Cubillages on cyclic zonotopes, membranes, and higher separation

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Abstract. We study certain structural properties of fine zonotopal tilings, or *cubillages*, on cyclic zonotopes $Z(n,d)$ of an arbitrary dimension d and their relations to $(d-1)$-separated collections of subsets of a set $\{1,2,\dots,n\}$. (Collections of this sort are well known as *strongly separated* ones when $d=2$, and as *chord separated* ones when $d=3$.)

Keywords: zonotope, cubillage, higher Bruhat order, strongly separated sets, chord separated sets

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1 Introduction

We consider a generalization of the notions of strongly separated and chord separated set-systems. Let n be a positive integer and denote the set $\{1,2,\dots,n\}$ by $[n]$.

Definition. Let $r\in[n-1]$. Two sets $A,B\subseteq[n]$ are called *r-separated* (from each other) if there is no sequence $i_0<i_1<\dots<i_{r+1}$ of elements of $[n]$ such that the elements with even indices (namely, $i_0,i_2,\dots$) and the elements with odd indices $(i_1,i_3,\dots)$ belong to different sets among $A-B$ and $B-A$ (where $A'-B'$ denotes the set difference $\{i\colon A'\ni i\notin B'\}$). In other words, one can choose $r'\le r$ integers ("separating points") $a_1\le a_2\le\dots\le a_{r'}$ in $[n]$ such that the intervals $[a_i,a_{i+1}]$ with i even cover one of $A-B$ and $B-A$, while the ones with i odd cover the other of these sets, where $a_{r'+1}:=n$ and $[a,b]$ denotes $\{a,a+1,\dots,b\}$. Accordingly, a collection (set-system) $\mathcal{A}\subseteq 2^{[n]}$ is called r-separated if any two of its members are such.

We denote the set of all inclusion-wise maximal r-separated collections $\mathcal{A}$ in $2^{[n]}$ as $\mathbf{S}_{n,r+1}$, and the maximal size $|\mathcal{A}|$ of such an $\mathcal{A}$ by $s_{n,r+1}$ (for technical reasons, we prefer to use the subscript pair $(n,r+1)$ rather than (n,r)). When all collections in $\mathbf{S}_{n,r+1}$ are of the same size, $\mathbf{S}_{n,r+1}$ is said to be *pure*.

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In particular, $\mathbf{S}_{n,n}$ consists of the unique collection $2^{[n]}$ (since any two subsets of $[n]$ are $(n-1)$-separated), giving the simplest purity case.

The concept of 1-separation was introduced, under the name of *strong separation*, by Leclerc and Zelevinsky [5] who proved the important fact that

(1.1) for any $n \ge 2$, the set $\mathbf{S}_{n,2}$ is pure (and $s_{n,2}$ equals $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}=\frac{1}{2}n(n+1)+1$).

Recently an analogous purity result on 2-separation was shown by Galashin [3]:

(1.2) for any $n \ge 3$, the set $\mathbf{S}_{n,3}$ is pure (and $s_{n,3}=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}$).

(In [3], 2-separated sets $A, B \subseteq [n]$ are called *chord separated*, which is justified by the observation that if n points $1, 2, \dots, n$ are disposed on a circumference O, in this order cyclically, then there is a chord to O separating $A-B$ from $B-A$.)

However, a nice purity behavior as above for r-separated set-systems with $r=1,2$ is not extended in general to larger r's, as it follows from profound results on oriented matroids due to Galashin and Postnikov [4]. Being specified to r-separated set-systems, the following property is obtained.

Theorem 1.1 [4] $\mathbf{S}_{n,r+1}$ *is pure if and only if* $\min\{r, n-r\} \le 2$.

It should be noted that the proof given in [4] is rather sophisticated and long (even if one extracts merely those arguments that directly concern the sets $\mathbf{S}_{n,r+1}$), and one purpose of this paper is to present a shorter and more transparent proof of Theorem 1.1.

In fact, the content of this paper is wider. In particular, we are going to demonstrate representable cases of extendable and non-extendable set-systems. Here we say that $\mathcal{A} \subseteq 2^{[n]}$ is $(n, r+1)$-*extendable* if there exists a maximal by size r-separated collection in $2^{[n]}$ including $\mathcal{A}$. (So $\mathbf{S}_{n,r+1}$ is pure if and only if any r-separated set-system in $2^{[n]}$ is $(n, r+1)$-extendable.)

Our study of separated set-systems is based on a geometric approach whose theoretical grounds were originated in the classical work by Manin and Schechtman [6] where *higher Bruhat orders*, generalizing weak Bruhat ones, were introduced and well studied. (Recall that the higher Bruhat order for (n, d) compares certain (so-called "packet admissible") total orders on the set $\binom{[n]}{d}$ of d-elements subsets of $[n]$, and it turns into the weak one when $d=1$, which compares permutations on $[n]$.) Subsequently Voevodskij and Kapranov [7] and Ziegler [8] gave nice geometric interpretations and established additional important results.

Based on these sources, we deal with a *cyclic zonotope* $Z=Z(n,d)$, that is the Minkowski sum of n segments in $\mathbb{R}^d$ forming a *cyclic configuration*, and consider a fine zonotopal tiling, that is a subdivision Q of Z into parallelotopes; we call it a *cubillage* for short. The vertices of Q are associated, in a natural way, with subsets of $[n]$, forming a collection $\mathrm{Sp}(Q) \subseteq 2^{[n]}$, called the *spectrum* of Q. In the special case $d=2$, Q is viewed as a rhombus tiling on a zonogon, and it is well known due to [5] that the spectra of these are exactly the maximal strongly separated set-systems in $2^{[n]}$. Similarly, the spectra of cubillages on $Z(n,3)$ are exactly the maximal chord separated set-systems in

$2^{[n]}$, as is shown in [3]. It turned out that this phenomenon is extended to an arbitrary d: the cubillages Q on $Z(n,d)$ are bijective to the *maximal by size* $(d-1)$-separated set-systems $\mathcal{S}$ in $2^{[n]}$, with the equality $\mathrm{Sp}(Q)=\mathcal{S}$; see [4].

This paper is organized as follows. Section 2 reviews definitions and basic properties of cyclic zonotopes and cubillages. Section 3 gives a short proof of the non-purity of $\mathbf{S}_{6,4}$, which is the crucial case in our method of proof of Theorem 1.1. As a by-product, we obtain non-(n,4)-extendable 3-separated collections consisting of only three sets. To show the other non-purity cases in Theorem 1.1, we need to use additional notions and constructions, which generalize those exhibited in [1] for $d=3$ and are discussed in Section 4. Here we introduce a *membrane* in a cubillage Q on $Z(n,d)$, to be a special $(d-1)$-dimensional subcomplex M in Q (when the latter is regarded as the corresponding polyhedral complex). An important fact is that any cubillage on $Z(n,d-1)$ can be lifted as a membrane in some cubillage on $Z(n,d)$. Also we describe nice operations on cubillages on $Z(n,d)$ (called *contraction* and *expansion* ones) that produce cubillages on $Z(n-1,d)$ and $Z(n+1,d)$.

Section 5 utilizes this machinery to show relations between (n,d)-, $(n+1,d)$-, and $(n+1,d+1)$-extendable set-systems. As a consequence, we easily prove the remaining non-purity cases in Theorem 1.1, relying on the above result for $(n,d)=(6,4)$. Also we demonstrate in this section one interesting class of extendable set-systems (in Proposition 5.3) and raise two open questions. Section 6 is devoted to additional results involving *inversions* of membranes. These objects arise as a natural generalization of the classical notion of inversions for permutations, and their definition for an arbitrary (n,d) goes back to Manin and Schechtman [6]. In particular, we show that for two membranes M and M' in the same cubillage on $Z(n,d)$, if any inversion of M is an inversion of M', then $\mathrm{Sp}(M)\cup\mathrm{Sp}(M')$ is $(d-1)$-separated (see Theorem 6.4).

2 Cyclic zonotopes and cubillages

The objects that we deal with live in the euclidean space $\mathbb{R}^d$ of dimension $d>1$. A *cyclic configuration* of size $n\ge d$ is meant to be an ordered set Ξ of n vectors $\xi_1=\xi(t_i),\dots,\xi_n=\xi(t_n)$ in $\mathbb{R}^d$ lying on the Veronese curve $\xi(t)=(1,t,t^2,\dots,t^{d-1})$, $t\in\mathbb{R}$, and satisfying $t_1<\dots<t_n$. A useful property of Ξ is that

(2.1) any d vectors $\xi_{i(1)},\dots,\xi_{i(d)}$ with $i(1)<\dots<i(d)$ are independent and, moreover, $\det(A)>0$, where A is the matrix whose j-th column is $\xi_{i(j)}$.

In addition, we will also assume that Ξ is $\mathbb{Z}_2$-independent (i.e., all combinations of vectors of Ξ with coefficients 0,1 are different).

The configuration Ξ generates the (*cyclic*) *zonotope* $Z=Z(\Xi)$ in $\mathbb{R}^d$, the polytope represented as the Minkowski sum of line segments $[0,\xi_i]$, $i=1,\dots,n$. An object of our interest is a *fine zonotopal tiling* on Z, that is a subdivision Q of Z into d-dimensional parallelotopes of which any two intersecting ones share a common face, and each facet (a face of codimension 1) of the boundary $\partial(Z)$ of Z is contained in one of these parallelotopes. For brevity, we liberally refer to these parallelotopes as *cubes*, and to Q

as a *cubillage*. In fact, depending on the context, we may think of a cubillage Q in two ways: either as a set of d-dimensional cubes (and may write $C \in Q$ for a cube C in Q) or as the corresponding polyhedral complex. One can see that

(2.2) each cube in Q is viewed as

$$\sum\nolimits_{b\in X}\xi_b+\Big\{\sum(\lambda_{a(i)}\xi_{a(i)}:\ 0\le\lambda_{a(i)}\le 1,\ i=1,\dots,d)\Big\}$$

for some $a(1)<\cdots<a(d)$ and $X\subseteq[n]-a(1)a(2)\cdots a(d)$.

Hereinafter, for a subset $\{a,\dots,a'\}$ of $[n]$, we use the abbreviated notation $a\cdots a'$. When $X\subset[n]$ and $a\cdots a'$ are disjoint, their union may be denoted as $Xa\cdots a'$. Also for a set S and element $i\in S$, we may write $S-i$ for $S-\{i\}$.

For a cube C in (2.2), we say that the set $a(1)\cdots a(d)$ is the *type* of C, denoted as $\tau(C)$. Also, regarding the first coordinate x_1 of a vector (point) $x=(x_1,\dots,x_d)\in\mathbb{R}^d$ as its *height*, we denote the lowest point $\sum_{b\in X}\xi_b$ of C by $\mathrm{bt}(C)$, called the *bottom* of C. The cells of dimensions 0 and 1 in Q are called *vertices* and *edges*, respectively. When needed, each edge e is directed so as to be a parallel translation of corresponding generating vector ξ_i, and we say that e is an edge of *color* i, or an *i-edge*. This forms a directed graph on the vertices of Q, denoted as $G_Q=(V_Q,E_Q)$.

The subsets $X\subseteq[n]$ are naturally identified with the corresponding points $\sum_{b\in X}\xi_b$ in Z (which are different due the $\mathbb{Z}_2$-independence of Ξ). This represents each vertex of Q as a subset of $[n]$, and the collection of these subsets is called the *spectrum* of Q and denoted by $\mathrm{Sp}(Q)$.

Note that structural properties of cubillages depend on n and d, but the choice of a cyclic configuration Ξ for these parameters is not important in essence; so we may speak of cubillages Q on a generic cyclic zonotope, denoted as $Z(n,d)$. There are known a number of nice properties of Q. Among those, two rather elementary ones are as follows:

(2.3) all types $\tau(C)$ of cubes $C\in Q$ are different and range the set $\binom{[n]}{d}$ of d-element subsets of $[n]$ (so there are exactly $\binom{n}{d}$ cubes in Q); and

(2.4) $|\mathrm{Sp}(Q)|=\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{d}$.

One more useful property of cubillages (which is shown for $d=3$ in [1, Prop. 3.5]) and can be straightforwardly extended to an arbitrary d) is:

(2.5) suppose that for disjoint subsets X,A of $[n]$, a cubillage Q contains the vertices of the form $X\cup A'$ for all $A'\subseteq A$; then these vertices belong to a cube in Q.

In particular, if Q has vertices X and Xi, where $i\in[n]-X$, then Q contains the edge connecting these vertices.

A less trivial fact is shown in [4]; it says that

(2.6) the correspondence $Q \mapsto \mathrm{Sp}(Q)$ gives a bijection between the set $\mathbf{Q}_{n,d}$ of cubillages on $Z(n,d)$ and the set $\mathbf{S}^*_{n,d}$ of maximal *by size* $(d-1)$-separated collections in $2^{[n]}$ (in particular, $|\mathrm{Sp}(Q)| = s_{n,d}$).

In light of (1.1) and (1.2), $\mathbf{S}^*_{n,d} = \mathbf{S}_{n,d}$ when $d = 2, 3$, and in view of (2.6), all maximal strongly separated and chord separated set-systems in $2^{[d]}$ are represented by the spectra of corresponding cubillages; these facts were established for $d = 2$ in [5] (using equivalent terms of pseudo-line arrangements), and for $d = 3$ in [3]. On the other hand, Theorem 1.1 asserts that $\mathbf{S}^*_{n,d} \neq \mathbf{S}_{n,d}$ when $d \geq 4$ and $n \geq d + 2$.

In our further analysis we will use additional facts on the structure of the boundary $\partial(Z)$ of a zonotope $Z = Z(n,d)$. Let us say that a set $X \subseteq [n]$ is a *k-pieced cortege* if it is the union of k intervals (including the case $k = 0$). As a non-difficult exercise, one can obtain the following description for the collection $\mathrm{Sp}(Z)$ $(= \mathrm{Sp}(\partial(Z)))$ of subsets of $[n]$ represented by the vertices of Z:

(2.7) for $Z = Z(n,d)$, $\mathrm{Sp}(Z)$ consists of exactly those sets $X \subseteq [n]$ that are $(d-1)$-separated from any subset of $[n]$; specifically: when d is even, $\mathrm{Sp}(Z)$ is formed by all $d/2$-pieced corteges containing at least one of the elements 1 and n and all k-pieced corteges with $k < d/2$, while when d is odd, $\mathrm{Sp}(Z)$ is formed by all $(d+1)/2$-pieced corteges containing both 1 and n and all k-pieced corteges with $k \leq (d-1)/2$.

In particular, $\mathrm{Sp}(Z)$ is included in any collection in $\mathbf{S}_{n,d}$.

In case $n = d$, the zonotope Z turns into one cube and the purity of $\mathbf{S}_{n,n}$ is trivial. And in case $n = d+1$, one can conclude from (2.7) that there are exactly two subsets of $[n]$ that do not belong to $\mathrm{Sp}(Z)$, one being formed by the odd elements, and the other by the even elements of $[n]$, i.e., the sets $X = 135\ldots$ and $Y = 246\ldots$. Clearly they are not $(d-1)$-separated from each other. Therefore, $\mathbf{S}_{n,n-1}$ consists of two collections $\mathrm{Sp}(Z_{n,n-1}) \cup \{X\}$ and $\mathrm{Sp}(Z_{n,n-1}) \cup \{Y\}$, implying that $\mathbf{S}_{n,n-1}$ is pure. This together with (1.1) and (1.2) gives “if” part of Theorem 1.1.

The non-purity cases of this theorem (giving “only if” part) are discussed in Sections 3 and 5.

3 Case $(n,d) = (6,4)$

This case is crucial and will be used as a base to handle the other non-purity cases in Theorem 1.1 (in Sect. 5).

Consider $Z = Z(6,4)$. By (2.7), $\mathrm{Sp}(Z)$ consists of all intervals and all 2-pieced corteges containing 1 or 6. A direct enumeration shows that the number of these amounts to 52. Therefore, $2^6 - 52 = 12$ subsets of [6] are not in $\mathrm{Sp}(Z)$, namely:

(3.1) 24, 245, 25, 235, 35, 135, 1356, 136, 1346, 146, 1246, 246.

(Recall that $a \cdots b$ stands for $\{a, \ldots, b\}$.) Let A_i denote i-th member in this sequence (so $A_1 = 24$ and $A_{12} = 246$). Form the collection

$$\mathcal{A} := \mathrm{Sp}(Z) \cup \{A_1, A_5, A_9\}.$$

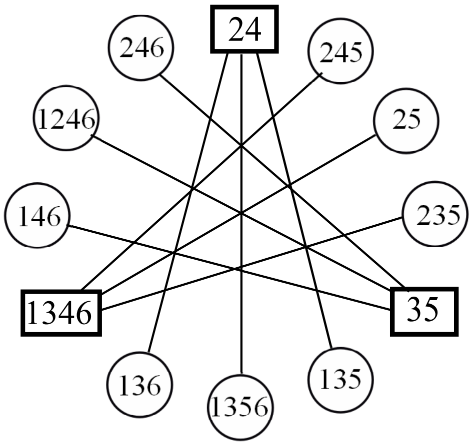
It consists of $52 + 3 = 55$ sets, whereas the number $s_{6,4}$ is equal to $\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 57$. Now the non-purity of $\mathbf{S}_{6,4}$ is implied by the following

Lemma 3.1 *$\mathcal{A}$ is a maximal 3-separated collection in $2^{[6]}$.*

Proof By (2.7), any two $X \in \mathrm{Sp}(Z)$ and $Y \in \mathcal{A}$ are 3-separated. Observe that $|A_{i-1} \triangle A_i| = 1$ for any $1 \le i \le 12$ (where $A_0 := A_{12}$ and $A \triangle B$ denotes the symmetric difference $(A - B) \cup (B - A)$). Then any $A, A' \in \{A_1, A_5, A_9\}$ satisfy $|A \triangle A'| \le 4$. This implies that A and A' are 3-separated. Therefore, the collection $\mathcal{A}$ is 3-separated.

The maximality of $\mathcal{A}$ follows from the observation that adding to $\mathcal{A}$ any member of $\{A_i : 1 \le i \le 12,\ i \ne 1, 5, 9\}$ would violate the 3-separation. Indeed, a routine verification shows that A_1 is not 3-separated from any of A_6, A_7, A_8, and similarly for A_5 and $\{A_{10}, A_{11}, A_{12}\}$, and for A_9 and $\{A_2, A_3, A_4\}$. ∎

Remark 1. To visualize a verification in the above proof, one can use the circular diagram illustrated in the picture below where the sets from (3.1) are disposed in the cyclic order. Here the sets A_1, A_5, A_9 are drawn in boxes and connected by lines with those sets where the 3-separation is violated. Note that, instead of A_1, A_5, A_9, one could take in the lemma any triple of the form A_i, A_{i+4}, A_{i+8} (taking indices modulo 12).



In conclusion of this section recall that a collection in $2^{[n]}$ is called (n,d)-*extendable* if it can be extended to a *maximal by size* $(d-1)$-separated collection in $2^{[n]}$, or, equivalently (in view of (2.6)), if there exists a cubillage on $Z(n,d)$ whose spectrum includes the given collection. An immediate consequence of Lemma 3.1 is the existence of small non-extendable 3-separated collections.

Corollary 3.2 *Any 3-separated triple of the form $\{A_i, A_{i+4}, A_{i+8}\}$, e.g. $\{24, 35, 1346\}$, is not (6,4)-extendable (defining $A_{i'}$ as above and taking indices modulo 12).*

(Here we take into account that any maximal 3-separated collection in $2^{[6]}$ includes $\mathrm{Sp}(Z(6,4))$.) Note that, in view of Lemma 5.1 in Sect. 5, this corollary implies that $\{A_i, A_{i+4}, A_{i+8}\}$ is not $(n,4)$-extendable for any $n \ge 6$.

4 Membranes and pies

For further purposes, we need additional notions, borrowing terminology from [1].

4.1 Membranes. Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection along the last coordinate vector, i.e., sending $x = (x_1, \dots, x_d)$ to $(x_1, \dots, x_{d-1})$. Then $\pi(\Xi) = (\pi(\xi_1), \dots, \pi(\xi_n))$ is again a cyclic configuration and $Z(\pi(\Xi))$ forms a $(d-1)$-dimensional cyclic zonotope. We may use notation $Z(n, d-1)$ for $Z(\pi(\Xi))$ and call it the projection of $Z(n, d)$.

Definition. Let Q be a cubillage on $Z(n, d)$. By a *membrane* in Q we mean a subcomplex M of Q that is bijectively projected by π to $Z(n, d-1)$.

In particular, M has dimension $(d-1)$, each facet of M is projected to a $(d-1)$-dimensional "cube" (represented as in (2.2) where each $\xi_\bullet$ should be replaced by $\pi(\xi_\bullet)$), and these "cubes" constitute a cubillage of $Z(n, d-1)$, whence the collection $\mathrm{Sp}(M)$ is $(d-2)$-separated (cf. (2.6)). A converse property says that any cubillage can be lifted as a membrane into some cubillage of the next dimension:

(4.1) for any cubillage Q' on $Z(n, d-1)$, there exists a cubillage Q on $Z(n, d)$ and a membrane M in Q such that $\pi(M) = Q'$.

(This fact can be deduced from results on higher Bruhat orders and related aspects in [6, 7, 8]. See also [1, Sec. 5] for a direct construction when $d = 3$.)

When the choice of Q as in (4.1) is not important for us, we may think of the (abstract) membrane M in $Z(n, d)$ representing a cubillage Q' on $Z(n, d-1)$, saying that M *is obtained by lifting* Q'. (To construct such an M, one should take the points in $Z(n, d)$ representing the same subsets of $[n]$ as those in $\mathrm{Sp}(Q')$ and then extend the corresponding 2^{d-1}-element subsets of these points into $(d-1)$-dimensional "cubes"; cf. (2.5)).

Two membranes are of an especial interest. For a closed set X of points in $Z = Z(n, d)$, let X^{fr} (resp. X^{rear}) be the subset of points of X "seen" in the direction of d-th coordinate vector e_d (resp. $-e_d$), i.e., such that for each $x' \in \pi(X)$, this subset contains the point $x \in \pi^{-1}(x') \cap X$ with the minimal (resp. maximal) value x_d. We call it the *front* (resp. *rear*) side of X. When $X = Z$, the sides Z^{fr} and Z^{rear} (respecting their cell structures) are membranes of any cubillage on Z. The subcomplex $Z^{\mathrm{fr}} \cap Z^{\mathrm{rear}}$ is called the *rim* of Z and denoted as Z^{rim}.

Borrowing terminology from [1], we refer to $\pi(Z^{\mathrm{fr}})$ and $\pi(Z^{\mathrm{rear}})$ are the *standard* and *anti-standard* cubillages on $Z(n, d-1)$, denoted as $Q^{\mathrm{st}}_{n,d-1}$ and $Q^{\mathrm{ant}}_{n,d-1}$, respectively.

As a refinement of (2.7), one can characterize the spectra of Z^{rim}, Z^{fr} and Z^{rear} for $Z = Z(n, d)$ as follows (a check-up is routine and we omit it here):

(4.2) (i) when d is even, $\mathrm{Sp}(Z^{\mathrm{rim}})$ consists of all $d/2$-pieced corteges containing both elements 1 and n, and all k-pieced corteges with $k < d/2$, whereas when d is odd, $\mathrm{Sp}(Z^{\mathrm{rim}})$ consists of all $(d-1)/2$-pieced corteges containing at least one of 1 and n, and all k-pieced corteges with $k < (d-1)/2$;

(ii) $\mathrm{Sp}(Z^{\mathrm{fr}}) - \mathrm{Sp}(Z^{\mathrm{rim}})$ consists of all $d/2$-pieced corteges containing the element 1 but not n when d is even, and consists of all $(d-1)/2$-pieced corteges containing none of 1 and n when d is odd;

(iii) $\mathrm{Sp}(Z^{\mathrm{rear}}) - \mathrm{Sp}(Z^{\mathrm{rim}})$ consists of all $d/2$-pieced corteges containing the element n but not 1 when d is even, and consists of all $(d+1)/2$-pieced corteges containing both 1 and n when d is odd.

4.2 Pies, contraction and expansion. For a cubillage Q on $Z(n,d)$ and $i \in [n]$, let $\Pi_i = \Pi_i(Q)$ be the subcomplex of Q formed by the cubes C having an edge of color i, called *i-cubes*. We refer to Π_i as the *i-pie* in Q. It has the following nice properties (which are shown by attracting standard topological reasonings):

(4.3) (i) Π_i is representable as the "direct Minkowski sum" $\{x = \beta + \alpha \colon \beta \in B_i,\ \alpha \in S_i\}$, where B_i is a subcomplex of Q homeomorphic to a $(d-1)$-dimensional ball and S_i is the segment $[0, \xi_i]$;

(ii) removing from Q the point set $\Pi - (B_i \cup B'_i)$, where $B'_i := B_i + \xi_i$, produces two connected components R and R' containing B_i and B'_i, respectively.

(iii) gluing R with R' shifted by $-\xi_i$, we obtain a cubillage on the zonotope $Z(\Xi - \xi_i)$; it is denoted as Q/i and called the *i-contraction* of Q;

(iv) $\mathrm{Sp}(Q/i)$ consists of all sets $X \subseteq [n] - i$ such that at least one of X, Xi is in $\mathrm{Sp}(Q)$.

When $i = n$, the pie structure becomes more transparent. Namely, the maximality of color n in the type of each cube in Π_n provides that the ball B_n (B'_n) is contained in the front (resp. rear) side of Π_n (a similar fact is also true for $i = 1$ but need not hold when $1 < i < n$). This implies that

(4.4) B_n is a membrane in the reduced cubillage (n-contraction) Q/n.

In other words, the n-contraction operation applied to Q, defined by (ii),(iii) in (4.3), transforms the pie Π_n into a membrane of Q/n. A converse operation blows a membrane into an n-pie.

More precisely, let M be a membrane in a cubillage Q' on the zonotope $Z' = Z(n-1, d)$. Define $Z^-(M)$ ($Z^+(M)$) to be the part of Z' between $(Z')^{\mathrm{fr}}$ and M (resp. between M and $(Z')^{\mathrm{rear}}$) and define $Q^-(M)$ ($Q^+(M)$) to be the subcubillage of Q' contained in $Z^-(M)$ (resp. $Z^+(M)$). The *n-expansion* operation for (Q', M) consists in shifting $Z^+(M)$ equipped with $Q^+(M)$ by the vector ξ_n and filling the "space between" M and $M + \xi_n$ by the corresponding set of n-cubes. More precisely, each $(d-1)$-dimensional cube C' (having type $\tau(C')$ and bottom vertex $\mathrm{bt}(C')$) in M generates the n-dimensional cube $C = C' + [0, \xi_n]$; so $\tau(C) = \tau(C') \cup \{n\}$ and $\mathrm{bt}(C) = \mathrm{bt}(C')$. Then combining $Q^-(M)$ with the shifted subcubillage $Q^+(M) + \xi_n$ and the "blowed membrane" $M + [0, \xi_n]$ (forming an n-pie), we obtain a cubillage on $Z(n,d)$. This cubillage is called the *n-expansion* of Q' using M and denoted as $Q_n(Q', M)$.

The n-contraction and n-expansion operations are naturally related to each other (as a straightforward generalization to an arbitrary d of Proposition 3.4 from [1]):

(4.5) (i) the correspondence $(Q', M) \mapsto Q_n(Q', M)$, where Q' is a cubillage on $Z(n-1, d)$ and M is a membrane in Q', gives a bijection between the set of such pairs (Q', M) in $Z(n-1, d)$ and the set $\mathbf{Q}_{n,d}$ of cubillages on $Z(n, d)$;

(ii) under this correspondence, Q' is the n-contraction Q/n of $Q = Q_n(Q', M)$ and M is the image of the n-pie in Q under the n-contraction operation.

5 Other non-purity cases

Return to proving "only if" part of Theorem 1.1. For any (n, d) with $\min\{d-1, n-d+1\} \geq 3$, we have $n-6 \geq d-4 \geq 0$. Therefore, "only if" part of Theorem 1.1 (with r replaced by $d-1$) will follow from Lemma 3.1 concerning $(n, d) = (6, 4)$ and the next two assertions on lifting set-systems in $2^{[n]}$.

Lemma 5.1 *Let $\mathcal{A} \subseteq 2^{[n]}$. Then $\mathcal{A}$ is (n, d)-extendable if and only if $\mathcal{A}$ is $(n+1, d)$-extendable.*

Lemma 5.2 *Let $\mathcal{A} \subseteq 2^{[n]}$, $n' := n+1$, and $\mathcal{A}' := \{Xn' \colon X \in \mathcal{A}\}$. Then $\mathcal{A}$ is (n, d)-extendable if and only if $\mathcal{A} \cup \mathcal{A}'$ is $(n+1, d+1)$-extendable.*

Proof of Lemma 5.1 Clearly $\mathcal{A}$ is $(d-1)$-separated relative to n if and only if it is $(d-1)$-separated relative to $n+1$.

If $\mathcal{A}$ is (n, d)-extendable, then $\mathcal{A} \subseteq \mathrm{Sp}(Q)$ for some cubillage Q on $Z = Z(n, d)$. Let Q' be the $(n+1)$-extension of Q using as a membrane the rear side Z^{rear} of Z (see Sect. 4.2 for definitions). Then $\mathcal{A} \subseteq \mathrm{Sp}(Q')$, implying that $\mathcal{A}$ is $(n+1, d)$-extendable.

Conversely, if $\mathcal{A}$ is $(n+1, d)$-extendable, then $\mathcal{A} \subseteq \mathrm{Sp}(Q')$ for some cubillage Q' on $Z(n+1, d)$. Let Q be the $(n+1)$-contraction of Q' (i.e., Q is a cubillage on $Z(n, d)$ obtained by shrinking the $(n+1)$-pie in Q', cf. (4.3)). Since the sets in $\mathcal{A}$ do not contain the element $n+1$, their corresponding vertices in Q' preserve under the $(n+1)$-contraction operation. So $\mathcal{A} \subseteq \mathrm{Sp}(Q)$ and therefore $\mathcal{A}$ is (n, d)-extendable. ∎

Proof of Lemma 5.2 Let $\mathcal{A}$ be (n, d)-extendable (in particular, $\mathcal{A}$ is $(d-1)$-separated). Take a cubillage Q on $Z(n, d)$ with $\mathcal{A} \subseteq \mathrm{Sp}(Q)$. By (4.1), there exist a cubillage Q' on $Z(n, d+1)$ and a membrane M in Q' such that $Q = \pi(M)$, where π is the projection $\mathbb{R}^{d+1} \to \mathbb{R}^d$ as in Sect. 4.1. Let Q'' be the n'-expansion of Q' using M. Then Q'' is a cubillage on $Z(n+1, d+1)$. Take the n'-pie $\Pi_{n'}$ in Q''. Then the side $B_{n'}$ of $\Pi_{n'}$ contains the vertices of M, whereas the side $B'_{n'}$ contains the vertices Xn' for $X \in \mathrm{Sp}(M)$ (cf. (4.3)(ii)). Since $\mathcal{A} \subseteq \mathrm{Sp}(M) = \mathrm{Sp}(B_{n'})$, we have $\mathcal{A}' \subseteq \mathrm{Sp}(B'_{n'})$, whence $\mathcal{A} \cup \mathcal{A}' \subseteq \mathrm{Sp}(Q'')$. Thus, $\mathcal{A} \cup \mathcal{A}'$ is $(n+1, d+1)$-extendable.

Conversely, let $\mathcal{A} \cup \mathcal{A}'$ be $(n+1, d+1)$-extendable (in particular, it is d-separated). Then $\mathcal{A} \cup \mathcal{A}' \subseteq \mathrm{Sp}(Q)$ for some cubillage Q on $Z(n+1, d+1)$. By (2.5), any $X \in \mathcal{A}$ must be connected with $Xn' \in \mathcal{A}'$ by an n'-edge in Q. This implies that $\mathcal{A} \cup \mathcal{A}'$ is contained in the n'-pie of Q (in which $\mathcal{A}$ and $\mathcal{A}'$ lie in the sides $B_{n'}$ and $B'_{n'}$ of $\Pi_{n'}$, respectively). Then the n'-contraction operation applied to Q transforms Q and $\Pi_{n'}$ into a cubillage Q' on $Z(n, d+1)$ and a membrane M in Q', preserving the vertices of $B_{n'}$. Hence $\mathcal{A} \subseteq \mathrm{Sp}(M)$, and now taking the cubillage $Q'' := \pi(M)$ on $Z(n,d)$, we obtain $\mathcal{A} \subseteq \mathrm{Sp}(Q'')$, as required. ∎

Note that Lemmas 5.1 and 5.2 can also be used to demonstrate an interesting class of extendable set-systems, as follows.

Proposition 5.3 *Let $k \in [n]$, $\mathcal{B} \subseteq 2^{[k]}$, $D \subseteq \{k+1,\dots,n\}$, and $d := |D|$. Let $\mathcal{C}$ consist of the sets of the form $B \cup D'$ for all $B \in \mathcal{B}$ and $D' \subseteq D$. Suppose that $\mathcal{B}$ is 2-separated. Then $\mathcal{C}$ is $(n, d+3)$-extendable.*

Proof We use induction on $n+d$. When $d = 0$, $\mathcal{C}$ becomes $\mathcal{B}$, and therefore it is $(n,3)$-extendable (by (1.2)).

So assume that $d \geq 1$. Let p be the maximal element in D. If $p < n$ then $\mathcal{C}$ is contained in $2^{[n-1]}$. By Lemma 5.1, $\mathcal{C}$ is $(n, d+3)$-extendable if and only if it is $(n-1, d+3)$-extendable, and we can apply induction.

Now let $p = n$. Form $\mathcal{C}' := \{X \in \mathcal{C} \colon n \notin X\}$ and $\mathcal{C}'' := \{X \in \mathcal{C} \colon n \in X\}$. Then $\mathcal{C}' \cap \mathcal{C}'' = \emptyset$, $\mathcal{C}' \cup \mathcal{C}'' = \mathcal{C}$, and the construction of $\mathcal{C}$ implies that $\mathcal{C}'' = \{Xn \colon X \in \mathcal{C}'\}$. Therefore, by Lemma 5.2, $\mathcal{C}$ is $(n, d+3)$-extendable if and only if $\mathcal{C}'$ is $(n-1, d+2)$-extendable. Since $\mathcal{C}'$ consists of the sets of the form $B \cup D'$ for all $B \in \mathcal{B}$ and $D' \subseteq D \cap [n-1]$, we can apply induction. ∎

As a special case in this proposition, we obtain the following

Corollary 5.4 *For any $D \subseteq [n]$ with $|D| \leq d$, the collection $\{X \subseteq D\}$ (forming the vertex set of a "cube") is (n,d)-extendable.*

(In this case one should take as $\mathcal{B}$ the "cube" on three smallest elements of D.)

Thus, one cube of dimension $\leq d$ within a zonotope $Z(n,d)$ can always be extended to a cubillage. In contrast, as we have seen earlier (cf. Corollary 3.2), a triple of (duly separated) "cubes" of dimension 0 need not be extendable. In light of these facts, we can address the following open question:

($O1$) : Whether or not any two "cubes" $\mathcal{C} = \{A \cup X \colon X \subseteq D\}$ and $\mathcal{C}' = \{A' \cup Y \colon Y \subseteq D'\}$, where $A, A', D, D' \subset [n]$, $|D|, |D'| \leq d$, and $\mathcal{C} \cup \mathcal{C}'$ is $(d-1)$-separated, can be extended to a cubillage on $Z(n,d)$?

A similar open question concerns a membrane and a cube:

($O2$) : Whether or not any pair consisting of a membrane M in $Z(n,d)$ and a "cube" $\mathcal{C} = \{A \cup X \colon X \subseteq D\}$, where $A, D \subset [n]$, $|D| \leq d$, and $\mathrm{Sp}(M) \cup \mathcal{C}$ is $(d-1)$-separated, can be extended to a cubillage on $Z(n,d)$?

6 Inversions

Inversions discussed in this section arise as a natural generalization of the classical notion of inversions in elements (permutations) of a symmetric group $\mathcal{S}_n$, inspired by the observation that a permutation of $[n]$ can be interpreted as a membrane in the zonogon $Z(n,2)$. We will use two ways to define inversions, which are shown to be equivalent. The first one is of a geometric flavor, as follows.

Definitions. Consider a cubillage Q on $Z = Z(n,d)$, a membrane M in Q, and the corresponding cubillage $Q' := \pi(M)$ on $Z(n,d-1)$. A d-tuple $K \in \binom{[n]}{d}$ is called *inversive*, or an *inversion*, for M, as well as for Q', if the cube C of Q having type K lies *before* M, i.e., in the region $Z^-(M)$ between $Z^{\rm fr}$ and M (see Sect. 4.1 for definitions). Otherwise (when the cube $C \in Q$ with $\tau(C) = K$ lies in the region $Z^+(M)$ between M and $Z^{\rm rear}$), we say that K is *straight* for M (and for Q').

An important fact is that the set of inversions for M does not depend on the choice of a cuballage Q on Z that contains M as a membrane (see Remark 2 below); we denote this set as $\mathrm{Inv}(M)$, or $\mathrm{Inv}(Q')$.

In particular, the smallest case $\mathrm{Inv}(M) = \emptyset$ (the largest case $\mathrm{Inv}(M) = \binom{[n]}{d}$) happens when $M = Z^{\rm fr}$ (resp. $M = Z^{\rm rear}$), or, in terms of cubillages, when $Q' = \pi(M)$ becomes the standard cubillage $Q^{\rm st}_{n,d-1}$ (resp. the anti-standard cubillage $Q^{\rm ant}_{n,d-1}$). (Note that [8] exhibits necessary and sufficient conditions on a collection in $\binom{[n]}{d}$ to be the set of inversions for a membrane, but we do not use this in what follows.)

The second way to define inversions relies on a natural binary relations on cubes of a cubillage. More precisely, given a cubillage $\widetilde{Q}$ on $Z(\widetilde{n},\widetilde{d})$, we say that a cube $C \in \widetilde{Q}$ *immediately precedes* a cube $C' \in \widetilde{Q}$ if their sides $C^{\rm rear}$ and $(C')^{\rm fr}$ share a facet (a $(\widetilde{d}-1)$-dimensional face). Then for $C, C' \in \widetilde{Q}$, we write $C \prec_{\widetilde{Q}} C'$ and say that C *precedes* C' if there is a sequence $C = C_0, C_1, \ldots, C_k = C'$ such that C_{i-1} immediately precedes C_i for each i. It can be shown rather easily that the relation $\prec_{\widetilde{Q}}$ is a partial order (arguing in spirit of the proof of Lemma 4.2 in [1] for $d = 3$).

A behavior of this order under contraction operations on pies (defined in Sect. 4.2) is featured as follows.

Lemma 6.1 *Let Q^* be the i-contraction of a cubillage $\widetilde{Q}$ on $Z(\widetilde{n},\widetilde{d})$, where $i \in [\widetilde{n}]$, and suppose that D, D' are cubes in Q^* such that $D \prec_{Q^*} D'$. Then the cubes $C, C' \in \widetilde{Q}$ with $\tau(C) = \tau(D)$ and $\tau(C') = \tau(D')$ satisfy $C \prec_{\widetilde{Q}} C'$.*

Proof Let B_i, R, R' be as in (4.3)(i),(ii) for the i-pie Π_i in $\widetilde{Q}$. It suffices to consider the case when D immediately precedes D'. Then four cases are possible: (a) both D, D' lie in the part R of Q^*, (b) both D, D' lie in the part $R' - \xi_i$ of Q^* corresponding to R' in $\widetilde{Q}$, (c) D lies in R, and D' in $R' - \xi_i$, or (d) D lies in $R' - \xi_i$, and D' in R.

In case (a), we have $C = D$ and $C' = D'$, while in case (b), $C = D + \xi_i$ and $C' = D' + \xi_i$. So in both cases C immediately precedes C'. In case (c), we have $C = D$ and $C' = D' + \xi_i$. Also $D^{\rm rear}$ and $(D')^{\rm fr}$ share a facet F contained in B_i. Therefore, the pie Π_i has the i-cube C'' that is the sum of F and the segment $[0, \xi_i]$. One can see that $C^{\rm rear} \cap (C'')^{\rm fr} = F$ and $(C'')^{\rm rear} \cap (C')^{\rm fr} = F + \xi_i$. Then C immediately precedes C'', and C'' immediately precedes C'. This implies $C \prec_{\widetilde{Q}} C'$. Finally, in case (d), $C = D + \xi_i$ and $C' = D'$. Also $D^{\rm rear}$ and $(D')^{\rm fr}$ share a facet F in B_i. Again, Π_i has the i-cube C'' that is the sum of F and the segment $[0, \xi_i]$. But now we have $C^{\rm rear} \cap (C'')^{\rm fr} = F + \xi_i$ and $(C'')^{\rm rear} \cap (C')^{\rm fr} = F$. Then $C \prec_{\widetilde{Q}} C'' \prec_{\widetilde{Q}} C'$, implying $C \prec_{\widetilde{Q}} C'$, as required. ∎

Now return to Q, M, Q' as above. Fix a cube $C \in Q$ and let $K := \tau(C)$. Suppose we apply to Q the i-contraction operation with $i \in [n] - K$. One can see that under this operation M turns into a membrane M' in the resulting cubillage Q/i, and comparing the location of C relative to M with that of the "image" of C relative to M', one can see that the status of K preserves, i.e., K is inversive for M' if and only if so is for M.

By applying, step by step, the i-contraction operations to all $i \in [n]-K$, we produce from Q the cubillage $\widehat{Q} := Q/([n] - K)$ consisting of a single cube $\widehat{C}$ having type K. Accordingly, M and Q' turn into the membrane $\widehat{M} := M/([n] - K)$ in $\widehat{Q}$ and its projection $\widehat{Q}' := \pi(\widehat{M})$, respectively. Since $\widehat{Q}$ has exactly two membranes, namely, $\widehat{C}^{\rm fr}$ and $\widehat{C}^{\rm rear}$, and since the status of K preserves during the contraction process, we can conclude that

(6.1) if K is inversive (straight) for M, then the reduced membrane $\widehat{M} := M/([n]-K)$ is isomorphic to the rear (resp. front) side of a cube of type K.

Assuming that K consists of elements $k_1 < \cdots < k_d$, we will write $Z(K, d-1)$ for the zonotope generated by $\xi'_p := \pi(\xi_{k_p})$, $p = 1, \ldots, d$ (where, as before, $\xi_\bullet$ is a generator from Ξ). In view of $|K| = d$, there exist exactly two cubillages on $Z' := Z(K, d-1)$, namely, the standard and anti-standard cubillages, denoted as $Q^{\rm st}_{K,d-1}$ and $Q^{\rm ant}_{K,d-1}$, respectively (which correspond to the standard and anti-standard cubillages in $Z(d, d-1)$).

Since $Q^{\rm st}_{K,d-1}$ and $Q^{\rm ant}_{K,d-1}$ are the projections of $C^{\rm fr}$ and $C^{\rm rear}$, respectively, where C is a cube of type K (viz. $C = Z(K, d)$), (6.1) implies that

(6.2) $K \in \binom{[n]}{d}$ is an inversion for a membrane M in $Z(n, d)$ if and only if $\pi(M/([n] - K))$ is the anti-standard cubillage $Q^{\rm ant}_{K,d-1}$ on $Z(K, d-1)$.

This and Lemma 6.1 lead to a description of inversions for M in terms of the partial order $\prec_M$, giving the second ("intrinsic") way to characterize ${\rm Inv}(M)$. Following [6], for $K \in \binom{[n]}{d}$, define ${\rm Pac}(K)$ to be the set $\{K - i \colon i \in K\}$ of $(d-1)$-element subsets of K, called the *packet* of K. Then each $K' \in {\rm Pac}(K)$ is the type of some cube in M. For convenience, we use the same notation $\prec_M$ for the corresponding types; so if $C, C' \in M$ and $C \prec_M C'$, we may write $\tau(C) \prec_M \tau(C')$.

Proposition 6.2 *Let $K \in \binom{[n]}{d}$ consist of elements $k_1 < \cdots < k_d$ and let M be a membrane in $Z(n, d)$. Then the elements of* ${\rm Pac}(K)$ *occur in the lexicographic order*

$$(K - k_d) \prec_M (K - k_{d-1}) \prec_M \cdots \prec_M (K - k_1)$$

if K is straight for M, and in the anti-lexicographic order

$$(K - k_1) \prec_M (K - k_2) \prec_M \cdots \prec_M (K - k_d)$$

if K is inversive for M.

Proof In view of Lemma 6.1, the required relations for $\prec_M$ would follow from similar relations for $\prec_{M'}$, where $M' := M/([n] - K)$. Moreover, since the relations $\prec_{M'}$ and $\prec_{\pi(M')}$ on ${\rm Pac}(K)$ are the same, it suffices to consider cubillages on $Z(K, d-1)$, or, equivalently, on $Z(d,d-1)$. In other words, we have to show that for the sets $A_i := [d]-i$, $i=1,\ldots,d$:

$$A_d \prec' A_{d-1} \prec' \cdots \prec' A_1, \quad \text{and} \tag{6.3}$$
$$A_1 \prec'' A_2 \prec'' \cdots \prec'' A_d, \tag{6.4}$$

where $\prec'$ ($\prec''$) denotes the order in $Q' := Q^{\rm st}_{d,d-1}$ (resp. $Q'' := Q^{\rm ant}_{d,d-1}$).

To show this, we first specify the spectra of Q' and Q''. Let X (Y) be the set of elements $i\in[d]$ with $d-i$ odd (resp. even). Note that X and Y are not $(d-2)$-separated from each other; so one of X,Y belongs to ${\rm Sp}(Q')$, and the other to ${\rm Sp}(Q'')$ (taking into account that each of ${\rm Sp}(Q')$ and ${\rm Sp}(Q'')$ is $(d-2)$-separated and that $|{\rm Sp}(Q')| = |{\rm Sp}(Q'')| = \binom{d}{0}+\binom{d}{1}+\cdots\binom{d}{d-1} = 2^d-1$; cf. (2.6) and (2.4)). Using ${\rm Sp}(Q') = {\rm Sp}(Z^{\rm fr})$ and ${\rm Sp}(Q'') = {\rm Sp}(Z^{\rm rear})$ for the "cube" $Z=Z(d,d)$ and considering (4.2)(ii),(iii), one can conclude that

$$X\in {\rm Sp}(Q') \quad \text{and} \quad Y\in {\rm Sp}(Q'').$$

Let us prove (6.3). The cubillage Q' is formed by d cubes $C_1,\ldots,C_d$ of types $A_1,\ldots,A_d$, respectively, each of which must contain the unique vertex of Q' lying in the interior of $Z'=Z(d,d-1)$, namely, the vertex X (viz. $\sum(\pi(\xi_i)\colon i\in X)$). It follows that in the digraph $G_{Q'}$ (defined in Sect. 2),

(6.5) the vertex X is incident to d edges $a_1,\ldots,a_d$ of $G_{Q'}$, where each a_i is an i-edge, and a_i *enters* (resp. *leaves*) X if $i\in X$ (resp. $i\in[d]-X$);

(6.6) for $i=1,\ldots,d$, the cube C_i contains all edges in $E:=\{a_1,\ldots,a_d\}$ except for a_i.

Consider "consecutive" cubes C_i,C_{i+1} ($1\le i<d$). They share a facet, namely, the one lying in the hyperplane H_i spanned by the edge set $E-\{a_i,a_{i+1}\}$. The required relation $A_{i+1}\prec' A_i$ in (6.3) can be reformulated as:

($*$) when seeing in the direction of the last coordinate vector, the cube C_i is located *behind* H_i (whereas C_{i+1} is located *before* H_i).

To see ($*$), for $j=1,\ldots,d$, denote $\pi(\xi_j)$ by φ_j, and define the vector $\overline\varphi_j$ to be $-\varphi_j$ if $j\in X$, and φ_j otherwise. We write $D(\beta,\ldots,\beta')$ for the determinant of the matrix formed by a sequence $\beta,\ldots,\beta'$ of $d-1$ column vectors in $\mathbb{R}^{d-1}$. Note that ($*$) says that the edge a_{i+1} of C_i is located behind H_i (and the edge a_i of C_{i+1} before H_i). One can realize that this location corresponds to the relation

($**$) $D := D(\varphi_1,\varphi_2,\ldots,\varphi_{i-1},\varphi_{i+2},\varphi_{i+3},\ldots,\varphi_d,\overline\varphi_{i+1})>0.$

Now validity of ($**$) follows from

$$D = (-1)^{d-i-1}D(\varphi_1,\ldots,\varphi_{i-1},\overline\varphi_{i+1},\varphi_{i+2},\ldots,\varphi_d)$$
$$= D(\varphi_1,\ldots,\varphi_{i-1},\varphi_{i+1},\varphi_{i+2},\ldots,\varphi_d)>0$$

(taking into account (2.1) and the fact that $\overline\varphi_{i+1}=-\varphi_{i+1}$ if and only if $d-i-1$ is odd).

To show (6.4), we argue in a similar way, replacing (6.5) by:

(6.7) in the graph $G_{Q''}$, the vertex Y is incident to d edges $b_1, \ldots, b_d$, where each b_i is an i-edge, and b_i enters (resp. leaves) Y if $i \in Y$ (resp. $i \in [d] - Y$).

Using this and the fact that Y if formed by elements $i \in [d]$ with $d - i$ even, one shows that for each i, the cube C_i of type A_i is located *before* the hyperplane separated C_i and C_{i+1} (cf. $(*)$), and (6.4) follows. ∎

Remark 2. The above proposition implies that the "geometric" definition of $\mathrm{Inv}(M)$ (given in the beginning of this section) does not depend on the choice of a cubillage containing M as a membrane. Next, for a membrane M in $Z(n, d)$, we alternatively could give a "packet" definition for straight and inversive tuples $\binom{[n]}{d}$ in a spirit of the statement in this proposition, and then come to the "geometric" characterization by reversing reasonings in the above proof. This alternative way to define $\mathrm{Inv}(M)$ matches the classical definition due to Manin and Schechtman (cf. Theorem 3 in [6]). Recall that they introduced a "packet admissible" total order $\prec$ on $\binom{[n]}{d-1}$, which means that for each tuple $K \in \binom{[n]}{d}$, the elements of $\mathrm{Pac}(K)$ become ordered by $\prec$ either lexicographically or anti-lexicographically, and in the latter case, K is said to be an inversion for $(\binom{[n]}{d-1}, \prec)$. (Compare $\prec$ with $\prec_M$.)

Note also that the method of proof of Proposition 6.2 enables us to reveal one more useful fact.

Proposition 6.3 *Let M be a membrane in $Z(n, d)$ and let $K \in \binom{[n]}{d}$ consist of elements $k_1 < \cdots < k_d$. Then:*

(i) *K is inversive for M if and only if there is $X \in \mathrm{Sp}(M)$ such that $X \cap K = \{k_i \colon d - i \text{ odd}\} =: K^{\mathrm{odd}}$;*

(ii) *K is straight for M if and only if there is $Y \in \mathrm{Sp}(M)$ such that $Y \cap K = \{k_i \colon d - i \text{ even}\} =: K^{\mathrm{even}}$.*

Proof Let $[n] - K = \{j_1, j_2, \ldots, j_{n-d}\}$ and form the sequence $M_0 = M, M_1, \ldots, M_{n-d}$ of membranes in the corresponding zonotopes, where $M_i = (M_{i-1})/j_i$. So $M' := M_{n-d}$ is a membrane in the final zonotope $Z' := Z(K, d)$ (a single cube). We know that if K is inversive for M, then $M' = (Z')^{\mathrm{rear}}$ and M' contains the vertex K^{odd}, whereas if K is straight for M, then $M' = (Z')^{\mathrm{fr}}$ and M' contains the vertex K^{even} (cf. the proof of Proposition 6.2). Now the result follows by observing that for $1 \le i \le n - d$, if the membrane M_i has a vertex A, then the previous membrane M_{i-1} has a vertex A' of the form A or $A \cup \{j_i\}$. ∎

As a consequence, we obtain the following result.

Theorem 6.4 *Let $M_1, \ldots, M_p$ be membranes in $Z(n, d)$ such that $\mathrm{Inv}(M_1) \subset \cdots \subset \mathrm{Inv}(M_p)$. Then the collection $\mathrm{Sp}(M_1) \cup \ldots \cup \mathrm{Sp}(M_p)$ is $(d-1)$-separated.*

Proof Suppose that this is not so. Then for some $i < j$, there exist $X \in \mathrm{Sp}(M_i)$ and $Y \in \mathrm{Sp}(M_j)$ that are not $(d-1)$-separated from each other. Therefore, there exist elements $i_1 < i_2 < \cdots < i_{d+1}$ of $[n]$ that alternate in $X - Y$ and $Y - X$. Let for definiteness the elements i_k with k odd are contained in $X - Y$ (and the other in $Y - X$). Consider the d-element sets $K := \{i_1, \ldots, i_d\}$ and $K' := \{i_2, \ldots, i_{d+1}\}$. Then, by Proposition 6.3, K is straight for one, and inversive for the other membrane among M_i, M_j. But the behavior of M_i, M_j relative to K' is opposite. Thus, neither $\mathrm{Inv}(M_i) \subset \mathrm{Inv}(M_j)$ nor $\mathrm{Inv}(M_j) \subset \mathrm{Inv}(M_i)$ is possible; a contradiction. ∎

We finish this section with two applications.

1) Let M, N be two membranes with $\mathrm{Inv}(M) \subset \mathrm{Inv}(N)$ in $Z = Z(n, d)$. By Theorem 6.4, the collection $\mathcal{C} := \mathrm{Sp}(M) \cup \mathrm{Sp}(N)$ is $(d-1)$-separated; so it is tempting to hope that $\mathcal{C}$ is extendable to a maximal by size $(d-1)$-separated set-system, or, equivalently, that there exists a cubillage Q on Z containing both membranes. We can try to construct such a Q by filling the region $Z^-(M)$ (between Z^{fr} and M) with a "partial" cubillage Q', and filling the region $Z^+(N)$ (between N and Z^{rear}) with a "partial" cubillage Q'' (such Q', Q'' exist by (4.1)). But what is about the rest of Z between M and N, denoted as $Z(M, N)$? (Note that $\mathrm{Inv}(M) \subset \mathrm{Inv}(N)$ provides that M lies within $Z^-(N)$.)

Let us say that M, N are *agreeable* if the collection $\mathrm{Sp}(M) \cup \mathrm{Sp}(N)$ is (n, d)-extendable, i.e., a cubillage on Z containing both M, N (equivalently, a "partial" cubillage filling $Z(M, N)$) does exist. Ziegler [8] explicitly constructed two membranes M, N in the zonotope $Z(8, 4)$ such that $\mathrm{Inv}(M) \subset \mathrm{Inv}(N)$ but M, N are not agreeable (in our terms) . This together with Theorem 6.4 implies that the set system $\mathbf{S}_{8,4}$ is not pure (the latter fact was omitted in [8]). (Compare $(O2)$ in the end of the previous section that considers the union of a membrane and a cube.)

2) In light of the above result for $d = 4$, Ziegler asked about the existence of two non-agreeable membranes in dimension 3. Answering this question, Felsner and Weil [2] proved that for an arbitrary n, any two membranes M, N with $\mathrm{Inv}(M) \subset \mathrm{Inv}(N)$ in $Z(n, 3)$ are agreeable. Note that the proof in [2] attracted a non-trivial combinatorial techniqies. An alternative proof immediately follows from Galashin's result in [3] (mentioned in (1.2)) and Theorem 6.4.

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