[n]}$ and its projection C' (defined as in Section 9) is a largest ws-collection in $2^{[n-1]}$. Let T be a g-tiling on Z_{n-1} whose spectrum B_T is C'. Clearly for $h > m$ (resp. $h < m - 1$), the set $C'_h := \{X \in C'\colon |X| = h\}$ consists of all intervals (resp. all co-intervals) of size h in $[n - 1]$.

As to level m, all sets $X \in C'_m$ are exactly the members of C not containing the element n (since there exists only one set $Y \in C^*$ with $|Y| > m$, namely, the interval $[n - m..n]$, whose projection occurs in C'_m, but the latter is the interval $[n - m..n - 1]$, which belongs to $C \cap C'_m$). And in level $m - 1$, all members of C'_{m-1} are exactly the projections of those members of C that contain n (since there exists only one set $Y \in C^*$ with $|Y| < m$, namely, the "co-interval" $[1..m - 1]$, that occurs in C'_{m-1}, but it is the projection of the co-interval $[1..m - 1] \cup \{n\}$, which belongs to C).

Now consider two possible cases.

Case 1. Let T have no feasible M-configuration of height $m - 1$ (defined as in Section 4.1). Then, by Proposition 5.3, C'_{m-1} contains only co-intervals. It is easy to see that, under the above projection map, the preimages in C of these co-intervals X (which are exactly the members of C containing n) are only intervals and sesquialteral intervals. Also the facts that C' is the straight extension of C'_m and that C' is a largest ws-collection in $2^{[n-1]}$ (by Corollary 9.2) imply $C'_m \in \mathcal{C}_{n-1}^m$. Since $C \neq \mathcal{IS}_n^m$, the subcollection C'_m of C differs from $\mathcal{IS}_{n-1}^m$. By induction C'_m admits a lowering 4-flip; this is just a required 4-flip for C.

Case 2. Let T have a feasible M-configuration $CM(X; i, j, k)$ of height $m - 1$, i.e., $C' = B_T$ contains sets Xi, Xj, Xk, Xij, Xjk with $|X| = m - 2$ and $i < j < k$. Since the first three sets among them belong to level $m - 1$, C contains the sets Xin, Xjn, Xkn. These together with the sets Xij, Xjk contained in C (since the latter ones are in level m of T) give the desired configuration (involving the quadruple $i < j < k < n$) for performing a lowering 4-flip in C.

This completes the proof of Theorem A.1. □

Remark 8. We can give an alternative method of proving the equality $\mathcal{B}_n^m = \mathcal{C}_n^m$ relying on results of Postnikov [10] on alternating strand diagrams, or, briefly, *as-diagrams*. One can outline the idea of this method as follows (omitting details). Given $C \in \mathcal{C}_n^m$, let T and W be, respectively, the g-tiling and proper wiring with $B_T = B_W = C^*$. Take the maximum zigzag paths P, P' in the principal components in the forests H_{m-1}, H_m of the graph G_T, respectively (see Section 8), both going from the vertex $[m]$ to the vertex $[n - m + 1..n]$. Then P' passes (as vertices) all interval of sizes m and $m + 1$, and one can see that the sequence of colors of its edges is $m + 1, 1, m + 2, 2, \dots, n, n - m$ (in this order on P'). In its turn, the reverse path P^{-1} of P passes all co-intervals of sizes $m - 1$ and m, and the sequence of colors of its edges is $n - m + 1, 1, n - m + 2, 2, \dots, n, m$. When considering the planar layout of G_T on the disc D_T, the concatenation (circuit) Q of P' and P^{-1} cuts out a smaller disc $\tilde{D}$ in D_T; it is the union of (the squares representing) the tiles of height m in T.

Take the parts w'_i of wires $w_i \in W$ going across $\tilde{D}$, and denote the beginning and end points of w'_i by u_i and v_i, respectively (both u_i, v_i are interior points of edges of color i in Q, by the construction in Section 6). Let S be the sequence of $2m$ points u_i, v_j along P^{-1}, and S'

the sequence of $2(n-m)$ points u_i, v_j along P'. Partition S (resp. S') into m (resp. $n-m$) consecutive pairs; then each pair contains one "source" u_i and one "sink" v_j of the wiring $W' = (w'_1, \dots, w'_n)$. Now extend each wire in W' within the boundary Q of $\tilde{D}$ so that, for the pairs $\{u_i, v_j\}$ as above, the beginning of w'_i coincide with the end of w'_j. One shows (a key) that the resulting wiring forms an as-diagram $\mathcal{D}$ of [10]. Moreover, the sequences of edge colors in P^{-1}, P' indicated above provide that the corresponding permutation on $[n]$ associated to the set of directed chords (u_i, v_i) (when the above n pairs are numbered clockwise) is the Grassmann permutation ω for m, n, namely, $\omega(i) = m + i$ for $i = 1, \dots, n-m$, and $\omega(i) = i - n + m$ otherwise. Finally, using properties of nonterminal vertices of T exhibited in Corollary 3.3, one shows that Postnikov's operations (M1) ("square moves") on $\mathcal{D}$ correspond to 4-flips on C. Then Theorem A.1 can be derived from a result (Theorem 13.4) in [10].

Next, the fact that the poset $(\mathcal{B}_n^m, \prec)$ (where $B \prec B'$ if B is obtained from B' by lowering 4-flips) has a unique minimal element (namely, $\mathcal{IS}_n^m$) easily implies that this poset has a unique maximal element. This is $\mathcal{I}_n^m \cup$ co-$\mathcal{S}_n^m$, where co-$\mathcal{S}_n^m$ consists of the m-sized sets representable as the union of two nonempty intervals $[p..q]$ and $[r..n]$ with $r > q+1$. Let us call this basis *co-standard* for Δ_n^m.

One more useful construction concerns an embedding of the set $\mathcal{B}_n$ of semi-normal TP-bases for the Boolean cube $2^{[n]}$ into TP-bases for some hyper-simplex. More precisely, consider a hyper-simplex $\Delta_{n+n'}^n$ with $n' \geqslant n$. For $X \subseteq [n]$, define X^Δ to be the union of X and the interval $[q..n+n']$ of size $n - |X|$ (which is empty when $X = [n]$). In addition, let $\mathcal{A}$ be the collection of all n-sized intervals $[p..q]$ with $n < q < n+n'$ and all n-sized sets $[p..q] \cup [r..n+n']$ with $n+1 < q+1 < r \leqslant n+n'$. Then the union of $\mathcal{A}$ and the collection $\{I^\Delta \colon I \in \mathcal{I}_n\}$ forms the co-standard basis for $\Delta_{n+n'}^n$. In a similar fashion, we associate to any $\mathcal{X} \subseteq 2^{[n]}$ the collection $\mathcal{A} \cup \{X^\Delta \colon X \in \mathcal{X}\}$, denoted as $\mathcal{X}^\Delta$.

Proposition A.3. *For any semi-normal TP-basis $B \in \mathcal{B}_n$, one holds $B^\Delta \in \mathcal{B}_{n+n'}^n$.*

Proof. Let $B \neq \mathcal{I}_n$. Then B contains sets Xi, Xk, Xij, Xik, Xjk for some $i < j < k$ and $X \subseteq [n] - \{i, j, k\}$. Under the transformation $Y \mapsto Y^\Delta$, these sets turn into the sets (respectively) $Xi \cup [\ell..n+n']$, $Xk \cup [\ell..n+n']$, $Xij \cup [\ell+1..n+n']$, $Xik \cup [\ell+1..n+n']$, $Xjk \cup [\ell+1..n+n']$ in $\Delta_{n+n'}^m$, where $\ell := n' + |X| + 2$. Then the lowering flip $Xik \rightsquigarrow Xj$ in B corresponds to the raising flip $X'ik \rightsquigarrow X'j\ell$ in B^Δ, where $X' := X \cup [\ell+1..n+n']$, and the result follows. □

The embedding $B \mapsto B^\Delta$ of the TP-bases for $2^{[n]}$ into TP-bases for $\Delta_{n+n'}^n$ can be regarded as a certain dual analog of the straight extension map $B' \mapsto (B')^*$ of the TP-bases for Δ_n^m to TP-bases for $2^{[n]}$.

We conclude this section with a generalization to a truncated Boolean cube $\Delta_n^{m,m'} := \{S \subseteq [n] \colon m \leqslant |S| \leqslant m'\}$, where $0 \leqslant m \leqslant m' \leqslant n$. In this case, one considers the class $\mathcal{TP}_n^{m,m'}$ of functions $f : \Delta_n^{m,m'} \to \mathbb{R}$ obeying relations (1.1) and (A.1) for all corresponding corteges where the six sets occurring as arguments belong to $\Delta_n^{m,m'}$. (In fact, it suffices to require that (1.1) be imposed everywhere but that (A.1) be explicitly imposed only for the corteges related to the lowest level, i.e., when $|X| = m-2$. Then (A.1) for the other corteges will follow; see [2].)

The class $\mathcal{TP}_n^{m,m'}$ has as a basis the set $\mathcal{IS}_n^{m,m'} := \mathcal{S}_n^m \cup \mathcal{I}_n^m \cup \mathcal{I}_n^{m+1} \cup \dots \cup \mathcal{I}_n^{m'}$ [2], called the *standard* basis for this case. We define $\mathcal{B}_n^{m,m'}$ to be the set of bases obtained from the standard one by a series of weak 3-flips (i.e., related to (1.1)) and 4-flips. (From the theorem below it follows that the latter ones are important only in level m.) On the other hand, we can consider the set $\mathcal{C}_n^{m,m'}$ of largest ws-collections in $\Delta_n^{m,m'}$ (whose cardinality is $\binom{n+1}{2} - \binom{n-m'+1}{2} - \binom{m+1}{2} = |\mathcal{IS}_n^{m,m'}|$). By explanations above, $\mathcal{B}_n^{m,m'} \subseteq \mathcal{C}_n^{m,m'}$.

Theorem A.4. *Let $C \in \mathcal{C}_n^{m,m'}$. Then $\mathcal{IS}_n^{m,m'}$ can be obtained from C by a series of lowering weak 3-flips followed by a series of lowering 4-flips in level m. Therefore, $\mathcal{B}_n^{m,m'} = \mathcal{C}_n^{m,m'}$.*

Proof. If C admits a lowering 3-flip, then performing such a flip produces a collection in $\mathcal{C}_n^{m,m'}$ with a smaller total size of its members. If such a flip is impossible, then all sets $X \in C$ with $|X| > m$ are intervals, by Proposition 5.3. Then $\{X \in C\colon |X| = m\}$ is a largest ws-collection for the hyper-simplex Δ_n^m, and the result follows from Theorem A.1. □

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