

Combined tilings and the purity phenomenon on separated set-systems

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Dedicated to the memory of Andrei Zelevinsky

Abstract. In 1998 Leclerc and Zelevinsky found a purely combinatorial characterization for quasi-commuting flag minors of a quantum matrix. It involves the notion of *weakly separated* collections of subsets of the ordered set $[n]$ of elements $1, 2, \dots, n$. Answering their conjectures on such collections, several sorts of domains $\mathcal{D} \subseteq 2^{[n]}$ have been revealed that possess the property of *purity*, in the sense that all inclusion-wise maximal weakly separated collections contained in \mathcal{D} have equal cardinalities. In particular, so are the full domain $2^{[n]}$ and the Boolean hyper-simplex $\{X \subseteq [n]: |X| = m\}$ for $m \in [n]$.

In this paper, generalizing those earlier results, we describe wide classes of pure domains. They are mostly obtained as consequences of our study of a new geometric model for weakly separated set-systems, so-called *combined (polygonal) tilings* on a zonogon, yielding a new insight in the area.

In parallel, we discuss a similar phenomenon with respect to the *strong* separation relation (which is easier).

Keywords: weakly separated sets, strongly separated sets, rhombus tiling, generalized tiling, combined tiling, lattice paths, Grassmann necklace

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1 Introduction

For a positive integer n , the set $\{1, 2, \dots, n\}$ with the usual order is denoted by $[n]$. For a subset $X \subseteq [n]$ formed by elements $x_1 < x_2 < \dots < x_k$, we use notation (x_1, \dots, x_k) for X , $\min(X)$ for x_1 , and $\max(X)$ for x_k , where $\min(X) = \max(X) := 0$ if $X = \emptyset$.

We will deal with several binary relations on the set $2^{[n]}$ of subsets of $[n]$. Namely, for distinct $A, B \subseteq [n]$, we write:

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- (1.1) (i) $A \prec B$ if $A = (a_1, \dots, a_k)$, $B = (b_1, \dots, b_m)$, $k \leq m$, and $a_i \leq b_i$ for $i = 1, \dots, k$ (*termwise dominating*);
- (ii) $A < B$ if $\max(A) < \min(B)$ (*global dominating*);
- (iii) $A \triangleleft B$ if $(A - B) < (B - A)$, where $A' - B'$ stands for the set difference $\{i' : A' \ni i' \notin B'\}$ (*global dominating after cancelations*);
- (iv) $A \triangleright B$ if $A - B \neq \emptyset$, and $B - A$ can be expressed as a disjoint union of nonempty subsets B', B'' so that $B' < (A - B) < B''$ (*splitting*).

(Note that relations (i),(ii) are transitive, whereas (iii),(iv) are not in general.) Relations (iii) and (iv) give rise to two important notions introduced by Leclerc and Zelevinsky in [4] (where these notions appear in characterizations of quasi-commuting flag minors of a generic q -matrix).

Definitions. Sets $A, B \subseteq [n]$ are called *strongly separated* (from each other) if $A \triangleleft B$ or $B \triangleleft A$ or $A = B$. Sets $A, B \subseteq [n]$ are called *weakly separated* if either they are strongly separated, or $A \triangleright B$ and $|A| \geq |B|$, or $B \triangleright A$ and $|B| \geq |A|$. Accordingly, a collection $\mathcal{F} \subseteq 2^{[n]}$ is called strongly (resp. weakly) separated if any two of its members are such. For brevity we refer to strongly and weakly separated collections as *s-collections* and *w-collections*, respectively.

In what follows we will distinguish one or another set-system $\mathcal{D} \subseteq 2^{[n]}$, referring to it as a *ground collection*, or a *domain*. We are interested in the situation when (strongly or weakly) separated collections in \mathcal{D} possess the property of *max-size purity*, which means the following.

Definitions. Let us say that \mathcal{D} is *s-pure* if all (inclusion-wise) maximal s-collections in \mathcal{D} have the same cardinality, which in this case is called the *s-rank* of \mathcal{D} and denoted by $r^s(\mathcal{D})$. Similarly, we say that \mathcal{D} is *w-pure* if all maximal w-collections in \mathcal{D} have the same cardinality, called the *w-rank* of \mathcal{D} and denoted by $r^w(\mathcal{D})$.

(The term “purity” is borrowed from topological areas where this is often used to distinguish the complexes in which all maximal cells have the same dimension. In our case we can interpret each s-collection (resp. w-collection) as a cell, forming an abstract simplicial complex (with \mathcal{D} as the set of 0-dimensional cells). This justifies our terms “s-pure” and “w-pure”.)

Leclerc and Zelevinsky [4] proved that the *full domain* $\mathcal{D} = 2^{[n]}$ is s-pure and conjectured that $2^{[n]}$ is w-pure as well (in which case there would be $r^w(2^{[n]}) = r^s(2^{[n]}) = \frac{n(n+1)}{2} + 1$). A sharper version of this conjecture involves an arbitrary permutation ω on $[n]$ and considers subsets $X \subseteq [n]$ satisfying the condition:

- (1.2) if $i < j$, $\omega(i) < \omega(j)$, and $j \in X$, then $i \in X$,

called *ω -chamber sets*. They conjectured that the domain $\mathcal{D}(\omega)$ formed by the *ω -chamber sets* is w-pure (in our terms), with the w-rank equal to $|\text{Inv}(\omega)| + n + 1$. Here $\text{Inv}(\omega)$ denotes the set of *inversions* of ω , i.e., pairs (i, j) in $[n]$ such that $i < j$ and $\omega(i) > \omega(j)$. The number $|\text{Inv}(\omega)|$ is called the *length* of ω . The *longest* permutation ω_0 (where $\omega_0(i) = n - i + 1$) gives $\mathcal{D}(\omega_0) = 2^{[n]}$.

Leclerc–Zelevinsky’s conjecture was proved affirmatively in [2]. The key part of that work consisted in showing the w-purity of $2^{[n]}$, and this result was extended to an arbitrary permutation ω , and more.

Theorem 1.1 ([2]) *The full domain $2^{[n]}$ is w-pure. As a consequence, the following domains \mathcal{D} are w-pure as well:*

- (i) $\mathcal{D} = \mathcal{D}(\omega)$ for any permutation ω on $[n]$;
- (ii) $\mathcal{D} = \mathcal{D}(\omega', \omega)$, where ω', ω are two permutations on $[n]$ with $\text{Inv}(\omega') \subset \text{Inv}(\omega)$, and $\mathcal{D}(\omega', \omega)$ is formed by the ω -chamber sets $X \subseteq [n]$ satisfying the additional condition: if $i < j$, $\omega'(i) > \omega'(j)$, and $i \in X$, then $j \in X$; furthermore, $r^w(\mathcal{D}(\omega, \omega')) = |\text{Inv}(\omega)| - |\text{Inv}(\omega')| + n + 1$;
- (iii) $\mathcal{D} = \Delta_n^{m', m} := \{X \subseteq [n] : m' \leq |X| \leq m\}$ for any $m' \leq m$; furthermore, $r^w(\Delta_n^{m', m}) = \binom{n+1}{2} - \binom{n-m+1}{2} - \binom{m'+1}{2} + 1$ (this turns into $m(n-m) + 1$ when $m' = m$).

Note that (ii) generalizes (i) since $\mathcal{D}(\omega) = \mathcal{D}(\text{id}, \omega)$, where id is the identical permutation ($\text{id}(i) = i$). The domain $\Delta_n^{m', m}$ in (iii), which generalizes the Boolean hyper-simplex (or *discrete Grassmannian*) $\Delta_n^m := \Delta_n^{m, m}$, can be called a *restricted discrete flag*. As is explained in [2], cases (i) and (ii) can be reduced to $\mathcal{D} = 2^{[n]}$; also in these two cases the domains turn out to be s-pure as well, and the w- and s-ranks are equal. (Note that in general a domain \mathcal{D} may be w-pure but not s-pure (e.g. for $\mathcal{D} = \Delta_5^2$), and vice versa; also when both w- and s-ranks exist, they may be different.) Using simple observations from [4], one can reduce case (iii) to $2^{[n]}$ as well. In its turn, the proof of w-purity for $2^{[n]}$ given in [2] is direct and essentially relies on a mini-theory of *generalized tilings* (briefly, *g-tilings*) developed in [1].

Subsequently Oh, Postnikov, and Speyer [5] gave another proof for Δ_n^m , using a machinery of *plabic graphs* and *alternating strand diagrams* elaborated in [6]. Moreover, [5] established the w-purity for certain domains $\mathcal{D} \subseteq \Delta_n^m$ related to so-called *Grassmann necklaces* and *positroids* (see also [3] for a sharper result). They also explained that the w-purity of such necklace domains implies the w-purity for the above-mentioned ω -chamber domains $\mathcal{D}(\omega)$. (At the same time, the w-purity of $\mathcal{D}(\omega', \omega)$ does not follow directly from results in [5].)

The purpose of this paper is to extend the known w-purity results by demonstrating wide classes of domains whose w-purity follows from the w-purity of $2^{[n]}$.

One sort of such domains \mathcal{D} (generalizing (iii) in Theorem 1.1) is described in terms of lattice paths in the so-called *steep ladder diagrams* in the plane (defined in Section 3). Our method of handling such a \mathcal{D} borrows one idea used in [2] to prove the w-purity for $\mathcal{D}(\omega)$ as in Theorem 1.1(i). More precisely, we will consider a w-pure domain \mathcal{L} including \mathcal{D} and find a certain w-collection $\mathcal{C} \subset \mathcal{L}$, called a *checker* for \mathcal{D} . It has the property that for any $X \in \mathcal{L} - \mathcal{C}$, the following are equivalent: (a) $X \in \mathcal{D}$, and (b) X is weakly separated from \mathcal{C} . Then for any maximal w-collection \mathcal{F} in \mathcal{D} , $\mathcal{F} \cup \mathcal{C}$ is a maximal w-collection in \mathcal{L} , and now the w-purity of \mathcal{D} follows from that of \mathcal{L} .

Domains of a more general type are generated by *cyclic patterns (sequences)* $\mathcal{S} = (S_1, S_2, \dots, S_r = S_0)$ consisting of different and pairwise weakly separated subsets of $[n]$ such that for each i , $|S_i - S_{i-1}| \leq 1$ and $|S_{i-1} - S_i| \leq 1$; we call \mathcal{S} *simple* if $|S_{i-1}| \neq |S_i|$ holds for each i , and *generalized* otherwise. We use a well-known representation of subsets of $[n]$ as certain points in a zonogon Z in the plane and associate to \mathcal{S} the piecewise linear curve $\zeta_{\mathcal{S}}$ in Z by connecting consecutive “points” of \mathcal{S} by straight-line segments. It turns out that $\zeta_{\mathcal{S}}$ is non-self-intersecting when \mathcal{S} is simple, and we point out necessary and sufficient conditions to be non-self-intersecting in the generalized case. This allows us to represent the collection $\mathcal{D}_{\mathcal{S}}$ of all subsets of $[n]$ weakly separated from \mathcal{S} as the union of two domains $\mathcal{D}_{\mathcal{S}}^{\text{in}}$ and $\mathcal{D}_{\mathcal{S}}^{\text{out}}$ consisting of those elements of $\mathcal{D}_{\mathcal{S}}$ that are located inside and outside $\zeta_{\mathcal{S}}$, respectively. Domains of type $\mathcal{D}_{\mathcal{S}}^{\text{in}}$ generalize those exposed in Theorem 1.1 and those arising in the necklace constructions of [5].

We prove that both domains $\mathcal{D}_{\mathcal{S}}^{\text{in}}$ and $\mathcal{D}_{\mathcal{S}}^{\text{out}}$ are w-pure. Moreover, one shows that these domains form a *complementary pair*, in the sense that any $X \in \mathcal{D}_{\mathcal{S}}^{\text{in}}$ and any $Y \in \mathcal{D}_{\mathcal{S}}^{\text{out}}$ are weakly separated from each other; this proves a conjecture on generalized necklaces in [3]. In fact, this result is obtained as a consequence of our proper study of a new graphical model for w-collections. We introduce a new sort of polygonal complexes on a zonogon that we call *combined tilings*, or *c-tilings* for short; this is viewed as an “intermediate construction” between usual rhombus tilings and g-tilings of [1] (where the former (latter) are known to model maximal s-collections (resp. w-collections)). Taking advantages from a nice planar structure of c-tilings (which is less sophisticated than the structure of g-tilings), we demonstrate a series of combinatorial results on these objects. Of most importance is that the sets of vertices of c-tilings give maximal w-collections in $2^{[n]}$, and vice versa. The declared w-purity of the domains $\mathcal{D}_{\mathcal{S}}^{\text{in}}$, $\mathcal{D}_{\mathcal{S}}^{\text{out}}$ follows from the observation that a c-tiling whose vertex set contains the given pattern \mathcal{S} can be split into two subtilings, one lying inside, and the other outside $\zeta_{\mathcal{S}}$.

The most general case of w-pure domains that we present is obtained when the role of patterns is played by *planar graphs* \mathcal{H} , embedded in a zonogon, whose vertices are represented by subsets of $[n]$ giving a w-collection \mathcal{S} , and edges are formed by pairs similar to those in cyclic patterns. Then instead of two domains $\mathcal{D}_{\mathcal{S}}^{\text{in}}$, $\mathcal{D}_{\mathcal{S}}^{\text{out}}$ as above, we deal with a set of domains $\mathcal{D}_{\mathcal{S}}(F)$, each being related to a face F of \mathcal{H} , and we prove that any two of them form a complementary pair. In particular, each $\mathcal{D}_{\mathcal{S}}(F)$ is w-pure.

Note that for cyclic patterns consisting of strongly separated sets, we can consider corresponding pairs of domains concerning the strong separation relation. One shows that such domains are s-pure (which is relatively easy) and that the corresponding s-ranks and w-ranks are equal.

This paper is organized as follows. Section 2 uses a graph theoretic approach to demonstrate basic properties of checkers and complementary pairs. Section 3 is devoted to domains generated by steep ladder diagrams. Section 4 introduces simple cyclic patterns \mathcal{S} and related domains $\mathcal{D}_{\mathcal{S}}^{\text{in}}$ and $\mathcal{D}_{\mathcal{S}}^{\text{out}}$ and state a theorem on their w-purity (Theorem 4.2). Section 5 starts with a brief review on rhombus tilings and then introduce the construction of combined tilings. The most content of this section is devoted to a description of transformations of c-tilings, so-called *raising* and *lowering flips*, which correspond to standard mutations for maximal w-collections. As a result, we establish a bijection between the c-tilings on n -zonogon and the maximal w-collections in $2^{[n]}$.

Section 6 further develops our “mini-theory” of c-tilings by describing operations of *contraction* and *expansion* (analogous to those given for g-tilings in [1, Sec. 8] but looking much simpler for c-tilings); they transform c-tilings on n -zonogon to ones on $(n - 1)$ -zonogon, and conversely. This enables us to conduct proofs by induction on n , and using this and results from Section 5, we finish the proof of Theorem 4.2. The concluding Section 7 is devoted to special cases, illustrations and generalizations. Here we extend the w-purity result to generalized cyclic patterns (Theorem 7.3), and finish with the most general case, which involves planar graph patterns (Theorem 7.4).

Additional notations. An *interval* in $[n]$ is a set of the form $\{p, p + 1, \dots, q\}$, and a *co-interval* is the complement of an interval to $[n]$. For $p \leq q$, we denote by $[p..q]$ the interval $\{p, p + 1, \dots, q\}$.

When sets $A, B \subseteq [n]$ are weakly (strongly) separated, we write $A \overline{\text{weak}} B$ (resp. $A \overline{\text{str}} B$). We use similar notation $A \overline{\text{weak}} \mathcal{B}$ (resp. $A \overline{\text{str}} \mathcal{B}$) when a set $A \subseteq [n]$ is weakly (resp. strongly) separated from all members of a collection $\mathcal{B} \subseteq 2^{[n]}$.

For a set $X \subset [n]$, distinct elements $i, \dots, j \in [n] - X$, and an element $k \in X$, we abbreviate $X \cup \{i\} \cup \dots \cup \{j\}$ as $Xi \dots j$, and $X - \{k\}$ as $X - k$.

By a *path* in a directed graph we mean a sequence $P = (v_0, e_1, v_1, \dots, e_k, v_k)$, where each e_i is an edge connecting vertices v_{i-1} and v_i . An edge e_i is called *forward* (*backward*) if it goes from v_{i-1} to v_i (resp. from v_i to v_{i-1}), and we write $e_i = (v_{i-1}, v_i)$ (resp. $e_i = (v_i, v_{i-1})$). The path is called *directed* if all its edges are forward. Sometimes we will use abbreviated notation for P via vertices, writing $P = v_0 v_1 \dots v_k$.

Also we will use the following simple fact for $A, B \subset [n]$:

(1.3) if $A \overline{\text{weak}} B$ and $|A| \leq |B|$, then relations $A \prec B$ and $A \triangleleft B$ are equivalent.

2 Checkers and related objects

In fact, the idea of “checkering” mentioned in the Introduction is applicable wider, due to the following simple method of constructing a representable class of pure domains. This is convenient to be described in graph theoretical terms. More precisely, we can associate to a domain $\mathcal{D} \subseteq 2^n$ the undirected graph $G_{\mathcal{D}} = (V, E)$ whose vertices are the elements of \mathcal{D} and whose edges are the weakly (resp. strongly) separated pairs $A, B \in \mathcal{D}$, $A \neq B$. Then each w-collection (resp. s-collection) in \mathcal{D} corresponds to a *clique* of $G_{\mathcal{D}}$, a subset of vertices in which any two vertices are adjacent (connected by edge) in $G_{\mathcal{D}}$. Accordingly, the w-purity (resp. s-purity) means that all *maximal cliques* in $G_{\mathcal{D}}$ have the same size.

In a general setting, we can consider an arbitrary undirected graph $G = (V, E)$ and a vertex subset $D \subseteq V$ (playing the role of “domain”). We write $G[D]$ for the subgraph of G induced by D (i.e. of the form (D, E') with E' maximal). We say that D is *pure* (w.r.t. cliques) if all maximal cliques in $G[D]$ have the same size.

Lemma 2.1 *Let C be a clique in a graph $G = (V, E)$ and let $C' \subseteq C$. Define D to be the set of vertices v of G such that $v \notin C - C'$ and $\{v\} \cup C$ is a clique. Suppose that the whole set V is pure. Then D is pure as well.*

Proof Consider a maximal clique X in $G[D]$. Then $X \cap C = C'$, and $Y := X \cup C$ is a clique of G . Moreover, Y is a maximal clique of G . Indeed, suppose that there is a vertex $v \notin Y$ such that $Y \cup \{v\}$ is a clique. Then $v \notin C$ and $\{v\} \cup C$ is a clique. This implies $v \in X$, leading to a contradiction. Now the purity of D follows from that of V and the equality $|Y| = |X| + |C - C'|$. \blacksquare

Corollary 2.2 *Let $\mathcal{L} \subseteq 2^{[n]}$ be a w-pure domain. Let $\mathcal{C} \subseteq \mathcal{L}$ be a weakly separated collection and let $\mathcal{C}' \subseteq \mathcal{C}$. Define $\mathcal{D}_{\mathcal{C},\mathcal{C}'}^{\mathcal{L}}$ to be the set of $X \in \mathcal{L}$ such that $X \notin \mathcal{C} - \mathcal{C}'$ and $X \overline{\text{weak}} \mathcal{C}$. Then the domain $\mathcal{D}_{\mathcal{C},\mathcal{C}'}^{\mathcal{L}}$ is w-pure. A similar assertion is valid for the strong separation.*

When a domain $\mathcal{D} \subseteq \mathcal{L}$ is representable in the form $\mathcal{D}_{\mathcal{C},\mathcal{C}'}^{\mathcal{L}}$, where \mathcal{L} and \mathcal{C} are as in the corollary, and $\mathcal{C}' = \mathcal{D} \cap \mathcal{C}$, we say that \mathcal{C} is a *checker* for \mathcal{D} *within* \mathcal{L} (regarding w- or s-purity). If we take as \mathcal{L} the entire set $2^{[n]}$, we abbreviate $\mathcal{D}_{\mathcal{C},\mathcal{C}'}^{\mathcal{L}}$ to $\mathcal{D}_{\mathcal{C},\mathcal{C}'}$.

It should be noted that all particular domains whose purity has been known to us so far are just checker-possessing ones within 2^n . In light of this, one may ask: whether *every* w-pure domain $\mathcal{D} \subset 2^{[n]}$ has a checker within 2^n ? However, this is not so, as can be shown by a rather simple counterexample.

For completeness of our description we now outline explicit constructions of checkers for the domains exposed in Theorem 1.1.

Example 1. Consider the domain $\Delta_n^{m',m}$ as in case (iii) of this theorem, where $m' \leq m$. Let \mathcal{C} consist of all intervals of size $\geq m$ and all co-intervals of size $\leq m'$. One can check that \mathcal{C} is a w-collection. By simple observations in [4], one shows that $X \overline{\text{weak}} \mathcal{C}$ for each $X \in \Delta_n^{m',m}$, and that $X \in 2^{[n]} - \Delta_n^{m',m}$ together with $X \overline{\text{weak}} \mathcal{C}$ implies $X \in \mathcal{C}$. Therefore, \mathcal{C} is a checker for $\Delta_n^{m',m}$, and we have $\Delta_n^{m',m} = \mathcal{D}_{\mathcal{C},\mathcal{C}'}$, where \mathcal{C}' is formed by the intervals of size m and the co-intervals of size m' .

Example 2. For a permutation ω on $[n]$, consider the domain $\mathcal{D}(\omega)$ consisting of the ω -chamber sets (defined by (1.2)). As is shown in [2, Theorem 2.1], $\mathcal{D}(\omega)$ has as a checker the following set-system

$$\mathcal{C}(\omega) := \{\omega^{-1}[k] \cap [j..n] : k, j \in [n]\} \cup \{\emptyset\}$$

(where possible repeated sets are ignored and where $\omega^{-1}[k] = \{i : \omega(i) \in [k]\}$). This implies that $\mathcal{D}(\omega) = \mathcal{D}_{\mathcal{C}(\omega),\mathcal{C}'}$ with $\mathcal{C}' := \{\omega^{-1}[k] : k \in [n]\}$.

Example 3. Consider the domain $\mathcal{D}(\omega', \omega)$ defined in case (ii) of Theorem 1.1. It follows from a description in [2, Sec. 7] that $\mathcal{D}(\omega', \omega)$ has as a checker the set-system

$$\mathcal{C}(\omega) \cup \{(\omega')^{-1}[k] \cap [j] : k, j \in [n]\},$$

where $\mathcal{C}(\omega)$ is defined as in the previous example.

(Note that verifications of the checkers in Examples 2,3 are not straightforward; in particular, a proof for $\mathcal{D}(\omega', \omega)$ in [2] uses a machinery of generalized tilings.)

In the rest of this section we outline one more useful construction; it deals with “complementary pairs” of pure domains. We again prefer the language of graphs.

For a graph $G = (V, E)$ and a vertex subset $V' \subset V$, let $r(V')$ denote the maximal size of a clique in V' . For $v \in V$, we denote the set of vertices adjacent to v by $N(v)$, and denote $\{v\} \cup N(v)$ by $N^+(v)$.

Lemma 2.3 *Let subsets $D, D' \subseteq V$ be such that: (a) $D \subseteq N^+(v)$ for each $v \in D'$, and $D' \subseteq N^+(v)$ for each $v \in D$; and (b) the set $D \cup D'$ is pure. Then each of D, D' is pure and $r(D) + r(D') = r(D \cup D') + r(D \cap D')$.*

Proof W.l.o.g., one may assume that $D \cup D' = V$. Fix a maximal clique C in D' and let $C' := C \cap D$. One can see that D is exactly the set of $v \in V - (C - C')$ such that $\{v\} \cup C$ is a clique. Then D is pure by Lemma 2.1. Similarly D' is pure. The required equality is obvious. \blacksquare

Corollary 2.4 *Let domains $\mathcal{D}, \mathcal{D}' \subset 2^{[n]}$ be such that any $X \in \mathcal{D}$ and $Y \in \mathcal{D}'$ are weakly separated from each other, and $\mathcal{D} \cup \mathcal{D}'$ is w-pure. Then each of $\mathcal{D}, \mathcal{D}'$ is w-pure. A similar assertion is valid for the strong separation.*

We say that such domains $\mathcal{D}, \mathcal{D}'$ form a *complementary pair*.

3 Steep ladder diagrams

In this section we show the w-purity for domains generated by one class of lattice paths.

By the (full square) *grid* we mean the directed graph Γ whose vertices are the points in \mathbb{Z}^2 and whose edges are the unit-length segments directed up or to the right. So each vertex (point) (i, j) has one outgoing *horizontal* edge $((i, j), (i + 1, j))$, denoted by $\text{hor}_{i,j}$, and one outgoing *vertical* edge $((i, j), (i, j + 1))$, denoted by $\text{vert}_{i,j}$.

Each finite directed path P in Γ beginning at the origin $(0, 0)$ encodes a finite subset $S(P)$, namely:

(3.1) for $P = v_0 v_1 \dots v_k$ with $v_0 = (0, 0)$, $S(P)$ consists of the elements $i \in [k]$ such that the edge (v_{i-1}, v_i) is horizontal.

In particular, the set of directed paths P of length n is bijective to $2^{[n]}$, and the set of directed paths P ending at $(m, n - m)$ is bijective to Δ_n^m (where P begins at $(0, 0)$).

We consider a certain finite part of Γ . It is determined by a sequence $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_k)$ of weakly decreasing nonnegative integers, i.e., $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0$ (a $(k + 1)$ -*partition*). Define $m := \lambda_0$ and $n := m + k$. The subgraph $\Gamma_\lambda = (V_\lambda, E_\lambda)$ of Γ induced by the set of vertices

$$V_\lambda := \cup_{j=0}^k \{(i, j) : 0 \leq i \leq \lambda_j\}$$

is called the *ladder* determined by λ . Its north-east boundary $L = L_\lambda$ is formed by a (non-directed) path from $(m, 0)$ to $(0, k)$ in which the vertical (horizontal) edges are traversed in the forward (resp. backward) direction. Two examples are drawn in Fig. 1 where the paths L_λ are indicated in bold.

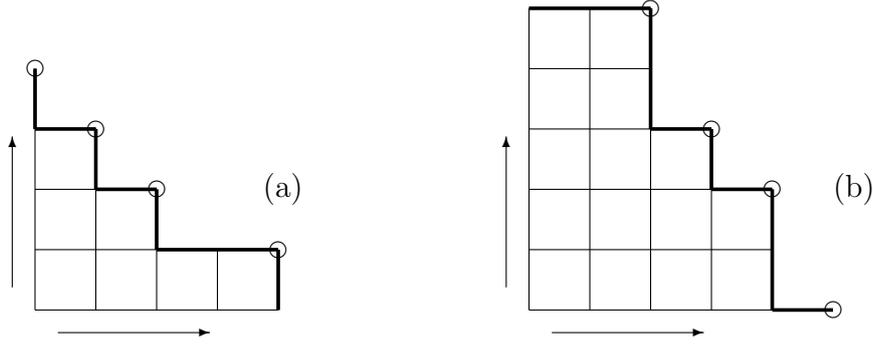


Figure 1: (a) $\Gamma_{4,4,2,1,0}$; (b) $\Gamma_{5,4,4,3,2,2}$

Clearly for each vertex v of Γ_λ , any directed path P in Γ going from $(0,0)$ to v is entirely contained in Γ_λ . Also the set $S(P)$ does not change under extending P from the end by any number of vertical edges. We define:

- (i) T_λ to be the set of vertices (i,j) in L_λ that are “seen from north and from east”, or “forming outer corners”, i.e., such that neither $\text{hor}_{i,j}$ nor $\text{vert}_{i,j}$ belongs to Γ_λ ;
- (ii) \mathcal{P}_λ to be the set of directed paths from $(0,0)$ to T_λ ;
- (iii) \mathcal{D}_λ to be the collection $\{S(P) : P \in \mathcal{P}_\lambda\}$.

The domain \mathcal{D}_λ is just of our interest in this section. Note that the map $P \mapsto S(P)$ is injective on \mathcal{P}_λ .

Example 4. When $k = m$ and $\lambda_i = k - i$ for $i = 0, \dots, k$ (see Fig. 2(a)), we have $\mathcal{D}_\lambda = 2^{[k]}$.

Example 5. When $\lambda_0 = \lambda_1 = \dots = \lambda_k (= m)$ (see Fig. 2(b)), Γ_λ spans the rectangle between $(0,0)$ and (m,k) , T_λ consists of the single vertex (m,k) , and $\mathcal{D}_\lambda = \Delta_{m+k}^m$. This ladder is denoted by $\Gamma^{m,k}$.

Example 6 (generalizing Examples 4,5). Let $\lambda_0 = \lambda_1 = \dots = \lambda_{k'} (= m)$ and $\lambda_j = \lambda_{j-1} - 1$ for $j = k' + 1, \dots, k$ (see Fig. 2(c)). Then $\mathcal{D}_\lambda = \Delta_n^{m',m}$ for $n := m + k'$ and $m' := m + k' - k$.

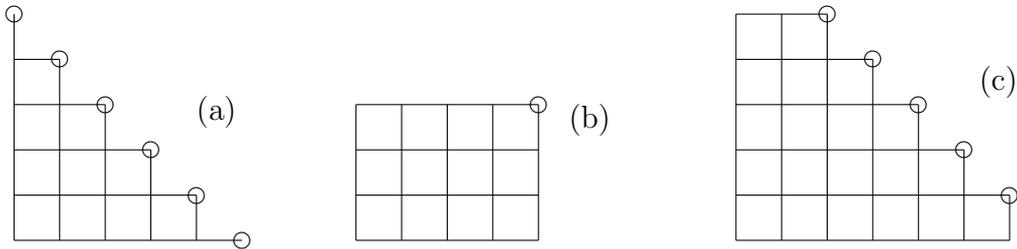


Figure 2: (a) $\lambda = (5, 4, 3, 2, 1, 0)$, $\mathcal{D}_\lambda = 2^{[5]}$; (b) $\lambda = (4, 4, 4, 4)$, $\Gamma_\lambda = \Gamma^{4,3}$, $\mathcal{D}_\lambda = \Delta_7^4$; (c) $\lambda = (6, 6, 5, 4, 3, 2)$, $\mathcal{D}_\lambda = \Delta_7^{2,6}$

Definition. A ladder Γ_λ is called *steep* if $\lambda_i \geq \lambda_{i-1} - 1$ for all $i = 1, \dots, k$.

In particular, the ladders in Examples 4–6 and in Fig. 1(b) are steep, but the one

in Fig. 1(a) is not. The domains \mathcal{D}_λ in Examples 4–6 are pure by Theorem 1.1, and as a generalization of case (iii) of that theorem, we show the following

Theorem 3.1 *For a steep ladder $\Gamma_{\lambda=(\lambda_0, \dots, \lambda_k)}$, the domain \mathcal{D}_λ is w-pure.*

Remark 1. \mathcal{D}_λ need not be w-pure when Γ_λ is not steep. Indeed, let λ be as in Fig. 1(a). Then $T_\lambda = \{(0, 4), (1, 3), (2, 2), (4, 1)\}$, and one can compute that $\mathcal{D}_\lambda = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 1234, 1235, 1245, 1345, 2345\}$. Since the intervals $\emptyset, 1, 12, 1234, 2345$ are weakly separate from any subset in the interval $[5]$, we can consider the rest in \mathcal{D}_λ . It has maximal w-collections of sizes 4 and 5, e.g. $\{2, 4, 24, 1245\}$ and $\{2, 3, 4, 23, 34\}$, whence \mathcal{D}_λ is not w-pure.

Proof of Theorem 3.1 As before, m stands for λ_0 , and we set

$$m' := \lambda_k \quad \text{and} \quad n := k + m'.$$

Define the partition $\lambda' = (\lambda'_0, \dots, \lambda'_k)$ by

$$\lambda'_0 := n \quad \text{and} \quad \lambda'_j := \lambda'_{j-1} - 1 \quad \text{for } j = 1, \dots, k.$$

This λ' is as in Example 6 (with $k' = 0$), and we have $\mathcal{D}_{\lambda'} = \Delta_n^{m', n}$. Since $\lambda'_k = m - k = m' = \lambda_k$, the steepness of Γ_λ implies $\lambda'_j \geq \lambda_j$ for all j . Therefore, Γ_λ is entirely contained in $\Gamma_{\lambda'}$, and (m', k) is a common vertex of these ladders. This gives $\mathcal{D}_\lambda \subseteq \Delta_n^{m', n}$. Indeed, for $X \in \mathcal{D}_\lambda$, if P is the path in \mathcal{P}_λ with $S(P) = X$, then extending P at the end by vertical edges (if needed), we obtain a path P' in $\mathcal{P}_{\lambda'}$ such that $S(P') = S(P)$. We call P' the *extension* of P (into $\Gamma_{\lambda'}$).

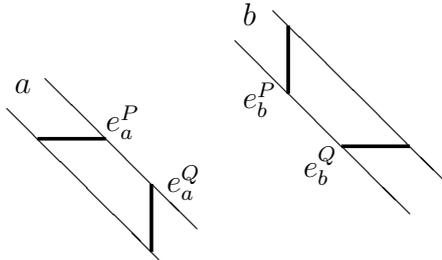
By Theorem 1.1(iii), the domain $\Delta_n^{m', n}$ is w-pure. We are going to show the w-purity of \mathcal{D}_λ by constructing a checker for \mathcal{D}_λ within $\Delta_n^{m', n}$; then the result will follow from Corollary 2.2.

To this aim, we first reformulate the condition of weak separation in $\mathcal{D}_{\lambda'}$ in graphical terms as suggested in [7]. We denote i -th edge of a path P by e_i^P .

Definition. Two directed paths P, Q in $\mathcal{P}_{\lambda'}$ are said to be *conflicting* if

(3.2) there are $1 < a < b \leq n$ such that, up to renaming P and Q , the edges e_a^P, e_b^Q are horizontal, the edges e_b^P, e_a^Q are vertical, e_a^Q is below e_a^P , and e_b^Q is below e_b^P .

Here for edges e, e' leaving vertices (i, j) and (i', j') , respectively, with $i + j = i' + j'$, we say that e is (located) *below* e' if either $j < j'$, or $j = j'$ and e is horizontal, whereas e' is vertical. A pair (a, b) as in (3.2) is called *critical*. See the picture.



Claim Let $P, Q \in \mathcal{P}_{\lambda'}$, $X = S(P)$, and $Y = S(Q)$. The sets X, Y are weakly separated if and only if P, Q are non-conflicting.

This fact was established in [7]. To make our description self-contained (and for the reason that the text of [7] is not accessible at present), we give a proof.

Proof of the Claim Suppose that P, Q are conflicting, and let (a, b) be a critical pair for them. Then (up to renaming P, Q), $e_a^P = \text{hor}_{i,j}$, $e_a^Q = \text{vert}_{i',j'}$, $e_b^P = \text{vert}_{p,q}$, $e_b^Q = \text{hor}_{p',q'}$, where $i + j = i' + j' = a - 1$, $i < i'$, $p + q = p' + q' = b - 1$, $p \leq p'$. It follows that $b, c \in Y - X$ and $a \in X - Y$ for some $c < a$ (where c exists because of $|X \cap [a - 1]| = i < i' = |Y \cap [a - 1]|$). These relations together with $|X \cap [b]| = |X \cap [b - 1]| = p < p' + 1 = |Y \cap [b]|$ imply that X, Y are not weakly separated.

Conversely, suppose that X, Y are not weakly separated. Let $X - Y = X_1 \cup X_2 \cup \dots \cup X_\alpha$ and $Y - X = Y_1 \cup Y_2 \cup \dots \cup Y_\beta$, where all X_i, Y_j are nonempty and, up to renaming X, Y , one has $Y_1 < X_1 < Y_2 < X_2 < \dots$ (with $<$ defined in (1.1)(ii)). Then one of the following takes place: (i) $\beta \geq 2$ and $|X_1| < |Y_1 \cup Y_2|$, and (ii) $\alpha, \beta \geq 2$ and $|X_1| \geq |Y_1 \cup Y_2|$. Define $a := \min(X_1)$ and $b := \max(Y_2)$. Then $a < b$; e_a^P, e_b^Q are horizontal; e_a^Q, e_b^P are vertical; and e_a^Q is below e_a^P (since $|Y \cap [a - 1]| - |X \cap [a - 1]| = |Y_1| > 0$). Also, in case (i), e_b^Q is below e_b^P (since $|Y \cap [b]| - |X \cap [b]| = |Y_1 \cup Y_2| - |X_1| > 0$), whence (a, b) is critical and P, Q are conflicting. And in case (ii), for $d := \min(X_2)$, we have: $b < d$; e_b^Q, e_d^P are horizontal; e_b^P, e_d^Q are vertical; e_b^P is below e_b^Q (since $|X \cap [b - 1]| > |Y \cap [b - 1]|$); and e_d^P is below e_d^Q . So (b, d) is critical and P, Q are again conflicting. \blacksquare

Next we construct the desired checker. Its members are induced by certain paths in $\mathcal{P}_{\lambda'}$, as follows. For a vertex $(i, j) \in V_{\lambda'}$, let $H_{i,j}$ be the directed path formed by the vertical path P_1 from $(0, 0)$ to $(0, j)$, followed by the horizontal path P_2 from $(0, j)$ to (i, j) , followed by the vertical path P_3 from (i, j) to the vertex (i, j') of $\Gamma_{\lambda'}$ with j' maximum (some of P_1, P_2, P_3 may be degenerate). Such an $H_{i,j}$ is called a *double hook* in $\Gamma_{\lambda'}$, and we say that it is *essential* for Γ_λ if the *horizontal* edge $\text{hor}_{i,j}$ does not belong to Γ_λ (note that $\text{hor}_{i,j}$ need not belong to $\Gamma_{\lambda'}$ either). In particular, an essential $H_{i,j}$ ends in $T_{\lambda'}$ (and therefore belongs to $\mathcal{P}_{\lambda'}$), and the vertex (i, j) either is not in Γ_λ , or belongs to the boundary L_λ and is “seen from east”.

We assert that the collection \mathcal{C} of sets $S(H)$ over all essential double hooks H for Γ_λ is a checker for \mathcal{D}_λ within $\mathcal{D}_{\lambda'}$.

Indeed, first of all it is easy to see that any two double hooks are non-conflicting; so \mathcal{C} is a w-collection by the Claim.

Consider an essential double hook $H = H_{i,j}$ and its corresponding concatenation into P_1, P_2, P_3 (where P_1, P_3 are vertical and P_2 is horizontal). Suppose that some path $Q \in \mathcal{P}_{\lambda'}$ is conflicting to H . Then there is a critical pair (a, b) such that (taking into account the construction of H): $e_a^H \in P_2$, $e_b^H \in P_3$, e_a^Q is vertical and lies below e_a^H , and e_b^Q is horizontal and lies below e_b^H . Let $e_b^H = \text{vert}_{i',j'}$ and $e_b^Q = \text{hor}_{p,q}$. Since $e_b^H \in P_3$, we have $i' = i$, $j' \geq j$, and $i + j \leq i + j' = b - 1$. At the same time, $p \geq i' = i$ (since e_b^Q is below e_b^H) and $p + q = b - 1$. Now the facts that Γ_λ is steep and that $\text{hor}_{i,j}$ is not in Γ_λ imply that the edge e_b^Q is not in Γ_λ either. This means that Q cannot be the extension of any path in $\mathcal{P}_{\lambda'}$, and hence $S(Q)$ is not in \mathcal{D}_λ .

Conversely, suppose that a path $Q \in \mathcal{P}_{\lambda'}$ is not the extension of any path in \mathcal{P}_{λ} . Let $e = \text{hor}_{p,q}$ be the last horizontal edge of Q . Then e does not belong to Γ_{λ} . Let Q' be the part of Q from $(0,0)$ to $(p+1,q)$. Two cases are possible.

(i) Q' is the concatenation of the vertical path from $(0,0)$ to $(0,q)$ and the horizontal path from $(0,q)$ to $(p+1,q)$. Then Q is nothing else than the double hook $H_{p+1,q}$. Moreover, Q is essential since the vertex $(p+1,q)$ (and therefore the edge $\text{hor}_{p+1,q}$) is not in Γ_{λ} .

(ii) Q' is not as in (i). Let $e' = (i',j')$ be the last vertical edge of Q' . Then $0 < i' \leq p$ and $j' = q-1$. Take the double hook $H = H_{p,q}$. It is essential since $\text{hor}_{p,q}$ is not in Γ_{λ} . Also (p,q) cannot be the endvertex of H (since the fact that $(p+1,q)$ is a vertex of $\Gamma_{\lambda'}$ implies $(p,q) \notin T_{\lambda'}$). Therefore, H contains the edge $\text{vert}_{p,q}$. Now define $a := i' + q$ and $b := p + q + 1$. Then $a < b$, the edge e_a^H is horizontal (namely, $\text{hor}_{i'-1,q}$) and lies above $e_a^Q = e' = \text{vert}_{i',q-1}$, and the edge e_b^H is vertical (namely, $\text{vert}_{p,q}$) and lies above $e_b^Q = \text{hor}_{p,q}$. Hence Q and H are conflicting, implying that the sets $S(Q)$ and $S(H) \in \mathcal{C}$ are not weakly separated.

Thus, \mathcal{C} is indeed a checker for \mathcal{D}_{λ} within $\mathcal{D}_{\lambda'} = \Delta_n^{m',m}$ (and \mathcal{D}_{λ} is represented as $\mathcal{D}_{\mathcal{C},\mathcal{C}'}$, where \mathcal{C}' consists of essential double hooks $H_{i,j}$ with (i,j) contained in the boundary L_{λ} and seen from east). Then \mathcal{D}_{λ} is w-pure by Corollary 2.2. \blacksquare

In Section 7.3 we explain that Theorem 3.1 can also be obtained from a general w-purity result.

4 Simple cyclic patterns and their geometric interpretation

Recall that by a *simple cyclic pattern* we mean a sequence \mathcal{S} of subsets $S_1, S_2, \dots, S_r = S_0$ of $[n]$ such that $|S_{i-1} \Delta S_i| = 1$ for $i = 1, \dots, r$, where $A \Delta B$ denotes the symmetric difference $(A - B) \cup (B - A)$. Unless otherwise is explicitly said, we also assume that \mathcal{S} satisfies the following conditions:

(C1) All sets in \mathcal{S} are different;

(C2) \mathcal{S} is weakly separated.

(Condition (C1) can be slightly weakened, as we explain in Section 7.1.) We associate to \mathcal{S} the collection

$$\mathcal{D}_{\mathcal{S}} := \{X \subseteq [n] : X \overline{\text{weak}} \mathcal{S}\}$$

and extract from $\mathcal{D}_{\mathcal{S}}$ two domains $\mathcal{D}_{\mathcal{S}}^{\text{in}}$ and $\mathcal{D}_{\mathcal{S}}^{\text{out}}$. These domains can be defined in two ways: by explicit expressions and by a geometric construction. In the former way, let us say that an element S_i of \mathcal{S} is a *slope* if $|S_{i-1}| \neq |S_{i+1}|$ (i.e., one of these quantities is equal to $|S_i| - 1$, and the other to $|S_i| + 1$). For $h = 0, \dots, n$, the set of slopes S_i with $|S_i| = h$ is denoted by \mathcal{A}_h . Define

$$\begin{aligned} \mathcal{X}_h &:= \{X \in \mathcal{D}_{\mathcal{S}} : |X| = h \text{ and } S_i \prec X \text{ for an odd number of } S_i \in \mathcal{A}_h\}, \\ \mathcal{Y}_h &:= \{X \in \mathcal{D}_{\mathcal{S}} : |X| = h \text{ and } S_i \prec X \text{ for an even number of } S_i \in \mathcal{A}_h\} \end{aligned}$$

(using relation \prec from (1.1)(i)). Then the domains of our interest are defined as follows:

$$\mathcal{D}_S^{\text{in}} := \mathcal{S} \cup \mathcal{X}_0 \cup \mathcal{X}_1 \cup \dots \cup \mathcal{X}_n, \quad (4.1)$$

$$\mathcal{D}_S^{\text{out}} := \mathcal{S} \cup \mathcal{Y}_0 \cup \mathcal{Y}_1 \cup \dots \cup \mathcal{Y}_n. \quad (4.2)$$

Clearly $\mathcal{D}_S^{\text{in}} \cap \mathcal{D}_S^{\text{out}} = \mathcal{S}$ and $\mathcal{D}_S^{\text{in}} \cup \mathcal{D}_S^{\text{out}} = \mathcal{D}_S$. The terms “in” and “out” in our notation for these domains are justified by the second way to define them: by a geometric construction that we now describe.

In the upper half-plane $\mathbb{R} \times \mathbb{R}_{>0}$, fix n vectors $\xi_1 = (x_1, y_1), \dots, \xi_n = (x_n, y_n)$, called *generators*, so that

(4.3) ξ_1, \dots, ξ_n follow in this order clockwise around the origin $(0, 0)$ (i.e., $y_1, \dots, y_n > 0$ and $x_1/y_1 < \dots < x_n/y_n$) and are \mathbb{Z} -independent (i.e., all integer combinations of these vectors are different).

Then the set

$$Z = Z_n := \{\lambda_1 \xi_1 + \dots + \lambda_n \xi_n : \lambda_i \in \mathbb{R}, 0 \leq \lambda_i \leq 1, i = 1, \dots, n\}$$

is a $2n$ -gon. Moreover, Z is a *zonogon*, as it is the sum of line-segments $\{\lambda \xi_i : 1 \leq \lambda \leq 1\}$, $i = 1, \dots, n$. Also it is the image by a linear projection π of the solid n -cube $\text{conv}(2^{[n]}) \subset \mathbb{R}^{[n]}$ into the plane, defined by $\pi(x) := x_1 \xi_1 + \dots + x_n \xi_n$ for $x \in \mathbb{R}^{[n]}$. The *boundary* $bd(Z)$ of Z consists of two parts: the *left boundary* $lbd(Z)$ formed by the sequence of points (vertices) $z_i^\ell := \xi_1 + \dots + \xi_i$ ($i = 0, \dots, n$) connected by the line-segments $z_{i-1}^\ell z_i^\ell$ congruent to ξ_i , and the *right boundary* $rbd(Z)$ formed by the sequence of points $z_i^r := \xi_{n-i+1} + \dots + \xi_n$ ($i = 0, \dots, n$) connected by the line-segments $z_{i-1}^r z_i^r$ congruent to ξ_{n-i+1} . Then $z_0^\ell = z_0^r = (0, 0)$ is the bottom(most) vertex, and $z_n^\ell = z_n^r = \xi_1 + \dots + \xi_n$ is the top(most) vertex of Z .

We identify a subset $X \subseteq [n]$ with the corresponding point $\sum_{i \in X} \xi_i$ in Z . Due to the \mathbb{Z} -independence, all such points in Z are different.

For purposes of the desired geometric construction, we need to impose one more requirement to the generators (which is caused by features of combinatorial tilings behind the construction). Namely, in addition to (4.3), we assume that

(4.4) all vectors ξ_i have the *same euclidean length*, say, $\|\xi_i\| = 1$.

Remark 2. In fact, it suffices to impose the concavity condition: each point ξ_i lies strictly above the line-segment connecting ξ_{i-1} and ξ_{i+1} , $1 < i < n$.

We draw the closed piecewise linear curve ζ_S by concatenating the line-segments connecting consecutive points S_{i-1} and S_i for $i = 1, \dots, r$. The following properties are important.

Proposition 4.1 (i) *For a simple cyclic pattern \mathcal{S} , the curve ζ_S is non-self-intersecting, and therefore it subdivides the zonogon Z into two closed regions R_S^{in} and R_S^{out} such that $R_S^{\text{in}} \cap R_S^{\text{out}} = \mathcal{S}$, $R_S^{\text{in}} \cup R_S^{\text{out}} = Z$, and $bd(Z) \subseteq R_S^{\text{out}}$ (so R_S^{in} is a disc, i.e., it is homeomorphic to $\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$).*

(ii) $\mathcal{D}_S^{\text{in}} = \mathcal{D}_S \cap R_S^{\text{in}}$ and $\mathcal{D}_S^{\text{out}} = \mathcal{D}_S \cap R_S^{\text{out}}$ (justifying notation $\mathcal{D}_S^{\text{in}}$ and $\mathcal{D}_S^{\text{out}}$).

This proposition will be proved in Section 6 relying on properties of combined tilings introduced in the next section. Moreover, using such tilings, we will prove the following w-purity result (where the definition of a complementary pair is given in Section 2).

Theorem 4.2 *For a simple cyclic pattern \mathcal{S} , the domains $\mathcal{D}_{\mathcal{S}}^{\text{in}}$ and $\mathcal{D}_{\mathcal{S}}^{\text{out}}$ form a complementary pair. As a consequence, both $\mathcal{D}_{\mathcal{S}}^{\text{in}}$ and $\mathcal{D}_{\mathcal{S}}^{\text{out}}$ are w-pure.*

This theorem has a natural strong separation counterpart.

Theorem 4.3 *Let \mathcal{S} be a cyclic pattern consisting of different pairwise strongly separated sets in $[n]$. Define $\widehat{\mathcal{D}}_{\mathcal{S}}^{\text{in}}, \widehat{\mathcal{D}}_{\mathcal{S}}^{\text{out}}$ as in (4.1), (4.2) with $\mathcal{D}_{\mathcal{S}}$ replaced by $\widehat{\mathcal{D}}_{\mathcal{S}} := \{X \subseteq [n] : X \overline{\text{str}} \mathcal{S}\}$. Then $X \overline{\text{str}} Y$ for any $X \in \widehat{\mathcal{D}}_{\mathcal{S}}^{\text{in}}$ and $Y \in \widehat{\mathcal{D}}_{\mathcal{S}}^{\text{out}}$. Hence both $\widehat{\mathcal{D}}_{\mathcal{S}}^{\text{in}}$ and $\widehat{\mathcal{D}}_{\mathcal{S}}^{\text{out}}$ are s-pure. Also the s- and w-ranks of $\widehat{\mathcal{D}}_{\mathcal{S}}^{\text{in}}$ are equal, and similarly for $\widehat{\mathcal{D}}_{\mathcal{S}}^{\text{out}}$.*

5 Tilings

We know that maximal s-collections (w-collections) are representable as spectra of rhombus (resp. generalized) tilings on the zonogon $Z = Z_n$. In this section we start with a short review on pure (rhombus) tilings and then introduce the notion of *combined tilings* on Z and show that the latter objects behave similarly to g-tilings: their spectra represent maximal w-collections. At the same time, this new combinatorial model is viewed significantly simpler than the one of g-tilings and we will take advantage from this to obtain a rather transparent proof of Theorem 4.2.

5.1 Rhombus tilings

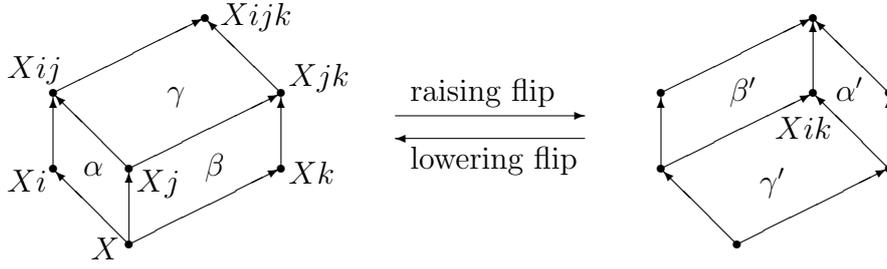
By a *planar tiling* on the zonogon Z we mean a collection T of convex polygons, called *tiles*, such that: (i) the union of tiles is Z , and (ii) any two tiles either are disjoint or intersect by a common vertex or by a common side (in particular, each boundary edge of Z belongs to exactly one tile). Then the set of vertices and the set of edges (sides) of tiles in T , not counting multiplicities, form a planar graph, denoted by $G_T = (V_T, E_T)$. Usually the edges of G_T are equipped with certain directions. Speaking of vertices or edges of T , we mean those in G_T .

A *pure tiling* is a planar tiling T in which all tiles are parallelograms and each edge e is congruent to some generator ξ_i and directed accordingly (upward). More precisely, each tile τ is of the form $X + \{\lambda\xi_i + \lambda'\xi_j : 0 \leq \lambda, \lambda' \leq 1\}$ for some $i < j$ and some subset $X \subset [n] - \{i, j\}$ (regarded as a point in Z). We call τ a tile of *type ij* and denote by $\tau(X; i, j)$. By a natural visualization of τ , its vertices X, Xi, Xj, Xij are called the *bottom, left, right, top* vertices of τ and denoted by $b(\tau), \ell(\tau), r(\tau), t(\tau)$, respectively.

The vertex set V_T of G_T is also called the *spectrum* of T . When the generators ξ_i satisfy (4.4), all tiles in T become rhombi, and T is said to be a *rhombus tiling*.

Pure tilings admit a sort of *mutations*, called strong raising and lowering flips, which transform one tiling T into another, and back. More precisely, suppose that T contains a hexagon H formed by three tiles $\alpha := \tau(X; i, j)$, $\beta := \tau(X; j, k)$, and $\gamma := \tau(Xj; i, k)$. The *strong raising flip* replaces α, β, γ by the other possible combination of three tiles

in H , namely, by $\alpha' := \tau(Xk; i, j)$, $\beta' := \tau(Xi; j, k)$, and $\gamma' := \tau(X; i, k)$. The *strong lowering flip* acts conversely: it replaces α', β', γ' by α, β, γ . See the picture.



In terms of sets, a strong flip is applicable when for some X and $i < j < k$, there are six sets $X, Xi, Xk, Xij, Xjk, Xijk$ and one more set $Y \in \{Xj, Xik\}$. The flip (“in the presence of six witnesses”, in terminology of [4]) replaces Y by the other member of $\{Xj, Xik\}$; the replacement $Xj \rightsquigarrow Xik$ means that the flip is raising, and $Xik \rightsquigarrow Xj$ that the flip is lowering. Leclerk and Zelevinsky described in [4] important properties of strongly separated set-systems. Among those are the following.

(i) If $\mathcal{F} \subseteq 2^{[n]}$ is a maximal s-collection and if \mathcal{F}' is obtained by a strong flip from \mathcal{F} , then \mathcal{F}' is a maximal s-collection as well.

(ii) Let \mathbf{S}_n be the set of maximal s-collections in $2^{[n]}$, and for $\mathcal{F}, \mathcal{F}' \in \mathbf{S}_n$, let us write $\mathcal{F} \prec_s \mathcal{F}'$ if \mathcal{F}' is obtained from \mathcal{F} by a series of strong raising flips. Then the poset (\mathbf{S}_n, \prec_s) has a unique minimal element and a unique maximal element, which are the collection \mathcal{I}_n of all intervals, and the collection $\text{co-}\mathcal{I}_n$ of all co-intervals in $[n]$, respectively.

Moreover, they established a correspondence between maximal s-collections and so-called commutation classes of pseudo-line arrangements. In terms of pure tilings (which are dual to pseudo-line arrangements, in a sense), this is viewed as follows.

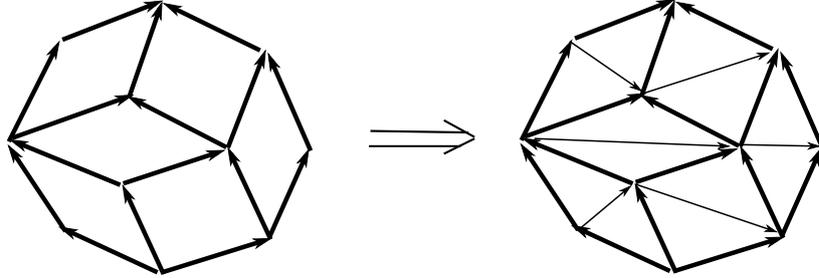
Theorem 5.1 [4] For each pure (rhombus) tiling T , its spectrum V_T is a maximal s-collection (regarding a vertex as a subset of $[n]$). This correspondence gives a bijection between the set of pure (rhombus) tilings on Z_n and \mathbf{S}_n .

5.2 Combined tilings

We assume that the generators ξ_i satisfy (4.3) and (4.4). In addition, we will deal with vectors $\epsilon_{ij} := \xi_j - \xi_i$, where $1 \leq i < j \leq n$. A *combined tiling*, abbreviated as a *c-tiling*, or simply a *combi*, is a sort of planar tilings generalizing rhombus tilings in essence (and defined later) in which each edge is congruent to either ξ_i for some i or ϵ_{ij} for some $i < j$; we say that e has type i in the former case, and type ij in the latter case.

Remark 3. In fact, we are free to choose arbitrary basic vectors ξ_i (subject to (4.3) and (4.4)) without affecting our model (in the sense that corresponding structures and set-systems remain equivalent when the generators vary). Sometimes, to simplify visualization, it is convenient to think of vectors $\xi_i = (x_i, y_i)$ as “almost vertical” ones (i.e., assume that $|x_i| = o(1)$). Then the vectors ϵ_{ij} become “almost horizontal”. For this reason, we will refer to edges congruent to ξ_\bullet as *V-edges*, and those congruent to ϵ_\bullet as *H-edges*. Also we say that V-edges go *upward*, and H-edges go *to the right*.

The simplest case of combies arises from arbitrary rhombus tilings T by subdividing each rhombus τ of T into two isosceles triangles Δ and ∇ , where the former (the “upper” triangle) uses the vertices $\ell(\tau), t(\tau), r(\tau)$, and the latter (the “lower” triangle) the vertices $\ell(\tau), b(\tau), r(\tau)$. Then the resulting combi has as V-edges all edges of T and has as H-edges the diagonals $(\ell(\tau), r(\tau))$ of rhombi τ of T . We refer to such a combi as a *semi-rhombus tiling*.



The above picture illustrates the transformation of a rhombus tiling into a combi; here V-edges (H-edges) are drawn by thick (resp. thin) lines.

In a general case, a combi is a planar tiling K formed by tiles of three sorts: Δ -tiles, ∇ -tiles, and lenses. As before, the vertices of K (or G_K) represent subsets of $[n]$.

A Δ -tile is an isosceles triangle $\Delta = \Delta(A|BC)$ with vertices A, B, C and edges $(B, A), (C, A), (B, C)$, where (B, C) is an H-edge, while (B, A) and (C, A) are V-edges (so (B, C) is the *base* side and A is the *top* vertex of Δ).

A ∇ -tile is symmetric. It is an isosceles triangle $\nabla = \nabla(A'|B'C')$ with vertices A', B', C' and edges $(A', B'), (A', C'), (B', C')$, where (B', C') is an H-edge, while (A', B') and (A', C') are V-edges (so (B', C') is the *base* side and A' is the *bottom* vertex of ∇). The picture illustrates Δ - and ∇ -tiles.



We say that a Δ - or ∇ -tile τ has type ij if its base edge has this type (and therefore the V-edges of τ have types i and j).

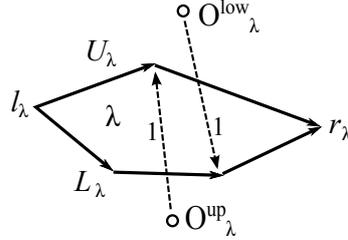
In a *lens* λ , the boundary is formed by two directed paths U_λ and L_λ with at least two edges each. The paths U_λ and L_λ have the same beginning vertex ℓ_λ and the same end vertex r_λ (called the *left* and *right* vertices of λ , respectively), contain merely H-edges, and form the *upper* and *lower* boundaries of λ , respectively. More precisely,

- (5.1) the path $U_\lambda = (\ell_\lambda = X_0, e_1, X_1, \dots, e_q, X_q = r_\lambda)$ (where $q \geq 2$) is associated with a sequence $i_0 < i_1 < \dots < i_q$ in $[n]$ so that for $p = 1, \dots, q$, the edge e_p (going from the vertex X_{p-1} to the vertex X_i) is of type $i_{p-1}i_p$.

This implies that the vertices of U_λ are expressed as $X_p = X_{i_p}$ for one and the same set $X \subset [n]$ (equal to $X_p \cap X_{p'}$ for any $p \neq p'$). In other words, the point X_p of Z is obtained from X by adding the vector ξ_{i_p} , and in view of (4.4), all vertices of U_λ lie on the upper half of the circumference of radius 1 centered at X . We call X the *center* of U_λ and denote by O_λ^{up} . In its turn,

(5.2) the path $L_\lambda = (\ell_\lambda = X'_0, e'_1, X'_1, \dots, e'_{q'}, X'_{q'} = r_\lambda)$ (with $q' \geq 2$) is associated with a sequence $j_0 > j_1 > \dots > j_{q'}$ in $[n]$ so that for $p = 1, \dots, q'$, the edge e'_p is of type $j_p j_{p-1}$.

Then the vertices of L_λ are expressed as $X'_p = Y - j_p$ for the same set $Y \subseteq [n]$ (equal to $X'_p \cup X'_{p'}$ for any $p \neq p'$). So the vertices of L_λ lie on the lower half of the circumference of radius 1 centered at Y , called the *center* of L_λ and denoted by O_λ^{low} . See the picture (where $q = 2$ and $q' = 3$).

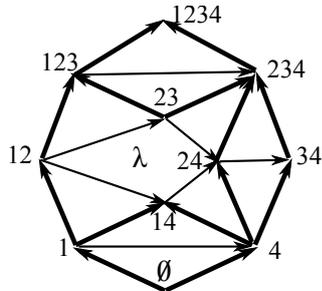


Note that $\ell_\lambda = Xi_0 = Y - j_0$ and $r_\lambda = Xi_q = Y - j_{q'}$ imply $i_0 = j_{q'}$ and $i_q = j_0$. We say that the lens λ has type $i_0 i_q$. The intersection of the circumferences of radius 1 centered at O_λ^{up} and O_λ^{low} is called the *rounding* of λ and denoted by Ω_λ . Observe that all vertices of λ lie on the boundary of Ω_λ . It is not difficult to realize that none of the vertices of K lies in the interior (i.e., strictly inside) of Ω_λ .

Clearly all vertices of a lens λ have the same size (regarding a vertex as a subset of $[n]$); we call it the *level* of λ . For an H-edge e , consider the pair of tiles in K containing e . Then either both of them are lenses, or one is a lens and the other is a triangle, or both are triangles (Δ - and ∇ -tiles), in which case we regard e as a *degenerate lens*.

The union of lenses of level h (including degenerate lenses) forms a closed simply connected region meeting $lbd(Z)$ at the vertex $z_h^\ell = [h]$, and $rbd(Z)$ at the vertex $z_h^r = [(n - h + 1)..n]$; we call it h -th *girdle* and denote by Λ_h . A vertex v in Λ_h having both entering and leaving V-edges is called *critical*; it splits the girdle into two (left and right) closed sets, and the part of Λ_h between two consecutive critical vertices is either a single H-edge or a disc (being the union of some non-degenerate lenses). The region between the upper boundary of Λ_h and the lower boundary of Λ_{h+1} is filled up by triangles; we say that these triangles have level $h + \frac{1}{2}$.

The picture below illustrates a combi for $n = 4$ having one lens λ ; here the bold (thin) arrows indicate V-edges (resp.H-edges); note that the girdle Λ_2 is the union of the lens λ and the edge $(24, 34)$.



$$V_K = \{ \emptyset, 1, 4, 12, 14, 23, 24, 34, \\ 123, 234, 1234 \}$$

5.3 Flips in c-tilings

Now our aim is to show that the spectrum (viz. vertex set) V_K of a combi K is a maximal w-collection and that any maximal w-collection is obtained in this way. To show this, we elaborate a technique of flips on combies (which, due to the planarity of a combi, looks simpler than a similar technique on the language of g-tilings in [1]).

We will rely on the following two statements.

Proposition 5.2 *Suppose that a combi K contains two vertices of the form X and Xi (where $i \in [n] - X$). Then K has edge from X to Xi .*

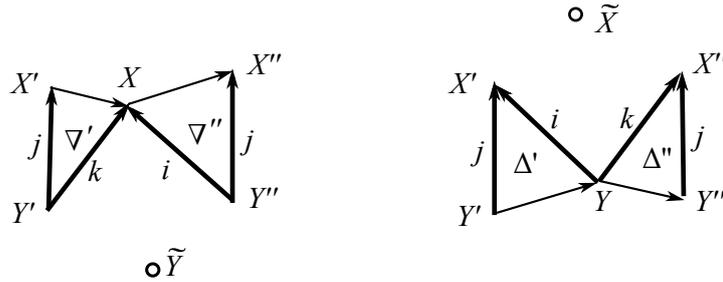
Proposition 5.3 *Suppose that for some $i < j'' \leq j' < k$, a combi K contains H-edges $e = (A, B)$ and $e' = (B, C)$ of types $j'k$ and ij'' , respectively. Then $j' = j''$ and the edges e, e' belong to either the lower boundary of one lens of K , or two Δ -tiles with the same top vertex, namely, $\Delta(D|AB)$ and $\Delta(D|BC)$, where $D = Ak = Bj' = Ci$. Symmetrically, if e, e' as above have types ij'' and $j'k$, respectively, then $j' = j''$ and e, e' belong to either the upper boundary of one lens of K , or two ∇ -tiles with the same bottom vertex, namely, $\nabla(D'|AB)$ and $\nabla(D'|BC)$, where $D' = A - i = B - j' = C - k$.*

These propositions will be proved in Section 6. Assuming their validity, we show the following.

Theorem 5.4 *For a combi K , the spectrum V_K is a maximal w-collection.*

Proof First of all we introduce W- and M-configurations in K (their counterparts for g-tilings are described in [1, Sec. 4]).

A *W-configuration* is formed by two ∇ -tiles $\nabla' = \nabla(Y'|X'X)$ and $\nabla'' = \nabla(Y''|XX'')$ (resembling letter W), as indicated in the left fragment of the picture.



Here for some $i < j < k$, the left edge (Y', X') of ∇' and the right edge (Y'', X'') of ∇'' have type j , the right edge (Y', X) of ∇' has type k , and the left edge (Y'', X) of ∇'' has type i . In other words, letting $\tilde{Y} := Y' \cap Y''$ (which need not be a vertex of K), we have

$$Y' = \tilde{Y}i, \quad Y'' = \tilde{Y}k, \quad X' = \tilde{Y}ij, \quad X = \tilde{Y}ik, \quad X'' = \tilde{Y}jk, \quad (5.3)$$

and denote such a W-configuration as $W(\tilde{Y}; i, j, k)$.

In its turn, an M-configuration is formed by two Δ -tiles $\Delta' = \Delta(X'|Y'Y)$ and $\Delta'' = \Delta(X''|YY'')$ in K (resembling letter M), as indicated in the right fragment of the above picture. Here for $i < j < k$, the V-edges $(Y', X'), (Y, X'), (Y, X''), (Y'', X'')$ have

types j, i, k, j , respectively. For $\tilde{Y} := Y' \cap Y''$ (as before), the vertices Y', X', Y'', X'' are expressed as in (5.3), and

$$Y = \tilde{Y}j. \quad (5.4)$$

We denote such an M-configuration as $M(\tilde{Y}; i, j, k)$.

When K has a W-configuration (M-configuration), we can make a *weak lowering* (resp. *raising*) *flip* to transform K into another combi K' . (This is somewhat similar to “strong” flips in pure tilings but now the flip is performed “in the presence of four (rather than six) witnesses”, in terminology of Leclerk and Zelevinsky [4], namely, the vertices X', X'', Y', Y'' as above.) Roughly speaking, the weak lowering flip applied to $W = W(\tilde{Y}; i, j, k)$ replaces the “middle” vertex $X = \tilde{Y}ik$ by the new vertex Y as in (5.4), updating the tile structure in a neighborhood of X . In particular, W is replaced by the M-configuration $M(\tilde{Y}; i, j, k)$. Weak raising flips act conversely.

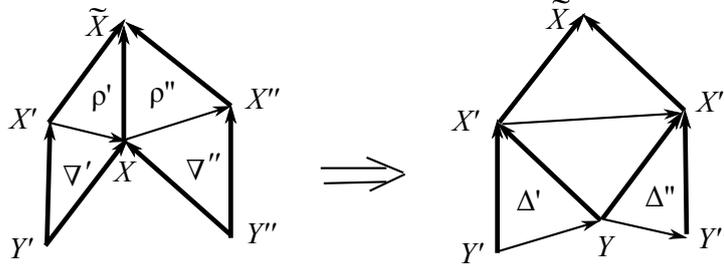
In what follows, speaking of a flip, we default mean a weak flip.

Next we describe the lowering flip for W in detail (using notation as above). Define $\tilde{X} := X' \cup X'' (= \tilde{Y}ijk)$. Two cases are possible.

Case 1: \tilde{X} is a vertex of K . Note that $\tilde{X} = X'k = Xj = X''i$. Therefore, by Proposition 5.2, K has the edges $e' = (X', \tilde{X})$, $e = (X, \tilde{X})$, and $e'' = (X'', \tilde{X})$. This implies that K contains the Δ -tiles $\rho' := \Delta(\tilde{X}|X'X)$ and $\rho'' := \Delta(\tilde{X}|XX'')$. We replace $\nabla', \nabla'', \rho', \rho''$ by the triangles

$$\Delta(X'|Y'Y), \quad \Delta(X''|YY''), \quad \Delta(\tilde{X}|X'X''), \quad \nabla(Y|X'X''),$$

as illustrated in the picture.

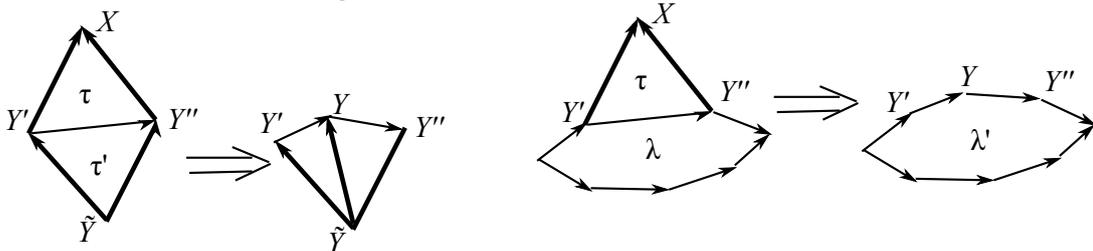


The tile structure of K in a neighborhood of X below the edges (Y', X) and (Y'', X) can be of three possibilities.

Subcase 1a: The V-edges (Y', X) and (Y'', X) belong to one and the same Δ -tile $\tau = \Delta(X|Y'Y'')$, and the base (Y', Y'') is shared by τ and a ∇ -tile τ' . Since $Y' = \tilde{Y}i$ and $Y'' = \tilde{Y}k$ (cf. (5.3)), τ' is of the form $\nabla(\tilde{Y}|Y'Y'')$. We replace τ, τ' by the ∇ -tiles

$$\nabla(\tilde{Y}|Y'Y) \quad \text{and} \quad \nabla(\tilde{Y}|YY''),$$

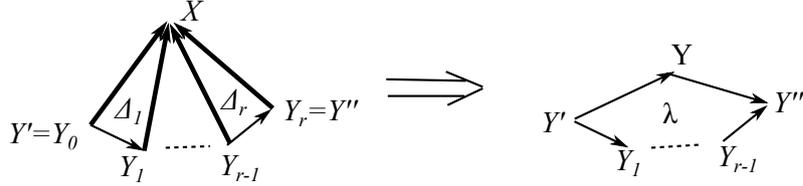
as illustrated in the left fragment of the picture.



(Therefore, taking together, the six triangles in the hexagon with the vertices $\tilde{Y}, Y', X', \tilde{X}, X'', Y''$ are replaced by another combination of six triangles so as to change the inner vertex X to Y . This matches lowering flips in pure tilings.)

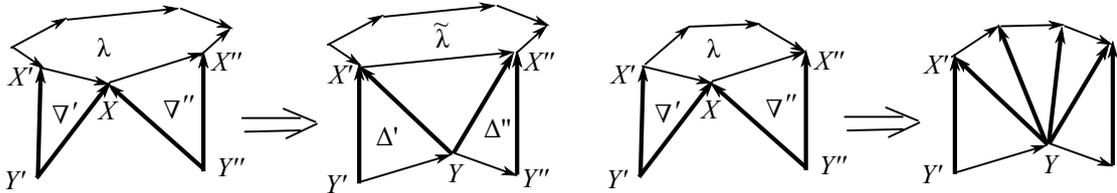
Subcase 1b: (Y', X) and (Y'', X) belong to the same Δ -tile $\tau = \Delta(X|Y'Y'')$, but the edge $e = (Y', Y'')$ is shared by τ and a lens λ . Then e belongs to the upper boundary of λ and the center of U_λ is just \tilde{Y} (since $Y' \cap Y'' = \tilde{Y}$). We replace the edge e by the path with two edges (Y', Y) and (Y, Y'') , thus transforming λ into a larger lens λ' (which is correct since $L_{\lambda'} = L_\lambda$, $|U_{\lambda'}| = |U_\lambda| + 1$, and $Y = \tilde{Y}j$). The transformation is illustrated in the right fragment of the above picture.

Subcase 1c: The edges (Y', X) and (Y'', X) belong to different Δ -tiles. Then the “angle” between these edges is filled by a sequence of two or more consecutive Δ -tiles $\Delta_1 = \Delta(X|Y_0Y_1)$, $\Delta_2 = \Delta(X|Y_1Y_2), \dots, \Delta_r = \Delta(X|Y_{r-1}Y_r)$, where $r \geq 2$, $Y_0 = Y'$, and $Y_r = Y''$. We replace these triangles by one lens λ with U_λ formed by the path $Y'Y''$ and with L_λ formed by the path $Y_0Y_1 \dots Y_r$ (using path notation via vertices). This transformation is illustrated in the picture.



Case 2: \tilde{X} is not a vertex of K . Then the H-edges $e' = (X', X)$ and $e'' = (X, X'')$ belong to the lower boundary of some lenses λ' and λ'' , respectively. Since e' has type jk and e'' has type ij with $i < j < k$, it follows from Proposition 5.3 that $\lambda' = \lambda'' =: \lambda$. Two cases are possible.

Subcase 2a: $|L_\lambda| \geq 3$ (i.e. L_λ contains e', e'' and at least one more edge). We replace e', e'' by one H-edge $\tilde{e} = (X', X'')$, which has type ik . This reduces λ to lens $\tilde{\lambda}$ with $U_{\tilde{\lambda}} = U_\lambda$ and $L_{\tilde{\lambda}} = (L_\lambda - \{e', e''\}) \cup \{\tilde{e}\}$. (It is indeed a lens since $|L_{\tilde{\lambda}}| = |L_\lambda| - 1 \geq 2$ and $X' \cup X'' = \tilde{X}$, thus keeping the lower center: $O_{\tilde{\lambda}}^{\text{low}} = O_\lambda^{\text{low}}$.) Also, acting as in Case 1, we remove the ∇ -tiles ∇', ∇'' and add the ∇ -tile $\nabla(Y|X'X'')$ and the two Δ -tiles $\Delta(X'|Y'Y)$ and $\Delta(X''|YY'')$. The transformation is illustrated in the left fragment of the picture (where $|L_\lambda| = 4$ and $|U_\lambda| = 3$).



Subcase 2b: $|L_\lambda| = 2$. Then $\ell_\lambda = X', r_\lambda = X''$, and L_λ is formed by the edges e', e'' . Let $X_0X_1 \dots X_q$ be the upper boundary U_λ , where $X_0 = X'$ and $X_q = X''$. We replace λ by the sequence of ∇ -tiles $\nabla(Y|X_0X_1), \dots, \nabla(Y|X_{q-1}X_q)$ (taking into account that $Y = O_\lambda^{\text{up}}$). See the right fragment of the above picture.

In addition, in both Subcases 2a,2b we should perform an appropriate update of the tile structure in a neighborhood of X below the edges (Y', X) and (Y'', X) . This is done in the same way as described in Subcases 1a–1c.

A straightforward verification shows that in all cases we obtain a correct combi K' . As a result of the flip, the vertex $X = \tilde{Y}ik$ is replaced by $Y = \tilde{Y}j$, and the sum of the sizes of vertices decreases by 1. The arising Δ -tiles $\Delta(X'|Y'Y)$ and $\Delta(X''|YY'')$ give an M-configuration M in K' , and making the corresponding raising flip with M , we return the initial K . (The description of raising flip is “symmetric” to that of lowering flip given above. In fact, a raising flip is equivalent to the corresponding lowering flip in the combi whose vertices are the complements $[n] - X$ of the vertices X of K .)

In what follows we use some results from [1, 2, 4].

(A) Let $\mathcal{F} \subset 2^{[n]}$ be a maximal w-collection. Suppose that for some triple $i < j < k$ in $[n]$ and some set $A \subseteq [n] - \{i, j, k\}$, \mathcal{F} contains the four sets Ai, Ak, Aij, Ajk and one set $B \in \{Aj, Aik\}$ (both Aj, Aik cannot be simultaneously in \mathcal{F} as they are not weakly separated from each other). Then replacing in \mathcal{F} the set B by the other set from $\{Aj, Aik\}$ (a *weak flip* on set-systems) makes a maximal w-collection as well [4].

(B) Let $\mathcal{P} = (\mathbf{W}_n, \prec_w)$ be the poset where \mathbf{W}_n is the set of maximal w-collections for $[n]$ and we write $\mathcal{F} \prec_w \mathcal{F}'$ if \mathcal{F}' can be obtained from \mathcal{F} by a series of (weak) raising flips. Then \mathcal{P} has a unique minimal element and a unique maximal element, which are the set \mathcal{I}_n of intervals and the set $\text{co-}\mathcal{I}_n$ of co-intervals in $[n]$, respectively [1, 2].

Now we finish the proof of the theorem as follows. We associate to a combi K the parameter $\eta(K) := \sum(|X| : X \in V_K)$. Then a lowering (raising) flip applied to K decreases (resp. increases) η by 1. We assert that if K admits no lowering flip, then $V_K = \mathcal{I}_n$.

Indeed, suppose that K has a (non-degenerate) lens λ . Let λ be chosen so that L_λ be entirely contained in the lower boundary of the girdle Λ_h , where h is the level of λ (see Section 5.2); one easily shows that such a λ does exist. Take two consecutive edges e, e' in L_λ . Then e has type jk and e' has type ij for some $i < j < k$. Furthermore, e, e' are the bases of some ∇ -tiles ∇, ∇' , respectively (lying in level $h - \frac{1}{2}$). But then ∇, ∇' form a W-configuration. So K admits a lowering flip; a contradiction.

Hence, K is a semi-rhombus tiling. Let T be its underlying rhombus tiling. We know (see Section 5.1) that T admits no strong lowering flip (concerning a hexagon) if and only if $V_T = \mathcal{I}_n$. Since $V_T = V_K$ and a (strong) lowering flip in T is translated as a (weak) lowering flip in K , we obtain the desired assertion.

It follows that starting with an arbitrary combi K and making a finite number of lowering flips, we can always reach the combi K_0 with $V_{K_0} = \mathcal{I}_n$ (taking into account that each application of lowering flips decreases η). Then K is obtained from K_0 by a series of raising flips. Each of such flips changes the spectrum of the current combi in the same way as described for flips concerning w-collections in (A). Thus, V_K is a maximal w-collection.

This completes the proof of the theorem. ■

A converse property is valid as well.

Theorem 5.5 *For any maximal w-collection \mathcal{F} in $2^{[n]}$, there exists a combi K such that $V_K = \mathcal{F}$. Moreover, such a K is unique.*

Proof To see the existence of K with $V_K = \mathcal{F}$, it suffices to show that flips in combies and in maximal w-collections are consistent (taking into account (A),(B) in the proof of Theorem 5.4). This is provided by the following:

(5.5) for a combi K , if V_K contains the sets Ai, Ak, Aij, Ajk, Aik for some $i < j < k$ and $A \subseteq [n] - \{i, j, k\}$, then K contains the ∇ -tiles $\nabla = \nabla(Ai|Aij, Aik)$ and $\nabla' = \nabla(Ak|Aik, Ajk)$.

(So ∇, ∇' form a W-configuration and we can make a lowering flip in K to obtain a combi K' with $V_{K'} = (V_K - \{Aik\}) \cup \{Aj\}$. This matches the flip $Aik \rightsquigarrow Aj$ in the corresponding w-collection. The assertion on raising flips is symmetric.)

Indeed, Proposition 5.2 ensures the existence of V-edges $e_1 = (Ai, Aij)$, $e_2 = (Ai, Aik)$, $e_3 = (Ak, Aik)$, $e_4 = (Ak, Ajk)$ in K . Obviously, the angle between the edges e_1 and e_2 is covered by one or more ∇ -tiles; let $\tilde{\nabla} = \nabla(Ai|Aij', Aik)$ be the rightmost tile among them (namely, the one containing the edge e_2). Similarly, let $\tilde{\nabla}' = \nabla(Ak|Aik, Aj''k)$ be the leftmost tile among the ∇ -tiles lying between the edges e_3 and e_4 . Clearly $j \leq j' < k$ and $i < j'' \leq j$. By Proposition 5.3, $j' = j = j''$. Therefore, $\tilde{\nabla} = \nabla$ and $\tilde{\nabla}' = \nabla'$, as required in (5.5).

Finally, K is determined by its spectrum V_K (yielding the required uniqueness). Indeed, using Proposition 5.2, we can uniquely restore the V-edges of K . This determines the set of Δ - and ∇ -tiles of K . Now for $h = 1, \dots, n-1$, consider the region (girdle) Λ_h in Z bounded from below by the directed path L_h formed by the base edges of ∇ -tiles of level $h - \frac{1}{2}$, and bounded from above by the directed path U_h formed by the base edges of Δ -tiles of level $h + \frac{1}{2}$. Then Λ_h is the union of lenses of level h , and we have to show that these lenses are restored uniquely.

It suffices to show this for the disc D in Λ_h between two consecutive critical vertices u, v (see Section 5.2). The lower boundary L_D of D is the part of L_h beginning at u and ending at v ; let $L_D = (u = X_0, e_1, X_1, \dots, e_r, X_r = v)$. Each edge $e_p = (X_{p-1}, X_p)$ belongs to the lower boundary of some lens λ with center $O_\lambda^{\text{low}} = X_{p-1} \cup X_p$ (and we have $X_{p-1} = O_\lambda^{\text{low}} - k$ and $X_p = O_\lambda^{\text{low}} - j$ for some $j < k$). From Proposition 5.3 it follows that consecutive edges e_p, e_{p-1} belong to the same lens if and only if $X_{p-1} \cup X_p = X_p \cup X_{p+1}$. Also there exists a lens λ such that L_λ is entirely contained in L_D . Such a λ is characterized by the following conditions:

- (i) L_λ is a maximal part $X_p X_{p+1} \dots X_q$ of L_D satisfying $q \geq p+2$ and $X_p \cup X_{p+1} = \dots = X_{q-1} \cup X_q$, and
- (ii) there exists a vertex $Y \neq X_p, X_q$ in D such that $X_p \cap Y = Y \cap X_q = X_p \cap X_q$.

In this case $\ell_\lambda = X_p$, $r_\lambda = X_q$, $L_\lambda = X_p X_{p+1} \dots X_q$, and U_λ is formed by X_p, X_q and all those vertices Y that satisfy (ii). (Conditions (i),(ii) are necessary for a lens λ with $L_\lambda \subseteq L_D$. To see the sufficiency, suppose that $L_\lambda \cap L_D = X_p X_{p+1} \dots X_q$ but $L_\lambda \not\subseteq L_D$, i.e., either $\ell_\lambda \neq X_p$ or $r_\lambda \neq X_q$ or both. Consider the rounding Ω_λ of λ (see Section 5.2). One can realize that a point Y as in (ii) is located in the *interior* of Ω_λ , which is impossible.)

Once a lens λ satisfying (i),(ii) is chosen, we “remove” it from D and repeat the procedure with the lower boundary of the updated (reduced) D , and so on. Upon termination of the process for all h , we obtain the list of lenses of K , and this list is constructed uniquely. ■

6 Proofs

In this section we prove the assertions stated but left unproved in Sections 4, 5, namely, Propositions 4.1, 5.2, 5.3 and Theorem 4.2.

We will use a technique of contractions and expansions (transforming combies on Z_n into ones on Z_{n-1} , and back) which is analogous to that developed for g-tilings in [1, Sec. 8] (see also [2, Sec. 6]), but now this is arranged simpler because of the planarity of objects we deal with.

6.1 n -contraction

The n -contraction operation is, in fact, well-known and rather transparent when we deal with a rhombus tiling T on the zonogon $Z = Z_n$. In this case we take the sequence Q of rhombi of type $*n$ (where $*$ means any type i different from n) in which the first rhombus contains the first edge (having type n) in $rbd(Z)$, the last rhombus does the last edge in $lbd(Z)$, and each pair of consecutive rhombi share an edge of type n . The operation consists in shrinking each rhombus $\tau = \tau(X; i, n)$ in Q into the only edge (X, Xi) (cf. Section 5.1). Under this operation, the right boundary of Q is shifted by $-\xi_n$, getting merged with the left one, each vertex Y of T containing the element n (i.e., Y lies on the right from Q) turns into $Y - n$, and the resulting set T' of rhombi forms a correct rhombus tiling on the zonogon Z_{n-1} (generated by ξ_1, \dots, ξ_{n-1}). This T' is called the n -contraction of T .

In case of c-tilings, the n -contraction operation becomes less trivial since, besides triangles (“semi-rhombi”), it should involve lenses of type $*n$. Below we describe this in detail.

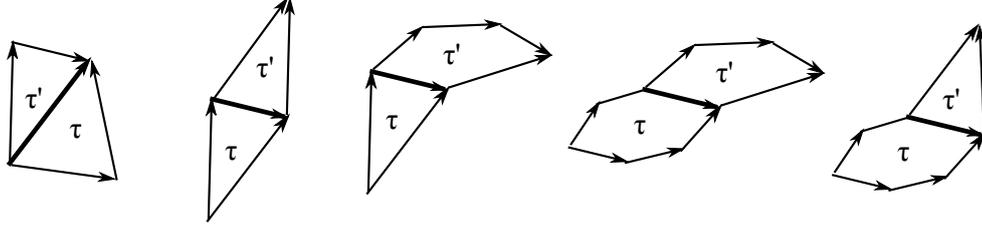
Consider a combi K on Z . Let \mathcal{T}^n be the set of tiles of types $*n$ in K ; it consists of the Δ -tiles $\Delta(A|BC)$ whose left edge (B, A) is of type n , the ∇ -tiles $\nabla(A'|B'C')$ whose right edge (A', C') is of type n , and the lenses λ of type $*n$. Note that the first edge of L_λ has type jn , and the last edge of U_λ has type $j'n$ for some j, j' . Let E^n be the set of edges of type n or $*n$ in K .

By an n -strip we mean a maximal sequence $Q = (\tau_0, \tau_1, \dots, \tau_N)$ of tiles in \mathcal{T}^n such that for any two consecutive $\tau = \tau_{p-1}$ and $\tau' = \tau_p$, their intersection $\tau \cap \tau'$ consists of an edge $e \in E^n$, and:

- (i) if e has type n , then τ is a Δ -tile and τ' is a ∇ -tile;
- (ii) if e has type $*n$, then τ lies below e and τ' lies above e .

In case (ii), there are four possibilities: (a) τ is a ∇ -tile, τ' is a Δ -tile, and e is their common base; (b) τ is a ∇ -tile, τ' is a lens, and e is the base of τ and the first edge of $L_{\tau'}$; (c) τ, τ' are lenses, and e is the last edge of U_τ and the first edge of $L_{\tau'}$; and (d) τ

is a lens, τ' is a Δ -tile, and e is the last edge of U_τ and the base of τ'). The possible cases of τ, τ' are illustrated in the picture, where e is drawn bold.



The following property is useful:

- (6.1) there is only one n -strip Q as above; each tile of \mathcal{T}^n occurs in Q exactly once; Q begins with the ∇ -tile in K containing the first edge $e_1^r = (z_0^r, z_1^r)$ of $rbd(Z)$, and ends with the Δ -tile in K containing the last edge $e_n^\ell = (z_{n-1}^\ell, z_n^\ell)$ of $lbd(Z)$.

This follows from three facts: (i) each tile in \mathcal{T}^n contains exactly two edges from E^n ; (ii) each edge in E^n belongs to exactly two tiles from \mathcal{T}^n , except for the edges e_1^r and e_n^ℓ , which belong to one tile each; and (iii) Q cannot be cyclic. To see validity of (iii), observe that if τ_{p-1}, τ_p are triangles, then either the level of τ_p is greater than that of τ_{p-1} (when τ_{p-1} is a ∇ -tile), or the level of the *base* of τ_p is greater than that of τ_{p-1} (when τ_{p-1} is a Δ -tile). As to traversing across lenses, we rely on the fact that the relation: a lens λ is “higher” than a lens λ' if L_λ and $U_{\lambda'}$ share an edge, induces a poset on the set of all lenses in K (which in turn appeals to evident topological reasonings using the facts that the lenses are convex and all H-edges are “directed to the right”).

Note that for each tile $\tau \in \mathcal{T}^n$ and its edges $e, e' \in E^n$, $bd(\tau) - \{e, e'\}$ consists of two directed paths, *left* and *right* ones (of which one is a single vertex when τ is a triangle). The concatenation of the left (resp. right) paths along Q gives a directed path from $z_0^r = (0, 0)$ to z_{n-1}^ℓ (resp. from z_1^r to $z_n^\ell = z_n^r$); we call this path the *left* (resp. *right*) *boundary* of the strip Q and denote as P^{left} (resp. P^{right}).

Let Z^{left} (Z^{right}) be the region in Z bounded by P^{left} and the part of $lbd(Z)$ from $(0, 0)$ to z_{n-1}^ℓ (resp. bounded by P^{right} and the part of $rbd(Z)$ from z_1^r to z_n^r). Accordingly, the graph $(V_K, E_K - E^n)$ consists of two connected components, the *left subgraph* G^{left} and the *right subgraph* G^{right} (lying in Z^{left} and Z^{right} , respectively), and we denote by K^{left} and K^{right} the corresponding subtilings in $K - Q$. It is easy to see that $n \notin X$ (resp. $n \in X$) holds for each vertex X of G^{left} (resp. G^{right}).

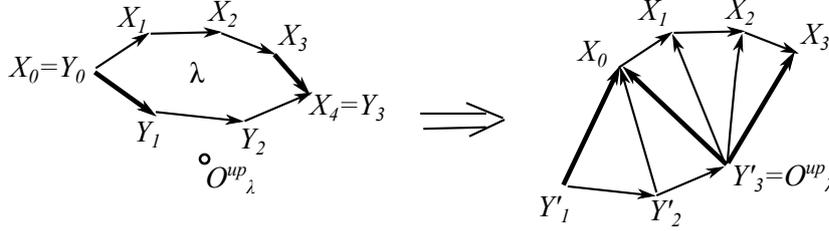
The n -contraction operation applied to K makes the following.

(Q1) The region Z^{left} preserves, while Z^{right} is shifted by the vector $-\xi_n$. As a result, the right boundary of the shifted Z^{right} becomes the right boundary of the zonogon Z_{n-1} (while the left boundary of Z^{left} coincides with $lbd(Z_{n-1})$). Accordingly, the subtiling K^{left} preserves, and K^{right} is shifted so that each vertex X becomes $X - n$.

(Q2) The edges in E^n and the triangles in \mathcal{T}^n vanish.

(Q3) The lenses λ in \mathcal{T}^n are transformed as follows. Let $U_\lambda = X_0 X_1 \dots X_p$ and $L_\lambda = Y_0 Y_1 \dots Y_q$ (using path notation via vertices); so $X_0 = Y_0 = \ell_\lambda$, $X_p = Y_q = r_\lambda$, and $(Y_0, Y_1), (X_{p-1}, X_p) \in E^n$. Under the above shift, each vertex Y_r becomes $Y'_r := Y_r - n$. Since $r_\lambda = O_\lambda^{\text{up}} + \xi_n$ and $Y'_q = Y_q - n$, we obtain $Y'_q = O_\lambda^{\text{up}}$; hence

Y'_q becomes the center for X_0, \dots, X_{p-1} . In its turn, the equality $\ell_\lambda = O_\lambda^{\text{low}} - \xi_n$ implies that X_0 becomes the center for Y'_1, \dots, Y'_q . We fill the space between the paths $X_0 \dots X_{p-1}$ and $Y'_1 \dots Y'_q$ with new Δ -tiles $\Delta(X_0|Y'_1Y'_2), \dots, \Delta(X_0|Y'_{q-1}Y'_q)$ and new ∇ -tiles $\nabla(Y'_q|X_0X_1), \dots, \nabla(Y'_q|X_{p-2}X_{p-1})$. See the picture for $p = 4$ and $q = 3$.



We associate to the lens λ as above in K the zigzag path $Y'_1X_0Y'_qX_{p-1}$, denoted as P_λ , in the resulting object (it is drawn in bold in the right fragment of the above picture). One may say that the transformation described in (Q3) “replaces the lens λ by the zigzag path P_λ with two inscribed fillings” (consisting of Δ - and ∇ -tiles), and we refer to this as an *L-Z transformation*.

Let K' be the set of tiles occurring in K^{left} and in the shifted K^{right} plus those Δ - and ∇ -tiles that arise instead of the lenses in \mathcal{T}^n as described in (Q3). A routine verification shows the following.

Proposition 6.1 *The resulting K' is a correct combi on the zonogon Z_{n-1} . ■*

Acting in a similar fashion w.r.t. the set \mathcal{T}^1 of tiles of types 1^* in K , one can construct the corresponding *1-contraction* of K , which is a combi on the $(n-1)$ -zonogon generated by the vectors ξ_2, \dots, ξ_n . This corresponds to the n -contraction of the combi \widehat{K} that is the mirror-reflection of K w.r.t. the vertical axis. More precisely, \widehat{K} is obtained from K by changing each generator $\xi_i = (x_i, y_i)$ to $\widehat{\xi}_{n-i+1} = (-x_i, y_i)$.

6.2 n -expansion

Now we describe a converse operation, called the *n -expansion one*. It is easy when we deal with a rhombus tiling T' on the zonogon $Z' = Z_{n-1}$. In this case we take a *directed* path P in $G_{T'}$ going from the bottom $(0, 0)$ to the top z_{n-1} of Z' . Such a P splits Z' and T' into the corresponding left and right parts. We shift the right region of Z' by the vector ξ_n (and shift the right subtiling of T' by adding the element n to its vertices), and then fill the space between P and its shifted copy with corresponding rhombi of type $*n$. This results in a corrected rhombus tiling T on the zonogon Z_n in which the new rhombi form its n -strip, and the n -contraction operation applied to T returns T' .

In case of c-tilings, the operation becomes somewhat more involved since now we should deal with paths P in Z' which are not necessarily directed.

More precisely, we consider a combi K' on Z' and a path $P = (A_0, e_1, A_1, \dots, e_M, A_M)$ in the graph $G_{K'}$ such that:

(P1) P goes from the bottom to the top of Z' , i.e., $A_0 = \emptyset$ and $A_M = [n-1]$, and all edges of P are V-edges;

(P2) for any two consecutive edges of P , at least one is a forward edge, i.e., $|A_{d-1}| > |A_d|$ implies $|A_d| < |A_{d+1}|$;

(P3) any zigzag subpath in P goes to the right; in other words, if $|A_{d-1}| = |A_{d+1}| \neq |A_d|$, then either $A_{d-1} = A_d i$ and $A_{d+1} = A_d j$ for some $i < j$, or $A_{d-1} = A_d - i'$ and $A_{d+1} = A_d - j'$ for some $i' > j'$.

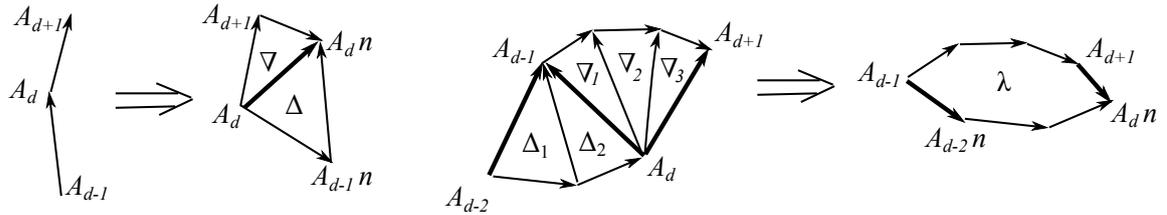
Borrowing terminology in [1, Sec. 8], we call such a P a *legal* path for K' . It splits Z' into two closed simply connected regions R_1, R_2 (the *left* and *right* ones, respectively), where $R_1 \cup R_2 = Z'$, $R_1 \cap R_2 = P$, R_1 contains $lbd(Z')$, and R_2 contains $rbd(Z')$. Let G_i and K_i denote, respectively, the subgraph of $G_{K'}$ and the subtiling of K' contained in R_i , $i = 1, 2$.

We call a vertex A_d of P a *slope* if $|A_{d-1}| < |A_d| < |A_{d+1}|$, a *peak* if $|A_{d-1}| = |A_{d+1}| < |A_d|$, and a *pit* if $|A_{d-1}| = |A_{d+1}| > |A_d|$. (By (P2), the case $|A_{d-1}| > |A_d| > |A_{d+1}|$ is impossible.) When A_d is a peak, the angle between the edges e_d and e_{d+1} is covered by a sequence of Δ -tiles $\Delta(A_d|Y_{r-1}Y_r)$, $r = 1, \dots, q$, ordered from left to right, where $Y_0 = A_{d-1}$ and $Y_q = A_{d+1}$. We call this sequence the (*lower*) *filling* at A_d w.r.t. P . Similarly, when A_d is a pit, the angle between e_d and e_{d+1} is covered by a sequence of ∇ -tiles $\nabla(A_d|X_{r-1}X_r)$, $r = 1, \dots, p$, called the (*upper*) *filling* at A_d w.r.t. P .

The n -*expansion operation* applied to K' and P constructs a combi K on Z_n as follows.

(E1) K inherits all tiles of K_1 except for those in the fillings of pits of P . For each tile τ of K_2 not contained in the filling of any peak of P , K receives the shifted tile $\tau + \xi_n$. Accordingly, the vertex set of K consists of the vertices of G_1 except for the pits of P , and the vertices of the form Xn for all vertices X of G_2 except for the peaks of P . In particular, each slope X of P (and only these vertices of K') creates two vertices in K , namely, X and Xn .

(E2) Each slope A_d creates two additional tiles in K : the ∇ -tile $\nabla(A_d|A_{d+1}, A_d n)$ of type in and the Δ -tile $\Delta(A_d n|A_d, A_{d-1} n)$ of type jn , where i, j are the types of the edges (A_d, A_{d+1}) and (A_{d-1}, A_d) , respectively. See the left fragment of the picture.



In addition, the first and last vertices of P create two extra tilings: $\nabla(A_0 = \emptyset|A_1, A_0 n = z_1^r)$ and $\Delta(A_M n = [n]|A_M, A_{M-1} n)$.

(E3) Each backward edge $e_d = (A_d, A_{d-1})$ of P (equivalently, each peak-pit pair A_{d-1}, A_d) creates a new lens λ in K by the following rule. Let $\Delta(A_{d-1}|Y_{r-1}Y_r)$, $r = 1, \dots, q$, be the filling at A_{d-1} w.r.t. P , and let $\nabla(A_d|X_{r-1}X_r)$, $r = 1, \dots, p$, be the filling at A_d (so $Y_0 = A_{d-2}$, $Y_q = A_d$, $X_0 = A_{d-1}$, and $X_p = A_{d+1}$). Then λ is such that: $\ell_\lambda = A_{d-1}$, $r_\lambda = A_d n$, U_λ has the vertex sequence $X_0, X_1, \dots, X_p, A_d n$, and L_λ has the vertex sequence $A_{d-1}, Y_0 n, Y_1 n, \dots, Y_q n$. See the right fragment of the above picture (where $p = 3$ and $q = 2$). We refer to (E3) as a *Z-L transformation* (replacing

a three-edge zigzag of P by a lens); this is converse to an L-Z transformation described in (Q3) of Section 6.1.

One can check that K obtained in this way is indeed a correct combi on Z_n . It is called the n -expansion of K' w.r.t. the legal path P . The corresponding sequence Q of tiles of types $*n$ in K is just formed by the Δ - and ∇ -tiles induced by the slopes of P (together with A_0, A_M) as described in (E2), and by the lenses induced by the backward edges (or the three-edge zigzags) of P as described in (E3).

A straightforward examination shows that the n -contraction operation applied to K returns the initial K' (and the legal path P in it is obtained by a natural deformation of the left boundary of Q); details are left to the reader. As a result, we conclude with the following.

Theorem 6.2 *The correspondence $(K', P) \mapsto K$, where K' is a c -tiling on Z_{n-1} , P is a legal path for K' , and K is the n -expansion of K' w.r.t. P , gives a bijection between the set of such pairs (K', P) and the set of c -tillings on Z_n . Under this correspondence, K' is the n -contraction K/n of K .*

Note that when we consider in $G_{K'}$ a path P' defined similarly to P with the only difference (in (P3)) that any zigzag subpath in P' goes *to the left*, then, duly modifying the expansion operation described in (E1)–(E3), we obtain what is called the 1 -expansion of K' ; this gives an analogue of Theorem 6.2 concerning type 1. (Again, to clarify the construction we can make the mirror-reflection w.r.t. the vertical axis.)

6.3 Proofs

We first prove Propositions 5.2 and 5.3, thus completing the proof of Theorem 5.4.

Proof of Proposition 5.2. Let X and Xi be vertices of a combi K on the zonogon Z_n . We use induction on n and, assuming w.l.o.g. that $i \neq n$, consider the n -contraction K' of K (in case $i = n$, we could consider the 1-contraction $K/1$ and argue in a similar way). Also we use terminology, notation and constructions from the previous subsections. Two cases are possible.

(i) Let $n \notin X$. Then X and Xi belong to the left subgraph G^{left} of G_K (w.r.t. the n -strip Q). Hence X, Xi are vertices of K' , and by induction K' has edge $e = (X, Xi)$ (which is a V-edge). Let P be the legal path in K' such that the n -expansion of K' w.r.t. P gives K (i.e., P is the “image” of the strip Q in K'). Then e is a V-edge of the left subgraph G_1 of $G_{K'}$ (w.r.t. P). The only situation when e could be destroyed under the n -expansion operation is that e belongs to a ∇ -tile in the filling at some pit A_d of P , implying $X = A_d$. But this is impossible since A_d induces only one vertex $A_d n$ in K .

(ii) Let $n \in X$. Then X, Xi belong to G^{right} . So K' has vertices $X' = X - n$ and $X'' = Xi - n$, and by induction K' has V-edge $e = (X', X'')$. For the corresponding legal path P in K' , the edge e cannot generate the edge $(X'n, X''n) = (X, Xi)$ in K only if e belongs to a Δ -tile in the filling at some peak A_d of P , implying $X'' = A_d$. But A_d induces only one vertex A_d in K (instead of the required vertex $A_d n = Xi$); a contradiction.

Thus, in all cases (X, Xi) is an edge of K . ■

Proof of Proposition 5.3. Let $e = (A, B)$, $e' = (B, C)$, and $i < j'' \leq j' < k$ be as in the hypotheses of the first statement in this proposition. Consider two cases.

Case 1. Suppose that the vertex B has at least one outgoing V-edge. Let $\tilde{e} = (B, D)$ and $\tilde{e}' = (B, D')$ be the leftmost and rightmost edges among such V-edges, respectively. Then there exist Δ -tiles of the form $\Delta := \Delta(D|A'B)$ and $\Delta' := \Delta(D'|BC')$ (i.e., lying on the left of \tilde{e} and on the right of \tilde{e}' , respectively). Let the V-edges $(A', D), (B, D), (B, D'), (C', D')$ be of types $\tilde{k}, \tilde{j}', \tilde{j}'', \tilde{i}$, respectively. By the choice of \tilde{e} and \tilde{e}' , we have $j' \leq j''$.

Next, the base (A', B) of Δ has type $\tilde{j}'\tilde{k}$ and should lie above the edge (A, B) (admitting the equality $(A', B) = (A, B)$). This implies $j' \leq \tilde{j}'$ (and $k \leq \tilde{k}$), by comparing Δ with the abstract Δ -tile with the base (A, B) , which, obviously, has type $j'k$. For similar reason, since the base (B, C') of Δ' lies above (B, C) , we have $\tilde{j}'' \leq j''$. Therefore, $\tilde{j}'' \leq j'' \leq j' \leq \tilde{j}' \leq \tilde{j}''$. This gives equality throughout, implying that $\Delta = \Delta(D|AB)$ and $\Delta' = \Delta(D|BC)$, as required.

Case 2. Let B have no outgoing V-edges. Then B is an intermediate vertex in the lower boundary of some lens λ . Let $\tilde{e} = (A', B)$ and $\tilde{e}' = (B, C')$ be the edges of L_λ entering and leaving B , respectively. Then \tilde{e}, \tilde{e}' have types $\tilde{j}\tilde{k}$ and $\tilde{i}\tilde{j}$ for some $\tilde{i} < \tilde{j} < \tilde{k}$. Since the edge \tilde{e} lies above e , and \tilde{e}' lies above e' (admitting equalities), we have $j' \leq \tilde{j} \leq j''$. This together with $j'' \leq j'$ gives $e = \tilde{e}$ and $e' = \tilde{e}'$, and the result follows.

The second assertion in the proposition is symmetric. ■

This completes the proof of Theorem 5.4. ■■

Now we return to a simple cyclic pattern $\mathcal{S} = (S_1, \dots, S_r = S_0)$, i.e., such that $|S_{i-1} \Delta S_i| = 1$ for all i , and the sets S_i are different and weakly separated from each other (conditions (C1), (C2) in Section 4). By Theorem 5.5, \mathcal{S} is included in the spectrum (vertex set) V_K of some combi K on Z_n . Proposition 5.2 implies that each pair of consecutive vertices S_{i-1}, S_i is connected in G_K by a V-edge directed from the smaller set to the bigger one. This gives the corresponding cycle in G_K ; we identify it with the curve $\zeta_{\mathcal{S}}$ in Z_n (defined in Section 4). The planarity of G_K implies that $\zeta_{\mathcal{S}}$ is non-self-intersecting, as required in part (i) of Proposition 4.1.

We prove (ii) in Proposition 4.1 as follows. Consider $X \subseteq [n]$ with $X \overline{\text{weak}} \mathcal{S}$ and a slope S_i with $|S_i| = |X|$. We assert that X and S_i are *strongly* separated. Indeed, let for definiteness $|S_{i-1}| < |S_i| < |S_{i+1}|$, $S_{i-1} = S_i - p$, and $S_{i+1} = S_i + q$ for some $p, q \in [n]$. Suppose, for a contradiction, that X and S_i are not strongly separated. Since $X \overline{\text{weak}} S_i$, two cases are possible, where we set $A := X - S_i$ and $B := S_i - X$.

Case 1: $B' < A < B''$, where $B' \sqcup B'' = B$ and $B', B'' \neq \emptyset$. Compare X and $S_{i+1} = S_i + q$. If $q \in X$, then $S_{i+1} - X = B$ (as before) and $X - S_{i+1} = A - q =: \tilde{A}$. Hence \tilde{A} splits B . But $|\tilde{A}| < |B|$, contradicting $X \overline{\text{weak}} S_{i+1}$. And if $q \notin X$, then the replacement of S_i by S_{i+1} either increases one of B', B'' , or splits A by the element q . In all cases X and S_{i+1} are not weakly separated.

Case 2: $A' < B < A''$, where $A' \sqcup A'' = A$ and $A', A'' \neq \emptyset$. We compare X and

$S_{i-1} = S_i - p$ and argue as in the previous case. More precisely, when we replace S_i by S_{i-1} , case $p \notin X$ leads to decreasing B (while preserving A), whereas case $p \in X$ leads to either increasing one of A', A'' or splitting the middle set by p . So in all cases X and S_{i-1} are not weakly separated.

Thus, $X \overline{\text{str}} \mathcal{A}_h$, where $h := |X|$ and \mathcal{A}_h is the set of slopes S_i with $|S_i| = h$. By (1.3), for any $S_i \in \mathcal{A}_h$, $S_i \prec X$ if and only if $S_i \prec X$. In particular, the elements of \mathcal{A}_h are linearly ordered by \prec , say, $S_{i(1)} \prec S_{i(2)} \prec \dots \prec S_{i(k)}$, where $k = |\mathcal{A}_h|$.

Consider the girdle Λ_h for K (defined in Section 5.2). The elements of \mathcal{A}_h are critical vertices in Λ_h and they subdivide Λ_h into $k + 1$ closed simply connected sets $\omega_1, \omega_2, \dots, \omega_{k+1}$ so that $\omega_j \cap \omega_{j+1}$ consists of the single point $S_{i(j)}$. By easy topological reasonings, ω_j is entirely contained in R_S^{in} (resp. R_S^{out}) if j is odd (resp. even). We have shown that any vertex X of K with $|X| = h$ is comparable by \prec with each critical vertex of level h . It follows that X lies in ω_j if and only if $S_{i(j-1)} \preceq X \preceq S_{i(j)}$ (letting $S_{i(0)} := z_h^{\ell}$ and $S_{i(k+1)} := z_h^r$). This implies (ii) in Proposition 4.1. \blacksquare

Now we finish the proof of Theorem 4.2 as follows. Consider arbitrary $X \in \mathcal{D}_S^{\text{in}}$ and $Y \in \mathcal{D}_S^{\text{out}}$. Since $\mathcal{S} \cup \{X\}$ is weakly separated, it is included in the spectrum V_K of some combi K , by Theorem 5.5. Let K^{in} be the subtiling of K formed by the tiles occurring in R_S^{in} . Then X is a vertex of K^{in} . In its turn, $\mathcal{S} \cup \{Y\} \subseteq V_{K'}$ for some combi K' , and Y is a vertex of the subtiling K'^{out} of K' formed by the tiles occurring in R_S^{out} (or Y is a vertex in $bd(Z_n)$). Since $R_S^{\text{in}} \cap R_S^{\text{out}} = \zeta_S$, the union $K^{\text{in}} \cup K'^{\text{out}}$ gives a correct combi \tilde{K} on Z_n containing both X and Y . Since $V_{\tilde{K}}$ is a w-collection (by Theorem 5.4), we have $X \overline{\text{weak}} Y$, as required. Therefore, $\mathcal{D}_S^{\text{in}}, \mathcal{D}_S^{\text{out}}$ form a complementary pair (their union \mathcal{D} is w-pure). This implies the w-purity of both $\mathcal{D}_S^{\text{in}}, \mathcal{D}_S^{\text{out}}$, by Corollary 2.4. \blacksquare

The strong separation counterpart of this theorem, namely, Theorem 4.3, is obtained in a similar way. Given a cyclic pattern \mathcal{S} consisting of different and pairwise strongly separated sets, the result immediately follows from Theorem 5.1 and the fact that any pure tiling on R_S^{in} and any pure tiling on R_S^{out} can be combined to form a pure tiling on Z_n . The equality $r^s(\mathcal{D}) = r^w(\mathcal{D})$ is obvious, where $\mathcal{D} = \widehat{\mathcal{D}}_S^{\text{in}}$ or $\widehat{\mathcal{D}}_S^{\text{out}}$.

7 Special cases and generalizations

In this section we discuss possible ways to extend the w-purity result to more general cyclic patterns. Along the way we demonstrate some representative special cases.

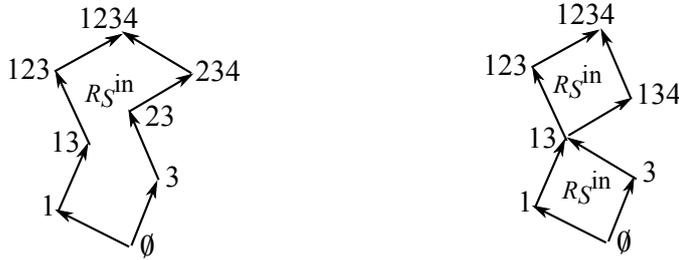
7.1 Semi-simple cyclic patterns

We can slightly generalize Theorem 4.2 by weakening condition (C1) in Section 4. Now we admit, with a due care, cyclic patterns \mathcal{S} having some repeated elements $C_i = C_j$. More precisely, in the corresponding closed curve ζ_S in $Z = Z_n$ we allow only touchings but not crossings, which is equivalent to saying that under a “very small” deformation the curve becomes non-self-intersecting. We refer to \mathcal{S} satisfying this requirement and condition (C2) as *semi-simple*. The definitions of regions $R_S^{\text{in}}, R_S^{\text{out}}$ and domains $\mathcal{D}_S^{\text{in}}, \mathcal{D}_S^{\text{out}}$ are modified in a natural way. The desired generalization reads as follows:

For a semi-simple cyclic pattern \mathcal{S} , the domains $\mathcal{D}_\mathcal{S}^{\text{in}}$ and $\mathcal{D}_\mathcal{S}^{\text{out}}$ form a complementary pair; as a consequence, both $\mathcal{D}_\mathcal{S}^{\text{in}}$ and $\mathcal{D}_\mathcal{S}^{\text{out}}$ are w -pure. This is proved in a similar way as Theorem 4.2 and we omit it here.

One can see that domains of types $\mathcal{D}(\omega)$ and $\mathcal{D}(\omega', \omega)$ exposed in Theorem 1.1(i),(ii) are representable as $\mathcal{D}_\mathcal{S}^{\text{in}}$ for special simple or semi-simple cyclic patterns \mathcal{S} . (See also Examples 2,3 in Section 2.)

In particular, for the domain $\mathcal{D} = \mathcal{D}(\omega', \omega)$ with permutations $\omega', \omega : [n] \rightarrow [n]$ such that $\text{Inv}(\omega') \subset \text{Inv}(\omega)$, the cyclic pattern \mathcal{S} involves the sets \emptyset , $[n]$, and $\omega'^{-1}[i]$ and $\omega^{-1}[i]$ for $i = 1, \dots, n-1$; such an \mathcal{S} is simple if $\omega'^{-1}[i] \neq \omega^{-1}[i]$ for all i , and semi-simple otherwise. The picture below illustrates \mathcal{S} for $\mathcal{D}(\omega', \omega)$ with $n = 4$ in the case $\omega' = 1324$ and $\omega = 3241$ (left), and in the case $\omega' = 1324$ and $\omega = 3142$ (right).



7.2 Generalized cyclic patterns

Next we are going to discuss another, more important way to extend the obtained w -purity results, namely, we consider a *generalized* cyclic pattern $\mathcal{S} = (S_1, \dots, S_r = S_0)$ in $2^{[n]}$ defined in Section 4. As before, \mathcal{S} obeys conditions (C1) and (C2), and now we assume that

$$(7.1) \text{ for each } p = 1, \dots, r, \text{ either } |S_{p-1} \triangle S_p| = 1, \text{ or } |S_{p-1} \triangle S_p| = 2 \text{ and } |S_{p-1}| = |S_p|.$$

In the former (latter) case, we say that $\{S_{p-1}, S_p\}$ is a *1-distance pair* (resp. *2-distance pair*) in \mathcal{S} . In the latter case, $S_{p-1} - S_p$ and $S_p - S_{p-1}$ consist of some singletons i and j , respectively, and setting $X := S_{p-1} \cap S_p$ and $Y := S_{p-1} \cup S_p$, we have

$$S_{p-1} = Xi = Y - j \quad \text{and} \quad S_p = Xj = Y - i.$$

We rely on the following assertion. (A property of a similar flavor for plabic graphs is established in [5].)

Proposition 7.1 *Suppose that a combi K has two vertices of the form $A = Xi$ and $B = Xj$. Then at least one of the following takes place:*

- (i) K contains the vertex X (and therefore the edges (X, A) and (X, B));
- (ii) K contains the vertex Xij (and therefore the edges (A, Xij) and (B, Xij));
- (iii) both A, B belong to the lower boundary of some lens in K ;
- (iv) both A, B belong to the upper boundary of some lens in K .

Proof Let for definiteness $i < j$. Like the proof of Proposition 5.2, we use induction on n and consider the n -contraction K' of K . Then K is the n -expansion of K' w.r.t. a legal path P in $G_{K'}$. We first assume that $j < n$ and consider two possible cases.

Case 1. Let $n \notin X$. Then A, B are vertices of K' contained in the left subgraph G_1 of $G_{K'}$ (w.r.t. P). By induction either (a) both A, B belong to one boundary (L_λ or U_λ) of some lens λ in K' , or (b) K' contains $X = A \cap B$, or (c) K' contains $Xij = A \cup B$. In case (a), λ continues to be a lens in K , yielding (iii) or (iv) in the proposition. In case (b), if X is not a pit of P , then X is a vertex of K , as required in (i). And if X is a pit of P , then A, B belong to ∇ -tiles in the filling at X w.r.t. P , and therefore A, B become vertices of U_λ for the lens λ arising in K in place of the three-edge zigzag P_λ of P containing X ; this yields (iv). In case (c), Xij cannot be a pit of P (otherwise A, B would be not in G_1). Hence Xij is a vertex of K , as required in (ii).

Case 2. Let $n \in X$. Then $A' := A - n$ and $B' := B - n$ are vertices of K' contained in the right subgraph G_2 of $G_{K'}$. By induction either (a') both A', B' belong to one of L_λ, U_λ for some lens λ of K' , or (b') K' contains $A' \cap B'$, or (c') K' contains $A' \cup B'$. Cases (a') and (b') are easy, as well as (c') unless $A' \cup B'$ is a peak of P . And if $A' \cup B'$ is a peak of P , then A', B' belong to Δ -tiles in the filling at $A' \cup B'$ w.r.t. P , implying that $A = A'n$ and $B = B'n$ are vertices of L_λ for the corresponding lens λ of K .

Now assume that $j = n$. Then $n \notin X$ and $n \notin A$. Hence A is a vertex of the subgraph G_1 , and $B \cap [n - 1] = X$ is a vertex of G_2 of K' . Also $A = Xi$, and by Proposition 5.2, K' has the V-edge (X, A) of type i . Consider two cases.

Case 3a. X is a slope or a peak of P . Then X is a vertex of K , yielding (i) in the proposition.

Case 3b. X is a pit of P . Then A belongs to a ∇ -tile in the filling at X w.r.t. P , whence A is a vertex of U_λ for the corresponding lens λ of K . The right vertex r_λ of λ is just of the form $Xn = B$, and we obtain (iv) in the proposition. \blacksquare

Like simple cyclic patterns in Section 4, the sets S_p are identified with the corresponding points in the zonogon $Z = Z_n$, and we connect each pair S_{p-1}, S_p by line segment e_p , obtaining the closed piecewise linear curve ζ_S in Z . We direct each e_p so as to be congruent to the corresponding generator ξ_i or vector ϵ_{ij} .

A reasonable question arises: when ζ_S is non-self-intersecting? Proposition 7.1 enables us to find necessary and sufficient conditions in terms of “forbidden quadruples” in \mathcal{S} . These conditions are as follows.

(C3) S contains no quadruple $S_{p-1}, S_p, S_{q-1}, S_q$ such that either $\{S_{p-1}, S_p\} = \{Xi, Xk\}$ and $\{S_{q-1}, S_q\} = \{Xj, X\ell\}$, or $\{S_{p-1}, S_p\} = \{X-i, X-k\}$ and $\{S_{q-1}, S_q\} = \{X-j, X-\ell\}$, where $i < j < k < \ell$.

(C4) S contains no quadruple $S_{p-1}, S_p, S_{q-1}, S_q$ such that either $\{S_{p-1}, S_p\} = \{Xi, Xk\}$ and $\{S_{q-1}, S_q\} = \{X, Xj\}$, or $\{S_{p-1}, S_p\} = \{X-i, X-k\}$ and $\{S_{q-1}, S_q\} = \{X, X-j\}$, where $i < j < k$.

To prove the next assertion and for further purposes, we need to refine some definitions. Fix a combi K and consider a vertex A in it. By the (full) upper filling at A

we mean the sequence $\nabla(A|X_0X_1), \dots, \nabla(A|X_{q-1}X_q)$ of all ∇ -tiles having the bottom A and ordered from left to right (i.e., $X_0X_1 \dots X_q$ is a directed path in G_K). The union of these tiles is called the *upper sector* at A and denoted by Σ_A^{up} , and the path $X_0X_1 \dots X_q$ is called the *upper boundary* of Σ_A^{up} and denoted by U_A . Symmetrically, the (full) *lower filling* at A is the sequence $\Delta(A|Y_0Y_1), \dots, \Delta(A|Y_{q'-1}Y_{q'})$ of all Δ -tiles having the top A and ordered from left to right, the *lower sector* Σ_A^{low} at A is the union of these tiles, and the *lower boundary* L_A of Σ_A^{low} is the directed path $Y_0Y_1 \dots Y_{q'}$. Note that one of these fillings or both may be empty.

Proposition 7.2 *For a generalized cyclic pattern \mathcal{S} , the curve $\zeta_{\mathcal{S}}$ is non-self-intersecting if and only if \mathcal{S} satisfies (C1)–(C4).*

Proof It is easy to see that if \mathcal{S} has a quadruple as in (C3) or (C4), then the corresponding segments (edges) e_p and e_q are crossing (have a common interior point), and therefore $\zeta_{\mathcal{S}}$ is self-intersecting.

Conversely, suppose that ζ is self-intersecting. To show the existence of a pair as in (C3) or (C4), fix a combi K with V_K including \mathcal{S} . Since all sets in \mathcal{S} are different, $\zeta_{\mathcal{S}}$ has two crossing segments e_p and e_q . The edges of K are non-crossing (since K is planar); therefore, at least one of e_p, e_q is not an edge of K . Let for definiteness e_p be such, i.e., $\{S_{p-1}, S_p\}$ is a 2-distance pair.

By Proposition 7.1, at least one of the following takes place: (i) $X := S_{p-1} \cap S_p \in V_K$; (ii) $Y := S_{p-1} \cup S_p \in V_K$; (iii) both S_{p-1}, S_p belong to the same boundary, either U_λ or L_λ , for some lens λ of K . In case (iii), the only possibility for e_q to cross e_p is when $\{S_{q-1}, S_q\}$ is a 2-distance pair occurring in the same boundary (either U_λ or L_λ) of λ where S_{p-1}, S_p lie; moreover, the elements of these two pairs should be intermixing in this boundary. This gives a quadruple as in (C3).

In case (i), both vertices S_{p-1}, S_p (being of the form Xi, Xk for some i, k) lie in the boundary U_X of the upper sector Σ_X^{up} at X . Then e_q can cross e_p only in two cases: (a) the pair $\{S_{q-1}, S_q\}$ lies in U_X as well and, moreover, its elements and those of $\{S_{p-1}, S_p\}$ are intermixing in U_X (yielding a quadruple as in (C3)); and (b) one of S_{q-1}, S_q is just X while the other belongs to U_X and, moreover, the latter lies between S_{p-1} and S_p (yielding a quadruple as in (C4)). The case (ii) is symmetric to (i) and we argue in a similar way. ■

To extend Theorem 4.2 to a generalized cyclic pattern $\mathcal{S} = (S_1, \dots, S_r)$, we will consider one or another combi K with $\mathcal{S} \subseteq V_K$. Unlike the case of simple cyclic patterns, it now becomes less trivial to split K into two subtilings K^{in} and K^{out} (lying in the regions $R_{\mathcal{S}}^{\text{in}}$ and $R_{\mathcal{S}}^{\text{out}}$, respectively). A trouble is that some 2-segments $e_p = [S_{p-1}, S_p]$ may cut some tiles of K . Hereinafter we refer to the line segment $[S_{p-1}, S_p]$ connecting points S_{p-1}, S_p in the zonogon as a *1-segment* (resp. *2-segment*) if these points form a 1-distance (resp. 2-distance) pair.

We overcome this trouble by use of the *splitting method* described below. It works somewhat differently for lenses and for triangles.

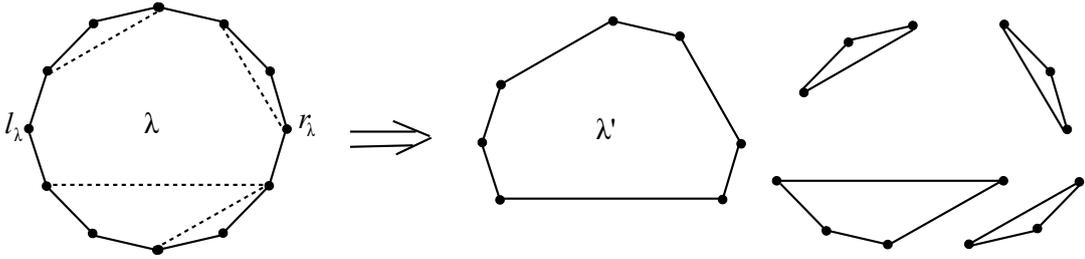
I. First we consider a lens λ of K such that $\zeta_{\mathcal{S}}$ cuts λ (otherwise there is no problem with λ at all). The curve $\zeta_{\mathcal{S}}$ may go across λ several times; let $e_{p(1)}, \dots, e_{p(d)}$ be the 2-segments cutting λ . These segments are pairwise non-crossing (by (C3)) and subdivide

λ into $d + 1$ polygons D_1, \dots, D_{d+1} ; so each $e_{p(i)}$ is $D_j \cap D_{j'}$ for some j, j' (and one of $D_j, D_{j'}$ lies in R_S^{in} , and the other in R_S^{out}).

If $\{S_{p-1}, S_p\} = \{\ell_\lambda, r_\lambda\}$, we say that $e_{p(i)}$ is the *central segment*. Otherwise both ends of $e_{p(i)}$ belong to either U_λ or L_λ ; in the former (latter) case, $e_{p(i)}$ is called an *upper* (resp. *lower*) *segment*, and we associate to it the path $P_{p(i)}$ in U_λ (resp. L_λ) connecting $S_{p(i)-1}$ and $S_{p(i)}$. In view of (C3), such paths form a *nested family*, i.e., for any $i \neq i'$, either $P_{p(i)} \cap P_{p(i')} = \emptyset$ or $P_{p(i)} \subset P_{p(i')}$ or $P_{p(i)} \supset P_{p(i')}$.

Accordingly, polygons D_j can be of three sorts. When D_j has all vertices in U_λ , its lower boundary is formed by exactly one (upper or central) segment (while its upper boundary is formed by some edges of $P_{p(i)}$ and segments $e_{p(i')}$); we call such a D_j an *upper semi-lens*. Symmetrically, when D_j has all vertices in L_λ , its upper boundary is formed by exactly one (lower or central) segment, and we call D_j a *lower semi-lens*. Besides, when the central segment does not exist, there appears one more polygon D_j ; it is viewed as an (abstract) lens λ' with $\ell_{\lambda'} = \ell_\lambda$ and $r_{\lambda'} = r_\lambda$, and we call it (when exists) a *secondary lens*.

A possible splitting of a lens λ is illustrated in the picture, where $d = 4$ and the four cutting segments are indicated by dotted lines.

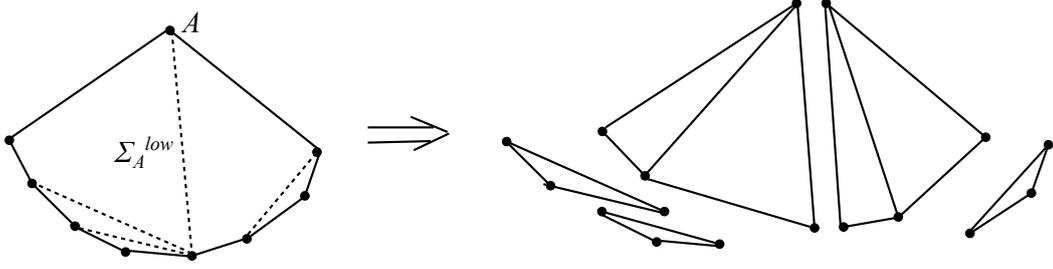


II. Next suppose that ζ_S cuts some Δ -tile $\Delta(A|BC)$ of K . Then the lower sector Σ_A^{low} at A is cut by some 2-segments in ζ_S ; let $e_{p(1)}, \dots, e_{p(d)}$ be these 2-segments. (Besides, Σ_A^{low} can be cut by some 1-segments.) For each segment $e_{p(i)}$, its ends $S_{p(i)-1}$ and $S_{p(i)}$ belong to the lower boundary L_A of the sector, and we denote by $P_{p(i)}$ the directed path in L_A connecting these vertices. Such paths form a nested family, by (C3).

The sector Σ_A^{low} is subdivided by the above 2-segments into $d + 1$ polygons D_1, \dots, D_{d+1} . Among these, d polygons are lower semi-lenses, each being associated with the segment $e_{p(i)}$ forming its upper boundary. The remaining polygon contains the vertex A and is viewed as a lower sector Σ' whose lower boundary $L(\Sigma')$ is formed by the segments $e_{p(i)}$ with $P_{p(i)}$ maximal and the edges of L_A between these segments. We fill Σ' with the corresponding Δ -tiles (so each edge (B, C) of $L(\Sigma')$ generates one Δ -tile, namely, $(A|BC)$). We call them *secondary Δ -tiles*.

In view of (C4), each 1-segment e_i cutting Σ_A^{low} connects some vertex of $L(\Sigma')$ and the top A , and therefore e_i coincides with the common V-edge of some two neighboring Δ -tiles in the filling of Σ' .

A possible splitting of a sector by three 2-segments and one 1-segment (drawn by dotted lines) is illustrated in the picture.



When $\zeta_{\mathcal{S}}$ cuts some ∇ -tile $\nabla(A|BC)$, we consider the upper sector Σ_A^{up} and makes a splitting in a similar way (which subdivides Σ_A^{up} into corresponding upper semi-lenses and secondary ∇ -tiles).

Let \widehat{K} be the resulting set of tiles upon termination of the splitting process for K (it consists of the tiles of K not cut by $\zeta_{\mathcal{S}}$ and the appeared semi-lenses and secondary tiles). Then the desired K^{in} (K^{out}) is defined to be the set of tiles of \widehat{K} lying in $R_{\mathcal{S}}^{\text{in}}$ (resp. $R_{\mathcal{S}}^{\text{out}}$). We refer to \widehat{K} , K^{in} , K^{out} as *quasi-combies* on Z , $R_{\mathcal{S}}^{\text{in}}$, $R_{\mathcal{S}}^{\text{out}}$, respectively, *agreeable with \mathcal{S}* .

Now we are ready to generalize Theorem 4.2. As before, the domain $\mathcal{D}_{\mathcal{S}}^{\text{in}}$ ($\mathcal{D}_{\mathcal{S}}^{\text{out}}$) consists of the sets (points) $X \subseteq [n]$ such that $X \text{ (weak) } \mathcal{S}$ and X lies in $R_{\mathcal{S}}^{\text{in}}$ (resp. $R_{\mathcal{S}}^{\text{out}}$).

Theorem 7.3 *Let \mathcal{S} be a generalized cyclic pattern satisfying (C1)–(C4). Then the domains $\mathcal{D}_{\mathcal{S}}^{\text{in}}$ and $\mathcal{D}_{\mathcal{S}}^{\text{out}}$ form a complementary pair. As a consequence, both $\mathcal{D}_{\mathcal{S}}^{\text{in}}$ and $\mathcal{D}_{\mathcal{S}}^{\text{out}}$ are w-pure.*

Proof Given arbitrary $X \in \mathcal{D}_{\mathcal{S}}^{\text{in}}$ and $Y \in \mathcal{D}_{\mathcal{S}}^{\text{out}}$, take a combi K with V_K including $\mathcal{S} \cup \{X\}$ and a combi K' with $V_{K'}$ including $\mathcal{S} \cup \{Y\}$. Split K into the corresponding quasi-combies K^{in} and K^{out} on $R_{\mathcal{S}}^{\text{in}}$ and $R_{\mathcal{S}}^{\text{out}}$, respectively, and similarly, split K' into quasi-combies K'^{in} and K'^{out} . Then X is a vertex in K^{in} , and Y is a vertex in K'^{out} . Their union $\widetilde{K} := K^{\text{in}} \cup K'^{\text{out}}$ is a quasi-combi on Z . We transform \widetilde{K} , step by step, in order to obtain a correct combi with the same vertex set.

More precisely, in a current \widetilde{K} choose a semi-lens λ . If λ is a lower semi-lens with $\ell_{\lambda} = X$ and $r_{\lambda} = Y$, then its upper boundary is formed by the unique edge $e = (X, Y)$, and this edge belongs to another tile τ in \widetilde{K} . Three cases are possible: (a) τ is a Δ -tile (and e is its base); (b) τ is a lens or a lower semi-lens (and e belongs to its lower boundary L_{τ}); and (c) τ is an upper semi-lens (and e forms its lower boundary). We remove the edge e , combining λ and τ into one polygon ρ .

In case (a), ρ looks like an upper sector, and we fill it with the corresponding Δ -tiles. In case (b), ρ is again a lens or a lower semi-lens (like τ). In case (c), ρ is a lens. The new \widetilde{K} is a quasi-combi on Z , and the number of semi-lenses becomes smaller.

If λ is an upper semi-lens, it is treated symmetrically.

We repeat the procedure for the current \widetilde{K} , and so on, until we get rid of all semi-lenses. Then the eventual \widetilde{K} is a correct combi containing both X, Y , and the result follows. ■

7.3 Planar graph patterns

We can further extend the w-purity result by considering an arbitrary graph $\mathcal{H} = (\mathcal{S}, \mathcal{E})$ with the following properties:

(H1) the vertex set \mathcal{S} is a w-collection in $2^{[n]}$;

(H2) each edge $e \in \mathcal{E}$ is formed by a 1- or 2-distance pair in \mathcal{S} ;

(H3) the edges of \mathcal{H} obey (C3) and (C4), in the sense that there are no quadruple of vertices of \mathcal{H} that can be labeled as $S_{p-1}, S_p, S_{q-1}, S_q$ so that both $\{S_{p-1}, S_p\}$ and $\{S_{q-1}, S_q\}$ be edges of \mathcal{H} and they behave as indicated in (C3) or (C4).

Representing the vertices of \mathcal{H} as corresponding points in $Z = Z_n$, and the edges as line-segments, we observe from (H3) and the proof of Proposition 7.2 that the graph \mathcal{H} is planar (has a planar layout in Z). Let \mathcal{F} be the set of its (closed 2-dimensional) faces. (W.l.o.g., we may assume that \mathcal{H} includes the boundary of Z , since $bd(Z)$ is weakly separated from any subset of $[n]$.) For a face $F \in \mathcal{F}$, the set of elements of $\mathcal{D}_{\mathcal{S}}$ contained in F is denoted by $\mathcal{D}_{\mathcal{S}}(F)$.

Theorem 7.4 *Let \mathcal{H} be a graph satisfying (H1)–(H3). Then for any two different faces F, F' of \mathcal{H} , the domains $\mathcal{D}_{\mathcal{S}}(F)$ and $\mathcal{D}_{\mathcal{S}}(F')$ form a complementary pair. As a consequence, $\mathcal{D}_{\mathcal{S}}(F)$ is w-pure for each face F .*

(When \mathcal{H} is a simple cycle, this turns into Theorem 7.3. Also this generalizes the result on semi-simple cyclic patterns in Section 7.1.)

Proof If sets (points) X, Y lie in F, F' , respectively, then they are separated by the curve corresponding to some simple cycle \mathcal{C} in \mathcal{H} . This \mathcal{C} is, in fact, a generalized cyclic pattern obeying conditions (C1)–(C4). Also one of X, Y lies in the region $R_{\mathcal{C}}^{\text{in}}$, and the other in $R_{\mathcal{C}}^{\text{out}}$. Now the result follows from Theorem 7.3. \blacksquare

One can reformulate this theorem as follows: for each face F of \mathcal{H} , take an arbitrary maximal w-collection \mathcal{X}_F in $\mathcal{D}_{\mathcal{S}}(F)$; then for any set \mathcal{F}' of faces of \mathcal{H} , $\cup(\mathcal{X}_F : F \in \mathcal{F}')$ is a maximal w-collection in $\cup(\mathcal{D}_{\mathcal{S}}(F) : F \in \mathcal{F}')$.

We conclude this paper with two important special cases.

A. A special case of generalized cyclic patterns is a *Grassmann necklace* \mathcal{N} of [5], which is a sequence $(S_1, S_2, \dots, S_n = S_0)$ of sets in Δ_n^m such that $S_{i+1} - S_i = \{i\}$ for each i . One can check that such an \mathcal{N} is a w-collection satisfying (C3). As is shown in [5], the domain $\mathcal{D}_{\mathcal{N}}^{\text{in}}$ is w-pure. A sharper result is given in [3]; it says that the domains $\mathcal{D}_{\mathcal{N}}^{\text{in}}$ and $\mathcal{D}_{\mathcal{N}}^{\text{out}} \cap \Delta_n^m$ form a complementary pair within the hyper-simplex Δ_n^m . This is a special case of Theorem 7.4. Indeed, take as \mathcal{H} the union of the (natural) cycle \mathcal{C} on \mathcal{N} and the cycle \mathcal{C}_0 on the “maximal” necklace (formed by the intervals and co-intervals if size m in $[n]$). Then \mathcal{H} has the face surrounded by \mathcal{C} (giving the domain $\mathcal{D}_{\mathcal{N}}^{\text{in}}$) and the face (or the union of several faces) “lying between” \mathcal{C} and \mathcal{C}_0 (giving the domain $\mathcal{D}_{\mathcal{N}}^{\text{out}} \cap \Delta_n^m$).

B. Next we show that Theorem 3.1 on steep ladders can be deduced from Theorem 7.3. Given a steep ladder $\Gamma_{\lambda=(\lambda_0, \dots, \lambda_k)}$, our goal is to construct a generalized cyclic pattern

\mathcal{S} in $2^{[n]}$ such that $\mathcal{D}_\lambda = \mathcal{D}_S^{\text{in}}$, letting $m := \lambda_0$, $m' := \lambda_k$, and $n := k + m'$. In fact, we use the representation of \mathcal{D}_λ as $\cup(\Delta_{i+j}^i: (i, j) \in T_\lambda)$. Define the partition $\lambda' = (\lambda'_0, \dots, \lambda'_k)$ as in the proof of Theorem 3.1; then $\mathcal{D}_{\lambda'} = \Delta_n^{m', n}$. The desired \mathcal{S} will be the concatenation of three sequences $\mathcal{S}^+, \mathcal{S}^-, \mathcal{S}^0$ defined below.

The sequence \mathcal{S}^+ is determined by the essential double hooks $H_{p,q}$ in $\Gamma_{\lambda'}$ such that the set $S(H_{p,q})$ belongs to \mathcal{D}_λ . More precisely, $\mathcal{S}^+ = (S_0^+, S_1^+, \dots, S_k^+)$, where $S_i^+ := S(H_{\lambda_i, i})$, i.e., S_i^+ is the interval $i + [\lambda_i]$ ($= [(i+1)..(i+\lambda_i)]$). The steepness of Γ_λ implies $(\lambda_{i-1} - \lambda_i) \in \{0, 1\}$, $i = 1, \dots, k$, whence

$$S_i^+ = \begin{cases} (S_{i-1}^+ - \{i\}) \cup \{i + \lambda_i\} & \text{if } \lambda_i = \lambda_{i-1}; \\ S_{i-1}^+ - \{i\} & \text{if } \lambda_i = \lambda_{i-1} - 1. \end{cases}$$

Then each pair S_{i-1}^+, S_i^+ in \mathcal{S}^+ is a 1- or 2-distance pair, S_0^+ is the point $[\lambda_0] = [m]$ ($= z_m^\ell$) on the left boundary of Z_n , while S_k^+ is the point $[(k+1)..n]$ ($= z_{m'}^r$) on the right boundary. Concatenating the 1- or 2-segments $[S_{i-1}^+, S_i^+]$, we obtain a piecewise linear curve ζ^+ from z_m^ℓ to $z_{m'}^r$ (which will give the upper boundary of the region R_S^{in}).

To define \mathcal{S}^- , we take, for $i = 0, \dots, m'$, the directed path Q_i in \mathcal{P}_λ formed by the horizontal path from $(0, 0)$ to $(i, 0)$, followed by the vertical path from $(i, 0)$ to (i, k) , followed by the horizontal path from (i, k) to (m', k) . Observe that $S(Q_i)$ is the co-interval $[i] \cup [(i+k)..n]$ of size m' in $[n]$, and that Q_0 coincides with the double hook $H_{\lambda_k, k}$, whence $S(Q_0) = S_k^+$. We define $\mathcal{S}^- = (S(Q_0), \dots, S(Q_{m'}))$. Each consecutive pair in \mathcal{S}^- is a 2-distance pair, and concatenating the corresponding 2-segments, we obtain a piecewise linear curve ζ^- from $z_{m'}^r$ to z_m^ℓ (giving the lower boundary of R_S^{in}).

To define \mathcal{S}^0 , we take, for $i = 0, \dots, m - m'$, the directed path L_i in \mathcal{P}_λ formed by the horizontal path from $(0, 0)$ to $(m' + i, 0)$, followed by the vertical path from $(m' + i, 0)$ to the vertex in T_λ of the form $(m' + i, j)$. Then $S(L_i)$ is the point $[m' + i]$ ($= z_{m'+i}^\ell$) in $\text{lbd}(Z_n)$, and we have $L_0 = Q_{m'}$ and $S(L_{m-m'}) = S_0^+$. So \mathcal{S}^0 induces the path in $\text{lbd}(Z_n)$ from $z_{m'}^\ell$ to z_m^ℓ (giving the left boundary of R_S^{in}).

The concatenation of $\mathcal{S}^+, \mathcal{S}^-, \mathcal{S}^0$ is just the desired generalized cyclic pattern \mathcal{S} . It is not difficult to check that \mathcal{D}_λ coincides with the domain $\mathcal{D}_S^{\text{in}}$ consisting of the sets (points) X weakly separated from \mathcal{S} and lying in R_S^{in} . (We can argue as follows. The checker \mathcal{C} of \mathcal{D}_λ within $\Delta_n^{m', n}$, constructed in the proof of Theorem 3.1, consists of all intervals in $[n]$ lying in Z_n above ζ^+ (including ζ^+). They form the vertex set of a semi-rhombus tiling covering the region above ζ^+ . Adding to it the set of co-intervals of size $\leq m'$ gives a checker for \mathcal{D}_λ within $2^{[n]}$ (cf. Example 1 in Section 2). These co-intervals form the vertex set of a semi-rhombus tiling covering the region below ζ^- . It follows that \mathcal{D}_λ is the vertex set of a quasi-combi subdividing the remaining region R_S^{in} .) Thus, \mathcal{D}_λ is w-pure, as required.

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