Journal of Algebra 488 (2017) 145–200

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On universal quadratic identities for minors of quantum matrices

Vladimir I. Danilov [a], Alexander V. Karzanov [b,*]

[a] *Central Institute of Economics and Mathematics of the RAS, 47, Nakhimovskii Prospect, 117418 Moscow, Russia*
[b] *Institute for System Analysis at FRC Computer Science and Control of the RAS, 9, Prospect 60 Let Oktyabrya, 117312 Moscow, Russia*

A R T I C L E   I N F O

Article history:
Received 15 December 2016
Available online 17 June 2017
Communicated by Masaki Kashiwara

MSC:
16T99
05C75
05E99

Keywords:
Quantum matrix
Quadratic identity for minors
Planar graph
Cauchon diagram
Path matrix
Lindström's Lemma

A B S T R A C T

We give a complete combinatorial characterization of homogeneous quadratic relations of "universal character" valid for minors of quantum matrices (more precisely, for minors in the quantized coordinate ring $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ of $m \times n$ matrices over a field $\mathbb{K}$, where $q \in \mathbb{K}^*$). This is obtained as a consequence of a study of quantized minors of matrices generated by paths in certain planar graphs, called *SE-graphs*, generalizing the ones associated with Cauchon diagrams. Our efficient method of verifying universal quadratic identities for minors of quantum matrices is illustrated with many appealing examples.

* Corresponding author.
E-mail addresses: danilov@cemi.rssi.ru (V.I. Danilov), sasha@cs.isa.ru, akarzanov7@gmail.com (A.V. Karzanov).

http://dx.doi.org/10.1016/j.jalgebra.2017.05.034
0021-8693/

1. Introduction

The idea of quantization has proved its importance to bridge the commutative and noncommutative versions of certain algebraic structures and promote better understanding various aspects of the latter versions. One popular structure studied for the last three decades (as an important part of the study of algebraic quantum groups) is the quantized coordinate ring $\mathcal{R} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ of $m \times n$ matrices over a field $\mathbb{K}$, where q is a nonzero element of $\mathbb{K}$; it is usually called the *algebra of $m \times n$ quantum matrices*. Here $\mathcal{R}$ is the $\mathbb{K}$-algebra generated by the entries (indeterminates) of an $m \times n$ matrix X subject to the following (quasi)commutation relations due to Manin [12]: for $1 \le i < \ell \le m$ and $1 \le j < k \le n$,

$$x_{ij}x_{ik} = qx_{ik}x_{ij}, \qquad x_{ij}x_{\ell j} = qx_{\ell j}x_{ij}, \tag{1.1}$$
$$x_{ik}x_{\ell j} = x_{\ell j}x_{ik} \quad \text{and} \quad x_{ij}x_{\ell k} - x_{\ell k}x_{ij} = (q - q^{-1})x_{ik}x_{\ell j}.$$

This paper is devoted to quadratic identities for minors of quantum matrices (usually called quantum minors or quantized minors or q-minors). For representative cases, aspects and applications of such identities, see, e.g., [6–10,14,15] (where the list is incomplete). We present a novel, and rather transparent, combinatorial method which enables us to completely characterize and efficiently verify homogeneous quadratic identities of universal character that are valid for quantum minors.

The identities of our interest can be written as

$$\sum (s_i q^{\delta_i} [I_i|J_i]_q \, [I'_i|J'_i]_q \colon i = 1, \dots, N) = 0, \tag{1.2}$$

where $\delta_i \in \mathbb{Z}$, $s_i \in \{+1, -1\}$, and $[I|J]_q$ denotes the quantum minor whose rows and columns are indexed by $I \subseteq [m]$ and $J \subseteq [n]$, respectively. (Hereinafter, for a positive integer n', we write $[n']$ for $\{1, 2, \dots, n'\}$.) The homogeneity means that each of the sets $I_i \cup I'_i$, $I_i \cap I'_i$, $J_i \cup J'_i$, $J_i \cap J'_i$ does not depend i, and the term "universal" means that (1.2) should be valid independently of $\mathbb{K}, q$ and a q-matrix (a matrix whose entries obey Manin's relations and, possibly, additional ones). Note that any quadruple $(I|J, I'|J')$, referred to as a *cortege* later on, may be repeated in (1.2) several times.

Our approach is based on two sources. The first one is the *flow-matching method* elaborated in [4] to characterize quadratic identities for usual minors (viz. for $q = 1$). In that case the identities are viewed simpler than (1.2), namely, as

$$\sum (s_i [I_i|J_i] \, [I'_i|J'_i] \colon i = 1, \dots, N) = 0. \tag{1.3}$$

(In fact, [4] deals with natural analogs of (1.3) over commutative semirings, e.g. the tropical semiring $(\mathbb{R}, +, \max)$.) In the method of [4], each cortege $S = (I|J, I'|J')$ is associated with a certain set $\mathcal{M}(S)$ of *feasible matchings* on the set $(I \triangle I') \sqcup (J \triangle J')$ (where $A \triangle B$ denotes the symmetric difference $(A - B) \cup (B - A)$, and $A \sqcup B$ the disjoint

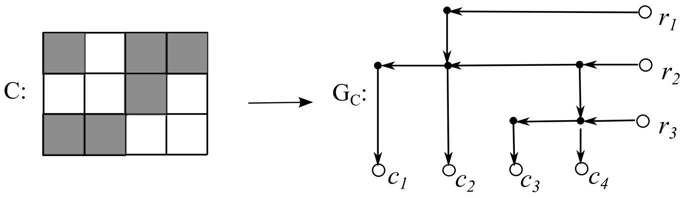


Fig. 1. An example of Cauchon diagrams (left) and the related graph (right).

union of sets A, B). The main theorem in [4] asserts that (1.3) is valid (universally) if and only if the families $\mathcal{I}^+$ and $\mathcal{I}^-$ of corteges S_i with signs $s_i = +$ and $s_i = -$, respectively, are *balanced*, in the sense that the total families of feasible matchings for corteges occurring in $\mathcal{I}^+$ and in $\mathcal{I}^-$ are equal.

The main result of this paper gives necessary and sufficient conditions for the quantum version (in Theorems 7.1 and 5.1). It says that (1.2) is valid (universally) if and only if the families of corteges $\mathcal{I}^+$ and $\mathcal{I}^-$ along with the function δ are *q-balanced*, which now means the existence of a bijection between the feasible matchings for $\mathcal{I}^+$ and $\mathcal{I}^-$ that is agreeable with δ in a certain sense. The proof of necessity (Theorem 7.1) considers non-q-balanced $\mathcal{I}^+, \mathcal{I}^-, \delta$ and explicitly constructs a certain graph determining a q-matrix for which (1.2) is violated when $\mathbb{K}$ is a field of characteristic 0 and q is transcendental.

The second source of our approach is the path method due to Casteels [1,2]. He associated with an $m \times n$ Cauchon diagram C of [3] a directed planar graph $G = G_C$ with $m + n$ distinguished vertices $r_1, \dots, r_m, c_1, \dots, c_n$ in which the remaining vertices correspond to white cells (i, j) in the diagram C and are labeled as t_{ij}. An example is illustrated in Fig. 1.

The labels t_{ij}, regarded as indeterminates, are assumed to (quasi)commute as

$$
\begin{aligned}
t_{ij}t_{i'j'} &= q t_{i'j'}t_{ij} \quad \text{if either } i = i' \text{ and } j < j', \text{ or } i < i' \text{ and } j = j', \\
&= t_{i'j'}t_{ij} \quad \text{otherwise}
\end{aligned}
\tag{1.4}
$$

(which is viewed “simpler” than (1.1)). The labels t_{ij} determine weights of edges and, further, weights of paths of G. The latter give rise to the *path matrix* P_G of size $m \times n$, of which (i, j)-th entry is the sum of weights of paths starting at r_i and ending at c_i.

The path matrix $P_G = (p_{ij})$ has three important properties. (i) It is a q-matrix, and therefore, $x_{ij} \mapsto p_{ij}$ gives a homomorphism of $\mathcal{R}$ to the corresponding algebra $\mathcal{R}_G$ generated by the p_{ij}. (ii) P_G admits an analog of Lindström's Lemma [11]: for any $I \subseteq [m]$ and $J \subseteq [n]$ with $|I| = |J|$, the minor $[I|J]_q$ of P_G can be expressed as the sum of weights of systems of *disjoint paths* from $\{r_1 : i \in I\}$ to $\{c_j : j \in J\}$ in G. (iii) From Cauchon's Algorithm [3] interpreted in graph terms in [1,2] it follows that: if the diagram C is maximal (i.e., has no black cells), then P_G becomes a *generic q-matrix*, see Corollary 3.2.5 in [2].

In this paper we consider a more general class of planar graphs G with horizontal and vertical edges, called *SE-graphs*, and show that they satisfy the above properties (i)–(ii) as well. Our goal is to characterize quadratic identities just for the class of path matrices of SE-graphs G. Since this class contains a generic q-matrix, the identities are automatically valid in $\mathcal{R}$.

We take an advantage from the representation of q-minors of path matrices via systems of disjoint paths, or *flows* in our terminology, and the desired results are obtained by applying a combinatorial machinery of handling flows in SE-graphs. Our method of establishing or verifying one or another identity admits a rather transparent implementation and we illustrate the method by enlightening graphical diagrams.

The paper is organized as follows. Sect. 2 contains basic definitions and backgrounds. Sect. 3 defines flows and path matrices for SE-graphs and states Lindström's type theorem for them. Sect. 4 is devoted to crucial ingredients of the method. It describes *exchange operations* on *double flows* (pairs of flows related to corteges $(I|J, I'|J')$) and expresses such operations on the language of planar matchings. The main working tool of the whole proof, stated in this section and proved in Appendix B, is Theorem 4.4 giving a q-relation between double flows before and after an ordinary exchange operation. Using this, Sect. 5 proves the sufficiency in the main result: (1.2) is valid if the corresponding $\mathcal{I}^+, \mathcal{I}^-, \delta$ are q-balanced (Theorem 5.1).

Sect. 6 is devoted to illustrations of our method. It explains how to obtain, with the help of the method, rather transparent proofs for several representative examples of quadratic identities, in particular:

(a) the pure commutation of $[I|J]_q$ and $[I'|J']_q$ when $I' \subset I$ and $J' \subset J$;

(b) a quasicommutation of *flag* q-minors $[I]_q$ and $[J]_q$ as in Leclerc–Zelevinsky's theorem [10];

(c) identities on flag q-minors involving triples $i < j < k$ and quadruples $i < j < k < \ell$;

(d) Dodgson's type identity;

(e) two general quadratic identities on flag q-minors from [9,15] occurring in descriptions of quantized Grassmannians and flag varieties.

In Sect. 7 we prove the necessity of the q-balancedness condition for validity of quadratic identities (Theorem 7.1); here we adapt a corresponding construction from [4] to obtain, in case of the non-q-balancedness, an SE-graph G such that the identity for its path matrix is false (in a special case of $\mathbb{K}$ and q). Sect. 8 poses the problem: when an identity in the commutative case, such as (1.3), can be turned, by choosing an appropriate δ, into the corresponding identity for the quantized case? For example, this is impossible for the trivial identity $[I]\,[J] = [J]\,[I]$ with usual flag minors when I, J are not weakly separated, as is shown in [10]. Also this section applies our method to obtain a relatively simple proof of Scott's result [14] on quasicommuting general (not necessarily flag) q-minors, and contains additional results.

Finally, Appendix A exhibits several auxiliary lemmas needed to us and proves the above-mentioned Lindström's type result for SE-graphs, and Appendix B gives the proof of Theorem 4.4 (which is rather technical).

2. Preliminaries

2.1. Paths in graphs

Throughout, by a *graph* we mean a directed graph. A *path* in a graph $G = (V, E)$ (with vertex set V and edge set E) is a sequence $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ such that each e_i is an edge connecting vertices v_{i-1}, v_i. An edge e_i is called *forward* if it is directed from v_{i-1} to v_i, denoted as $e_i = (v_{i-1}, v_i)$, and *backward* otherwise (when $e_i = (v_i, v_{i-1})$). The path P is called *directed* if it has no backward edge, and *simple* if all vertices v_i are different. When $k > 0$ and $v_0 = v_k$, P is called a *cycle*, and called a *simple cycle* if, in addition, $v_1, \dots, v_k$ are different. When it is not confusing, we may use for P the abbreviated notation via vertices: $P = v_0 v_1 \dots v_k$, or edges $P = e_1 e_2 \dots e_k$.

Also, using standard terminology in graph theory, for a directed edge $e = (u, v)$, we say that e *leaves* u and *enters* v, and that u is the *tail* and v is the *head* of e.

2.2. Quantum matrices

It will be convenient for us to visualize matrices in the Cartesian form: for an $m \times n$ matrix $A = (a_{ij})$, the row indices $i = 1, \dots, m$ are assumed to increase upwards, and the column indices $j = 1, \dots, n$ from left to right.

As mentioned above, we deal with the *quantized coordinate ring* $\mathcal{R} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ generated by indeterminates x_{ij} satisfying relations (1.1), called the algebra of $m \times n$ *quantum matrices*. A somewhat “simpler” object is the *quantum affine space*, the $\mathbb{K}$-algebra generated by indeterminates t_{ij} ($i \in [m]$, $j \in [n]$) subject to relations (1.4).

2.3. q-Minors

For an $m \times n$ matrix $A = (a_{ij})$, we denote by $A(I|J)$ the submatrix of A whose rows are indexed by $I \subseteq [m]$, and columns by $J \subseteq [n]$. Let $|I| = |J| =: k$, and let I consist of $i_1 < \dots < i_k$ and J consist of $j_1 < \dots < j_k$. Then the *q-determinant* of $A(I|J)$, or the *q-minor* of A for $(I|J)$, is defined as

$$[I|J]_{A,q} := \sum_{\sigma \in S_k} (-q)^{\ell(\sigma)} \prod_{d=1}^{k} a_{i_d j_{\sigma(d)}}, \tag{2.1}$$

where, in the noncommutative case, the product under $\prod$ is ordered (from left to right) by increasing d, and $\ell(\sigma)$ is the *length* (number of inversions) of a permutation σ. The terms A and/or q in $[I|J]_{A,q}$ may be omitted when they are clear from the context.

2.4. SE-graphs

A graph $G = (V, E)$ of this sort (also denoted as $(V, E; R, C)$) satisfies the following conditions:

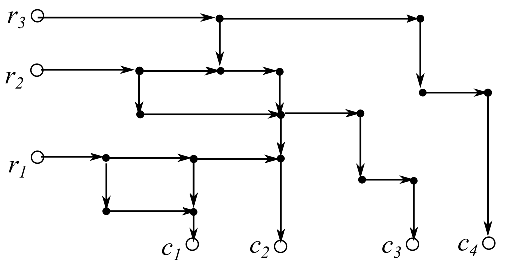
(SE1) G is planar (with a fixed layout in the plane);

(SE2) G has edges of two types: *horizontal* edges, or *H-edges*, which are directed to the right, and *vertical* edges, or *V-edges*, which are directed downwards (so each edge points to either *south* or *east*, justifying the term "SE-graph");

(SE3) G has two distinguished subsets of vertices: set $R = \{r_1, \dots, r_m\}$ of *sources* and set $C = \{c_1, \dots, c_n\}$ of *sinks*; moreover, $r_1, \dots, r_m$ are disposed on a vertical line, in this order upwards, and $c_1, \dots, c_n$ are disposed on a horizontal line, in this order from left to right; the sources (sinks) are incident only with H-edges (resp. V-edges);

(SE4) each vertex of G belongs to a directed path from R to C.

We denote by $W = W_G$ the set $V - (R \cup C)$ of *inner* vertices of G. An example of SE-graphs with $m = 3$ and $n = 4$ is drawn in the picture:



Remark 1. A special case of SE-graphs is formed by those corresponding to *Cauchon graphs* introduced in [1] (which are associated with Cauchon diagrams [3]). In this case, $R = \{(0, i) : i \in [m]\}$, $C = \{(j, 0) : j \in [n]\}$, and $W \subseteq [m] \times [n]$. (The correspondence with the definition in [1] is given by $(i, j) \mapsto (m + 1 - i, n + 1 - j)$ and $q \mapsto q^{-1}$.) When $W = [m] \times [n]$ (equivalently: when the Cauchon diagram has no black cells), we refer to such a graph as the *extended* (m, n)*-grid* and denote it by $\Gamma_{m,n}$.

We assign the weight $w(e)$ to each edge $e = (u, v) \in E$ in a way similar to that for Cauchon graphs in [1], namely:

(2.2) (i) $w(e) := v$ if $u \in R$;
(ii) $w(e) := u^{-1}v$ if e is an H-edge and $u, v \in W$;
(iii) $w(e) := 1$ if e is a V-edge.

This gives rise to defining the weight $w(P)$ of a directed path $P = e_1 e_2 \dots e_k$ (written in the edge notation) in G, to be the ordered (from left to right) product

$$w(P) = w(e_1)w(e_2) \cdots w(e_k). \tag{2.3}$$

Then $w(P)$ is a Laurent monomial in elements of W. Note that when P begins in R and ends in C, its weight can also be expressed in the following useful form; cf. [2, Prop. 3.1.8]. Let $u_1, v_1, u_2, v_2, \ldots, u_{d-1}, v_{d-1}, u_d$ be the sequence of vertices where P makes turns; namely, P changes the horizontal direction to the vertical one at each u_i, and conversely at each v_i. Then (due to the "telescopic effect" caused by (2.2)(ii)),

$$w(P) = u_1 v_1^{-1} u_2 v_2^{-1} \cdots u_{d-1} v_{d-1}^{-1} u_d. \tag{2.4}$$

We assume that the elements of W obey (quasi)commutation laws somewhat similar to those in (1.4); namely, for distinct $u, v \in W$,

(G1) if there is a directed *horizontal* path from u to v in G, then $uv = qvu$;

(G2) if there is a directed *vertical* path from u to v in G, then $vu = quv$;

(G3) otherwise $uv = vu$.

3. Path matrix and flows

As mentioned in the Introduction, it is shown in [1] that the path matrix associated with a Cauchon graph G has a nice property of Lindström's type, saying that each q-minor of this matrix corresponds to a certain set of collections of disjoint paths in G. We will show that this property is extended to the SE-graphs.

Let $G = (V, E)$ be an SE-graph with sources $R = (r_1, \ldots, r_m)$ and sinks $C = (c_1, \ldots, c_n)$, and let $w = w_G$ denote the edge weights in G defined by (2.2).

Definition. The *path matrix* $\mathrm{Path} = \mathrm{Path}_G$ associated with G is the $m \times n$ matrix whose entries are defined by

$$\mathrm{Path}(i|j) := \sum\nolimits_{P \in \Phi_G(i|j)} w(P), \qquad (i,j) \in [m] \times [n], \tag{3.1}$$

where $\Phi_G(i|j)$ is the set of directed paths from r_i to c_j in G. In particular, $\mathrm{Path}(i|j) = 0$ if $\Phi_G(i|j) = \emptyset$.

Thus, the entries of Path_G belong to the $\mathbb{K}$-algebra $\mathcal{L}_G$ of Laurent polynomials generated by the inner vertices $v \in W$ of G subject to relations (G1)–(G3).

Definition. Let $\mathcal{E}^{m,n}$ denote the set of pairs $(I|J)$ such that $I \subseteq [m]$, $J \subseteq [n]$ and $|I| = |J|$. Borrowing terminology from [4], for $(I|J) \in \mathcal{E}^{m,n}$, a set ϕ of pairwise disjoint directed paths from the source set $R_I := \{r_i : i \in I\}$ to the sink set $C_J := \{c_j : j \in J\}$ in G is called an $(I|J)$-*flow*.

The set of $(I|J)$-flows ϕ in G is denoted by $\Phi(I|J) = \Phi_G(I|J)$. We usually assume that the paths forming a flow ϕ are ordered by increasing the source indices. Namely, if I consists of $i(1) < i(2) < \cdots < i(k)$ and J consists of $j(1) < j(2) < \cdots < j(k)$, then ℓ-th path P_ℓ in ϕ begins at $r_{i(\ell)}$, and therefore, P_ℓ ends at $c_{j(\ell)}$ (which easily follows from the planarity of G, the orderings of sources and sinks in the boundary of G and the fact that the paths in ϕ are disjoint). We write $\phi = (P_1, P_2, \dots, P_k)$ and (similar to path systems in [1]) define the weight of ϕ to be the ordered product

$$w(\phi) = w(P_1)w(P_2)\cdots w(P_k). \tag{3.2}$$

Then the desired q-analog of Lindström's Lemma expresses q-minors of path matrices via flows as follows.

Theorem 3.1. *For the path matrix* Path $=$ Path$_G$ *of an* (m, n) *SE-graph* G *and for any* $(I|J) \in \mathcal{E}^{m,n}$*, there holds*

$$[I|J]_{\mathrm{Path},q} = \sum\nolimits_{\phi\in\Phi(I|J)} w(\phi). \tag{3.3}$$

A proof of this theorem, which is close to that in [1], is given in Appendix A.

An important fact is that the entries of Path$_G$ obey the (quasi)commutation relations similar to those for the canonical generators x_{ij} of the quantum algebra $\mathcal{R}$ given in (1.1). It is exhibited in the following assertion, which is known for the path matrices of Cauchon graphs due to [1] (where it is proved by use of the "Cauchon's deleting derivation algorithm in reverse" [3]).

Theorem 3.2. *For an SE-graph* G*, the entries of its path matrix* Path$_G$ *satisfy Manin's relations.*

We will show this in Sect. 6.3 as an easy application of our flow-matching method. This assertion implies that the map $x_{ij} \mapsto \mathrm{Path}_G(i|j)$ determines a homomorphism of $\mathcal{R}$ to the subalgebra $\mathcal{R}_G$ of $\mathcal{L}_G$ generated by the entries of Path$_G$, i.e., Path$_G$ is a q-matrix for any SE-graph G. In one especial case of G, a sharper result, attributed to Cauchon and Casteels, is as follows.

Theorem 3.3 *([3,2]). If* $G = \Gamma_{m,n}$ *(the extended* $m \times n$*-grid defined in Remark 1), then* Path$_G$ *is a generic q-matrix, i.e.,* $x_{ij} \mapsto \mathrm{Path}_G(i|j)$ *gives an injective map of* $\mathcal{R}$ *to* $\mathcal{L}_G$*.*

Due to this important property, the quadratic relations that are valid (universally) for q-minors of path matrices of SE-graphs turn out to be automatically valid for the algebra $\mathcal{R}$ of quantum matrices, and vice versa.

4. Double flows, matchings, and exchange operations

Quadratic identities of our interest in this paper involve products of quantum minors of the form $[I|J][I'|J']$, where $(I|J), (I'|J') \in \mathcal{E}^{m,n}$. This leads us to a proper study of ordered pairs of flows $\phi \in \Phi(I|J)$ and $\phi' \in \Phi(I'|J')$ in an SE-graph G (in light of Theorem 3.1).

We need some definitions and conventions, borrowing terminology from [4]. Given $I, J, I', J', \phi, \phi'$ as above, we call the pair (ϕ, ϕ') a *double flow* in G. Let

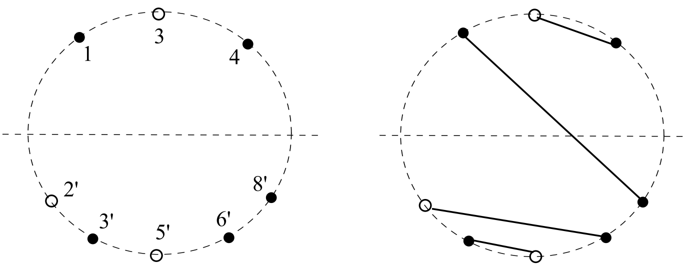
$$I^\circ := I - I', \quad J^\circ := J - J', \quad I^\bullet := I' - I, \quad J^\bullet := J' - J, \tag{4.1}$$
$$Y^{\mathrm{r}} := I^\circ \cup I^\bullet \quad \text{and} \quad Y^{\mathrm{c}} := J^\circ \cup J^\bullet.$$

Note that $|I| = |J|$ and $|I'| = |J'|$ imply that $|Y^{\mathrm{r}}| + |Y^{\mathrm{c}}|$ is even and

$$|I^\circ| - |I^\bullet| = |J^\circ| - |J^\bullet|. \tag{4.2}$$

We refer to the quadruple $(I|J, I'|J')$ as above as a *cortege*, and to $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ as the *refinement* of $(I|J, I'|J')$, or as a *refined cortege*.

It is convenient for us to interpret I° and $I^\bullet$ as the sets of *white* and *black* elements of Y^{r}, respectively, and similarly for $J^\circ, J^\bullet, Y^{\mathrm{c}}$, and visualize these objects by use of a *circular diagram* D in which the elements of Y^{r} (resp. Y^{c}) are disposed in the increasing order from left to right in the upper (resp. lower) half of a circumference O. For example if, say, $I^\circ = \{3\}$, $I^\bullet = \{1, 4\}$, $J^\circ = \{2', 5'\}$ and $J^\bullet = \{3', 6', 8'\}$, then the diagram is viewed as in the left fragment of the picture below. (Sometimes, to avoid a possible mess between elements of Y^{r} and Y^{c}, and when it leads to no confusion, we denote elements of Y^{c} with primes.)



Let M be a partition of $Y^{\mathrm{r}} \sqcup Y^{\mathrm{c}}$ into 2-element sets (recall that $A \sqcup B$ denotes the disjoint union of sets A, B). We refer to M as a *perfect matching* on $Y^{\mathrm{r}} \sqcup Y^{\mathrm{c}}$, and to its elements as *couples*. More specifically, we say that $\pi \in M$ is: an *R-couple* if $\pi \subseteq Y^{\mathrm{r}}$, a *C-couple* if $\pi \subseteq Y^{\mathrm{c}}$, and an *RC-couple* if $|\pi \cap Y^{\mathrm{r}}| = |\pi \cap Y^{\mathrm{c}}| = 1$ (as though π "connects" two sources, two sinks, and one source and one sink, respectively).

Definition. A (perfect) matching M as above is called a *feasible* matching for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ (and for $(I|J, I'|J')$) if:

(4.3) (i) for each $\pi = \{i, j\} \in M$, the elements i, j have different colors if π is an R- or C-couple, and have the same color if π is an RC-couple;

(ii) M is *planar*, in the sense that the chords connecting the couples in the circumference O are pairwise non-crossing.

The set of feasible matchings for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ is denoted by $\mathcal{M}_{I^\circ, I^\bullet, J^\circ, J^\bullet}$ and may also be denoted as $\mathcal{M}(I|J, I'|J')$. This set is nonempty unless $Y^{\mathrm{r}} \sqcup Y^{\mathrm{c}} = \emptyset$. (A proof: a feasible matching can be constructed recursively as follows. Let for definiteness $|I^\circ| \ge |I^\bullet|$. If $I^\bullet \ne \emptyset$, then choose $i \in I^\circ$ and $j \in I^\bullet$ with $|i-j|$ minimum, form the R-couple $\{i,j\}$ and delete i,j. And so on until $I^\bullet$ becomes empty. Act similarly for J° and $J^\bullet$. Eventually, in view of (4.2), we obtain $I^\bullet = J^\bullet = \emptyset$ and $|I^\circ| = |J^\circ|$. Then we form corresponding white RC-couples.)

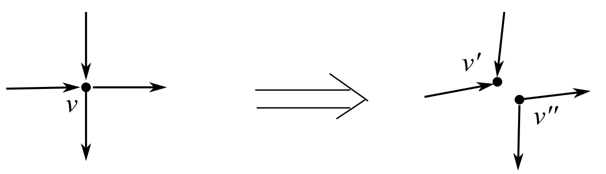
The right fragment of the above picture illustrates an instance of feasible matchings.

Return to a double flow (ϕ, ϕ') as above. Our aim is to associate to it a feasible matching for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$.

To do this, we write V_ϕ and E_ϕ, respectively, for the sets of vertices and edges of G occurring in ϕ, and similarly for ϕ'. An important role will be played by the subgraph $\langle U \rangle$ of G induced by the set of edges

$$U := E_\phi \triangle E_{\phi'}$$

(where $A \triangle B$ denotes $(A-B) \cup (B-A)$). Note that a vertex v of $\langle U \rangle$ has degree 1 if $v \in R_{I^\circ} \cup R_{I^\bullet} \cup C_{J^\circ} \cup C_{J^\bullet}$, and degree 2 or 4 otherwise. We slightly modify $\langle U \rangle$ by splitting each vertex v of degree 4 in $\langle U \rangle$ (if any) into two vertices v', v'' disposed in a small neighborhood of v so that the edges entering (resp. leaving) v become entering v' (resp. leaving v''); see the picture.



The resulting graph, denoted as $\langle U \rangle'$, is planar and has vertices of degree only 1 and 2. Therefore, $\langle U \rangle'$ consists of pairwise disjoint (non-directed) simple paths $P'_1, \dots, P'_k$ (considered up to reversing) and, possibly, simple cycles $Q'_1, \dots, Q'_d$. The corresponding images of $P'_1, \dots, P'_k$ (resp. $Q'_1, \dots, Q'_d$) give paths $P_1, \dots, P_k$ (resp. cycles $Q_1, \dots, Q_d$) in $\langle U \rangle$. When $\langle U \rangle$ has vertices of degree 4, some of the latter paths and cycles may be self-intersecting and may "touch", but not "cross", each other.

Lemma 4.1. (i) *$k = (|I^\circ| + |I^\bullet| + |J^\circ| + |J^\bullet|)/2$;*

(ii) *the set of endvertices of $P_1, \dots, P_k$ is $R_{I^\circ \cup I^\bullet} \cup C_{J^\circ \cup J^\bullet}$; moreover, each P_i connects either R_{I° and $R_{I^\bullet}$, or C_{J° and $C_{J^\bullet}$, or R_{I° and C_{J°, or $R_{I^\bullet}$ and $C_{J^\bullet}$;*

(iii) *in each path P_i, the edges of ϕ and the edges of ϕ' have different directions (say, the former edges are all forward, and the latter ones are all backward).*

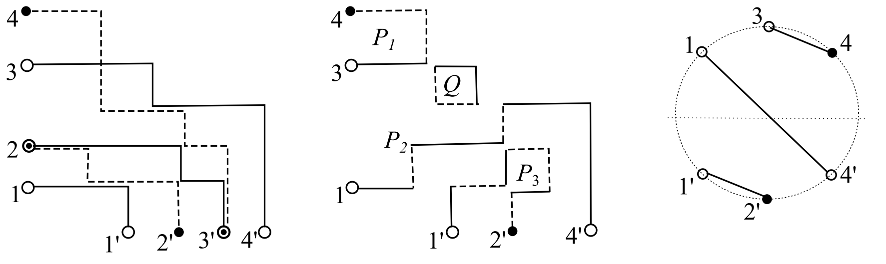


Fig. 2. Flows ϕ and ϕ' (left); $\langle E_\phi \triangle E_{\phi'} \rangle$ (middle); $M(\phi, \phi')$ (right).

Proof. (i) is trivial, and (ii) follows from (iii) and the fact that the sources r_i (resp. sinks c_j) have merely leaving (resp. entering) edges. In its turn, (iii) easily follows by considering a common inner vertex v of a directed path K in ϕ and a directed path L in ϕ'. Let e, e' (resp. u, u') be the edges of K (resp. L) incident to v. Then: if $\{e, e'\} = \{u, u'\}$, then v vanishes in $\langle U \rangle$. If $e = u$ and $e' \neq u'$, then either both e', u' enter v, or both e', u' leave v; whence e', u' are consecutive and differently directed edges of some path P_i or cycle Q_j. A similar property holds when $\{e, e'\} \cap \{u, u'\} = \emptyset$, as being a consequence of splitting v into two vertices as described. □

Thus, each P_i is represented as a concatenation $P_i^{(1)} \circ P_i^{(2)} \circ \ldots \circ P_i^{(\ell)}$ of forwardly and backwardly directed paths which are alternately contained in ϕ and ϕ', called the *segments* of P_i. We refer to P_i as an *exchange path* (by a reason that will be clear later). The endvertices of P_i determine, in a natural way, a pair of elements of $Y^{\mathrm{r}} \sqcup Y^{\mathrm{c}}$, denoted by π_i. Then $M := \{\pi_1, \ldots, \pi_k\}$ is a perfect matching on $Y^{\mathrm{r}} \sqcup Y^{\mathrm{c}}$. Moreover, it is a feasible matching, since (4.3)(i) follows from Lemma 4.1(ii), and (4.3)(ii) is provided by the fact that $P'_1, \ldots, P'_k$ are pairwise disjoint simple paths in $\langle U \rangle'$.

We denote M as $M(\phi, \phi')$, and for $\pi \in M$, denote the exchange path P_i corresponding to π (i.e., $\pi = \pi_i$) by $P(\pi)$.

Corollary 4.2. $M(\phi, \phi') \in \mathcal{M}_{I^\circ, I^\bullet, J^\circ, J^\bullet}$.

Fig. 2 illustrates an instance of (ϕ, ϕ') for $I = \{1, 2, 3\}$, $J = \{1', 3', 4'\}$, $I' = \{2, 4\}$, $J' = \{2', 3'\}$. Here ϕ and ϕ' are drawn by solid and dotted lines, respectively (in the left fragment), the subgraph $\langle E_\phi \triangle E_{\phi'} \rangle$ consists of three paths and one cycle (in the middle), and the circular diagram illustrates $M(\phi, \phi')$ (in the right fragment).

Flow exchange operation. It rearranges a given double flow (ϕ, ϕ') for $(I|J, I'|J')$ into another double flow (ψ, ψ') for some cortege $(\widetilde{I}|\widetilde{J}, \widetilde{I}'|\widetilde{J}')$, as follows. Fix a submatching $\Pi \subseteq M(\phi, \phi')$, and combine the exchange paths concerning Π, forming the set of edges

$$\mathcal{E} := \cup(E_{P(\pi)} \colon \pi \in \Pi)$$

(where E_P denotes the set of edges in a path P).

Lemma 4.3. *Let $V_\Pi := \cup(\pi \in \Pi)$. Define*

$$\widetilde{I} := I\triangle(V_\Pi \cap Y^{\rm r}),\ \ \widetilde{I}' := I'\triangle(V_\Pi \cap Y^{\rm r}),\ \ \widetilde{J} := J\triangle(V_\Pi \cap Y^{\rm c}),\ \ \widetilde{J}' := J'\triangle(V_\Pi \cap Y^{\rm c}).$$

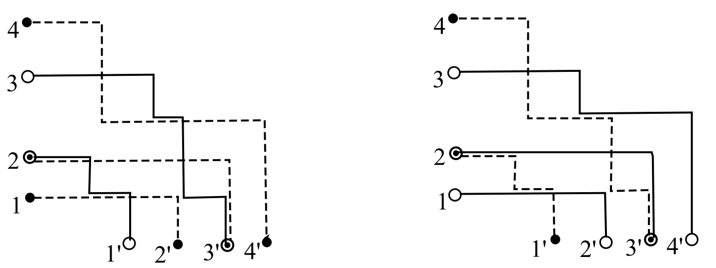
Then the subgraph ψ induced by $E_\phi\triangle\mathcal{E}$ gives a $(\widetilde{I}|\widetilde{J})$-flow, and the subgraph ψ' induced by $E_{\phi'}\triangle\mathcal{E}$ gives a $(\widetilde{I}'|\widetilde{J}')$-flow in G. Furthermore, $E_\psi \cup E_{\psi'} = E_\phi \cup E_{\phi'}$, $E_\psi\triangle E_{\psi'} = E_\phi\triangle E_{\phi'}$ $(= U)$, and $M(\psi,\psi') = M(\phi,\phi')$.

Proof. Consider a path $P = P(\pi)$ for $\pi \in \Pi$, and let P consist of segments $P^{(1)}, P^{(2)}, \dots, P^{(\ell)}$. Let for definiteness the segments $P^{(d)}$ with d odd concern ϕ, and denote by v_d the common endvertex of $P^{(d)}$ and $P^{(d+1)}$. Under the operation $E_\phi \mapsto E_\phi\triangle E_P$ the pieces $P^{(1)}, P^{(3)}, \dots$ in ϕ are replaced by $P^{(2)}, P^{(4)}, \dots$. In its turn, $E_{\phi'} \mapsto E_{\phi'}\triangle E_P$ replaces the pieces $P^{(2)}, P^{(4)}, \dots$ in ϕ' by $P^{(1)}, P^{(3)}, \dots$.

By Lemma 4.1(iii), for each d, the edges of $P^{(d)}, P^{(d+1)}$ incident to v_d either both enter or both leave v_d. Also each intermediate vertex of any segment $P^{(d)}$ occurs in exactly one flow among ϕ, ϕ'. These facts imply that under the above operations with P the flow ϕ (resp. ϕ') is transformed into a set of pairwise disjoint directed paths (a flow) going from $R_{I\triangle(\pi\cap Y^{\rm r})}$ to $C_{J\triangle(\pi\cap Y^{\rm c})}$ (resp. from $R_{I'\triangle(\pi\cap Y^{\rm r})}$ to $C_{J'\triangle(\pi\cap Y^{\rm c})}$).

Doing so for all $P(\pi)$ with $\pi \in \Pi$, we obtain flows ψ, ψ' from $R_{\widetilde{I}}$ to $C_{\widetilde{J}}$ and from $R_{\widetilde{I}'}$ to $C_{\widetilde{J}'}$, respectively. The equalities in the last sentence of the lemma are easy. □

We call the transformation $(\phi,\phi') \overset{\Pi}{\longmapsto} (\psi,\psi')$ in this lemma the *flow exchange operation* for (ϕ,ϕ') using $\Pi \subseteq M(\phi,\phi')$ (or using $\{P(\pi) \colon \pi \in \Pi\}$). Clearly the exchange operation applied to (ψ,ψ') using the same Π returns (ϕ,ϕ'). The picture below illustrates flows ψ,ψ' obtained from ϕ,ϕ' in Fig. 2 by the exchange operations using the single path P_2 (left) and the single path P_3 (right).



So far our description has been close to that given for the commutative case in [4]. From now on we will essentially deal with the quantum version. The next theorem will serve the main working tool in our arguments; its proof appealing to a combinatorial techniques on paths and flows is given in Appendix B.

Theorem 4.4. *Let ϕ be an $(I|J)$-flow, and ϕ' an $(I'|J')$-flow in G. Let (ψ,ψ') be the double flow obtained from (ϕ,ϕ') by the flow exchange operation using a single couple $\pi = \{f,g\} \in M(\phi,\phi')$. Then:*

(i) *when* π *is an* R- *or* C-*couple and* $f < g$,

$$w(\phi)w(\phi') = qw(\psi)w(\psi') \quad \textit{in case} \quad f \in I \cup J;$$
$$w(\phi)w(\phi') = q^{-1}w(\psi)w(\psi') \quad \textit{in case} \quad f \in I' \cup J';$$

(ii) *when* π *is an* RC-*couple,* $w(\phi)w(\phi') = w(\psi)w(\psi')$.

An immediate consequence from this theorem is the following

Corollary 4.5. *For an* $(I|J)$-*flow* ϕ *and an* $(I'|J')$-*flow* ϕ', *let* (ψ, ψ') *be obtained from* (ϕ, ϕ') *by the flow exchange operation using a set* $\Pi \subseteq M(\phi, \phi')$. *Then*

$$w(\phi)w(\phi') = q^{\zeta^\circ - \zeta^\bullet} w(\psi)w(\psi'), \tag{4.4}$$

where $\zeta^\circ = \zeta^\circ(I|J, I'|J'; \Pi)$ *(resp.* $\zeta^\bullet = \zeta^\bullet(I|J, I'|J'; \Pi)$*) is the number of* R- *or* C-*couples* $\pi = \{f, g\} \in \Pi$ *such that* $f < g$ *and* $f \in I \cup J$ *(resp.* $f \in I' \cup J'$*).*

Indeed, the flow exchange operation using the whole Π reduces to performing, step by step, the exchange operations using single couples $\pi \in \Pi$ (taking into account that for any current double flow (η, η') occurring in the process, the sets $E_\eta \cup E_{\eta'}$ and $E_\eta \triangle E_{\eta'}$, as well as the matching $M(\eta, \eta')$, do not change; cf. Lemma 4.3). Then (4.4) follows from Theorem 4.4.

5. Quadratic relations

As before, we consider an SE-graph $G = (V, E; R, C)$ and the weight function w which is initially defined on the edges of G by (2.2) and then extended to paths and flows according to (2.3) and (3.2). This gives rise to the q-minor function $[I|J]_q$ on the set $\mathcal{E}^{m,n} = \{(I|J) \colon I \subseteq [m],\ J \in [n],\ |I| = |J|\}$. In this section, based on Corollary 4.5 describing the transformation of the weights of double flows under the exchange operation, and developing a q-version of the flow-matching method elaborated for the commutative case in [4], we establish sufficient conditions on quadratic relations for q-minors of the matrix Path_G, to be valid independently of G (and some other objects, see Remark 2 below). Relations of our interest are of the form

$$\sum\nolimits_{\mathcal{I}} q^{\alpha(I|J, I'|J')}[I|J][I'|J'] = \sum\nolimits_{\mathcal{K}} q^{\beta(K|L, K'|L')}[K|L][K'|L'], \tag{5.1}$$

where α, β are integer-valued, $\mathcal{I}$ is a family of corteges $(I|J, I'|J') \in \mathcal{E}^{m,n} \times \mathcal{E}^{m,n}$ (with possible multiplicities), and similarly for $\mathcal{K}$. Cf. (1.2). We usually assume that $\mathcal{I}$ and $\mathcal{K}$ are *homogeneous*, in the sense that for any $(I|J, I'|J') \in \mathcal{I}$ and $(K|L, K'|L') \in \mathcal{K}$,

$$I \cup I' = K \cup K', \quad J \cup J' = L \cup L', \quad I \cap I' = K \cap K', \quad J \cap J' = L \cap L'. \tag{5.2}$$

Moreover, we shall see that only the refinements $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ and $(K^\circ, K^\bullet, L^\circ, L^\bullet)$ are important, whereas the sets $I \cap I'$ and $J \cap J'$ are, in fact, indifferent. (As before, I° means $I - I'$, $I^\bullet$ means $I' - I$, and so on.)

To formulate the validity conditions, we need some definitions and notation.

• We say that a tuple $(I|J, I'|J'; M)$, where $(I|J, I'|J') \in \mathcal{I}$ and $M \in \mathcal{M}_{I^\circ, I^\bullet, J^\circ, J^\bullet}$ (cf. (4.3)), is a *configuration* for $\mathcal{I}$. The family of all configurations for $\mathcal{I}$ is denoted by $\mathbf{C}(\mathcal{I})$. Similarly, we define the family $\mathbf{C}(\mathcal{K})$ of configurations for $\mathcal{K}$.

• Define $\mathbf{M}(\mathcal{I})$ to be the family of all matchings M occurring in the members of $\mathbf{C}(\mathcal{I})$, respecting multiplicities (i.e., $\mathbf{M}(\mathcal{I})$ is a multiset). Define $\mathbf{M}(\mathcal{K})$ similarly.

Definition. Families $\mathcal{I}$ and $\mathcal{K}$ are called *balanced* (borrowing terminology from [4]) if there exists a bijection $(I|J, I'|J'; M) \overset{\gamma}{\longmapsto} (K|K', L|L'; M')$ between $\mathbf{C}(\mathcal{I})$ and $\mathbf{C}(\mathcal{K})$ such that $M = M'$. In other words, $\mathcal{I}$ and $\mathcal{K}$ are balanced if $\mathbf{M}(\mathcal{I}) = \mathbf{M}(\mathcal{K})$.

Definition. We say that families $\mathcal{I}$ and $\mathcal{K}$ along with functions $\alpha : \mathcal{I} \to \mathbb{Z}$ and $\beta : \mathcal{K} \to \mathbb{Z}$ are *q-balanced* if there exists a bijection γ as above such that, for each $(I|J, I'|J'; M) \in \mathbf{C}(\mathcal{I})$ and for $(K|K', L|L'; M) = \gamma(I|J, I'|J'; M)$, there holds

$$\beta(K|K', L|L') - \alpha(I|J, I'|J') = \zeta^\circ - \zeta^\bullet. \tag{5.3}$$

(In particular, $\mathcal{I}, \mathcal{K}$ are balanced.) Here $\zeta^\circ, \zeta^\bullet$ are defined according to Corollary 4.5. Namely, $\zeta^\circ = \zeta^\circ(I|J, I'|J'; \Pi)$ and $\zeta^\bullet = \zeta^\bullet(I|J, I'|J'; \Pi)$, where Π is the set of couples $\pi \in M$ such that the white/black colors of the elements of π in the refined corteges $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ and $(K^\circ, K^\bullet, L^\circ, L^\bullet)$ are different. (Then ζ° ($\zeta^\bullet$) is the number of R- and C-couples $\{f, g\} \in \Pi$ with $f < g$ and $f \in I^\circ \cup J^\circ$ (resp. $f \in I^\bullet \cup J^\bullet$).) We say that $(K^\circ, K^\bullet, L^\circ, L^\bullet)$ is obtained from $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ by the *index exchange operation* using Π, and may write $\zeta^\circ(I^\circ, I^\bullet, J^\circ, J^\bullet; \Pi)$ for ζ°, and $\zeta^\bullet(I^\circ, I^\bullet, J^\circ, J^\bullet; \Pi)$ for $\zeta^\bullet$.

Theorem 5.1. *Let $\mathcal{I}$ and $\mathcal{K}$ be homogeneous families on $\mathcal{E}^{m,n} \times \mathcal{E}^{m,n}$, and let $\alpha : \mathcal{I} \to \mathbb{Z}$ and $\beta : \mathcal{K} \to \mathbb{Z}$. Suppose that $\mathcal{I}, \mathcal{K}, \alpha, \beta$ are q-balanced. Then for any SE-graph $G = (V, E; R, C)$, relation (5.1) is valid for q-minors of* Path_G.

Proof. It is close to the proof for the commutative case in [4, Proposition 3.2].

We fix G and denote by $\mathcal{D}(I|J, I'|J')$ the set of double flows for $(I|J, I'|J') \in \mathcal{I} \cup \mathcal{K}$ in G. A summand concerning $(I|J, J'|J') \in \mathcal{I}$ in the L.H.S. of (5.1) can be expressed via double flows as follows, ignoring the factor of $q^{\alpha(\cdot)}$:

$$\begin{aligned}
[I|J][I'|J'] &= \Big(\sum\nolimits_{\phi \in \Phi_G(I|J)} w(\phi)\Big) \times \Big(\sum\nolimits_{\phi' \in \Phi_G(I'|J')} w(\phi')\Big) \\
&= \sum\nolimits_{(\phi,\phi') \in \mathcal{D}(I|J, I'|J')} w(\phi) w(\phi') \\
&= \sum\nolimits_{M \in \mathcal{M}_{I^\circ, I^\bullet, J^\circ, J^\bullet}} \sum\nolimits_{(\phi,\phi') \in \mathcal{D}(I|J, I'|J') \,:\, M(\phi,\phi') = M} w(\phi) w(\phi').
\end{aligned} \tag{5.4}$$

The summand for $(K|L, K'|L') \in \mathcal{K}$ in the R.H.S. of (5.1) is expressed similarly.

Consider a configuration $S = (I|J, I'|J'; M) \in \mathbf{C}(\mathcal{I})$ and suppose that (ϕ, ϕ') is a double flow for $(I|J, I'|J')$ with $M(\phi, \phi') = M$ (if such a double flow in G exists). Since $\mathcal{I}, \mathcal{K}, \alpha, \beta$ are q-balanced, S is bijective to some configuration $S' = (K|L, K'|L'; M) \in \mathbf{C}(\mathcal{K})$ satisfying (5.3). As explained earlier, the cortege $(K|L, K'|L')$ is obtained from $(I|J, I'|J')$ by the index exchange operation using some $\Pi \subseteq M$. Then the flow exchange operation applied to (ϕ, ϕ') using this Π results in a double flow (ψ, ψ') for $(K|L, K'|L')$ which satisfies relation (4.4) in Corollary 4.5. Comparing (4.4) with (5.3), we observe that

$$q^{\alpha(I|J,I'|J')} w(\phi) w(\phi') = q^{\beta(K|K',L|L')} w(\psi) w(\psi').$$

Furthermore, such a map $(\phi, \phi') \mapsto (\psi, \psi')$ gives a bijection between all double flows concerning configurations in $\mathbf{C}(\mathcal{I})$ and those in $\mathbf{C}(\mathcal{K})$. Now the desired equality (5.1) follows by comparing the last term in expression (5.4) and the corresponding term in the analogous expression concerning $\mathcal{K}$. $\square$

As a consequence of Theorems 3.3 and 5.1, the following result is obtained.

Corollary 5.2. *If $\mathcal{I}, \mathcal{K}, \alpha, \beta$ as above are q-balanced, then relation (5.1) is valid for the corresponding minors in the algebra $\mathcal{R}$ of quantum $m \times n$ matrices.*

Remark 2. When speaking of a *universal quadratic identity* of the form (5.1) with homogeneous $\mathcal{I}$ and $\mathcal{K}$, abbreviated as a *UQ identity*, we mean that it depends neither on the graph G nor on the field $\mathbb{K}$ and element $q \in \mathbb{K}^*$, and that the index sets can be modified as follows. Given $(I|J, I'|J') \in \mathcal{I}$, let $A := I \triangle I'$, $B := J \triangle J'$, $S := I \cap I'$ and $T := J \cap J'$ (by the homogeneity, these sets do not depend on $(I|J, I'|J) \in \mathcal{I} \cup \mathcal{K}$). Take arbitrary $\widetilde{m} \geq |A|$ and $\widetilde{n} \geq |B|$ and replace A, B, S, T by disjoint sets $\widetilde{A}, \widetilde{S} \subseteq [\widetilde{m}]$ and disjoint sets $\widetilde{B}, \widetilde{T} \subseteq [\widetilde{n}]$ such that $|\widetilde{A}| = |A|$, $|\widetilde{B}| = |B|$ and $|\widetilde{S}| - |\widetilde{T}| = |S| - |T|$. Let $\nu : A \to \widetilde{A}$ and $\mu : B \to \widetilde{B}$ be the order preserving maps. Transform each $(I|J, I'|J') \in \mathcal{I}$ into $(\widetilde{I}|\widetilde{J}, \widetilde{I}'|\widetilde{J}')$, where

$$\widetilde{I} := \widetilde{S} \cup \nu(I - S), \quad \widetilde{I}' := \widetilde{S} \cup \nu(I' - S), \quad \widetilde{J} := \widetilde{T} \cup \mu(J - T), \quad \widetilde{J}' := \widetilde{T} \cup \mu(J' - T),$$

forming a new family $\widetilde{\mathcal{I}}$ on $\mathcal{E}^{\widetilde{m},\widetilde{n}} \times \mathcal{E}^{\widetilde{m},\widetilde{n}}$. Transform $\mathcal{K}$ into $\widetilde{K}$ in a similar way. One can see that if $\mathcal{I}, \mathcal{K}, \alpha, \beta$ are q-balanced, then so are $\widetilde{\mathcal{I}}, \widetilde{\mathcal{K}}$, keeping α, β. Therefore, if (5.1) is valid for $\mathcal{I}, \mathcal{K}$, then it is valid for $\widetilde{\mathcal{I}}, \widetilde{\mathcal{K}}$ as well.

Thus, the condition of q-balancedness is sufficient for validity of relation (5.1) for minors of any q-matrix. In Sect. 7 we shall see that this condition is necessary as well (Theorem 7.1).

One can say that identity (5.1), where all summands have positive signs, is written in the *canonical form*. Sometimes, however, it is more convenient to consider equivalent identities having negative summands in one or both sides (e.g. of the form (1.2)). Also one may simultaneously multiply all summands in (5.1) by the same degree of q.

Remark 3. A useful fact is that once we are given an instance of (5.1), we can form another identity by changing the white/black coloring in all refined corteges. More precisely, for a cortege $S = (I|J, I'|J')$, let us say that the cortege $S^{\rm rev} := (I'|J', I|J)$ is *reversed* to S. Given a family $\mathcal{I}$ of corteges, the *reversed* family $\mathcal{I}^{\rm rev}$ is formed by the corteges reversed to those in $\mathcal{I}$. Then the following property takes place.

Proposition 5.3. *Suppose that $\mathcal{I}, \mathcal{K}, \alpha, \beta$ are q-balanced. Then $\mathcal{I}^{\rm rev}, \mathcal{K}^{\rm rev}, -\alpha, -\beta$ are q-balanced as well. Therefore (by Theorem 5.1),*

$$\sum_{(I|J,I'|J')\in\mathcal{I}} q^{-\alpha(I|J,I'|J')}[I'|J'][I|J] = \sum_{(K|L,K'|L')\in\mathcal{K}} q^{-\beta(K|L,K'|L')}[K'|L'][K|L]. \tag{5.5}$$

Proof. Let $\gamma : \mathbf{C}(\mathcal{I}) \to \mathbf{C}(\mathcal{K})$ be a bijection in the definition of q-balancedness. Then γ induces a bijection of $\mathbf{C}(\mathcal{I}^{\rm rev})$ to $\mathbf{C}(\mathcal{K}^{\rm rev})$ (also denoted as γ). Namely, if $\gamma(S; M) = (T; M)$ for $S = (I|J, I'|J') \in \mathcal{I}$ and $T = (K|L, K'|L') \in \mathcal{K}$, then we define $\gamma(S^{\rm rev}; M) := (T^{\rm rev}; M)$. When coming from S to $S^{\rm rev}$, each R- or C-couple $\{i, j\}$ in M changes the colors of both elements i, j. This leads to swapping ζ° and $\zeta^\bullet$, i.e., $\zeta^\circ(S^{\rm rev}; \Pi) = \zeta^\bullet(S; \Pi)$ and $\zeta^\bullet(S^{\rm rev}; \Pi) = \zeta^\circ(S; \Pi)$ (where Π is the submatching in M involved in the exchange operation). Now (5.5) follows from relation (5.3). □

Another useful equivalent transformation is given by swapping row and column indices. Namely, for a cortege $S = (I|J, I'|J')$, the *transposed* cortege is $S^\top := (J|I, J'|I')$, and the family $\mathcal{I}^\top$ *transposed* to $\mathcal{I}$ consists of the corteges $S^\top$ for $S \in \mathcal{I}$, and similarly for $\mathcal{K}$. One can see that the corresponding values ζ° and $\zeta^\bullet$ preserve when coming from $\mathcal{I}$ to $\mathcal{I}^\top$ and from $\mathcal{K}$ to $\mathcal{K}^\top$, and therefore (5.3) implies the identity

$$\sum_{(I|J,I'|J')\in\mathcal{I}} q^{\alpha(I|J,I'|J')}[J|I][J'|I'] = \sum_{(K|L,K'|L')\in\mathcal{K}} q^{\beta(K|L,K'|L')}[L|K][L'|K']. \tag{5.6}$$

(Note also that (5.6) immediately follows from the known fact that any q-minor satisfies the symmetry relation $[J|I]_q = [J|I]_q$.)

We conclude this section with a rather simple algorithm which has as the input a corresponding quadruple $\mathcal{I}, \mathcal{K}, \alpha, \beta$ and recognizes the q-balanced for it. Therefore, in light of Theorems 5.1 and 7.1, the algorithm decides whether or not the given quadruple determines a UQ identity of the form (5.1).

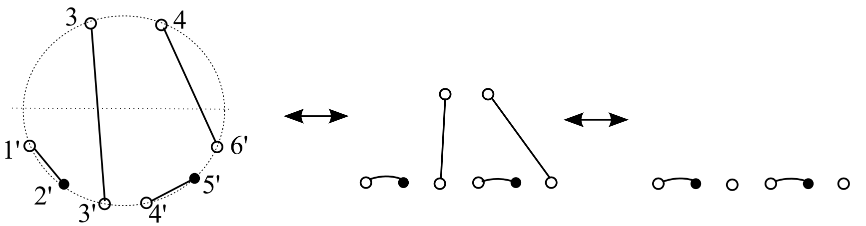
Algorithm. Compute the set $\mathcal{M}_{I^\circ,I^\bullet,J^\circ,J^\bullet}$ of feasible matchings M for each $(I|J, I'|J')$ $\in \mathcal{I}$, and similarly for $\mathcal{K}$. For each instance M occurring there, we extract the family $\mathbf{C}_M(\mathcal{I})$ of all configurations concerning M in $\mathbf{C}(\mathcal{I})$, and extract a similar family $\mathbf{C}_M(\mathcal{K})$ in $\mathbf{C}(\mathcal{K})$. If $|\mathbf{C}_M(\mathcal{I})| \neq |\mathbf{C}_M(\mathcal{K})|$ for at least one instance M, then $\mathcal{I}$ and

$\mathcal{K}$ are not balanced at all. Otherwise for each M, we seek for a required bijection $\gamma_M : \mathbf{C}_M(\mathcal{I}) \to \mathbf{C}_M(\mathcal{K})$ by solving the maximum matching problem in the corresponding bipartite graph H_M. More precisely, the vertices of H_M are the tuples $(I|J, I'|J'; M)$ and $(K|L, K'|L'; M)$ occurring in $\mathbf{C}_M(\mathcal{I})$ and $\mathbf{C}_M(\mathcal{K})$, and such tuples are connected by edge in H_M if they obey (5.3). Find a maximum matching N in H_M. (There are many fast algorithms to solve this classical problem; for a survey, see, e.g. [13].) If $|N| = |\mathbf{C}_M(\mathcal{I})|$, then N determines the desired γ_M in a natural way. Taking together, these γ_M give a bijection between $\mathbf{C}(\mathcal{I})$ and $\mathbf{C}(\mathcal{K})$ as required, implying that $\mathcal{I}, \mathcal{K}, \alpha, \beta$ are q-balanced. And if $|N| < |\mathbf{C}_M(\mathcal{I})|$ for at least one instance M, then the algorithm declares the non-q-balancedness.

6. Examples of universal quadratic identities

The flow-matching method described above is well adjusted to prove, relatively easily, classical or less known quadratic identities. In this section we give a number of appealing illustrations.

Instead of circular diagrams as in Sect. 4, we will use more compact, but equivalent, *two-level diagrams*. Also when dealing with a flag pair $(I|J)$, i.e., when I consists of the elements $1, 2, \dots, |J|$, we may use an appropriate *one-level diagram*, which leads to no loss of generality. For example, the refined cortege ($I^\circ = \{3, 4\}$, $I^\bullet = \emptyset$, $J^\circ = \{1', 3', 4', 6'\}$, $J^\bullet = \{2', 5'\}$) with the feasible matching $\{1'2', 4'5', 33', 46'\}$ can be visualized in three possible ways as:



A couple $\{i, j\}$ may be denoted as ij. Also for brevity we write $Xi \dots j$ for $X \cup \{i, \dots, j\}$, where X and $\{i, \dots, j\}$ are disjoint.

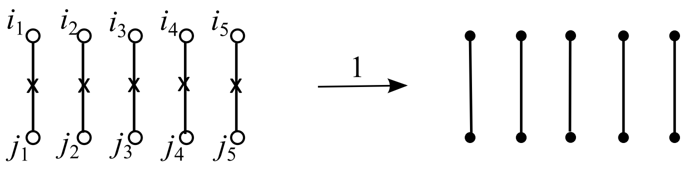
As before, we use notation $[I|J]$ for the corresponding q-minor of the path matrix Path_G (defined in Sect. 3). In the flag case $[I|J]$ is usually abbreviated to $[J]$.

6.1. Commuting minors

We start with a simple illustration of our method by showing that q-minors $[I|J]$ and $[I'|J']$ "purely" commute when $I' \subset I$ and $J' \subset J$. (This matches the known fact that a minor of a q-matrix commutes with any of its subminors, or that the q-determinant of a square q-matrix is a central element of the corresponding algebra.)

Let $I^\circ = I - I'$ consist of $i_1 < \ldots < i_k$, and $J^\circ = J - J'$ consist of $j_1 < \ldots < j_k$. Since $I^\bullet = I' - I = \emptyset$ and $J^\bullet = J' - J = \emptyset$, there is only one feasible matching M for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$; namely, the one formed by the RC-couples $\pi_\ell = i_\ell j_\ell$, $\ell = 1, \ldots, k$. The index exchange operation applied to $(I|J, I'|J')$ using the whole M produces the cortege $(K|L, K'|L')$ for which $K^\circ = I^\bullet = \emptyset$, $K^\bullet = I^\circ$, $L^\circ = J^\bullet = \emptyset$, $L^\bullet = J^\circ$ (and $K \cap K' = I \cap I'$, $L \cap L' = J \cap J'$). Since M consists of RC-couples only, we have $\zeta^\circ(I^\circ, I^\bullet, J^\circ, J^\bullet; M) = \zeta^\bullet(I^\circ, I^\bullet, J^\circ, J^\bullet; M) = 0$. So the (one-element) families $\mathcal{I} = \{(I|J, I'|J')\}$ and $\mathcal{K} = \{(K|L, K'|L')\}$ along with $\alpha = \beta = 0$ are q-balanced, and Theorem 5.1 gives the desired equality $[I|J][I'|J'] = [I'|J'][I|J]$.

This is illustrated in the picture with two-level diagrams (in case $k = 5$). Hereinafter we indicate by crosses the couples that are involved in the applied index exchange operation (i.e., the couples where the colors of elements are changed).



6.2. Quasicommuting minors

Recall that two sets $I, J \subseteq [n]$ are called *weakly separated* if, up to renaming I and J, there holds: $|I| \geq |J|$, and $J - I$ has a partition $J_1 \cup J_2$ such that $J_1 < I - J < J_2$ (where we write $X < Y$ if $x < y$ for any $x \in X$ and $y \in Y$). Leclerc and Zelevinsky proved the following

Theorem 6.1 *([10]). Two flag minors* $[I]$ *and* $[J]$ *of a quantum matrix* quasicommute, *i.e., satisfy*

$$[I][J] = q^c[J][I] \tag{6.1}$$

for some $c \in \mathbb{Z}$*, if and only if the column sets* I, J *are weakly separated. Moreover, when* $|I| \geq |J|$ *and* $J_1 \cup J_2$ *is a partition of* $J - I$ *with* $J_1 < I - J < J_2$*, the number* c *in (6.1) is equal to* $|J_2| - |J_1|$.

(In case $I \cap J = \emptyset$, "if" part is due to Krob and Leclerc [8]). We explain how to obtain "if" part of Theorem 6.1 by use of the flow-matching method.

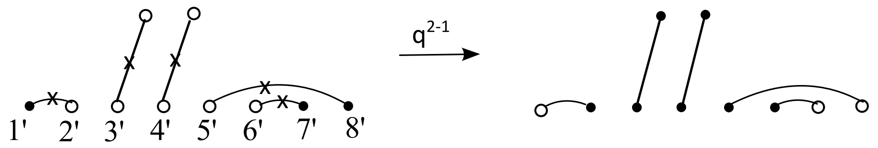
Let $A := \{1, \ldots, |I|\}$, $B := \{1, \ldots, |J|\}$, and (assuming $|I| \geq |J|$) define

$$A^\circ := A - B, \quad B^\bullet := B - A \; (= \emptyset), \quad I^\circ := I - J, \quad J^\bullet := J - I.$$

One can see that $(A^\circ, B^\bullet, I^\circ, J^\bullet)$ has exactly one feasible matching M; namely, J_1 is coupled with the first $|J_1|$ elements of I°, J_2 is coupled with the last $|J_2|$ elements of I° (forming all C-couples), and the rest of I° is coupled with A° (forming all RC-couples).

Observe that the index exchange operation applied to $(A|I, B|J)$ using the whole M swaps $A|I$ and $B|J$ (since it changes the colors of all elements in A°, I°, $J^\bullet$). Also M consists of $|J_1| + |J_2|$ C-couples and $|A^\circ|$ RC-couples. Moreover, the C-couples are partitioned into $|J_1|$ couples ij with $i < j$ and $i \in J_1$, and $|J_2|$ couples ij with $i < j$ and $j \in J_2$. This gives $\zeta^\circ = |J_2|$ and $\zeta^\bullet = |J_1|$. Hence the (one-element) families $\{(A|I, B|J)\}$ and $\{(B|J, A|I)\}$ along with $\alpha(A|I, B|J) = 0$ and $\beta(B|J, A|I) = |J_2| - |J_1|$ are q-balanced. Now Theorem 5.1 implies (6.1) with $c = |J_2| - |J_1|$.

The picture with two-level diagrams illustrates the case $|I - J| = 5$, $|J - I| = 3$, $|J_1| = 1$ and $|J_2| = 2$.

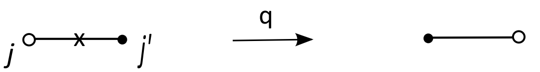


"Only if" part of Theorem 6.1 will be discussed in Sect. 8. Also we will discuss there a generalization of this theorem that characterizes the pairs of quasicommuting general q-minors.

6.3. *Manin's relations in path matrices*

Next we prove Theorem 3.2.

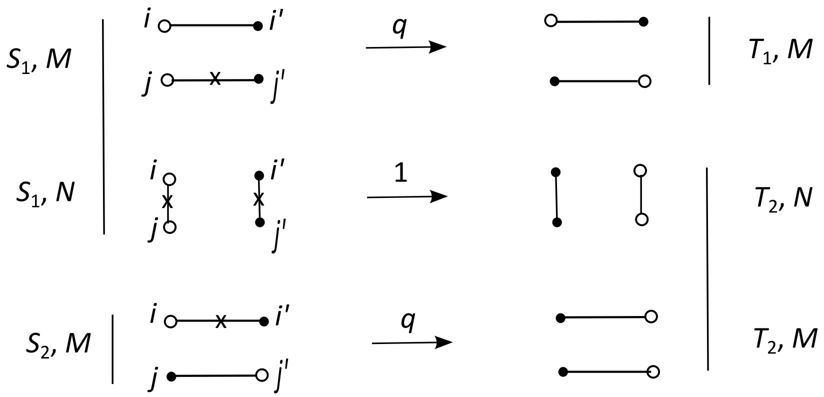
(a) Consider entries $[i|j]$ and $[i|j']$ with $j < j'$ in Path_G. The cortege $S = (i|j, i|j')$ admits a unique feasible matching; it consists of the single C-couple $\pi = jj'$. The index exchange operation using π transforms S into $T = (i|j', i|j)$; see the picture with one-level diagrams:



We observe that $\{S\}$ and $\{T\}$ along with $\alpha = 0$ and $\beta = 1$ $(= \zeta^\circ - \zeta^\bullet)$ are q-balanced, and Theorem 5.1 yields $[i|j][i|j'] = q[i|j'][i|j]$, as required.

(b) For a 2×1 submatrix of Path_G, the argument is similar.

(c) Consider a 2×2 submatrix $\binom{c\ d}{a\ b}$ of Path_G, where $a = [i|j]$, $b = [i|j']$, $c = [i'|j]$, $d = [i'|j']$ (then $i < i'$ and $j < j'$). Let $\mathcal{I}$ consist of two corteges $S_1 = (i|j, i'|j')$, $S_2 = (i|j', i'|j)$, and $\mathcal{K}$ consist of two corteges $T_1 = (i|j', i'|j)$, $T_2 = (i'|j', i|j)$ (note that $S_2 = T_1$). Observe that S_1 admits two feasible matchings, namely, $M = \{ii', jj'\}$ and $N = \{ij, i'j'\}$, while S_2 admits only one feasible matching M. In their turn, $\mathcal{M}(T_1) = \{M\}$ and $\mathcal{M}(T_2) = \{M, N\}$. Hence we can form the bijection between $\mathbf{C}(\mathcal{I})$ and $\mathbf{C}(\mathcal{K})$ that sends $(S_1; M)$ to $(T_1; M)$, (S_1, N) to $(T_2; N)$, and (S_2, M) to $(T_2; M)$. This bijection is illustrated in the picture (where, as before, we indicate the submathings involved in the exchange operations with crosses).



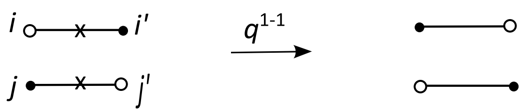
Assign $\alpha(S_1)=0$, $\alpha(S_2)=-1$, $\beta(T_1)=1$ and $\beta(T_2)=0$.

One can observe from the above diagrams that $\mathcal{I},\mathcal{K},\alpha,\beta$ are q-balanced. We obtain

$$[i|j][i'|j']+q^{-1}[i|j'][i'|j]=q[i|j'][i'|j]+[i'|j'][i|j],$$

yielding $ad-da=(q-q^{-1})bc$, as required.

Finally, to see $bc=cb$, take the 1-element families $\{S'=(i|j',i'|j)\}$ and $\{T'=(i'|j,i|j')\}$; then $\{ii',jj'\}$ is the only feasible matching for each of S',T'. The above families along with $\alpha=\beta=0$ are q-balanced, as is seen from the picture:



This gives $[i|j'][i'|j]=[i'|j][i|j']$, or $bc=cb$, as required.

6.4. Relations with triples and quadruples

In the commutative case (when dealing with the commutative coordinate ring of $m\times n$ matrices over a field), the simplest examples of quadratic identities on flag minors are presented by the classical Plücker relations involving 3- and 4-element sets of columns. More precisely, for $A\subseteq[n]$, let $g(A)$ denote the flag minor with the set A of columns of a matrix. Then for any three elements $i<j<k$ in $[n]$ and a set $X\subseteq[n]-\{i,j,k\}$, there holds

$$g(Xik)g(Xj)=g(Xij)g(Xk)+g(Xjk)g(Xi), \tag{6.2}$$

and for any $i<j<k<\ell$ and $X\subseteq[n]-\{i,j,k,\ell\}$,

$$g(Xik)g(Xj\ell)=g(Xij)g(Xk\ell)+g(Xj\ell)g(Xjk). \tag{6.3}$$

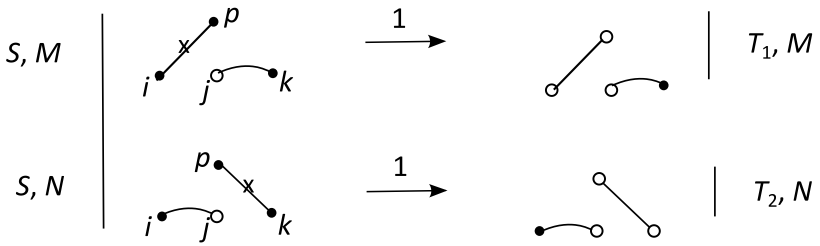
There are two quantized counterparts of (6.2) (concerning flag q-minors). One of them is viewed as

$$[Xj][Xik] = [Xij][Xk] + [Xjk][Xi], \tag{6.4}$$

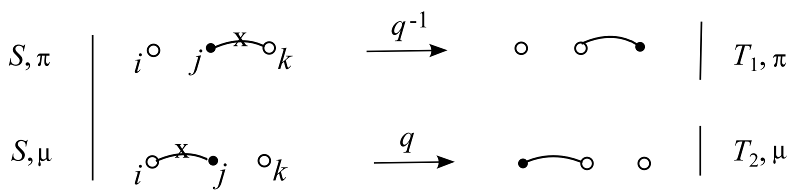
and the other as

$$[Xik][Xj] = q^{-1}[Xij][Xk] + q[Xjk][Xi]. \tag{6.5}$$

To see (6.4), associate to Xj the white pair $(I^\circ, J^\circ) = (\emptyset|\{j\})$, and to Xik the black pair $(I^\bullet|J^\bullet) = (\{p\}|\{i,k\})$, where p is the last row index for $[Xik]$ (i.e., $p = |X| + 2$). Then $\mathcal{M}_{I^\circ,I^\bullet,J^\circ,J^\bullet}$ consists of two feasible matchings: $M = \{pi, jk\}$ and $N = \{ij, pk\}$. Now (6.4) is seen from the following picture with two-level diagrams, where we write S for the cortege $([p-1]\,|Xj, [p]\,|Xik)$, T_1 for $([p]\,|Xij, [p-1]\,|Xk)$, and T_2 for $([p]\,|Xjk, [p-1]\,|Xi)$:



As to (6.5), it suffices to consider one-level diagrams (as we will not use RC-couples in the exchange operations). The "white" object is the column set $J^\circ = \{i,k\}$ and the "black" object is $J^\bullet = \{j\}$. Then $\mathcal{M}_{\{p\},\emptyset,J^\circ,J^\bullet}$ consists of two feasible matchings, one using the C-couple $\pi = jk$, and the other using the C-couple $\mu = ij$. Now (6.5) can be seen from the picture, where we write S for the flag cortege (Xik, Xj), T_1 for (Xij, Xk), and T_2 for (Xjk, Xi).

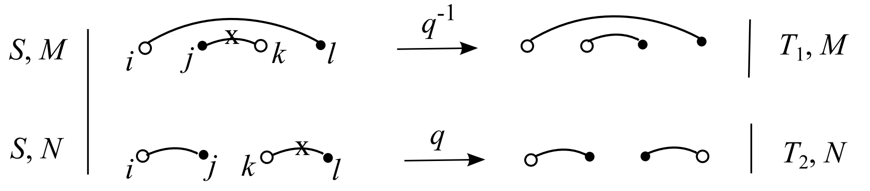


Next we demonstrate the following quantized counterpart of (6.3):

$$[Xik][Xj\ell] = q^{-1}[Xij][Xk\ell] + q[Xi\ell][Xjk]. \tag{6.6}$$

To see this, we use one-level diagrams and consider the column sets $J^\circ = \{i,k\}$ and $J^\bullet = \{j,\ell\}$. Then $\mathcal{M}_{\emptyset,\emptyset,J^\circ,J^\bullet}$ consists of two feasible matchings: $M = \{i\ell, jk\}$ and

$N = \{ij, k\ell\}$. Identity (6.6) can be seen from the picture, where $S = (Xik, Xj\ell)$, $T_1 = (Xij, Xk\ell)$ and $T_2 = (Xi\ell, Xjk)$.



Remark 4. Note that, if wished, one can produce more identities from (6.4) and (6.5), using the fact that Xij and Xk (as well as Xjk and Xi) are weakly separated, and therefore their corresponding flag q-minors quasicommute (see Sect. 6.2). In contrast, Xj and Xik are not weakly separated. Next, subtracting from (6.5) identity (6.4) multiplied by q results in the identity of the form

$$[Xik][Xj] = q[Xj][Xik] - (q - q^{-1})[Xij][Xk],$$

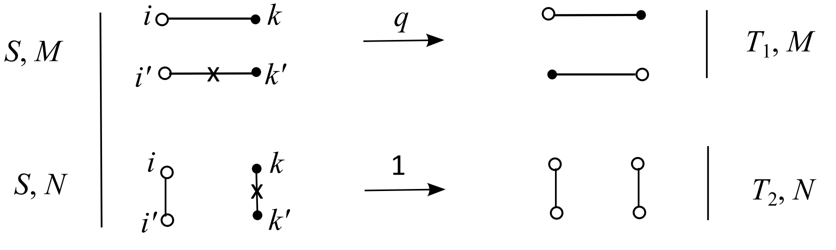
which is in spirit of *commutation relations* for quantum minors studied in [6,7].

6.5. Dodgson's type identity

As one more simple illustration of our method, let us consider a q-analogue of the classical Dodgson's condensation formula for usual minors [5]. It can be stated as follows: for elements $i < k$ of $[m]$, a set $X \subseteq [m] - \{i, k\}$, elements $i' < k'$ of $[n]$, and a set $X' \subseteq [n] - \{i', k'\}$ (with $|X'| = |X|$),

$$[Xi|X'i'][Xk|X'k'] = q[Xi|X'k'][Xk|X'i'] + [Xik|X'i'k'][X|X']. \tag{6.7}$$

In this case we deal with the cortege $S = (I|J, I'|J') = (Xi|X'i', Xk|X'k')$ and its refinement $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ of the form (i, k, i', k'). The latter admits two feasible matchings: $M = \{ik, i'k'\}$ and $N = \{ii', kk'\}$. Now (6.7) can be concluded by examining the picture below, where T_1 stands for $(Xi|X'k', Xk|X'i')$, and T_2 for $(Xik|X'i'k', X|X')$:



6.6. Two general quadratic identities

Two quadratic identities of a general form were established for quantum flag minors in [9,15].

The first one considers column subsets $I, J \subset [n]$ with $|I| \le |J|$ and is viewed as

$$[I][J] = \sum_{\mu \subseteq J-I,\ |\mu|=|J|-|I|} (-q)^{Inv(J-\mu,\,\mu)-Inv(I,\,\mu)}[I \cup \mu][J-\mu], \tag{6.8}$$

where $Inv(A, B)$ denotes the number of pairs $(a, b) \in A \times B$ with $a > b$. Observe that (6.4) is a special case of (6.8) in which the roles of I and J are played by Xj and Xik, respectively. Indeed, in this case μ ranges over the singletons $\{i\}$ and $\{k\}$, and we have $Inv(Xk, i) - Inv(Xj, i) = 0$ and $Inv(Xi, k) - Inv(Xj, k) = 0$. (For brevity, we write $Inv(\cdot, i')$ for $Inv(\cdot, \{i'\})$.)

The second one considers $I, J \subset [n]$ with $|I| - |J| \ge 2$ and is viewed as

$$\sum\nolimits_{a \in I-J} (-q)^{Inv(a,I-a)-Inv(a,J)}[Ja][I-a] = 0 \tag{6.9}$$

(where we write Ja for $J \cup \{a\}$, and $I - a$ for $I - \{a\}$). A special case is (6.6) (with $I = Xjk\ell$ and $J = Xi$).

We explain how (6.8) and (6.9) can be proved using the flow-matching method.

Proof of (6.8). The pair (I, J) corresponds to the cortege $S := ([p]\,|I,\ [p+k]\,|J)$ and its refinement $R := (\emptyset,\ Q := \{p+1, \dots, p+k\},\ I^\circ := I - J,\ J^\bullet := J - I)$, where $p := |I|$ and $k := |J| - |I|$. In its turn, each term $(I \cup \mu)|(J - \mu)$ occurring in the R.H.S. of (6.8) corresponds to the cortege $S_\mu := ([p+k]\,|(I_\mu := I \cup \mu),\ [p]\,|(J_\mu := J - \mu))$ and its refinement $R_\mu := (Q,\ \emptyset,\ I^\circ_\mu := I^\circ \cup \mu,\ J^\bullet_\mu := J^\bullet - \mu)$.

So we deal with the set

$$\mathcal{F} := \{S\} \cup \{S_\mu \colon \mu \subset J^\bullet,\ |\mu| = k\},$$

of corteges and the related set $\mathbf{C}(\mathcal{F})$ of configurations (of the form $(S; M)$ or $(S_\mu; M)$), and our aim is to construct an involution $\gamma : \mathbf{C}(\mathcal{F}) \to \mathbf{C}(\mathcal{F})$ which is agreeable with matchings, signs and q-factors figured in (6.8). (Under reducing (6.8) to the canonical form, $\mathcal{F}$ splits into two families $\mathcal{I}$ and $\mathcal{K}$, and γ determines the q-balancedness for $\mathcal{I}, \mathcal{K}$ with corresponding α, β.)

Consider a refined cortege $R_\mu = (Q, \emptyset, I^\circ_\mu, J^\bullet_\mu)$ and a feasible matching M for it. Note that M consists of $k = |Q|$ RC-couples (connecting Q and I°_μ) and $|J^\bullet_\mu| = |I^\circ|$ C-couples (connecting I°_μ and $J^\bullet_\mu$). Two cases are possible.

Case 1: Each C-couple connects $J^\bullet_\mu$ and I°. Then all RC-couples in M connect Q and μ. Therefore, the exchange operation applied to S_μ using the set Π of all RC-couples of M produces the “initial” cortege S (corresponding to the refinement

$R = (\emptyset, Q, I^\circ, J^\bullet)$). Clearly M is a feasible matching for S and the exchange operation applied to S using Π returns S_μ. We link $(S; M)$ and $(S_\mu; M)$ by γ.

Note that for each C-couple $\pi = ij \in M - \Pi$ and for each $r \in \mu$, either $r < i, j$ or $r > i, j$ (otherwise the RC-couple containing r would "cross" π, contrary to the planarity requirement (4.3)(ii) for M). This implies $Inv(J_\mu, \mu) = Inv(I, \mu)$, whence the terms $[I][J]$ in the L.H.S. and $(-q)^0[I_\mu][J_\mu]$ in the R.H.S. of (6.8) are q-balanced.

Case 2: There is a C-couple in M connecting $J^\bullet_\mu$ and μ. Among such couples, choose the couple $\pi = ij$ with $i < j$ such that: (a) $j - i$ is minimum, and (b) i is minimum subject to (a). From (4.3) and (a) it follows that

(6.10) if a couple $\pi' \in M$ has an element (strictly) between i and j, then π' connects I° and $J^\bullet_\mu$, and the other element of π' is between i and j as well.

Let $S_{\mu'}$ be obtained by applying to S_μ the exchange operation using the single couple π. Then $\mu' = \mu \triangle \pi$, $I^\circ_{\mu'} = I^\circ_\mu \triangle \pi$ and $J^\bullet_{\mu'} = J^\bullet_\mu \triangle \pi$. The matching M is feasible for $S_{\mu'}$, we are in Case 2 with $S_{\mu'}$ and M, and one can see that the couple $\pi' \in M$ chosen for $S_{\mu'}$ according to the above rules (a), (b) coincides with π. Based on these facts, we link $(S_\mu; M)$ and $(S_{\mu'}; M)$ by γ.

Now we compute and compare the numbers $a := Inv(J^\bullet_{\mu'} = J - \mu', \mu') - Inv(J^\bullet_\mu = J - \mu, \mu)$ and $b := Inv(I, \mu') - Inv(I, \mu)$. Let d be the number of elements of I° between i and j (recall that $\pi = ij$ and $i < j$). Property (6.10) ensures that the number of elements of $J^\bullet_\mu$ (as well as of $J^\bullet_{\mu'}$) between i and j is equal to d too. Consider two possibilities.

Subcase 2a: $i \in \mu$ (and $j \in J^\bullet_\mu$). Then $i \in J^\bullet_{\mu'}$ and $j \in \mu'$. This implies that $a = Inv(J^\bullet_{\mu'}, j) - Inv(J^\bullet_\mu, i) = d + 1$ and $b = Inv(I^\circ, j) - Inv(I^\circ, i) = d$.

Subcase 2b: $i \in J^\bullet_\mu$ (and $j \in \mu$). Then $i \in \mu'$ and $j \in J^\bullet_{\mu'}$, yielding $a = -d - 1$ and $b = -d$.

Finally, let $(-q)^\alpha$ and $(-q)^\beta$ be the multipliers to the terms $[I_\mu][J_\mu]$ and $[I_{\mu'}][J_{\mu'}]$ in (6.8), respectively. Then $\beta - \alpha = a - b$, which is equal to 1 in Subcase 2a and -1 in Subcase 2b. In both cases this amounts to the value $\zeta^\circ - \zeta^\bullet$ for the exchange operation applied to S_μ using π, and validity of (6.8) follows from Theorem 5.1. □

Remark 5. Sometimes it is useful to consider the identity formed by the corteges reversed to those in (6.8); by Proposition 5.3, it is viewed as

$$[J][I] = \sum_{\mu \subseteq J - I,\, |\mu| = |J| - |I|} (-q)^{Inv(I,\, \mu) - Inv(J - \mu,\, \mu)} [J - \mu][I \cup \mu].$$

Proof of (6.9). Let $p := |J|$, $k := |I| - |J|$, $Q := [p + k - 1] - [p + 1]$, $J^\circ := J - I$ and $I^\bullet := I - J$. For $a \in I^\bullet$, the term $(Ja|I - a)$ occurring in (6.9) corresponds to the cortege $S_a := ([p+1]\,|Ja,\ [p + k - 1]\,|(I - a))$ and its refinement $R_a := (\emptyset, Q,\ J^\circ a,\ I^\bullet_a := I^\bullet - a)$ (using the fact that $k \geq 2$).

We deal with the set $\mathcal{F} := \{S_a : a \in I^\bullet\}$ of corteges and the set $\mathbf{C}(\mathcal{F})$ of configurations $(S_a; M)$, and like the previous proof, our aim is to construct an appropriate involution $\gamma : \mathbf{C}(\mathcal{F}) \to \mathbf{C}(\mathcal{F})$.

Consider a refined cortege $R_a = (\emptyset, Q, J^\circ a, I^\bullet_a)$ and a feasible matching M for it. Take the couple in M containing a, say, $\pi = \{a, b\}$. Note that π is a C-couple and $b \in I^\bullet_a$ (since a is white, and Q and $I^\bullet_a$ are black). The exchange operation applied to S_a using π produces the member S_b of $\mathcal{F}$, and we link S_a and S_b by γ.

It remains to estimate the multipliers $(-q)^\alpha$ and $(-q)^\beta$ to the terms $[Ja][I - a]$ and $[Jb][I - b]$ in (6.9), respectively.

Let d be the number of elements of $I^\bullet$ between a and b. It is equal to the number of elements of J° between a and b (since, in view of (4.3), the elements of $I^\bullet \cup J^\circ$ between a and b must be partitioned into C-couples in M). This implies that if $a < b$, then $Inv(b, I - b) - Inv(a, I - a) = d + 1$ and $Inv(b, J) - Inv(a, J) = d$. Therefore, $\beta - \alpha = (d + 1) - d = 1$. And if $a > b$, then $Inv(b, I - b) - Inv(a, I - a) = -d - 1$ and $Inv(b, J) - Inv(a, J) = -d$, whence $\beta - \alpha = -1$. In both cases, $\beta - \alpha$ coincides with the corresponding value of $\zeta^\circ - \zeta^\bullet$, and the result follows. □

7. Necessity of the q-balancedness

In this section we give a converse assertion to Theorem 5.1, thus obtaining a complete characterization for the UQ identities on quantum minors. This characterization, given in terms of the q-balancedness, justifies the algorithm of recognizing UQ identities described in the end of Sect. 5. As before, we deal with homogeneous families of corteges in $\mathcal{E}^{m,n} \times \mathcal{E}^{m,n}$.

Theorem 7.1. *Let $\mathbb{K}$ be a field of characteristic zero and let $q \in \mathbb{K}^*$ be transcendental over $\mathbb{Q}$. Suppose that $\mathcal{I}, \mathcal{K}, \alpha, \beta$ (as in Sect. 5) are not q-balanced. Then there exists (and can be explicitly constructed) an SE-graph G for which relation (5.1) is violated.*

Proof. We essentially use an idea and construction worked out for the commutative version in [4, Sect. 5].

Recall that the homogeneity of $\mathcal{F} := \mathcal{I} \sqcup \mathcal{K}$ means the existence of $X^{\rm r}, Y^{\rm r} \subseteq [m]$ and $X^{\rm c}, Y^{\rm c} \subseteq [n]$ such that any cortege $(I|J, I'|J') \in \mathcal{F}$ satisfies

$$I \cap I' = X^{\rm r}, \quad I \triangle I' = Y^{\rm r}, \quad J \cap J' = X^{\rm c}, \quad J \triangle J' = Y^{\rm c} \tag{7.1}$$

(cf. (5.2)). For a perfect matching M on $Y^{\rm r} \sqcup Y^{\rm c}$, let us denote by $\mathcal{I}_M$ the set of corteges $S = (I|J, I'|J') \in \mathcal{I}$ for which M is feasible (see (4.3)), and denote by $\mathcal{K}_M$ a similar set for $\mathcal{K}$. The q-balancedness of $\mathcal{I}, \mathcal{K}, \alpha, \beta$ would mean that, for any $M \in \mathbf{M}(\mathcal{F})$, there exists a bijection $\gamma_M : \mathcal{I}_M \to \mathcal{K}_M$ respecting (5.3). That is, for any $S = (I|J, I'|J') \in \mathcal{I}_M$ and for $T = (K|L, K'|L') = \gamma_M(S)$, there holds

$$\beta(T) - \alpha(S) = \zeta^\circ(\Pi_{S,T}) - \zeta^\bullet(\Pi_{S,T}). \tag{7.2}$$

Here: $\Pi = \Pi_{S,T}$ is the subset of M such that the refined cortege $(K^\circ, K^\bullet, L^\circ, L^\bullet)$ is obtained from $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ by the index exchange operation using Π, and $\zeta^\circ(\Pi)$ (resp. $\zeta^\bullet(\Pi)$) is the number of R- and C-couples $\{i,j\} \in \Pi$ with $i<j$ and $i \in I^\circ \cup J^\circ$ (resp. $i \in I^\bullet \cup J^\bullet$). The following assertion is crucial.

Proposition 7.2. *Let M be a perfect planar matching on $Y^{\rm r} \sqcup Y^{\rm c}$. Then there exists (and can be explicitly constructed) an SE-graph $G=(V,E)$ with the following properties: for each cortege $S=(I|J, I'|J') \in \mathcal{E}^{m,n} \times \mathcal{E}^{m,n}$ satisfying (7.1),*

(P1) *if M is feasible for S, then G has a unique $(I|J)$-flow and a unique $(I'|J')$-flow;*
(P2) *if M is not feasible for S, then at least one of $\Phi_G(I|J)$ and $\Phi_G(I'|J')$ is empty.*

We will prove this proposition later, and now, assuming that it is valid, we complete the proof of the theorem.

Let $\mathcal{I}, \mathcal{K}, \alpha, \beta$ be not q-balanced. Then there exists a matching $M \in \mathbf{M}(\mathcal{F})$ that admits no bijection γ_M as mentioned above between $\mathcal{I}_M$ and $\mathcal{K}_M$ (and therefore at least one of $\mathcal{I}_M$ and $\mathcal{K}_M$ is nonempty). We fix one M of this sort and consider a graph G as in Proposition 7.2 for this M.

Our aim is to show that relation (5.1) is violated for q-minors of Path_G (yielding the theorem). Suppose, for a contradiction, that (5.1) is valid. By (P2) in the proposition, we have $[I|J][I'|J']=0$ for each cortege $(I|J, I'|J') \in \mathcal{F} - \mathcal{F}_M$, denoting $\mathcal{F}_M := \mathcal{I}_M \sqcup \mathcal{K}_M$. On the other hand, (P1) implies that if $(I|J, I'|J') \in \mathcal{F}_M$, then

$$[I|J][I'|J'] = w(\phi_{I|J})\, w(\phi_{I'|J'}),$$

where $\phi_{I|J}$ (resp. $\phi_{I'|J'}$) denotes the unique $(I|J)$-flow (resp. $(I'|J')$-flow) in G. Thus, (5.1) can be rewritten as

$$\sum\nolimits_{\mathcal{I}_M} q^{\alpha(I|J,I'|J')} w(\phi_{I|J}) w(\phi_{I'|J'}) = \sum\nolimits_{\mathcal{K}_M} q^{\beta(K|L,K'|L')} w(\phi_{K|L}) w(\phi_{K'|L'}). \qquad (7.3)$$

For each cortege $S=(I|J, I'|J') \in \mathcal{F}_M$, the weight $Q(S) := w(\phi_{I|J})\, w(\phi_{I'|J'})$ of the double flow $(\phi_{I|J}, \phi_{I'|J'})$ is a monomial in weights $w(e)$ of edges $e \in E$ (or a Laurent monomial in inner vertices of G); cf. (2.2), (2.3), (3.2). For any two corteges in $\mathcal{F}_M$, one can be obtained from the other by the index exchange operation using a submatching of M, and we know from the description in Sect. 4 that if one double flow is obtained from another by the flow exchange operation, then the (multi)sets of edges occurring in these double flows are the same (cf. Lemma 4.3).

Thus, the (multi)set of edges occurring in the weight monomial $Q(S)$ is the same for all corteges S in $\mathcal{F}_M$. Fix an arbitrary linear order ξ on E. Then the monomial $Q_\xi = Q_\xi(S)$ obtained from $Q(S)$ by a permutation of the entries so as to make them weakly decreasing w.r.t. ξ (from left to right) is the same for all $S \in \mathcal{F}_M$. Therefore, applying relations (G1)–(G3) on vertices of G (see Sect. 2.4), we observe that for $S \in \mathcal{F}_M$, the weight $Q(S)$ is expressed as

$$Q(S) = q^{\rho(S)} Q_\xi \tag{7.4}$$

for some $\rho(S) \in \mathbb{Z}$. Using such expressions, we rewrite (7.3) as

$$\sum\nolimits_{S\in\mathcal{I}_M} q^{\alpha(S)+\rho(S)} Q_\xi = \sum\nolimits_{T\in\mathcal{K}_M} q^{\beta(T)+\rho(T)} Q_\xi,$$

obtaining

$$\sum\nolimits_{S\in\mathcal{I}_M} q^{\alpha(S)+\rho(S)} = \sum\nolimits_{T\in\mathcal{K}_M} q^{\beta(T)+\rho(T)}. \tag{7.5}$$

Since q is transcendental, the polynomials in q in both sides of (7.5) are equal. Then $|\mathcal{I}_M| = |\mathcal{K}_M|$ and there exists a bijection $\widetilde{\gamma} : \mathcal{I}_M \to \mathcal{K}_M$ such that

$$\alpha(S) + \rho(S) = \beta(\widetilde{\gamma}(S)) + \rho(\widetilde{\gamma}(S)) \qquad \text{for each } S \in \mathcal{I}_M. \tag{7.6}$$

This together with relations of the form (7.4) gives

$$q^{\alpha(S)} Q(S) = q^{\beta(\widetilde{\gamma}(S))} Q(\widetilde{\gamma}(S)).$$

Now, for $S = (I|J, I'|J') \in \mathcal{I}_M$, let $T = (K|L, K'|L') := \widetilde{\gamma}(S)$ and let $\Pi := \Pi_{S,T}$. Using relation (4.4) from Corollary 4.5, we have

$$\begin{aligned} q^{\beta(T)-\alpha(S)} Q(T) &= Q(S) = w(\phi_{I|J})\, w(\phi_{I'|J'}) \\ &= q^{\zeta^\circ(\Pi)-\zeta^\bullet(\Pi)} w(\phi_{K|L})\, w(\phi_{K'|L'}) = q^{\zeta^\circ(\Pi)-\zeta^\bullet(\Pi)} Q(T), \end{aligned}$$

whence $\beta(T) - \alpha(S) = \zeta^\circ(\Pi) - \zeta^\bullet(\Pi)$. Thus, the bijection $\gamma_M := \widetilde{\gamma}$ satisfies (7.2). A contradiction. □

Proof of Proposition 7.2. We utilize the construction of a graph (which need not be an SE-graph) with properties (P1) and (P2) from [4, Sect. 5]; denote this graph by $H = (Z, U)$. We first outline essential features of that construction and then explain how to turn H into an equivalent SE-graph G. Transformations of H that we apply to obtain G consist of subdividing some edges $e = (u, v)$ (i.e., replacing e by a directed path from u to v) and parallel shifting some sets of vertices and edges in the plane (preserving the planar structure of the graph). Such transformations maintain properties (P1) and (P2), whence the result will follow.

Let $Y^{\rm r} \cup X^{\rm r} = \{1, 2, \dots, k\}$ and $Y^{\rm c} \cup X^{\rm c} = \{1', 2', \dots, k'\}$. Denote the sets of R-, C-, and RC-couples in M by $M^{\rm r}$, $M^{\rm c}$, and $M^{\rm rc}$, respectively. An R-couple $\pi = \{i, j\}$ with $i < j$ is denoted by ij, and we denote by $\prec$ the natural partial order on R-couples where $\pi' \prec \pi$ if $\pi' = pr$ is an R-couple with $i < p < r < j$. And similarly for C-couples. When $\pi' \prec \pi$ and there is no π'' between π and π' (i.e., $\pi' \prec \pi'' \prec \pi$), we say that π' is an immediate successor of π and denote the set of these by $\mathrm{ISuc}(\pi)$. Also for $\pi = ij \in M^{\rm r}$

and $d \in X^{\rm r}$, we say that d is *open* for π if $i < d < j$ and there is no $\pi' = pr \prec \pi$ with $p < d < r$; we denote the set of these by $\mathrm{Open}(\pi)$. And similarly for couples in $M^{\rm c}$ and elements of $X^{\rm c}$.

A current graph and its ingredients are identified with their images in the plane, and any edge in it is represented by a (directed) straight-line segment. We write (x_v, y_v) for the coordinates of a point v, and say that an edge $e = (u, v)$ *points down* if $y_u > y_v$.

The initial graph H has the following features (seen from the construction in [4]).

(i) The "sources" $1, \dots, k$ ("sinks" $1', \dots, k'$) are disposed in this order from left to right in the upper (resp. lower) half of a circumference O, and the graph H is drawn within the disk O^* surrounded by O. (Strictly speaking, the construction of H in [4] is a mirror reflection of what we describe; the latter is more convenient for us, without affecting the result.)

(ii) Each couple $\pi = ij \in M^{\rm r} \cup M^{\rm c}$ is extended to a chord between the points i and j, which is subdivided into a path L_π whose edges are alternately forward and backward ones. Let R_π denote the region in O^* between L_π and the paths $L_{\pi'}$ for all $\pi' \in \mathrm{ISuc}(\pi)$. Then each edge e of H (regarded as a line-segment) having a point in the interior of R_π connects a vertex in L_π with either a vertex in $L_{\pi'}$ for some $\pi' \in \mathrm{ISuc}(\pi)$ or some vertex $d \in \mathrm{Open}(\pi)$. Moreover, e is directed to L_π if $\pi \in M^{\rm r}$, and from L_π if $\pi \in M^{\rm c}$.

(iii) Let R^* be the region in O^* between the paths L_π for all maximal R- and C-couples π. Then any edge e of H having a point in the interior of R^* points down. Also if such an e has an incident vertex v lying on L_π for a maximal R-couple (resp. C-couple) π, then e leaves (resp. enters) v.

Using these properties, we transform H, step by step, keeping notation $H = (Z, U)$ for a current graph, and O^* for a current disk (which becomes a deformed circle) containing H. Iteratively applied steps (S1) and (S2), described below, aim to make a graph whose all edges point down.

(S1) Choose $\pi = ij \in M^{\rm r}$ and let R_π be the part of O^* above L_π. (Then R_π contains the paths $L_{\pi'}$ for all $\pi' \prec \pi$, and the elements $d \in X^{\rm r}$ with $i < d < j$.) We shift R_π upward by a sufficiently large distance $\lambda > 0$. More precisely, each vertex $v \in Z$ lying in R_π is replaced by vertex v' with $x_{v'} = x_v$ and $y_{v'} = y_v + \lambda$. Each edge $(u, w) \in U$ of the old graph induces the corresponding edge of the new one, namely: edge (u', w') if both u, w lie in R_π; edge (u, w) if $u, w \notin R_\pi$; and edge (u', w) if $u \in R_\pi$ and $w \in L_\pi$. (Case $u \in O^* - R_\pi$ and $w \in R_\pi$ is impossible.) As a result, the region O^* is enlarged by shifting the part R_π by $(0, \lambda)$ and filling the gap between L_π and $L_\pi + (0, \lambda)$ by the corresponding parallelogram.

One can realize that upon application of (S1) to all R-couples, the following property is ensured: for each $\pi \in M^{\rm r}$, all initial edges incident to exactly one vertex on L_π turn into edges pointing down. Moreover, since L_π is alternating and there is enough space (from below and from above) in a neighborhood of the current L_π, we can deform L_π into a zigzag path with all edges pointing down (by shifting each inner vertex v of L_π by a vector $(0, \epsilon)$ with an appropriate (positive or negative) $\epsilon \in \mathbb{R}$).

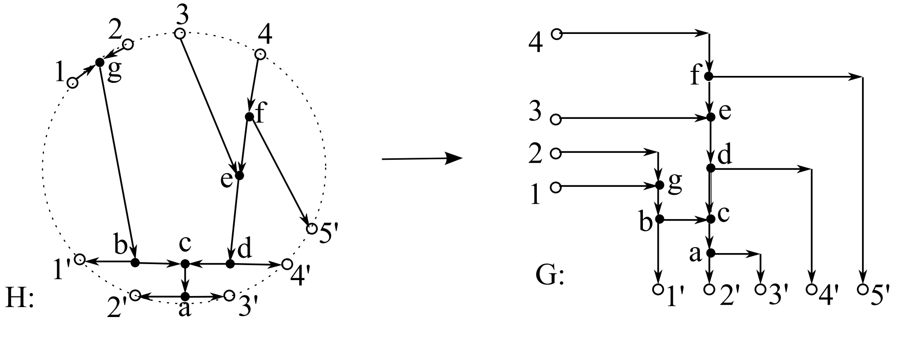
(S2) We choose $\pi \in M^{\rm c}$ and act similarly to (S1) with the differences that now R_π denotes the part of O^* below L_π and that R_π is shifted downward (by a sufficiently large $\lambda > 0$).

Upon termination of the process for all R- and C-couples, all edges of the current graph H (which is homeomorphic to the initial one) point down, as required. Moreover, H has one more useful property: the sources $1, \ldots, k$ are "seen from above" and the sinks $1', \ldots, k'$ are "seen from below". Hence we can add to H "long" vertical edges $h_1, \ldots, h_k$ entering the vertices $1, \ldots, k$, respectively, and "long" vertical edges $h_{1'}, \ldots, h_{k'}$ leaving the vertices $1', \ldots, k'$, respectively, maintaining the planarity of the graph. In the new graph one should transfer each source i into the tail of h_i, and each sink i' into the head of $h_{i'}$. One may assume that the new sources (sinks) lie within one horizontal line L (resp. L'), and that the rest of the graph lies between L and L'.

Now we get rid of the edges (u, v) such that $x_u > x_v$ (i.e. "pointing to the left"), by making the linear transformation $v \mapsto v'$ for the points v in H, defined by $x_{v'} = x_v - \lambda y_v$ and $y_{v'} = y_v$ with a sufficiently large $\lambda > 0$.

Thus, we eventually obtain a graph H (homeomorphic to the initial one) without edges pointing up or to the left. Also the sources and sinks are properly ordered from left to right in the horizontal lines L and L', respectively. Now it is routine to turn H into an SE-graph G as required in the proposition. $\square$

The transformation of H into G as in the proof is illustrated in the picture; here $X^{\rm r} = \{4\}$, $Y^{\rm r} = \{1, 2, 3\}$, $X^{\rm c} = \emptyset$, $Y^{\rm c} = \{1', \ldots, 5'\}$, and $M = \{12, 1'4', 2'3', 35'\}$.



8. Concluding remarks and additional results

8.1. An open question

It looks reasonable to ask: how narrow is the class of UQ identities for q-minors compared with the class of those in the commutative version. We know that the latter class is formed by balanced families $\mathcal{I}, \mathcal{K}$, whereas the former one is characterized via a stronger property of q-balancedness. So we can address the problem of characterizing the set of homogeneous balanced families $\mathcal{I}, \mathcal{K}$ of corteges $(I|J, I'|J') \in \mathcal{E}^{m,n} \times \mathcal{E}^{m,n}$

that admit functions $\alpha : \mathcal{I} \to \mathbb{Z}$ and $\beta : \mathcal{K} \to \mathbb{Z}$ such that the quadruple $\mathcal{I}, \mathcal{K}, \alpha, \beta$ is q-balanced.

In an algorithmic setting, we deal with the following problem (∗): given $\mathcal{I}, \mathcal{K}$ (as above), decide whether or not there exist corresponding α, β (as above). Concerning algorithmic complexity aspects, note that the number $|\mathbf{C}(\mathcal{I})| + |\mathbf{C}(\mathcal{K})|$ of configurations for $\mathcal{I}, \mathcal{K}$ may be exponentially large compared with $|\mathcal{I}| + |\mathcal{K}|$ (since a cortege of size N may have $2^{O(N)}$ feasible matchings). In light of this, it is logically reasonable to regard as the input of problem (∗) just the set $\mathbf{C}(\mathcal{I}) \sqcup \mathbf{C}(\mathcal{K})$ rather than $\mathcal{I} \sqcup \mathcal{K}$ (and measure the input size of (∗) accordingly). We conjecture that problem (∗) specified in this way is NP-hard and, moreover, it remains NP-hard even in the flag case.

8.2. Non-quasicommuting flag minors

The simplest example of balanced $\mathcal{I}, \mathcal{K}$ for which problem (∗) has answer "not" arises in the flag case with $\mathcal{I}, \mathcal{K}$ consisting of single corteges. That is, we deal with quantized flag minors $[I] = [A|I]$ and $[J] = [B|J]$, where $A := \{1, \dots, |I|\}$ and $B := \{1, \dots, |J|\}$, and consider the (trivially balanced) one-element families $\mathcal{I} = \{S := (A|I, B|J)\}$ and $\mathcal{K} = \{T := (B|J, A|I)\}$. By Leclerc–Zelevinsky's theorem (Theorem 6.1), $[I]$ and $[J]$ quasicommute if and only if the sets I, J are weakly separated. We have explained how to obtain "if" part of this theorem by use of the flow-matching method, and now we explain how to use this method to show, relatively easily, "only if" part (which has a more sophisticated proof in [10]).

So, assuming that I, J are not weakly separated, let us show that there do not exist $\alpha(S), \beta(T) \in \mathbb{Z}$ such that the equality

$$\beta(T) - \alpha(S) = \zeta^{\circ}(S; M) - \zeta^{\bullet}(S; M) \tag{8.1}$$

holds for all feasible matching M for S. The crucial observation is that

(8.2) $I, J \subset [n]$ are weakly separated if and only if S has exactly one feasible matching

(where "only if" part, mentioned in Sect. 6.2, is trivial). In fact, we need a sharper version of "if" part of (8.2): when $I, J \subset [n]$ are not weakly separated, there exist $M, M' \in \mathcal{M}(S)$ such that

$$\zeta^{\circ}(S; M) - \zeta^{\bullet}(S; M) \neq \zeta^{\circ}(S; M') - \zeta^{\bullet}(S; M'). \tag{8.3}$$

Then the fact that the exchange operation applied to S using M results in T, and similarly for M', implies that (8.1) cannot hold simultaneously for both M and M'.

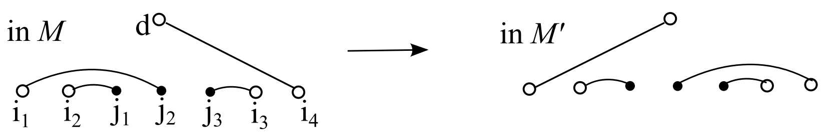
To construct the desired M and M', we argue as follows. Let for definiteness $|I| \geq |J|$ and let $I^{\circ} := I - J$ and $J^{\bullet} := J - I$. Since I, J are not weakly separated, one can see that there are $a, b \in [n]$ with $a < b$ such that the sets $\widetilde{I^{\circ}} := \{i \in I^{\circ} : a \leq i \leq b\}$ and

$\widetilde{J}^\bullet := \{j \in J^\bullet : a \le j \le b\}$ satisfy $|\widetilde{I}^\circ| - 1 = |\widetilde{J}^\bullet| =: k$, and $\widetilde{I}^\circ$ has a partition into nonempty sets I_1, I_2 satisfying $I_1 < \widetilde{J}^\bullet < I_2$. Let

$$I_1 = (i_1 < i_2 < \ldots < i_p), \quad I_2 = (i_{p+1} < \ldots < i_{k+1}), \quad \widetilde{J}^\bullet = (j_1 < \ldots < j_k)$$

(then $i_p < j_1$ and $j_k < i_{p+1}$). Choose an arbitrary matching $M \in \mathcal{M}(S)$, and consider the set $\varPi$ of couples in M containing elements of $\widetilde{J}^\bullet$; let $\varPi = \{\pi_1, \ldots, \pi_k\}$, where $j_\ell \in \pi_\ell$. Each π_ℓ is a C-couple (since it cannot be an RC- couple, in view of $B - A = \emptyset$), and condition (4.3) for M implies that only two cases are possible: (a) p couples in $\varPi$ meet I_1 and the remaining $k-p$ couples meet I_2, and (b) $p-1$ couples in $\varPi$ meet I_1 and the remaining $k-p+1$ couples meet I_2.

In case (a), we have $\pi_\ell = \{j_\ell, i_{p-\ell+1}\}$ for $\ell = 1, \ldots, p$, and $\pi_\ell = \{j_\ell, i_\ell\}$ for $\ell = p+1, \ldots, k$. An especial role is played by the couple in M containing the last element i_{k+1} of I_2, say, $\pi = \{i_{k+1}, d\}$ (note that d belongs to either $A - B$ or $J^\bullet - \widetilde{J}^\bullet$). We modify M by replacing the couple π by $\pi' := \{i_1, d\}$, and replacing $\pi_p = \{j_p, i_1\}$ by $\pi'_p := \{j_p, i_{k+1}\}$, forming matching M'. The picture illustrates the case $k = 3$, $p = 2$ and $d \in A - B$.



One can see that M' is feasible for S. Moreover, M and M' satisfy (8.3). Indeed, π_p contributes one unit to $\zeta^\circ(S; M)$ while π'_p contributes one unit to $\zeta^\bullet(S; M')$, the contributions from π and from π' are the same, and the rests of M and M' coincide.

Thus, in case (a), the one-element families $\{S\}$ and $\{T\}$ along with any numbers $\alpha(S), \beta(T)$ are not q-balanced. Then relation (6.1) (with any c) is impossible by Theorem 7.1. In case (b), the argument is similar. This yields the necessity ("only if" part) in Theorem 6.1. □

8.3. Quasicommuting general minors

Extending Leclerc–Zelevinsky's result (Theorem 6.1), Scott gave a characterization for the set of quasicommuting quantum minors in a general case.

Theorem 8.1 *([14]). Let $(I|J), (I'|J') \in \mathcal{E}^{m,n}$. The quantum minors $[I|J]$ and $[I'|J']$ quasicommute, i.e., $[I|J][I'|J'] = q^c[I'|J'][I|J]$ for some c, if and only if $S(I,J)$ and $S(I',J')$ are weakly separated subsets of $[m+n]$, where for $\widetilde{I} \subseteq [m]$ and $\widetilde{J} \subseteq [n]$, we write $S(\widetilde{I}, \widetilde{J})$ for the set $\{m+j : j \in \widetilde{J}\} \cup [m] - \{m-i+1 : i \in \widetilde{I}\}$. Furthermore, if $A := S(I,J)$ and $B := S(I',J')$ are weakly separated, $|A| \ge |B|$, and $B_1 \cup B_2$ is a partition of $B - A$ with $B_1 < (A - B) < B_2$, then c as above is equal to $|B_2| - |B_1| + |I| - |I'|$.*

Let us explain how to obtain a characterization of quasicommuting general q-minors by use of the flow-matching method. We state it in a slightly different form (leaving to the reader to check that the statement of Theorem 8.1 is equivalent to (i), (iii) in the next proposition).

Proposition 8.2. *Let $(I|J),(I'|J')\in\mathcal{E}^{m,n}$ and let $|I|\ge|I'|$. The following statements are equivalent:*

(i) *$[I|J][I'|J']=q^c[I'|J'][I|J]$ for some $c\in\mathbb{Z}$;*

(ii) *the cortege $S=(I|J,I'|J')$ admits exactly one feasible matching;*

(iii) *the sets I,I' are weakly separated, the sets J,J' are weakly separated, and for the refinement $(I^\circ,I^\bullet,J^\circ,J^\bullet)$ of S, one of the following takes place:*

- (a) *$|I^\bullet||J^\bullet|=0$; or*
- (b) *both sets $I^\bullet,J^\bullet$ are nonempty, and either $I^\circ<I^\bullet$ and $J^\bullet<J^\circ$, or $I^\bullet<I^\circ$ and $J^\circ<J^\bullet$.*

Also in case (iii) the number c is computed as follows: if $I^\bullet=\emptyset$, $J^\bullet=J_1\cup J_2$ and $J_1<J^\circ<J_2$, then $c=|J_2|-|J_1|$; (symmetrically) if $J^\bullet=\emptyset$, $I^\bullet=I_1\cup I_2$ and $I_1<I^\circ<I_2$, then $c=|I_2|-|I_1|$; if $I^\circ<I^\bullet$ and $J^\bullet<J^\circ$, then $c=|I^\bullet|-|J^\bullet|$; and (symmetrically) if $I^\bullet<I^\circ$ and $J^\circ<J^\bullet$, then $c=|J^\bullet|-|I^\bullet|$.

Proof. Implication (ii)→(i) is proved as in Sect. 6.2, and (iii)→(ii) is easy.

To show (i)→(iii), note that $|I^\circ|-|I^\bullet|=|J^\circ|-|J^\bullet|\ge 0$ (cf. (4.2)) and observe that a feasible matchings for S can be constructed by the following procedure (P) consisting of three steps. First, choose an arbitrary maximal feasible set $M^{\rm r}$ of R-couples in $Y^{\rm r}:=I^\circ\cup I^\bullet$. Here the feasibility means that the elements of each couple have different colors and there are neither couples $\{i,j\}$ and $\{p,r\}$ with $i<p<j<r$, nor a couple $\{i,j\}$ and an element $d\in Y^{\rm r}-\cup(\pi\in M^{\rm r})$ with $i<d<j$; cf. (4.3). Second, choose an arbitrary maximal feasible set $M^{\rm c}$ of C-couples in $Y^{\rm c}:=J^\circ\cup J^\bullet$. Third, when $|I|>|I'|$, the remaining elements of $Y^{\rm r}\sqcup Y^{\rm c}$ (which are all white) are coupled by a unique set $M^{\rm rc}$ of RC-couples. Then $M:=M^{\rm r}\cup M^{\rm c}\cup M^{\rm rc}$ is a feasible matching for S.

Suppose that (iii) is false and consider possible cases.

1) Let J,J' be not weakly separated. Then we construct $M^{\rm r},M^{\rm c},M^{\rm rc}$ by procedure (P) and work with the matching $\widetilde M:=M^{\rm c}\cup M^{\rm rc}$ in a similar way as in the above proof for the flag case (with non-weakly-separated column sets). This transforms $\widetilde M$ into $\widetilde M'$, and we obtain two different feasible matchings $M:=\widetilde M\cup M^{\rm r}$ and $M':=\widetilde M'\cup M^{\rm r}$ for S satisfying (8.3). This leads to a contradiction with (i) (as well as (ii)) in the theorem. When I,I' are not weakly separated, the argument is similar.

2) Assume that I,I' are weakly separated, and similarly for J,J'. Suppose that both $I^\bullet,J^\bullet$ are nonempty. Then I°,J° are nonempty as well, and for the matching M formed by procedure (P), $M^{\rm r}$ covers $I^\bullet$ and $M^{\rm c}$ covers $J^\bullet$.

Denote by a,a' (resp. b,b') the minimal and maximal elements in $Y^{\rm r}$ (resp. $Y^{\rm c}$), respectively. Suppose that both a,b are black. Then we can transform M into M' by replacing the R-couple containing a, say, ad, and the C-couple containing b, say, bf, by the two RC-couples ab and df. It is easy to see that M' is feasible and M, M' satisfy (8.3) (since under the transformation $M \to M'$ the value $\zeta^\circ - \zeta^\bullet$ increases by two), whence (i) is false. When both a', b' are black, we act similarly. So we may assume that each pair $\{a, b\}$ and $\{a', b'\}$ contains a white element. The case $a \in I^\circ$ and $b \in J^\circ$ is possible only if $|I^\circ| = |I^\bullet|$ (taking into account that $|I^\circ| \ge |I^\bullet| \ne 0$, $|J^\circ| \ge |J^\bullet| \ne 0$, and that $I^\circ, I^\bullet$, as well as $J^\circ, J^\bullet$, are weakly separated), implying $|J^\circ| = |J^\bullet|$. But then $M^{\rm r}$ covers I° and $M^{\rm c}$ covers J°; so we can construct a feasible matching $M' \ne M$ as in the previous case (after changing the colors everywhere). And similarly when both a', b' are white.

Thus, we may assume that a, b have different colors, and so are a', b'. Suppose that $a, a' \in I^\circ$ and $b, b' \in J^\bullet$ (the case $a, a' \in I^\bullet$ and $b, b' \in J^\circ$ is similar). This is possible only if $|I^\circ| = |I^\bullet|$ (since $|I| \ge |I'|$, and I, I' are weakly separated). Then the feasible matching M constructed by (P) consists of only R- and C-couples. Take the R-couple in M containing a and the C-couple containing b', say, $\pi = \{a, i\}$ and $\pi' = \{j, b'\}$; then both a, j are white and both i, b' are black. Replace π, π' by the RC-couples $\{a, j\}$ and $\{i, b'\}$. This gives a feasible matching $M' \ne M$ satisfying (8.3).

The remaining cases are just as in (a) or (b) of (iii), yielding (i)→(iii). □

Remark 6. Note that the situation when $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ has only one feasible matching can also be interpreted as follows. Let us change the colors of all elements in the upper half of the circumference O (i.e., I° becomes black and $I^\bullet$ becomes white). Then the quantities of white and black elements in O are equal and the elements of each color go in succession cyclically.

Remark 7. When minors $[I|J]$ and $[I'|J']$ quasicommute with $c = 0$, we obtain the situation of "purely commuting" quantum minors, such as those discussed in Sect. 6.1. The last assertion in Proposition 8.2 enables us to completely characterize the set of corteges $(I|J, I'|J')$ determining commuting q-minors, as follows.

Proposition 8.3. *The equality $[I|J][I'|J'] = [I'|J'][I|J]$ holds if and only if the refinement $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ satisfies at least one of the following:*

(C1) *$|I^\circ| = |J^\circ|$ (as well as $|I^\bullet| = |J^\bullet|$) and either $I^\circ < I^\bullet$ and $J^\bullet < J^\circ$, or, symmetrically, $I^\bullet < I^\circ$ and $J^\circ < J^\bullet$;*

(C2) *assuming for definiteness that $|I| \ge |I'|$, either $I^\bullet = \emptyset$ and $J^\bullet$ has a partition $J_1 \cup J_2$ such that $|J_1| = |J_2|$ and $J_1 < J^\circ < J_2$, or, symmetrically, $J^\bullet = \emptyset$ and $I^\bullet$ has a partition $I_1 \cup I_2$ such that $|I_1| = |I_2|$ and $I_1 < I^\circ < I_2$.*

Cases (C1) and (C2) are illustrated in the picture by two level diagrams.

8.4. Rotations

Return to a general UQ identity (5.1). In Sect. 5 we demonstrated two transformations of q-balanced $(\mathcal{I},\mathcal{K},\alpha,\beta)$ that preserve the q-balancedness (namely, the ones of *reversing* and *transposing*, which result in $(\mathcal{I}^{\rm rev},\mathcal{K}^{\rm rev},-\alpha,-\beta)$ and $(\mathcal{I}^\top,\mathcal{K}^\top,\alpha,\beta)$, respectively). Now we demonstrate one more interesting (and less trivial) transformation of $(\mathcal{I},\mathcal{K},\alpha,\beta)$ (in Theorem 8.4). We proceed in four steps.

First, for corresponding $X^{\rm r},Y^{\rm r}\subset[m]$ and $X^{\rm c},Y^{\rm c}\subset[n]$ (cf. (7.1)), let $Y^{\rm r}=(i_1<\cdots<i_k)$ and $Y^{\rm c}=(j_1<\cdots<j_{k'})$. Choose $g,h\in\mathbb{Z}$ such that

$$g+h\le k\ \text{ if } g,h\ge 0;\qquad |g|+|h'|\le k'\ \text{ if } g,h\le 0; \tag{8.4}$$
$$g\le k \text{ and } |h|\le k'\ \text{ if } g\ge 0\ge h;\qquad |g|\le k' \text{ and } h\le k\ \text{ if } g\le 0\le h.$$

Assuming that the numbers i_1, $m-i_k$, j_1, $n-j_{k'}$ are large enough, we take sets $A,B\subset[m]$ and $A',B'\subset[n]$ such that $|A|=|A'|=|g|$, $|B|=|B'|=|h'|$, $(A\cup B)\cap X^{\rm r}=\emptyset$, $(A'\cup B')\cap X^{\rm c}=\emptyset$, and

(8.5) (a) $A=\{i_1,\ldots,i_g\}$ and $A'<Y^{\rm c}$ if $g\ge 0$;
(a') $A<Y^{\rm r}$ and $A'=\{j_1,\ldots,j_{|g|}\}$ if $g\le 0$;
(b) $B=\{i_{k-h+1},\ldots,i_k\}$ and $B'>Y^{\rm c}$ if $h\ge 0$;
(b') $B>Y^{\rm r}$ and $B'=\{j_{k'-|h|+1},\ldots,j_{k'}\}$ if $h\le 0$.

Let ξ be the *order-reversing* bijection between A and A', i.e., ℓ-th element of A is bijective to $(|g|+1-\ell)$-th element of A', and η the order-reversing bijection between B and B'.

Second, we transform each cortege $S=(I|J,I'|J')\in\mathcal{I}\cup\mathcal{K}$ into cortege $S_{g,h}=(\widetilde I|\widetilde J,\widetilde I'|\widetilde J')$ such that $\widetilde I\cap\widetilde I'=X^{\rm r}$, $\widetilde J\cap\widetilde J'=X^{\rm c}$, and the refinement $(\widetilde{I^\circ},\widetilde{I^\bullet},\widetilde{J^\circ},\widetilde{J^\bullet})$ of $S_{g,h}$ is expressed via the refinement $(I^\circ,I^\bullet,J^\circ,J^\bullet)$ of S as follows:

(i) $\widetilde{I^\circ}\cup\widetilde{I^\bullet}=(Y^{\rm r}\pm A)\pm B=:Y^{\rm r}_{g,h}$ and $\widetilde{J^\circ}\cup\widetilde{J^\bullet}=(Y^{\rm c}\pm A')\pm B'=:Y^{\rm c}_{g,h}$ (where we write $P+Q$ for $P\cup Q$ in case $P\cap Q=\emptyset$, and write $P-Q$ for $P\setminus Q$ in case $P\supseteq Q$);
(ii) If $i\in I^\circ$ $(i\in I^\bullet)$ is not in $A\cup B$, then $i\in\widetilde{I^\circ}$ (resp. $i\in\widetilde{I^\bullet}$), and symmetrically, if $j\in J^\circ$ $(j\in J^\bullet)$ is not in $A'\cup B'$, then $j\in\widetilde{J^\circ}$ (resp. $j\in\widetilde{J^\bullet}$);
(iii) If $i\in I^\circ$ $(i\in I^\bullet)$ is in $A\cup B$, then the element bijective to i (by ξ or η) belongs to $\widetilde{J^\bullet}$ (resp. $\widetilde{J^\circ}$); and symmetrically, if $j\in J^\circ$ $(j\in J^\bullet)$ is in $A'\cup B'$, then the element bijective to j belongs to $\widetilde{I^\bullet}$ (resp. $\widetilde{I^\circ}$).

(In other words, ξ and η change the colors of elements occurring in A,B,A',B'.) We call $Y^{\rm r}_{g,h},Y^{\rm c}_{g,h},S_{g,h}$ the (g,h)-*rotations* of $Y^{\rm r},Y^{\rm c},S$, respectively. Accordingly, we say that $\{S_{g,h}\colon S\in\mathcal{I}\}$ is the (g,h)-rotation of $\mathcal{I}$, denoted as $\mathcal{I}^{\circlearrowleft}_{g,h}$, and similarly for $\mathcal{K}$.

(This terminology is justified by the observation that if $g=-h$, then each cortege S is transformed as though being rotated (by $|g|$ positions clockwise or counterclockwise)

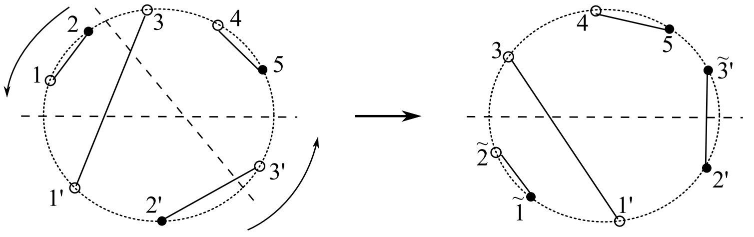


Fig. 3. An example of rotation with $k = 5$, $k' = 3$, $g = 2$ and $h = -1$.

on the circular diagram on $Y^{\rm r} \sqcup Y^{\rm c}$; thereby each element moving across the middle horizontal line of the diagram changes its color.)

Third, extend ξ and η to the bijection $\rho : Y^{\rm r} \sqcup Y^{\rm c} \to Y^{\rm r}_{g,h} \sqcup Y^{\rm c}_{g,h}$ so that ρ be identical on $Y^{\rm r} - (A \cup B)$ and on $Y^{\rm c} - (A' \cup B')$. Then a perfect matching M on $Y^{\rm r} \sqcup Y^{\rm c}$ induces the perfect matching $\{\rho(\pi) \colon \pi \in M\}$ on $Y^{\rm r}_{g,h} \sqcup Y^{\rm c}_{g,h}$, denoted as $M_{g,h}$. An important property (which is easy to check) is that

(8.6) if M is a feasible matching for $S \in \mathcal{I} \cup \mathcal{K}$, then $M_{g,h}$ is a feasible matching for $S_{g,h}$, and vice versa.

An example of rotation of S with $M \in \mathcal{M}(S)$ is illustrated in Fig. 3.

Fourth, for $S = (I|J, I'|J')$, define $\omega(S) := \delta_S(A) + \delta_S(A') + \delta_S(B) + \delta_S(B')$, where

$$\delta_S(A) := |A \cap I^\circ|, \qquad \delta_S(B) := -|B \cap I^\circ|, \tag{8.7}$$
$$\delta_S(A') := |A' \cap J^\circ|, \qquad \delta_S(B') := -|B' \cap J^\circ|.$$

Theorem 8.4. *Let $\mathcal{I}, \mathcal{K}, \alpha, \beta$ be q-balanced and let g and h be as in (8.4). Define $\alpha_{g,h}(S_{g,h}) := \alpha(S) + \omega(S)$ for $S \in \mathcal{I}$, and $\beta_{g,h}(T_{g,h}) := \beta(T) + \omega(T)$ for $T \in \mathcal{K}$. Then $\mathcal{I}^{\circlearrowleft}_{g,h}, \mathcal{K}^{\circlearrowleft}_{g,h}, \alpha_{g,h}, \beta_{g,h}$ are q-balanced.*

Proof. Let $\gamma : \mathbf{C}(\mathcal{I}) \to \mathbf{C}(\mathcal{K})$ be a bijection providing the q-balancedness of $\mathcal{I}, \mathcal{K}, \alpha, \beta$. By (8.6), γ induces a bijection $\gamma_{g,h} : \mathbf{C}(\mathcal{I}^{\circlearrowleft}_{g,h}) \to \mathbf{C}(\mathcal{K}^{\circlearrowleft}_{g,h})$. More precisely, for configurations $(S; M) \in \mathbf{C}(\mathcal{I})$ and $(T; M) = \gamma(S; M)$, $\gamma_{g,h}$ maps the configuration $(S_{g,h}; M_{g,h})$ to $(T_{g,h}; M_{g,h})$. We assert that $\gamma_{g,h}$ satisfies the corresponding equality of the form

$$\beta_{g,h}(T_{g,h}) - \alpha_{g,h}(S_{g,h}) = \zeta^\circ(S_{g,h}; \rho(\Pi)) - \zeta^\bullet(S_{g,h}; \rho(\Pi)) \tag{8.8}$$

(cf. (5.3)), yielding the result; here, as before, Π is the set of couples in M colored differently in the refinements of S and T.

For additivity reasons, it suffices to show (8.8) when $|g| + |h| = 1$. We will abbreviate corresponding $S_{g,h}, T_{g,h}, M_{g,h}$ as S', T', M'. (So T' is obtained from S' by the exchange operation using $\rho(\Pi) \subseteq M'$.) Let d denote the only element of $Y^{\rm r} \sqcup Y^{\rm c}$ that is not in $Y^{\rm r}_{g,h} \sqcup Y^{\rm c}_{g,h}$, and $\pi = \{d, f\}$ the couple in M containing d. Also we define $\Delta := \zeta^\circ(S; \Pi) - \zeta^\bullet(S; \Pi)$ and $\Delta' := \zeta^\circ(S'; \rho(\Pi)) - \zeta^\bullet(S'; \rho(\Pi))$.

Our aim is to show that $\omega(T)-\omega(S)=\Delta'-\Delta$; then (8.8) would immediately follow from (5.3). One can see that if $\pi\notin\Pi$, then $\Delta'=\Delta$, and $\delta_S(D)=\delta_T(D)$ holds for $D=A,A',B,B'$ (cf. (8.7)), implying $\omega(S)=\omega(T)$. So we may assume that $\pi\in\Pi$. Consider possible cases (where $S=(I|J,I'|J')$ and $T=(K|L,K'|L')$).

Case 1. Let $g=1$. Then $d=i_1$. First suppose that $d\in I^\circ$. Then $\omega(S)=\delta_S(A)=1$ and $\omega(T)=\delta_T(A)=0$ (since the exchange operation changes the color of d, i.e., $d\in K^\bullet$). If π is an R-couple for S, then π contributes 1 to Δ (since d is white and $d<f$), and $\rho(\pi)$ contributes 0 to Δ' (since $\rho(\pi)$ is an RC-couple for S'). Hence $\omega(T)-\omega(S)=-1=\Delta'-\Delta$, as required. And if π is an RC-couple for S, then π contributes 0 to Δ and $\rho(\pi)$ contributes -1 to Δ' (since $\rho(\pi)$ is a C-couple for S', $\rho(d)$ is black, $\rho(f)=f$ is white, and $\rho(d)<f$), giving again $\Delta'-\Delta=-1$.

When $d\in I^\bullet$, we argue "symmetrically" (as though the roles of S and T, as well as ζ° and $\zeta^\bullet$, are exchanged). Briefly, one can check that: $\omega(S)=0$ and $\omega(T)=1$; if π is an R-couple, then π contributes -1 to Δ, and $\rho(\pi)$ contributes 0 to Δ'; and if π is an RC-couple then π contributes 0 to Δ and $\rho(\pi)$ contributes 1 to Δ'. Thus, every time we obtain $\omega(T)-\omega(S)=1=\Delta'-\Delta$, as required.

Case 2. Let $h=1$. Then $d=i_k$. Suppose that $d\in I^\circ$. Then $\omega(S)=\delta_S(B)=-1$ and $\omega(T)=\delta_T(B)=0$. If π is an R-couple for S, then π contributes -1 to Δ (since d is white and $d>f$) and $\rho(\pi)$ contributes 0 to Δ' (since $\rho(\pi)$ is an RC-couple). And if π is an RC-couple for S, then π contributes 0 to Δ and $\rho(\pi)$ contributes 1 to Δ' (since $\rho(\pi)$ is a C-couple for S', $\rho(d)$ is black, and $\rho(d)>f$). In both cases, we obtain $\omega(T)-\omega(S)=1=\Delta'-\Delta$, as required. When $d\in I^\bullet$, we argue "symmetrically".

Finally, the cases $g=-1$ and $h=-1$ are "transposed" to Cases 1 and 2, respectively, and (8.8) follows by using relation (5.6). □

Acknowledgments

We thank Gleb Koshevoy for pointing out to us paper [1]. Also we thank the anonymous referee for remarks and useful suggestions.

Appendix A. Commutation properties of paths and a proof of Theorem 3.1

This section contains auxiliary lemmas that are used in the proof of Theorem 3.1 given in this section as well, and in the proof of Theorem 4.4 given in Appendix B. These lemmas deal with special pairs P,Q of paths in an SE-graph $G=(V,E;R,C)$ and compare the weights $w(P)w(Q)$ and $w(Q)w(P)$. Similar or close statements for Cauchon graphs are given in [1,2], and our method of proof is somewhat similar and rather straightforward as well.

We first specify some terminology, notation and conventions.

When it is not confusing, vertices, edges, paths and other objects in G are identified with their corresponding images in the plane. We assume that the set $R=\{r_1,\dots,r_m\}$

of sources and the set $C = \{c_1, \dots, c_n\}$ of sinks lie on the coordinate rays $(0, \mathbb{R}_{\ge 0})$ and $(\mathbb{R}_{\ge 0}, 0)$, respectively (then G is disposed within the nonnegative quadrant $\mathbb{R}^2_{\ge 0}$). The coordinates of a point v in $\mathbb{R}^2$ (in particular, a vertex v of G) are denoted as $(\alpha(v), \beta(v))$. It is convenient to assume that two vertices $u, v \in V$ have the same first (second) coordinate if and only if they belong to a vertical (resp. horizontal) path in G, in which case u, v are called *V-dependent* (resp. *H-dependent*); for we always can slightly perturb G to ensure such a property, without affecting the graph structure in essence. When u, v are V-dependent, i.e., $\alpha(u) = \alpha(v)$, we say that u is *lower* than v (and v is *higher* than u) if $\beta(u) < \beta(v)$. (In this case the commutation relation $uv = qvu$ takes place.)

Let P be a path in G. We denote: the first and last vertices of P by s_P and t_P, respectively; the *interior* of P (the set of points of $P - \{s_P, t_P\}$ in $\mathbb{R}^2$) by $\mathrm{Int}(P)$; the set of horizontal edges of P by E^H_P; and the projection $\{\alpha(x) : x \in P\}$ by $\alpha(P)$. Thus, if P is directed, then $\alpha(P)$ is the interval between $\alpha(s_P)$ and $\alpha(t_P)$.

For a directed path P, the following properties are equivalent: P is non-vertical; $E^H_P \neq \emptyset$; and $\alpha(s_P) \neq \alpha(t_P)$. We will refer to such a P as a *standard* path.

For a standard path P, we will take advantage from a compact expression for the weight $w(P)$. We call a vertex v of P *essential* if either P makes a turn at v (changing the direction from horizontal to vertical or conversely), or $v = s_P \notin R$ and the first edge of P is horizontal, or $v = t_P$ and the last edge of P is horizontal. If $u_0, u_1, \dots, u_k$ is the sequence of essential vertices of P in the natural order, then the weight of P can be expressed as

$$w(P) = u_0^{\sigma_0} u_1^{\sigma_1} \dots u_k^{\sigma_k}, \tag{A.1}$$

where $\sigma_i = 1$ if P makes a $\urcorner$-turn at u_i or if $i = k$, while $\sigma_i = -1$ if P makes a $\llcorner$-turn at u_i or if $i = 0$ and u_0 is the beginning of P. (Compare with (2.4) where a path from R to C is considered.) Note that if P does not begin in R, then its essential vertices are partitioned into H-dependent pairs.

Throughout the rest of the paper, for brevity, we denote q^{-1} by $\overline{q}$, and for an inner vertex $v \in W$ regarded as a generator, we may denote v^{-1} by $\overline{v}$.

A.1. Auxiliary lemmas

These lemmas deal with *weakly intersecting* directed paths P and Q, which means that

$$P \cap Q = \{s_P, t_P\} \cap \{s_Q, t_Q\}; \tag{A.2}$$

then $\mathrm{Int}(P) \cap \mathrm{Int}(Q) = \emptyset$. For such P, Q, we say that P is *lower* than Q if there are points $x \in P$ and $y \in Q$ such that $\alpha(x) = \alpha(y)$ and $\beta(x) < \beta(y)$ (then there are no $x' \in P$ and $y' \in Q$ with $\alpha(x') = \alpha(y')$ and $\beta(x') > \beta(y')$).

For paths P, Q, we define the value $\varphi = \varphi(P, Q)$ by the relation

$$w(P)w(Q) = \varphi w(Q)w(P).$$

Obviously, $\varphi(P, Q) = 1$ when P or Q is a V-path. In the lemmas below we default assume that both P, Q are standard.

Lemma A.1. *Let $\{\alpha(s_P), \alpha(t_P)\} \cap \{\alpha(s_Q), \alpha(t_Q)\} \cap \mathbb{R}_{>0} = \emptyset$. Then $\varphi(P, Q) = 1$.*

Proof. Consider an essential vertex u of P and an essential vertex v of Q. Then for any $\sigma, \sigma' \in \{1, -1\}$, we have $u^\sigma v^{\sigma'} = v^{\sigma'} u^\sigma$ unless u, v are dependent.

Suppose that u, v are V-dependent. From hypotheses of the lemma it follows that at least one of the following is true: $\alpha(s_P) < \alpha(u) < \alpha(t_P)$, or $\alpha(s_Q) < \alpha(v) < \alpha(t_Q)$. For definiteness assume the former. Then there is another essential vertex z of P such that $\alpha(z) = \alpha(u) = \alpha(v)$. Moreover, P makes a $\rceil$-turn an one of u, z, and a $\lfloor$-turn at the other. Since $P \cap Q = \emptyset$ (in view of (A.2)), the vertices u, z are either both higher or both lower than v. Let for definiteness u, z occur in this order in P; then $w(P)$ contains the terms u and $\overline{z}$. Let $w(Q)$ contain the term v^σ and let $uv^\sigma = \rho v^\sigma u$, where $\sigma \in \{1, -1\}$ and $\rho \in \{q, \overline{q}\}$. Then $\overline{z}v^\sigma = \overline{\rho} v^\sigma \overline{z}$, implying $u\overline{z}v^\sigma = v^\sigma u \overline{z}$. Hence the contributions to $w(P)w(Q)$ and $w(Q)w(P)$ from the pairs on generators u, z, v (namely, $\{u, v^\sigma\}$ and $\{\overline{z}, v^\sigma\}$) are equal.

Next suppose that u, v are H-dependent. One may assume that $\alpha(u) < \alpha(v)$. Then Q contains one more essential vertex $y \neq v$ with $\beta(y) = \beta(v) = \beta(u)$. Also $\alpha(u) < \alpha(v)$ and $P \cap Q = \emptyset$ imply $\alpha(u) < \alpha(y)$. Let for definiteness $\alpha(y) < \alpha(v)$. Then $w(Q)$ contains the terms $\overline{y}, v$, and we can conclude that the contributions to $w(P)w(Q)$ and $w(Q)w(P)$ from the pairs on generators u, y, v are equal (using the fact that $\alpha(u) < \alpha(y), \alpha(v)$).

These reasonings imply $\varphi(P, Q) = 1$. □

Lemma A.2. *Let $\alpha(s_P) = \alpha(s_Q) > 0$ and $\alpha(t_P) \neq \alpha(t_Q)$. Let P be lower than Q. Then $\varphi(P, Q) = q$.*

Proof. Let u and v be the first essential vertices in P and Q, respectively. Then $\alpha(s_P) = \alpha(s_Q) > 0$ implies $\alpha(u) = \alpha(s_P) = \alpha(s_Q) = \alpha(v)$. Since P is lower than Q, we have $\beta(u) \leq \beta(v)$. Moreover, this inequality is strong (since $\beta(u) = \beta(v)$ is impossible in view of (A.2) and the obvious fact that u, v are the tails of first H-edges in P, Q, respectively).

Now arguing as in the above proof, we can conclude that the discrepancy between $w(P)w(Q)$ and $w(Q)w(P)$ can arise only due to swapping the vertices u, v. Since u gives the term $\overline{u}$ in $w(P)$, and v the term $\overline{v}$ in $w(Q)$, the contribution from these vertices to $w(P)w(Q)$ and $w(Q)w(P)$ are expressed as $\overline{uv}$ and $\overline{vu}$, respectively. Since $\beta(u) < \beta(v)$, we have $\overline{uv} = q\overline{vu}$, and the result follows. □

Lemma A.3. *Let $\alpha(t_P) = \alpha(t_Q)$ and let either $\alpha(s_P) \neq \alpha(s_Q)$ or $\alpha(s_P) = \alpha(s_Q) = 0$. Let P be lower than Q. Then $\varphi(P, Q) = q$.*

Proof. We argue in spirit of the proof of Lemma A.2. Let u and v be the last essential vertices in P and Q, respectively. Then $\alpha(u) = \alpha(t_P) = \alpha(t_Q) = \alpha(v)$. Also $\beta(u) < \beta(v)$ (since P is lower than Q, and taking into account (A.2) and the fact that u, v are the heads of H-edges in P, Q, respectively). The condition on $\alpha(s_P)$ and $\alpha(s_Q)$ imply that the discrepancy between $w(P)w(Q)$ and $w(Q)w(P)$ can arise only due to swapping the vertices u, v (using reasonings as in the proof of Lemma A.1). Observe that $w(P)$ contains the term u, and $w(Q)$ the term v. So the generators u, v contribute uv to $w(P)w(Q)$, and vu to $w(Q)w(P)$. Now $\beta(u) < \beta(v)$ implies $uv = qvu$, and the result follows. □

Lemma A.4. *Let $\alpha(t_P) = \alpha(s_Q)$ and $\beta(t_P) \ge \beta(s_Q)$. Then $\varphi(P, Q) = q$.*

Proof. Let u be the last essential vertex in P and let v, z be the first and second essential vertices of Q, respectively (note that z exists because of $0 < \alpha(t_P) = \alpha(s_Q) < \alpha(t_Q)$). Then $\alpha(u) = \alpha(t_P) = \alpha(s_Q) = \alpha(v) < \alpha(z)$. Also $\beta(u) \ge \beta(t_P) \ge \beta(s_Q) \ge \beta(v) = \beta(z)$. Let Q' and Q'' be the parts of Q from s_Q to z and from z to t_Q, respectively. Then $\alpha(P) \cap \alpha(Q'') = \emptyset$, implying $\varphi(P, Q'') = 1$ (using Lemma A.1 when Q'' is standard). Hence $\varphi(P, Q) = \varphi(P, Q')$.

To compute $\varphi(P, Q')$, consider three possible cases.

(a) Let $\beta(u) > \beta(v)$. Then u, v form the unique pair of dependent essential vertices for P, Q'. Note that $w(P)$ contains the term u, and $w(Q')$ contains the term $\overline{v}$. Since $\beta(u) > \beta(v)$, we have $u\overline{v} = q\overline{v}u$, implying $\varphi(P, Q') = q$.

(b) Let $u = v$ and let u be the unique essential vertex of P (in other words, P is an H-path with $s_P \in R$). Note that $u = v$ and $\beta(t_P) \ge \beta(s_Q)$ imply $t_P = u = v = s_Q$. Also $\alpha(u) < \alpha(z)$ and $\beta(u) = \beta(z)$; so u, z are H-dependent essential vertices for P, Q' and $uz = qzu$. We have $w(P) = u$ and $w(Q') = \overline{u}z$ (in view of $u = v$). Then $u\overline{u}z = \overline{u}uz = q\overline{u}zu$ implies $\varphi(P, Q') = q$.

(c) Now let $u = v$ and let y be the essential vertex of P preceding u. Then $t_P = u = v = s_Q$, $\beta(y) = \beta(u) = \beta(z)$, and $\alpha(y) < \alpha(u) < \alpha(z)$. Hence y, u, z are H-dependent, $w(P)$ contains $\overline{y}u$, and $w(Q') = \overline{u}z$. We have

$$\overline{y}u\overline{u}z = \overline{y}\overline{u}uz = (q\overline{u}\overline{y})(qzu) = q^2\overline{u}(\overline{q}z\overline{y})u = q\overline{u}z\overline{y}u,$$

again obtaining $\varphi(P, Q') = q$. □

Lemma A.5. *Let $\alpha(t_P) = \alpha(s_Q)$ and $\beta(t_P) < \beta(s_Q)$. Then $\varphi(P, Q) = \overline{q}$.*

Proof. Let u be the last essential vertex of P, and v the first essential vertex of Q. Then $\alpha(u) = \alpha(t_P) = \alpha(s_Q) = \alpha(v)$, and $\beta(t_P) < \beta(s_Q)$ together with (A.2) implies $\beta(u) < \beta(v)$. Also $w(P)$ contains u and $w(Q)$ contains $\overline{v}$. Now $u\overline{v} = \overline{q}\overline{v}u$ implies $\varphi(P, Q) = \overline{q}$. □

A.2. Proof of Theorem 3.1

It can be conducted as a direct extension of the proof of a similar Lindström's type result given by Casteels [1, Sect. 4] for Cauchon graphs. To make our description more self-contained, we outline the main ingredients of the proof, leaving the details where needed to the reader.

Let $(I|J) \in \mathcal{E}^{m,n}$, $I = \{i(1) < \cdots < i(k)\}$ and $J = \{j(1) < \cdots < j(k)\}$. Recall that an $(I|J)$-flow in an SE-graph G (with m sources and n sinks) consists of pairwise disjoint paths $P_1, \dots, P_k$ from the source set $R_I = \{r_{i(1)}, \dots, r_{i(k)}\}$ to the sink set $C_J = \{c_{j(1)}, \dots, c_{j(k)}\}$, and (because of the planarity of G) we may assume that each P_d begins at $r_{i(d)}$ and ends at $c_{j(d)}$. Besides, we are forced to deal with an arbitrary *path system* $\mathcal{P} = (P_1, \dots, P_k)$ in which for $i = 1, \dots, k$, P_d is a directed path in G beginning at $r_{i(d)}$ and ending at $c_{j(\sigma(d))}$, where $\sigma(1), \dots, \sigma(k)$ are different, i.e., $\sigma = \sigma_{\mathcal{P}}$ is a permutation on $[k]$. (In particular, $\sigma_{\mathcal{P}}$ is identical if $\mathcal{P}$ is a flow.)

We naturally partition the set of all path systems for G and $(I|J)$ into the set $\Phi(I|J)$ of $(I|J)$-flows and the rest $\Psi(I|J)$ (consisting of those path systems that contain intersecting paths). The following property easily follows from the planarity of G (cf. [1, Lemma 4.2]):

(A.3) For any $\mathcal{P} = (P_1, \dots, P_k) \in \Psi(I|J)$, there exist two *consecutive* intersecting paths P_d, P_{d+1}.

The *q-sign* of a permutation σ is defined by

$$\mathrm{sgn}_q(\sigma) := (-q)^{\ell(\sigma)},$$

where $\ell(\sigma)$ is the length of σ (see Sect. 2).

Now we start computing the q-minor $[I|J]$ of the matrix Path_G with the following chain of equalities:

$$\begin{aligned}
[I|J] &= \sum\nolimits_{\sigma \in S_k} \mathrm{sgn}_q(\sigma) \left(\prod\nolimits_{d=1}^{k} \mathrm{Path}_G(i(d)|j(\sigma(d))) \right) \\
&= \sum\nolimits_{\sigma \in S_k} \mathrm{sgn}_q(\sigma) \left(\prod\nolimits_{d=1}^{k} \left(\sum (w(P) \,:\, P \in \Phi_G(i(d)|j(\sigma(d))) \right) \right) \\
&= \sum (\mathrm{sgn}_q(\sigma_{\mathcal{P}}) w(\mathcal{P}) \,:\, \mathcal{P} \in \Phi(I|J) \cup \Psi(I|J)) \\
&= \sum (w(\mathcal{P}) \,:\, \mathcal{P} \in \Phi(I|J)) + \sum (\mathrm{sgn}_q(\sigma_{\mathcal{P}}) w(\mathcal{P}) \,:\, \mathcal{P} \in \Psi(I|J)).
\end{aligned}$$

Thus, we have to show that the second sum in the last line is zero. It will follow from the existence of an involution $\eta : \Psi(I|J) \to \Psi(I|J)$ without fixed points such that for each $\mathcal{P} \in \Psi(I|J)$,

$$\mathrm{sgn}_q(\sigma_{\mathcal{P}}) w(\mathcal{P}) = -\mathrm{sgn}_q(\sigma_{\eta(\mathcal{P})}) w(\eta(\mathcal{P})). \tag{A.4}$$

To construct the desired η, consider $\mathcal{P} = (P_1, \dots, P_k) \in \Psi(I|J)$, take the minimal i such that P_i and P_{i+1} meet, take the last common vertex v of these paths, represent P_i as the concatenation $K \circ L$, and P_{i+1} as $K' \circ L'$, so that $t_K = t_{K'} = s_L = s_{L'} = v$, and exchange the portions L, L' of these paths, forming $Q_i := K \circ L'$ and $Q_{i+1} := K' \circ L$. Then we assign $\eta(\mathcal{P})$ to be obtained from $\mathcal{P}$ by replacing P_i, P_{i+1} by Q_i, Q_{i+1}. It is routine to check that η is indeed an involution (with $\eta(\mathcal{P}) \neq \mathcal{P}$) and that

$$\ell(\sigma_{\eta(\mathcal{P})}) = \ell(\sigma_{\mathcal{P}}) + 1, \tag{A.5}$$

assuming w.l.o.g. that $\sigma(i) < \sigma(i+1)$. On the other hand, applying to the paths K, L, K', L' Lemmas A.2 and A.4, one can obtain

$$\begin{aligned} w(P_i)w(P_{i+1}) &= w(K)w(L)w(K')w(L') = qw(K)w(L)w(L')w(K') \\ &= q^2 w(K)w(L')w(L)w(K') = qw(K)w(L')w(K')w(L) = qw(Q_i)w(Q_{i+1}), \end{aligned}$$

whence $w(\mathcal{P}) = qw(\eta(\mathcal{P}))$. This together with (A.5) gives

$$\mathrm{sgn}_q(\sigma_{\mathcal{P}})w(\mathcal{P}) + \mathrm{sgn}_q(\sigma_{\eta(\mathcal{P})})w(\eta(\mathcal{P})) = (-q)^{\ell(\sigma_{\mathcal{P}})} qw(\eta(\mathcal{P})) + (-q)^{\ell(\sigma_{\mathcal{P}})+1} w(\eta(\mathcal{P})) = 0,$$

yielding (A.4), and the result follows. □

Appendix B. Proof of Theorem 4.4

Using notation as in the hypotheses of this theorem, we first consider the case when

(C): $\pi = \{f, g\}$ is a C-couple in $M(\phi, \phi')$ with $f < g$ and $f \in J$.

(Then $f \in J^\circ$ and $g \in J^\bullet$.) We have to prove that

$$w(\phi)w(\phi') = qw(\psi)w(\psi') \tag{B.1}$$

The proof is given throughout Sects. B.1–B.5. The other possible cases in Theorem 4.4 will be discussed in Sect. B.6.

B.1. Snakes and links

Let Z be the exchange path determined by π (i.e., $Z = P(\pi)$ in notation of Sect. 4). It connects the sinks c_f and c_g, which may be regarded as the first and last vertices of Z, respectively. Then Z is representable as a concatenation $Z = \overline{Z}_1 \circ Z_2 \circ \overline{Z}_3 \circ \ldots \circ \overline{Z}_{k-1} \circ Z_k$, where k is even, each Z_i with i odd (even) is a directed path contained in ϕ (resp. ϕ'), and $\overline{Z}_i$ stands for the path reversed to Z_i. More precisely, let $z_0 := c_f$, $z_k := c_g$, and for $i = 1, \dots, k-1$, let z_i denote the common endvertex of Z_i and Z_{i+1}. Then each Z_i with

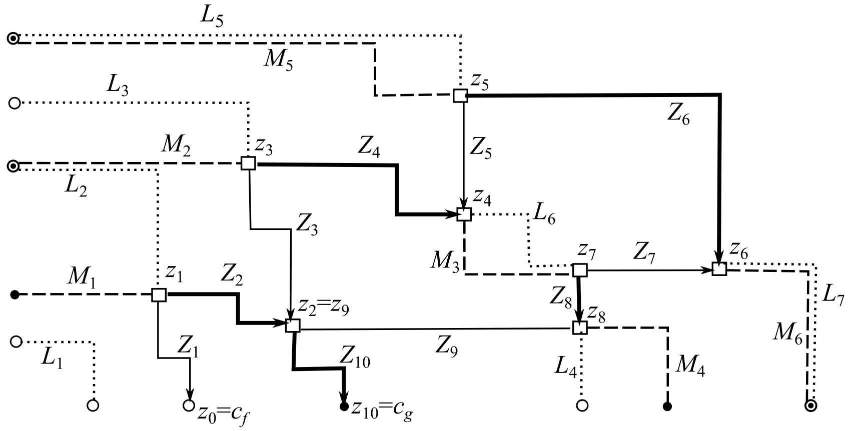


Fig. 4. Here: the bends $z_1, \ldots, z_9$ are marked by squares; the white and black snakes are drawn by thin and thick solid zigzag lines, respectively; the white links $L_1, \ldots, L_7$ are drawn by short-dotted lines, and the black links $M_1, \ldots, M_6$ by long-dotted lines.

i odd is a directed path from z_i to z_{i-1} in $\langle E_\phi - E_{\phi'}\rangle$, while each Z_i with i even is a directed path from z_{i-1} to z_i in $\langle E_{\phi'} - E_\phi\rangle$.

We refer to Z_i with i odd (even) as a *white* (resp. *black*) *snake*.

Also we refer to the vertices $z_1, \ldots, z_{k-1}$ as the *bends* of Z. A bend z_i is called a *peak* (a *pit*) if both path Z_i, Z_{i+1} leave (resp. enter) z_i; then $z_1, z_3, \ldots, z_{k-1}$ are the peaks, and $z_2, z_4, \ldots, z_{k-2}$ are the pits. Note that some peak z_i may coincide with some pit z_j; in this case we say that z_i, z_j are *twins*.

The rests of flows ϕ and ϕ' consist of directed paths that we call *white* and *black links*, respectively. More precisely, the white (black) links correspond to the connected components of the subgraph ϕ (resp. ϕ') from which the interiors of all snakes are removed. So a link connects either (a) a source and a sink (being a component of ϕ or ϕ'), or (b) a source and a pit, or (c) a peak and a sink, or (d) a peak and a pit. We say that a link is *unbounded* in case (a), *semi-bounded* in cases (b), (c), and *bounded* in case (d). Note that

(B.2) a bend z_i occurs as an endvertex in exactly four paths among snakes and links, namely: either in two snakes and two links (of different colors), or in four snakes $Z_i, Z_{i+1}, Z_j, Z_{j+1}$ (when z_i, z_j are twins).

We denote the sets of snakes and links (for ϕ, ϕ', π) by $\mathcal{S}$ and $\mathcal{L}$, respectively; the corresponding subsets of white and black elements of these sets are denoted as $\mathcal{S}^\circ$, $\mathcal{S}^\bullet$, $\mathcal{L}^\circ$, $\mathcal{L}^\bullet$. An example with $k = 10$ is drawn in Fig. 4.

The weight $w(\phi)w(\phi')$ of the double flow (ϕ, ϕ') can be written as the corresponding ordered product of the weights of snakes and links; let $\mathcal{N}$ be the string (sequence) of snakes and links in this product. The weight of the double flow (ψ, ψ') uses a string consisting of the same snakes and links but occurring in another order; we denote this string by $\mathcal{N}^*$.

We say that two elements among snakes and links are *invariant* if they occur in the same order in $\mathcal{N}$ and $\mathcal{N}^*$, and *permuting* otherwise. In particular, two links of different colors are invariant, whereas two snakes of different colors are always permitting.

For example, observe that the string $\mathcal{N}$ for the example in Fig. 4 is viewed as

$$L_1L_2Z_1L_3Z_3Z_9L_4L_5Z_5L_6Z_7L_7M_1Z_2Z_{10}M_2Z_4M_3Z_8M_4M_5Z_6M_6,$$

whereas $\mathcal{N}^*$ is viewed as

$$L_1L_2Z_2Z_{10}L_3Z_4L_6Z_8L_4L_5Z_6L_7M_1Z_1M_2Z_3Z_9M_4M_5Z_5M_3Z_7M_6.$$

For $A, B \in \mathcal{S} \cup \mathcal{L}$, we write $A \prec B$ (resp. $A \prec^* B$) if A occurs in $\mathcal{N}$ (resp. in $\mathcal{N}^*$) earlier than B. We define $\varphi_{A,B} = \varphi_{B,A} := 1$ if A, B are invariant, and define $\varphi_{A,B} = \varphi_{B,A}$ by the relation

$$w(A)w(B) = \varphi_{A,B}w(B)w(A) \tag{B.3}$$

if A, B are permuting and $A \prec B$. Note that $\varphi_{A,B}$ is defined somewhat differently than $\varphi(P,Q)$ in Sect. A.1.

For $A, B \in \mathcal{S} \cup \mathcal{L}$, we may use notation (A, B) when A, B are permuting and $A \prec B$ (and usually write $\{A, B\}$ when their orders by $\prec$ and $\prec^*$ are not important for us).

Our goal is to prove that in case (C),

$$\prod(\varphi_{A,B} \,:\, A, B \in \mathcal{S} \cup \mathcal{L}) = q, \tag{B.4}$$

whence (B.1) will immediately follow.

We first consider the *non-degenerate* case. This means the following restriction:

(B.5) all coordinates $\alpha(z_1), \dots, \alpha(z_{k-1}), \alpha(c_j)$, $j \in J \cup J'$, are different.

The proof of (B.4) subject to (B.5) will consist of three stages I, II, III where we compute the total contribution from the pairs of links, the pairs of snakes, and the pairs consisting of one snake and one link, respectively. As a consequence, the following three results will be obtained (implying (B.4)).

Proposition B.1. *In case (B.5), the product φ^I of the values $\varphi_{A,B}$ over all links $A, B \in \mathcal{L}$ is equal to 1.*

Proposition B.2. *In case (B.5), the product φ^{II} of the values $\varphi_{A,B}$ over all snakes $A, B \in \mathcal{S}$ is equal to q.*

Proposition B.3. *In case (B.5), the product φ^{III} of the values $\varphi_{A,B}$ where one of A, B is a snake and the other is a link is equal to 1.*

These propositions are proved in Sects. B.2–B.4. Sometimes it will be convenient for us to refer to a white (black) snake/link concerning ϕ, ϕ', π as a ϕ-snake/link (resp. a ϕ'-snake/link), and similarly for ψ, ψ', π.

B.2. Proof of Proposition B.1

Under the exchange operation for (ϕ, ϕ') using Z, any ϕ-link becomes a ψ-link and any ϕ'-link becomes a ψ'-link. The white links occur in $\mathcal{N}$ earlier than the black links, and similarly for $\mathcal{N}^*$. Therefore, if A, B are permuting links, then they are of the same color. This implies that $A \cap B = \emptyset$. Also each endvertex of any link either is a bend or belongs to $R \cup C$. Then (B.5) implies that the sets $\{\alpha(s_A), \alpha(t_A)\} \cap \mathbb{R}_{>0}$ and $\{\alpha(s_B), \alpha(t_B)\} \cap \mathbb{R}_{>0}$ are disjoint. Now Lemma A.1 gives $\varphi_{A,B} = 1$, and the proposition follows. $\square$

B.3. Proof of Proposition B.2

Consider two snakes $A = Z_i$ and $B = Z_j$, and let $A \prec B$. If $|i - j| > 1$ then $A \cap B = \emptyset$ and, moreover, $\{\alpha(s_A), \alpha(t_A)\} \cap \{\alpha(s_B), \alpha(t_B)\} = \emptyset$ (in view of (B.5) and since Z is simple). This gives $\varphi_{A,B} = 1$, by Lemma A.1.

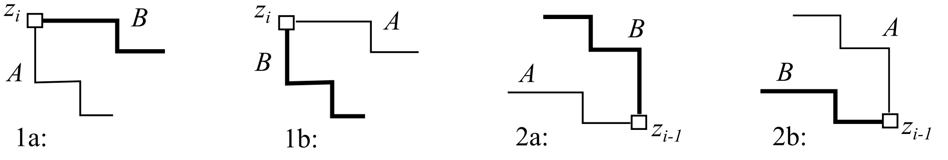
Now let $|i - j| = 1$. Then A, B have different colors; hence A is white and B is black (in view of $A \prec B$). So i is odd, and two cases are possible:

Case 1: $j = i + 1$ and z_i is a peak: $z_i = s_A = s_B$;
Case 2: $j = i - 1$ and z_{i-1} is a pit: $z_{i-1} = t_A = t_B$.

Each of these cases falls into two subcases (using the term "lower" from Appendix A).

Subcase 1a: $j = i + 1$ and A is lower than B.
Subcase 1b: $j = i + 1$ and B is lower than A.
Subcase 2a: $j = i - 1$ and A is lower than B.
Subcase 2b: $j = i - 1$ and B is lower than A.

These subcases are illustrated in the picture:



Under the exchange operation using Z, any snake changes its color; so A, B are permuting. Applying to A, B Lemmas A.2 and A.3, we obtain $\varphi_{A,B} = q$ in Subcases 1a, 2a, and $\varphi_{A,B} = \overline{q}$ in Subcases 1b, 2b.

It is convenient to associate with a bend z the number $\gamma(z)$ which is equal to $+1$ if, for the corresponding pair $A \in \mathcal{S}^\circ$ and $B \in \mathcal{S}^\bullet$ sharing z, A is lower than B (as in Subcases 1a, 2a), and equal to -1 otherwise (as in Subcases 1b, 2b). Define

$$\gamma_Z := \sum(\gamma(z) : z \text{ a bend of } Z). \tag{B.6}$$

Then $\varphi^{II} = q^{\gamma_Z}$. Thus, $\varphi^{II} = q$ is equivalent to

$$\gamma_Z = 1. \tag{B.7}$$

To show (B.7), we are forced to deal with a more general setting. More precisely, let us turn Z into simple cycle D by combining the directed path Z_1 (from z_1 to $z_0 = c_f$) with the horizontal path from c_f to c_g (to create the latter, we formally add to G the horizontal edges (c_j, c_{j+1}) for $j = f, \dots, g-1$). The resulting directed path $\widetilde{Z}_1$ from z_1 to $c_g = z_k$ is regarded as the new white snake replacing Z_1. Then $\widetilde{Z}_1$ shares the end z_k with the black path Z_k; so z_k is a pit of D, and $\widetilde{Z}$ is lower than Z_k. Thus, compared with Z, the cycle D acquires an additional bend, namely, z_k. We have $\gamma(z_k) = 1$, implying $\gamma_D = \gamma_Z + 1$. Then (B.7) is equivalent to $\gamma_D = 2$.

On this way, we come to a new (more general) setting by considering an arbitrary simple (non-directed) cycle D rather than a special path Z. Moreover, instead of an SE-graph as before, we can work with a more general directed planar graph G in which any edge $e = (u, v)$ points arbitrarily within the south-east angle, i.e., satisfies $\alpha(u) \le \alpha(v)$ and $\beta(u) \ge \beta(v)$. We call G of this sort a *weak SE-graph*.

So now we are given a colored simple cycle D in G, i.e., D is representable as a concatenation $\overline{D}_1 \circ D_2 \circ \dots \circ \overline{D}_{k-1} \circ D_k$, where each D_i is a directed path in G; a path ("snake") D_i with i odd (even) is colored white (resp. black). Let $d_1, \dots, d_k$ be the sequence of bends in D, i.e., d_i is a common endvertex of D_i and D_{i+1} (letting $D_{k+1} := D_1$). We assume that D is oriented according to the direction of D_i with i even. When this orientation is clockwise (counterclockwise) around the bounded region O_D of the plane surrounded by D, we say that D is *clockwise* (resp. *counterclockwise*). Then the cycle arising from the above path Z is clockwise.

Our goal is to prove the following

Lemma B.4. *Let D be a colored simple cycle in a weak SE-graph G. If D is clockwise then $\gamma_D = 2$. If D is counterclockwise then $\gamma_D = -2$.*

(Note that this need not hold for a self-intersecting colored closed curve D.)

Proof. We use induction on the number $\eta(D)$ of bends in D. It suffices to consider the case when D is clockwise (since for a counterclockwise cycle $D' = \overline{D}'_1 \circ D'_2 \circ \dots \circ \overline{D}'_{k-1} \circ D'_k$, the reversed cycle $\overline{D}' = \overline{D}'_k \circ D'_{k-1} \circ \dots \circ \overline{D}'_2 \circ D'_1$ is clockwise, and it is easy to see that $\gamma_{\overline{D}'} = -\gamma_{D'}$).

W.l.o.g., one may assume that the coordinates $\beta(d_i)$ of all bends d_i are different (as we can make, if needed, a due small perturbation of D, which does not affect γ).

If $\eta(D) = 2$, then $D = \overline{D}_1 \circ D_2$, and the clockwise orientation of D implies that the path D_1 is lower than D_2. So $\gamma(d_1) = \gamma(d_2) = 1$, implying $\gamma_D = 2$.

Now assume that $\eta(D) > 2$. Then at least one of the following is true:

(a) there exists a peak d_i such that the horizontal line through d_i meets D on the left of d_i, i.e., there is a point x in D with $\alpha(x) < \alpha(d_i)$ and $\beta(x) = \beta(d_i)$;

(b) there exists a pit d_i such that the horizontal line through d_i meets D on the right of d_i.

(This can be seen as follows. Let d_j be a peak with $\beta(d_j)$ maximum. Then the clockwise orientation of D implies that D_{j+1} lies on the right from D_j. If $\beta(d_{j-1}) < \beta(d_{j+1})$, then, by easy topological reasonings, either the pit d_{j+1} is as required in (b) (when d_{j+2} is on the right from D_{j+1}), or the peak d_{j+2} is as required in (a) (when d_{j+2} is on the left from D_{j+1}), or both. And if $\beta(d_{j-1}) > \beta(d_{j+1})$, then d_{j-1} is as in (b).)

We may assume that case (a) takes place (as case (b) is symmetric to (a), in a sense). Choose the point x as in (a) with $\alpha(x)$ maximum and draw the horizontal line-segment L connecting the points x and d_i. Then the interior of L does not meet D. Two cases are possible:

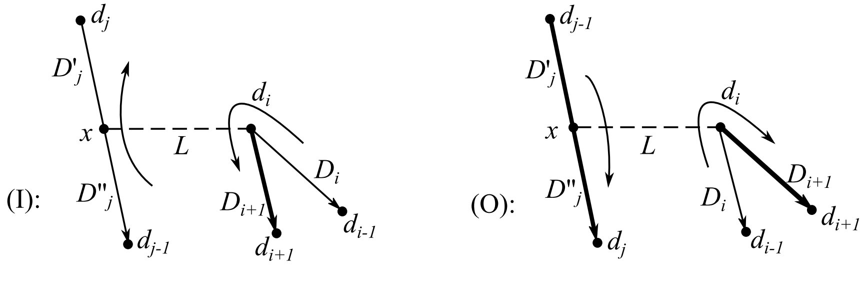
(I) Int(L) is contained in the region O_D; or

(O) Int(L) is outside O_D.

Since x cannot be a bend of D (in view of $\beta(x) = \beta(d_i)$ and $\beta(d_i) \neq \beta(d_{i'})$ for any $i' \neq i$), x is an interior point of some snake D_j; let D'_j and D''_j be the parts of D_j from s_{D_j} to x and from x to t_{D_j}, respectively. Using the facts that D is oriented clockwise and this orientation is agreeable with the forward (backward) direction of each black (resp. white) snake, one can realize that

(B.8) (a) in case (I), D_j is white and $\gamma(d_i) = -1$ (i.e., for the white snake D_i and black snake D_{i+1} that share the peak d_i, D_{i+1} is lower than D_i); and (b) in case (O), D_j is black and $\gamma(d_i) = 1$ (i.e., D_i is lower than D_{i+1}).

See the picture (where the orientation of D in each case is indicated):

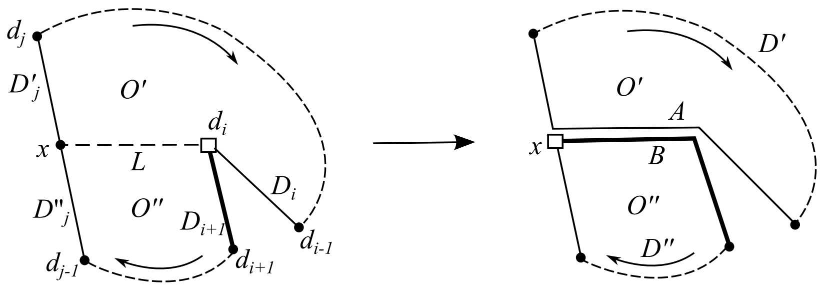


The points x and d_i split the cycle (closed curve) D into two parts ζ', ζ'', where the former contains D'_j and the latter does D''_j.

We first examine case (I). The line L divides the region O_D into two parts O' and O'' lying above and below L, respectively. Orienting the curve ζ' from x to d_i and adding to it the segment L oriented from d_i to x, we obtain closed curve D' surrounding O'. Note that D' is oriented clockwise around O'. We combine the paths D'_j, L (from x to d_i) and D_i into one directed path A (going from $s_{D'_j} = s_{D_j} = d_j$ to $t_{D_i} = d_{i-1}$). Then D' turns into a correctly colored simple cycle in which A is regarded as a white snake and the white/black snakes structure of the rest preserves (cf. (B.8)(a)).

In its turn, the curve ζ'' oriented from d_i to x plus the segment L (oriented from x to d_i) form closed curve D'' that surrounds O'' and is oriented clockwise as well. We combine L and D_{i+1} into one black snake B (going from x to d_{i+1}). Then D'' becomes a correctly colored cycle, and x is a peak in it. (The point x becomes a vertex of G.) We have $\gamma(x) = 1$ (since the white D''_j is lower than the black B).

The creation of D', D'' from D in case (I) is illustrated in the picture:



We observe that, compared with D, the pair $\{D', D''\}$ misses the bend d_i (with $\gamma(d_i) = -1$) but acquires the bend x (with $\gamma(x) = 1$). Then

$$\eta(D) = \eta(D') + \eta(D''), \tag{B.9}$$

implying $\eta(D'), \eta(D'') < \eta(D)$. Therefore, we can apply induction. This gives $\gamma_{D'} = \gamma_{D''} = 2$. Now, by reasonings above,

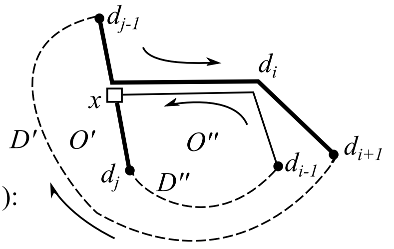
$$\gamma_D = \gamma_{D'} + \gamma_{D''} + \gamma(d_i) - \gamma(x) = 2 + 2 - 1 - 1 = 2,$$

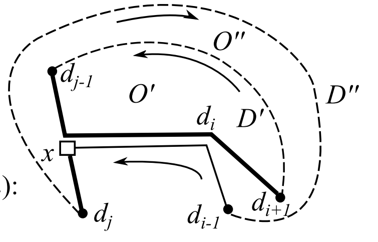
as required.

Next we examine case (O). The curve ζ' (containing D'_j) passes through the black snake D_{i+1}, and the curve ζ'' (containing D''_j) through the white snake D_i. Adding to each of ζ', ζ'' a copy of L, we obtain closed curves D', D'', respectively, each inheriting the orientation of D. They become correctly colored simple cycles when we combine the paths D'_j, L, D_{i+1} into one black snake (from d_{j-1} to d_{i+1}) in D', and combine the paths L, D_i into one white snake (from the new bend x to d_i) in D''. Let O', O'' be the bounded regions in the plane surrounded by D', D'', respectively. Two cases are possible (as illustrated in the picture below):

(O1) O' includes O'' (and O_D);

(O2) O'' includes O' (and O_D).

(O1): 

(O2): 

Observe that in case (O1), D' is clockwise and D'' is counterclockwise, whereas in case (O2) the behavior is converse. Also $\gamma(d_i) = 1$ and $\gamma(x) = -1$. Like case (I), relation (B.9) is true and we can apply induction. Then in case (O1), we have $\gamma_{D'} = 2$ and $\gamma_{D''} = -2$, whence

$$\gamma_D = \gamma_{D'} + \gamma_{D''} + \gamma(d_i) - \gamma(x) = 2 - 2 + 1 - (-1) = 2.$$

And in case (O2), we have $\gamma_{D'} = -2$ and $\gamma_{D''} = 2$, whence

$$\gamma_D = \gamma_{D'} + \gamma_{D''} + \gamma(d_i) - \gamma(x) = -2 + 2 + 1 - (-1) = 2.$$

Thus, in all cases we obtain $\gamma_D = 2$, yielding the lemma. □

This completes the proof of Proposition B.2. □

B.4. Proof of Proposition B.3

Consider a link L. By Lemma A.1, for any snake P, $\varphi_{L,P} \neq 1$ is possible only if L and P have a common endvertex v. Note that $v \notin R \cup C$.

First assume that $s_L \notin R$. Then there are exactly two snakes containing s_L, namely, a white snake A and a black snake B such that $s_L = t_A = t_B$. If L is white, then A and L belong to the same path in ϕ; therefore, $A \prec L \prec B$. Under the exchange operation A becomes black, B becomes white, and L continues to be white. Then B, L belong to the same path in ψ; this implies $B \prec^* L \prec^* A$. So both pairs (A, L) and (L, B) are permuting. Lemma A.4 gives $\varphi_{A,L} = q$ and $\varphi_{L,B} = \overline{q}$, whence $\varphi_{A,L}\varphi_{L,B} = 1$.

Now let L be black. Then $A \prec B \prec L$ and $B \prec^* A \prec^* L$. So both pairs $\{A, L\}$ and $\{B, L\}$ are invariant, whence $\varphi_{A,L} = \varphi_{B,L} = 1$.

Next we assume that $t_L \notin C$. Then there are exactly two snakes, a white snake A' and a black snake B', that contain t_L, namely: $t_L = s_{A'} = s_{B'}$. If L is white, then $L \prec A' \prec B'$ and $L \prec^* B' \prec^* A'$. Therefore, $\{L, A\}$ and $\{L, B'\}$ are invariant, yielding $\varphi_{L,A'} = \varphi_{L,B'} = 1$. And if L is black, then $A' \prec L \prec B'$ and $B' \prec^* L \prec^* A'$. So both (A', L) and (L, B') are permuting, and we obtain from Lemma A.4 that $\varphi_{A',L} = \overline{q}$ and $\varphi_{L,B'} = q$, yielding $\varphi_{A',L}\varphi_{L,B'} = 1$.

These reasonings prove the proposition. □

B.5. Degenerate case

We have proved relation (B.4) in a non-degenerate case, i.e., subject to (B.5), and now our goal is to prove (B.4) when the set

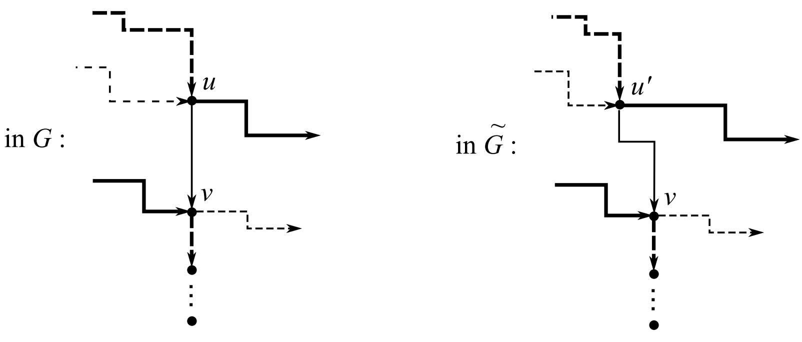
$$\mathcal{Z} := \{z_1, \dots, z_{k-1}\} \cup \{c_j : j \in J \cup J'\}$$

contains distinct elements u, v with $\alpha(u) = \alpha(v)$. We say that such u, v form a *defect pair*. A special defect pair is formed by twins z_i, z_j (i.e., bends satisfying $i \neq j$, $\alpha(z_i) = \alpha(z_j)$ and $\beta(z_i) = \beta(z_j)$). Another special defect pair is of the form $\{s_P, t_P\}$ when P is a *vertical* snake or link, i.e., $\alpha(s_P) = \alpha(t_P)$.

We will show (B.4) by induction on the number of defect pairs.

Let a be the *minimum* number such that the set $X := \{u \in \mathcal{Z} : \alpha(u) = a\}$ contains a defect pair. We denote the elements of X as $v_0, v_1, \dots, v_r$, where for each i, v_{i-1} is *higher* than v_i, which means that either $\beta(v_{i-1}) > \beta(v_i)$, or v_{i-1}, v_i are twins and v_{i-1} is a pit (while v_i is a peak) in the exchange path Z. The highest element v_0 is also denoted by u.

In order to conduct induction, we deform the graph G within a sufficiently narrow vertical strip $S = [a - \epsilon, a + \epsilon] \times \mathbb{R}$ (where $0 < \epsilon < \min\{|\alpha(z) - a| : z \in \mathcal{Z} - X\}$) to get rid of the defect pairs involving u in such a way that the configuration of snakes/links in the arising graph $\widetilde{G}$ remains "equivalent" to the initial one. More precisely, we shift the bend u at a small distance ($< \epsilon$) to the left, keeping the remaining elements of $\mathcal{Z}$; then the bend u' arising in place of u satisfies $\alpha(u') < \alpha(u)$ and $\beta(u') = \beta(u)$. The snakes/links with an endvertex at u are transformed accordingly; see the picture for an example.



Let Π and $\widetilde{\Pi}$ denote the L.H.S. value in (B.4) for the initial and deformed configurations, respectively. Under the deformation, the number of defect pairs becomes smaller, so we may assume by induction that $\widetilde{\Pi} = q$. Thus, we have to prove that

$$\Pi = \widetilde{\Pi}. \tag{B.10}$$

We need some notation and conventions. For $v \in X$, the set of (initial) snakes and links with an endvertex at v is denoted by $\mathcal{P}_v$. For $U \subseteq X$, $\mathcal{P}_U$ denotes $\cup(\mathcal{P}_v : v \in U)$.

Corresponding objects for the deformed graph $\widetilde{G}$ are usually denoted with tildes as well; e.g.: for a path P in G, its image in $\widetilde{G}$ is denoted by $\widetilde{P}$; the image of $\mathcal{P}_v$ is denoted by $\widetilde{\mathcal{P}}_v$ (or $\widetilde{\mathcal{P}}_{\widetilde{v}}$), and so on. The set of standard paths in $\mathcal{P}_U$ (resp. $\widetilde{\mathcal{P}}_U$) is denoted by $\mathcal{P}^{\rm st}_U$ (resp. $\widetilde{\mathcal{P}}^{\rm st}_U$). Define

$$\varPi_{u,X-u} := \prod(\varphi_{P,Q} \colon P \in \mathcal{P}_u,\ Q \in \mathcal{P}_{X-u}). \tag{B.11}$$

A similar product for $\widetilde{G}$ (i.e., with $\widetilde{\mathcal{P}}_u$ instead of $\mathcal{P}_u$) is denoted by $\widetilde{\varPi}_{u,X-u}$.

Note that (B.10) is equivalent to the equality

$$\varPi_{u,X-u} = \widetilde{\varPi}_{u,X-u}. \tag{B.12}$$

This follows from the fact that for any paths $P, Q \in \mathcal{S} \cup \mathcal{L}$ different from those involved in (B.11), the values $\varphi_{P,Q}$ and $\varphi_{\widetilde{P},\widetilde{Q}}$ are equal. (The only nontrivial case arises when $P, Q \in \mathcal{P}_u$ and Q is vertical (so $\widetilde{Q}$ becomes standard). Then $t_Q = v_1$. Hence $Q \in \mathcal{P}_{X-u}$, the pair P, Q is involved in $\varPi_{u,X-u}$, and the pair $\widetilde{P}, \widetilde{Q}$ in $\widetilde{\varPi}_{u,X-u}$.)

To simplify our description technically, one trick will be of use. Suppose that for each standard path $P \in \mathcal{P}^{\rm st}_X$, we choose a point (not necessarily a vertex) $v_P \in {\rm Int}(P)$ in such a way that $\alpha(s_P) < \alpha(v_P) < \alpha(t_P)$, and the coordinates $\alpha(v_P)$ for all such paths P are different. Then v_P splits P into two subpaths P', P'', where we denote by P' the subpath connecting s_P and v_P when $\alpha(s_P) = a$, and connecting v_P and t_P when $\alpha(t_P) = a$, while P'' is the rest. This provides the following property: for any $P, Q \in \mathcal{P}^{\rm st}_X$, $\varphi_{P',Q''} = \varphi_{Q',P''} = 1$ (in view of Lemma A.1). Hence $\varphi_{P,Q} = \varphi_{P',Q'}\varphi_{P'',Q''}$. Also $P'' = \widetilde{P}''$. It follows that (B.12) would be equivalent to the equality

$$\prod(\varphi_{P',Q'} \colon P \in \mathcal{P}_u,\ Q \in \mathcal{P}_{X-u}) = \prod(\varphi_{\widetilde{P}',\widetilde{Q}'} \colon P \in \mathcal{P}_u,\ Q \in \mathcal{P}_{X-u}).$$

In light of these observations, it suffices to prove (B.12) in the special case when

(B.13) any $P \in \mathcal{P}_u$ and $Q \in \mathcal{P}_{X-u}$ satisfy $\{\alpha(s_P), \alpha(t_P)\} \cap \{\alpha(s_Q), \alpha(t_Q)\} = \{a\}$.

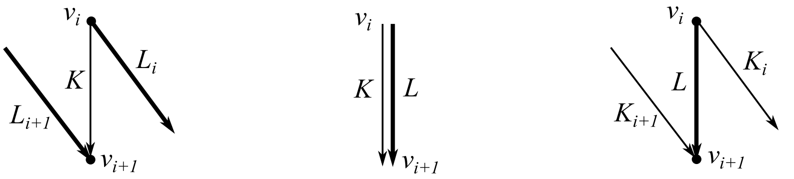
For $i = 0, \dots, r$, we denote by A_i, B_i, K_i, L_i, respectively, the white snake, black snake, white link, and black link that have an endvertex at v_i. Note that if v_{i-1}, v_i are twins, then the fact that v_{i-1} is a pit implies that A_{i-1}, B_{i-1} are the snakes entering v_{i-1}, and A_i, B_i are the snakes leaving v_i; for convenience, we formally define $K_{i-1}, K_i, L_{i-1}, L_i$ to be the same trivial path consisting of the single vertex v_i. Note that if $v_r \in C$, then some paths among A_r, B_r, K_r, L_r vanish (e.g., both snakes and one link).

When vertices v_i and v_{i+1} are connected by a (vertical) path in $\mathcal{S} \cup \mathcal{L}$, we denote such a path by P_i and say that the vertex v_i is *open*; otherwise v_i is said to be closed. Note that v_i, v_{i+1} can be connected by either one snake, or one link, or two links (namely, K_i, L_i); in the latter case, P_i is chosen arbitrarily among them. In particular, if v_i, v_{i+1} are twins, then v_i is open and the role of P_i is played by any of the trivial links K_i, L_i.

Obviously, in a sequence of vertical paths $P_i, P_{i+1}, \dots, P_j$, the snakes and links alternate. One can see that if P_i is a white snake, i.e., $P_i = A_i = A_{i+1} =: A$, then both black snakes B_i, B_{i+1} are standard, and we have $v_i = s_{B_i}$ and $v_{i+1} = t_{B_{i+1}}$. See the left fragment of the picture:

Symmetrically, if P_i is a black snake: $B_i = B_{i+1} =: B$, then the white snakes A_i, A_{i+1} are standard, $v_i = s_{A_i}$ and $v_{i+1} = t_{A_{i+1}}$; see the right fragment of the above picture.

In its turn, if P_i is a nontrivial white link, i.e., $P_i = K_i = K_{i+1}$, then two cases are possible: either the black links L_i, L_{i+1} are standard, $v_i = s_{L_i}$ and $v_{i+1} = t_{L_{i+1}}$, or $L_i = L_{i+1} = P_i$. And if P_i is a black link, the behavior is symmetric. See the picture:



Now we are ready to start proving equality (B.12). Note that the deformation of G preserves both orders $\prec$ and $\prec^*$.

We say that paths $P, P' \in \mathcal{P}_X^{\rm st}$ are *separated* (from each other) if they are not contained in the same path of any of the flows ϕ, ϕ', ψ, ψ'. The following observation will be of use:

(B.14) if $P, P' \in \mathcal{P}_X^{\rm st}$ have the same color (concerning ϕ, ϕ'), are separated, and P' is lower than P, then $P' \prec P$; and similarly w.r.t. $\psi, \psi', \prec^*$.

Indeed, suppose that P, P' are white, and let Q and Q' be the components of the flow ϕ containing P and P', respectively. Since P, P' are separated, the paths Q, Q' are different. Moreover, the fact that P' is lower than P implies that Q' is lower than Q (since Q, Q' are disjoint). Then Q' precedes Q in ϕ, yielding $P' \prec P$, as required. When P, P' concern one of ϕ', ψ, ψ', the argument is similar.

In what follows we will use the abbreviated notation A, B, K, L for the paths A_0, B_0, K_0, L_0 (respectively) having an endvertex at $u = v_0$. Also for $R \in \mathcal{P}_{X-u}$, we denote the product $\varphi_{A,R}\varphi_{B,R}\varphi_{K,R}\varphi_{L,R}$ by $\Pi(R)$, and denote by $\widetilde{\Pi}(R)$ a similar product for the paths $\widetilde{A}, \widetilde{B}, \widetilde{K}, \widetilde{L}, \widetilde{R}$ (in $\widetilde{G}$). One can see that $\Pi_{u,X-u}$ (resp. $\widetilde{\Pi}_{u,X-u}$) is equal to the product of the values $\Pi(R)$ (resp. $\widetilde{\Pi}(R)$) over $R \in \mathcal{P}_{X-u}$.

To show (B.12), we examine several cases. First we consider

Case (R1): the vertex u is closed; in other words, all paths A, B, K, L are standard.

Proposition B.5. *In case (R1), $\Pi(R) = \widetilde{\Pi}(R) = 1$ holds for any $R \in \mathcal{P}_{X-u}$. As a consequence, (B.12) is valid.*

Proof. Let $R \in \mathcal{P}_{v_p}$ for $p \geq 1$. Observe that (B.13) together with the fact that the vertex u moves to the left under the deformation $G \mapsto \widetilde{G}$ implies $\{\alpha(s_{\widetilde{P}}), \alpha(t_{\widetilde{P}})\} \cap \{\alpha(s_{\widetilde{R}}), \alpha(t_{\widetilde{R}})\} = \emptyset$ for any $P \in \mathcal{P}_u$. This gives $\widetilde{\Pi}(R) = 1$, by Lemma A.1.

To show $\Pi(R) = 1$, assume that R is standard (otherwise this equality is trivial). Since u is closed, A, B, K, L are separated from R.

Note that A, B, K, L, R are as follows: either (a) $t_A = t_B = s_K = s_L$ or (b) $s_A = s_B = t_K = t_L$, and either (c) $\alpha(s_R) = a$ or (d) $\alpha(t_R) = a$. Let us examine the possible cases when the combination of (a) and (d) takes place.

1) Let R be a white link, i.e., $R = K_p$. Since R is white and lower than A, B, K, L, we have $R \prec A, B, K, L$ (cf. (B.14)). The exchange operation preserves the color of R. Then $R \prec^* A, B, K, L$. Therefore, all pairs $\{P, R\}$ with $P \in \mathcal{P}_u$ are invariant, and $\Pi(R) = 1$ is trivial.

2) Let $R = L_p$. Since R is black, we have $A, K \prec R \prec B, L$. The exchange operation changes the colors of A, B and preserves the ones of K, L, R. Hence $B, K \prec^* R \prec^* A, L$, giving the permuting pairs (A, R) and (R, B). Lemma A.3 applied to these pairs implies $\varphi_{A,R} = \overline{q}$ and $\varphi_{R,B} = q$. Then $\Pi(R) = \varphi_{A,R}\varphi_{R,B} = \overline{q}q = 1$.

3) Let $R = A_p$. Then $R \prec A, B, K, L$ and $B, K \prec^* R \prec^* A, L$ (since the exchange operation changes the colors of A, B, R). This gives the permuting pairs (R, B) and (R, K). Then $\varphi_{R,B} = q$, by Lemma A.3, and $\varphi_{R,K} = \overline{q}$ by Lemma A.5, and we obtain $\Pi(R) = \varphi_{R,B}\varphi_{R,K} = 1$.

4) Let $R = B_p$. We have $A, K \prec R \prec B, L$ and $R \prec^* A, B, K, L$, giving the permuting pairs (A, R) and (K, R). Then $\varphi_{A,R} = \overline{q}$, by Lemma A.3, and $\varphi_{K,R} = q$, by Lemma A.5, whence $\Pi(R) = 1$.

The other combinations, namely, (a) and (c), (b) and (c), (b) and (d), are examined in a similar way (by appealing to appropriate lemmas from Appendix A), and we leave this to the reader as an exercise. □

Next we consider

Case (R2): u is open; in other words, at least one path among A, B, K, L is vertical (going from $u = v_0$ to v_1).

It falls into several subcases examined in propositions below.

Proposition B.6. *In case (R2), let $R \in \mathcal{P}^{\rm st}_{X-u}$ be separated from A, B, K, L. Then $\Pi(R) = \widetilde{\Pi}(R)$.*

Proof. We first assume that u and v_1 are connected by exactly one path P_0 (which is one of A, B, K, L) and give a reduction to the previous proposition, as follows.

Suppose that we replace P_0 by a standard path P' of the same color and type (snake or link) such that $s_{P'} = u$ (and $\alpha(t_{P'}) > a$). Then the set $\mathcal{P}'_u := (\{A, B, K, L\} - \{P_0\}) \cup \{P'\}$

becomes as in case (R1), and by Proposition B.5, the corresponding product $\Pi'(R)$ of values $\varphi_{R,Q}$ over $Q \in \mathcal{P}'_u$ is equal to 1. (This relies on the fact that R is separated from A, B, K, L.)

Compare the effects from P' and from $\widetilde{P}_0$. These paths have the same color and type, and both are separated from, and higher than R. Also $\alpha(s_{P'}) = \alpha(t_{\widetilde{P}_0}) = a$. Then using appropriate lemmas from Appendix A, one can conclude that $\{\varphi_{R,P'}, \varphi_{R,\widetilde{P}_0}\} = \{q, \overline{q}\}$. Therefore,

$$\widetilde{\Pi}(R) = \varphi_{R,\widetilde{P}_0} = \Pi'(R)\varphi^{-1}_{R,P'} = \Pi(R).$$

Now let u and v_1 be connected by two paths, namely, by K, L. We again can appeal to Proposition B.5. Consider $\mathcal{P}''_u := \{A, B, K'', L''\}$, where K'', L'' are standard links (white and black, respectively) with $s_{K''} = s_{L''} = u$. Then $\Pi''(R) := \prod(\varphi_{R,P} \colon P \in \mathcal{P}''_u) = 1$ and $\{\varphi_{R,K''}, \varphi_{R,\widetilde{K}}\} = \{\varphi_{R,L''}, \varphi_{R,\widetilde{L}}\} = \{q, \overline{q}\}$. We obtain

$$\widetilde{\Pi}(R) = \varphi_{R,\widetilde{K}}\varphi_{R,\widetilde{L}} = \Pi''(R)\varphi^{-1}_{R,K''}\varphi^{-1}_{R,L''} = \varphi_{R,A}\varphi_{R,B} = \Pi(R),$$

as required. □

Proposition B.7. *In case (R2), let R be a standard path in $\mathcal{P}_{v_p}$ with $p \geq 1$. Let R be not separated from at least one of A, B, K, L. Then $\Pi(R) = \widetilde{\Pi}(R)$.*

Proof. We first assume that P_0 is the unique vertical path connecting u and v_1 (in particular, u and v_1 are not twins). Then R is not separated from P_0.

Suppose that P_0 and R are contained in the same path of the flow ϕ; equivalently, both P_0, R are white and $P_0 \prec R$. Then neither ψ nor ψ' has a path containing both P_0, R (this is easy to conclude from the fact that one of R and P_{p-1} is a snake and the other is a link). Consider four possible cases for P_0, R.

(a) Let both P_0, R be links, i.e., $P_0 = K$ and $R = K_p$. Then $A, K \prec K_p \prec B, L$ and $K_p \prec^* B, K, A, L$ (since $K \prec^* K_p$ is impossible by the above observation). This gives the permuting pairs (A, K_p) and $(\widetilde{K}, K_p)$, yielding $\varphi_{A,K_p} = \varphi_{\widetilde{K},K_p}$.

(b) Let $P_0 = K$ and $R = A_p$. Then $A, K \prec A_p \prec B, L$ and $B, K \prec^* A_p \prec^* A, L$. This gives the permuting pairs (A, A_p) and (A_p, B), yielding $\varphi_{A,A_p}\varphi_{\widetilde{A}_p,B} = 1 = \varphi_{\widetilde{K},A_p}$.

(c) Let $P_0 = A$ and $R = K_p$. Then $K, A \prec K_p \prec L, B$ and $K_p \prec^* K, B, L, A$. This gives the permuting pairs (K, K_p) and $(\widetilde{A}, K_p)$, yielding $\varphi_{K,K_p} = \varphi_{\widetilde{A},K_p}$.

(d) Let $P_0 = A$ and $R = A_p$. Then $K, A \prec A_p \prec L, B$ and $K, B \prec^* A_p \prec^* L, A$. This gives the permuting pairs $(\widetilde{A}, A_p)$ and (A_p, B), yielding $\varphi_{\widetilde{A},A_p} = \varphi_{A_p,B}$.

In all cases, we obtain $\Pi(R) = \widetilde{\Pi}(R)$.

When P_0, R are contained in the same path in ϕ' (i.e., P_0, R are black and $P_0 \prec R$), we argue in a similar way. The cases with P_0, R contained in the same path of ψ or ψ' are symmetric.

A similar analysis is applicable (yielding $\Pi(R) = \widetilde{\Pi}(R)$) when u and v_1 are connected by two vertical paths (namely, K, L) and exactly one relation among $K \prec R$, $L \prec R$,

$K \prec^* R$ and $L \prec^* R$ takes place (equivalently: either K, R are separated or L, R are separated, not both).

Finally, let u and v_1 be connected by both K, L, and assume that K, R are not separated, and L, R are not separated as well. An important special case is when $p = 1$ and u, v_1 are twins. From the assumption it easily follows that R is a snake. If R is the white snake A_p, then we have $A, K \prec A_p \prec B, L$ and $B, K, A, L \prec^* A_p$. This gives the permuting pairs (A_p, B) and $(A_p, \widetilde{L})$, yielding $\varphi_{A_p,B} = \varphi_{A_p,\widetilde{L}}$ (since $\alpha(t_B) = \alpha(t_{\widetilde{L}})$). The case with $R = B_p$ is symmetric. In both cases, $\Pi(R) = \widetilde{\Pi}(R)$. □

Proposition B.8. *Let $R = P_0$ be the unique vertical path connecting u and v_1. Then $\Pi(R) = \widetilde{\Pi}(R) = 1$.*

Proof. The equality $\Pi(R) = 1$ is trivial. To see $\widetilde{\Pi}(R) = 1$, consider possible cases for R. If $R = K$, then $\widetilde{A} \prec \widetilde{K} \prec \widetilde{B}, \widetilde{L}$ and $\widetilde{B} \prec^* \widetilde{K} \prec^* \widetilde{A}, \widetilde{L}$, giving the permuting pairs $(\widetilde{A}, \widetilde{K})$ and $(\widetilde{K}, \widetilde{B})$ (note that $t_{\widetilde{A}} = t_{\widetilde{B}} = s_{\widetilde{K}} = \widetilde{u}$). If $R = L$, then $\widetilde{A}, \widetilde{K}, \widetilde{B} \prec \widetilde{L}$ and $\widetilde{B}, \widetilde{K}, \widetilde{A} \prec^* \widetilde{L}$; so all pairs involving $\widetilde{L}$ are invariant. If $R = A$, then $\widetilde{K} \prec \widetilde{A} \prec \widetilde{L}, \widetilde{B}$ and $\widetilde{K}, \widetilde{B}, \widetilde{L} \prec^* \widetilde{A}$, giving the permuting pairs $(\widetilde{A}, \widetilde{L})$ and $(\widetilde{A}, \widetilde{B})$ (note that $s_{\widetilde{A}} = s_{\widetilde{B}} = t_{\widetilde{L}} = \widetilde{u}$). And the case $R = B$ is symmetric to the previous one.

In all cases, using appropriate lemmas from Appendix A (and relying on the fact that all paths $\widetilde{A}, \widetilde{B}, \widetilde{K}, \widetilde{L}$ are standard), one can conclude that $\widetilde{\Pi}(R) = 1$. □

Proposition B.9. *Let both K, L be vertical. Then $\Pi(K)\Pi(L) = \widetilde{\Pi}(K)\widetilde{\Pi}(L) = 1$.*

Proof. The equality $\Pi(K)\Pi(L) = 1$ is trivial. To see $\widetilde{\Pi}(K)\widetilde{\Pi}(L) = 1$, observe that $\widetilde{A} \prec \widetilde{K} \prec \widetilde{B} \prec \widetilde{L}$ and $\widetilde{B} \prec^* \widetilde{K} \prec^* \widetilde{A} \prec^* \widetilde{L}$. This gives the permuting pairs $(\widetilde{A}, \widetilde{K})$ and $(\widetilde{K}, \widetilde{B})$. By Lemma A.4, $\varphi_{\widetilde{A},\widetilde{K}} = q$ and $\varphi_{\widetilde{K},\widetilde{B}} = \overline{q}$, and the result follows. □

Taken together, Propositions B.6–B.9 embrace all possibilities in case (R2). Adding to them Proposition B.5 concerning case (R1), we obtain the desired relation (B.12) in a degenerate case.

This completes the proof of Theorem 4.4 in case (C), namely, relation (B.1). □

B.6. Other cases

Let $(I|J), (I'|J'), \phi, \phi', \psi, \psi'$ and $\pi = \{f, g\}$ be as in the hypotheses of Theorem 4.4. We have proved this theorem in case (C), i.e., when π is a C-couple with $f < g$ and $f \in J$ (see the beginning of Appendix B). In other words, the exchange path $Z = P(\pi)$, used to transform the initial double flow (ϕ, ϕ') into the new double flow (ψ, ψ'), connects the sinks c_f and c_g covered by the "white flow" ϕ and the "black flow" ϕ', respectively.

The other possible cases in the theorem are as follows:

(C1) π is a C-couple with $f < g$ and $f \in J'$;
(C2) π is an R-couple with $f < g$ and $f \in I$;

(C3) π is an R-couple with $f < g$ and $f \in I'$;
(C4) π is an RC-couple with $f \in I$ and $g \in J$;
(C5) π is an RC-couple with $f \in I'$ and $g \in J'$.

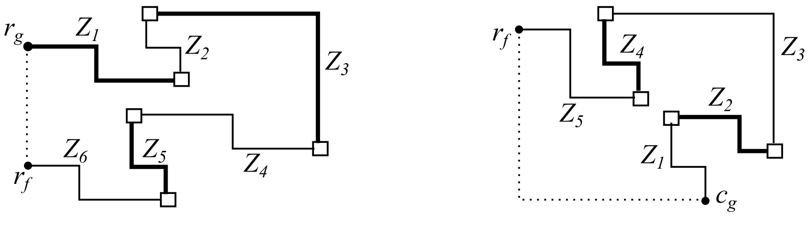
Case (C1) is symmetric to (C). This means that if double flows (ϕ, ϕ') and (ψ, ψ') are obtained from each other by applying the exchange operation using π (which changes the "colors" of both f and g), and if one double flow is subject to (C), then the other is subject to (C1). Rewriting $w(\phi)w(\phi') = qw(\psi)w(\psi')$ as $w(\psi)w(\psi') = q^{-1}w(\phi)w(\phi')$, we just obtain the required equality in case (C1) (where (ψ, ψ') and (ϕ, ϕ') play the roles of the initial and updated double flows, respectively).

For similar reasons, case (C3) is symmetric to (C2), and (C5) is symmetric to (C4). So it suffices to establish the desired equalities merely in cases (C2) and (C4).

To do this, we appeal to reasonings similar to those in Sects. B.2–B.5. More precisely, one can check that the descriptions in Sects. B.2 and B.4 (concerning link-link and snake-link pairs in $\mathcal{N}$) remain applicable and Propositions B.1 and B.3 are directly extended to cases (C2) and (C4). The method of getting rid of degeneracies developed in Sect. B.5 does work, without any troubles, for (C2) and (C4) as well.

As to the method in Sect. B.3 (concerning snake-snake pairs in case (C)), it should be modified as follows. We use terminology and notation from Sects. B.1 and B.3 and appeal to Lemma B.4.

When dealing with case (C2), we represent the exchange path $Z = P(\pi)$ as a concatenation $Z_1 \circ \overline{Z}_2 \circ Z_3 \circ \cdots \circ \overline{Z}_k$, where each Z_i with i odd (even) is a snake contained in the black flow ϕ' (resp. the white flow ϕ). Then Z_1 begins at the source r_g and Z_k begins at the source r_f. An example with $k = 6$ is illustrated in the left fragment of the picture:



The common vertex (bend) of Z_i and Z_{i+1} is denoted by z_i. As before, we associate with a bend z the number $\gamma(z)$ (equal to 1 if, in the pair of snakes sharing z, the white snake is lower that the black one, and -1 otherwise), and define γ_Z as in (B.6). We turn Z into simple cycle D by combining the directed path Z_k (from r_f to z_{k-1}) with the vertical path from r_g to r_f, which is formally added to G. (In the above picture, this path is drawn by a dotted line.) Then, compared with Z, the cycle D has an additional bend, namely, r_g. Since the extended white path $\widetilde{Z}_k$ is lower than the black path Z_1, we have $\gamma(r_g) = 1$, and therefore $\gamma_D = \gamma_Z + 1$.

One can see that the cycle D is oriented clockwise (where, as before, the orientation of D is agreeable with that of black snakes). So $\gamma_D = 2$, by Lemma B.4, implying $\gamma_Z = 1$. This is equivalent to the "snake-snake relation" $\varphi^{II} = q$, and as a consequence, we obtain the desired equality

$$w(\phi)w(\phi') = qw(\psi)w(\psi').$$

Finally, in case (C4), we represent the exchange path Z as the corresponding concatenation $\overline{Z}_1 \circ Z_2 \circ \overline{Z}_3 \circ \cdots \circ Z_{k-1} \circ \overline{Z}_k$ (with k odd), where the first white snake Z_1 ends at the sink c_g and the last white snake Z_k begins at the source r_f. See the right fragment of the above picture where $k = 5$. We turn Z into simple cycle D by adding a new "black snake" Z_{k+1} beginning at r_f and ending at c_g (it is formed by the vertical path from r_f to $(0, 0)$, followed by the horizontal path from $(0, 0)$ to c_g; see the above picture). Compared with Z, the cycle D has two additional bends, namely, r_f and c_g. Since the black snake Z_{k+1} is lower than each of Z_1, Z_k, we have $\gamma(r_f) = \gamma(c_g) = -1$, whence $\gamma_D = \gamma_Z - 2$. Note that the cycle D is oriented counterclockwise. Therefore, $\gamma_D = -2$, by Lemma B.4, implying $\gamma_Z = 0$. As a result, we obtain the desired equality $w(\phi)w(\phi') = w(\psi)w(\psi')$.

This completes the proof of Theorem 4.4. □

References

[1] K. Casteels, A graph theoretic method for determining generating sets of prime ideals in quantum matrices, J. Algebra 330 (2011) 188–205.
[2] K. Casteels, Quantum matrices by paths, Algebra Number Theory 8 (8) (2014) 1857–1912.
[3] G. Cauchon, Spectre premier de $O_q(M_n(k))$: image canonique et séparation normale, J. Algebra 260 (2) (2003) 519–569.
[4] V. Danilov, A. Karzanov, G. Koshevoy, Planar flows and quadratic relations over semirings, J. Algebraic Combin. 36 (2012) 441–474.
[5] C.L. Dodgson, Condensation of determinants, Proc. Royal Soc. Lond. 15 (1866) 150–155.
[6] R. Fioresi, Commutation relations among quantum minors in $O_q(M_n(k))$, J. Algebra 280 (2) (2004) 655–682.
[7] K.R. Goodearl, Commutation relations for arbitrary quantum minors, Pacific J. Math. 228 (1) (2006) 63–102.
[8] D. Krob, B. Leclerc, Minor identities for quasi-determinants and quantum determinants, Comm. Math. Phys. 169 (1995) 1–23.
[9] V. Lakshmibai, N. Reshetikhin, Quantum flag and Schubert schemes, Contemp. Math. 134 (1992) 145–181.
[10] B. Leclerc, A. Zelevinsky, Quasicommuting families of quantum Plücker coordinates, Amer. Math. Soc. Transl. Ser. 2 181 (1998) 85–108.
[11] B. Lindström, On the vector representations of induced matroids, Bull. Lond. Math. Soc. 5 (1973) 85–90.
[12] Yu.I. Manin, Quantum Groups and Non Commutative Geometry, vol. 49, Centre de Recherches Mathematiques, Montreal, 1988.
[13] A. Schrijver, Combinatorial Optimization, vol. A, Springer, Berlin, 2003.
[14] J.S. Scott, Quasi-commuting families of quantum minors, J. Algebra 290 (1) (2005) 204–220.
[15] E. Taft, J. Towber, Quantum deformation of flag schemes and Grassmann schemes. I. q-deformation of the shape-algebra for GL(n), J. Algebra 142 (1991) 1–36.