

An efficient algorithm for packing cuts and (2,3)-metrics in a planar graph with three holes

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Abstract

We consider a planar graph G in which the edges have nonnegative integer lengths such that the length of every cycle of G is even, and three faces are distinguished, called *holes* in G . It is known that there exists a packing of cuts and (2,3)-metrics with nonnegative integer weights in G which realizes the distances within each hole. We develop a purely combinatorial strongly polynomial-time algorithm to find such a packing.

Keywords: packing problem, strongly polynomial-time algorithm, planar graph, cut, (2,3)-metric, shortest path

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1 Introduction

An interesting class of problems in combinatorial optimization is formed by the ones on packing cuts and metrics that are related via a sort of polar duality to popular multi(commodity)flow demand problems in graphs. (For a discussion on such a relationship and some appealing examples, see, e.g., [3, Sect. 4], [9, Sects. 71, 72, 74]. The simplest example is the polar duality between the classical problems of finding an s - t flow of a given value in a capacitated graph G and finding a shortest s - t path in G with nonnegative lengths ℓ of edges, or, equivalently, finding a maximal packing of cuts separating the vertices s and t in (G, ℓ) .)

In this paper we consider a planar graph $G = (V, E)$ embedded in the plane in which the edges $e \in E$ have *nonnegative lengths* $\ell(e) \in \mathbb{R}_{\geq 0}$ and a subset \mathcal{H} of faces of G , called *holes*, is distinguished. Also we are given (implicitly) a certain set \mathcal{M} of metrics on V . Then the (fractional) *problem of packing metrics realizing the distances on the holes* for $G, \ell, \mathcal{H}, \mathcal{M}$ consists in the following:

PMP: Find metrics $m_1, \dots, m_k \in \mathcal{M}$ and weights $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geq 0}$ such that:

$$\lambda_1 m_1(e) + \dots + \lambda_k m_k(e) \leq \ell(e) \quad \text{for each } e \in E; \quad \text{and} \quad (1.1)$$

$$\lambda_1 m_1(st) + \dots + \lambda_k m_k(st) = \text{dist}_{G, \ell}(st) \quad \text{for all } s, t \in V_H, H \in \mathcal{H}. \quad (1.2)$$

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Hereinafter we use the following terminology, notation and conventions:

- (a) when it is not confusing, a pair (x, y) of vertices may be denoted as xy ;
- (b) for a face F , its *boundary* (regarded as a graph) is denoted by $\text{bd}(F) = (V_F, E_F)$;
- (c) a *metric* on V is meant to be a function $m : V \times V \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying $m(xx) = 0$, $m(xy) = m(yx)$, and $m(xy) + m(yz) \geq m(xz)$ for all $x, y, z \in V$ (admitting $m(xy) = 0$ when $x \neq y$);
- (d) $\text{dist}_{G,\ell}(uv)$ denotes the *distance* in (G, ℓ) between vertices $u, v \in V$, i.e., the minimum length $\ell(P) := \sum(\ell(e) : e \in E_P)$ of a path P connecting vertices u and v in G (where E_P is the set of edges in P); in particular, $\text{dist}_{G,\ell}$ is a metric.

Typically the class \mathcal{M} of metrics figured in PMP is described by fixing one or more graphs $K = (V_K, E_K)$ and ranging over arbitrary mappings $\gamma : V \rightarrow V_K$. Then each γ generates the metric m_γ on V by setting $m_\gamma(xy) := \text{dist}_{K,1}(\gamma(x)\gamma(y))$ for $x, y \in V$ (where 1 stands for the all-unit function on E_K). Two special cases of metrics are important for us:

- (i) when $K = K_2$ (the graph with two vertices and one edge), $m = m_\gamma$ is called a *cut-metric*; in other words, m is generated by a partition $\{V_1, V_2\}$ of V and establishes distance 0 inside each of V_1 and V_2 , and 1 between the elements of these subsets;
- (ii) when $K = K_{2,3}$ (the complete bipartite graph with parts consisting of two and three vertices), m_γ is called a *(2,3)-metric*; it is generated by a partition of V into five subsets S_1, S_2, S_3, T_1, T_2 and establishes distance 0 inside each of them, 1 between S_i and T_j ($i = 1, 2, 3, j = 1, 2$), and 2 otherwise.

Nontrivial integrality results on PMP, mentioned below, have been obtained when $|\mathcal{H}|$ is “small” and the edge length function ℓ is *cyclically even*, which means that ℓ is integer-valued and the length $\ell(E_C)$ of any cycle C in G is even. (For a function $f : S \rightarrow \mathbb{R}$ and a subset $S' \subseteq S$, we write $f(S')$ for $\sum(f(e) : e \in S')$.)

- (1.3) If ℓ is cyclically even and \mathcal{M} is the set of cut-metrics on V , then PMP has an *integer solution* (i.e., having integer weights of all metrics) when $|\mathcal{H}| = 1$ [1] and when $|\mathcal{H}| = 2$ [8].
- (1.4) If $|\mathcal{H}| = 3$, ℓ is cyclically even, and \mathcal{M} is formed by cut- and (2,3)-metrics on V , then PMP has an integer solution [4] (whereas if \mathcal{M} consists only of cut-metrics, then PMP need not have a solution at all).
- (1.5) If $|\mathcal{H}| = 4$, ℓ is cyclically even, and \mathcal{M} is formed by cut-, (2,3)- and $4f$ -metrics on V , then PMP has an integer solution, where a $4f$ -metric is generated by a mapping $\gamma : V \rightarrow V_K$ with K being a planar graph with four faces [4].

In fact, the existence of a solution to PMP with real-valued weights λ in cases $|\mathcal{H}| = 1, 2, 3$ can be immediately concluded, via polar duality, from solvability criteria for corresponding fractional multiflow demand problems given in [7],[6],[5], respectively, and the essence of (1.3)–(1.5) is just the existence of integer solutions when the lengths of edges are cyclically even (or, weaker, a half-integer solution when the lengths are

integer-valued). The proof for $|\mathcal{H}| \leq 2$ given in [8] is constructive and can be turned into a pseudo-polynomial algorithm. A strongly polynomial combinatorial algorithm for finding a solution formed by cut-metrics with integer weights when $|\mathcal{H}| \leq 2$ and ℓ is cyclically even is given in [2].

The purpose of this paper is to devise a strongly polynomial combinatorial algorithm for $|\mathcal{H}| = 3$.

Theorem 1.1 *Let $G = (V, E)$ be a planar graph with cyclically even lengths $\ell(e)$ of edges $e \in E$ and let \mathcal{H} be three distinguished faces of G . Then one can find, in strongly polynomial time, cut- or (2,3)-metrics m_1, \dots, m_k and nonnegative integers $\lambda_1, \dots, \lambda_k \in \mathbb{Z}_{\geq 0}$ satisfying (1.1)–(1.2).*

Note that this theorem is analogous, in a sense, to a result for arbitrary graphs in [3] where a strongly polynomial combinatorial algorithm is developed that, given a graph $G = (V, E)$ with cyclically even lengths of edges and a distinguished set $T \subset V$ of five terminals, finds an integer packing of cuts and (2,3)-metrics realizing the distance for each pair of terminals.

Our algorithm yielding Theorem 1.1 is given throughout Sects. 2–7. The main part of the algorithm involves three sorts of good reductions by cuts, called *Reductions I, II, III* and described in Sects. 3, 5, 6, respectively. Here by a *reduction by cuts*, we mean finding certain cut-metrics with integer weights and accordingly reducing the current lengths ℓ , and we say that the reduction is *good* if combining these cut-metrics with a solution to PMP for G, \mathcal{H} and the reduced lengths ℓ' , we obtain a solution for ℓ . During the algorithm, as soon as the length of some edge becomes zero, this edge is immediately contracted. We explain that Reductions I, II, III are implemented in strongly polynomial time (but do not care of precisely estimating and decreasing the time bound). It turns out that the sequence of reductions by cuts that we apply eventually produces a weighted graph (G, ℓ) whose structure is quite simple: it is formed by three paths connecting the same pair of vertices and having equal lengths. The distance in this (G, ℓ) is represented as the sum of weighted (2,3)-metrics and, possibly, one cut-metric, yielding the desired result.

Remark. In fact, our algorithm is self-contained (up to appealing to the algorithm of [2]) and leads to a proof of the existence theorem (1.4) alternative to that given in [4]. Note that the latter proof was nonconstructive and can hardly be turned into an algorithm finding an integer solution for (G, ℓ, \mathcal{H}) with $|\mathcal{H}| = 3$ and a cyclically even ℓ in pseudo polynomial time. The algorithm of this paper combines various combinatorial techniques and its description may look rather long. To compare: already a fractional analog of PMP with $|\mathcal{H}| = 3$ is a nontrivial task to solve. Indeed, a naive l.p. relaxation of the problem involves exponentially many variables. A more clever approach relies on the observation in [4, Sect. 3] that there exists a solution using special cut- and (2,3)-metrics; namely, those cut-metrics ((2,3)-metrics) for which the corresponding subdivision of G splits the boundary of each hole into at most two (resp. four) connected pieces. Using this and attracting certain fractional relaxations of cut-

and (2,3)-metrics, one can compose a consistent linear program that has $O(|V|^8)$ variables and $O(|V|^9)$ constraints; then a fractional solution μ can be found in strongly polynomial time (we omit the details here). However, it is not clear whether μ could somehow help to find (or approximate) an integer solution. It seems that a “fractional approach” of this or another sort cannot compete with the “natural” strongly combinatorial algorithm developed in this paper.

2 Initial reductions

We start with additional terminology, notation and conventions.

- Paths, cycles, subgraphs and other objects in a planar graph $G = (V, E)$ are naturally identified with their images in the plane or sphere. A path $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ from $x = v_0$ to $y = v_k$ is called an x - y *path*. When both x, y belong to (the boundary of) a hole $H \in \mathcal{H}$, we also say that P is an H -*path*. When it is not confusing, we may use notation $P = v_0 v_1 \dots v_k$ (via vertices) or $P = e_1 e_2 \dots e_k$ (via edges). For a simple path P , its subpath with endvertices u and v may be denoted as $P[u, v]$ (as a rule, but not always, we assume that u, v occur in this order in P). Also for paths $P = v_0 v_1 \dots v_k$ and $P' = v'_0 v'_1 \dots v'_k$ with $v_k = v'_0$, we write $P \cdot P'$ for the concatenated path $v_0 v_1 \dots v_k v'_1 \dots v'_k$. When $x = y$ and $|E_P| > 0$, P becomes a *cycle*. When needed, a path/cycle is identified with the one reverse to it.

- The set of faces of G is denoted by \mathcal{F}_G . A face F is regarded as a *closed* region in the plane or sphere (i.e., including the boundary $\text{bd}(F) = (V_F, E_F)$), and the *interior* $F - \text{bd}(F)$ of F is denoted as $\text{Int}(F)$. The boundary $\text{bd}(F)$ may be identified with the corresponding cycle. Usually the unbounded face of G is assumed to be a hole.

- We say that $V_{\mathcal{H}} := \cup(V_H : H \in \mathcal{H})$ is the set of *terminals*. The other vertices of G are called *inner*. Also we address the adjective *inner* to the faces in $\mathcal{F}_G - \mathcal{H}$ and to the edges not contained in the boundaries of holes.

- Usually we will abbreviate the distance function $\text{dist}_{G, \ell}$ to d . Note that the cyclic evenness of ℓ implies that for any edge $e = uv$ of G , the integers $\ell(e)$ and $d(uv)$ have the same parity. The distances $d(xy)$ for all $x, y \in V$ are computed in the beginning of the algorithm and updated when needed.

We assume that G, ℓ, \mathcal{H} satisfy the following conditions, which will simplify our description, leading to no loss of generality in essence:

(C1): G is connected and has no loops and parallel edges, and the cycle $\text{bd}(F)$ is simple for each face F ; in particular $|V_F| \geq 3$ for each face F .

(For otherwise we can make easy reductions of the problem, preserving the cyclic evenness.) The properties in (C1) will be default maintained during the algorithm. One more useful simplification is as follows.

(OP1) In the current graph, if there appears an edge $e = uv$ with $\ell(e) = 0$, then we immediately contract this edge (identifying the vertices u and v).

Also at the *preprocessing stage* of the algorithm, operations (OP2)–(OP5) described below are applied, step by step, in an arbitrary order. To describe them, we need additional definitions and notation.

For a face F , let Π_F denote the set of all pairs $x, y \in V_F$ of vertices in F . Due to condition (1.2), an important role is played by the set of terminal pairs $\cup(\Pi_H : H \in \mathcal{H})$, denoted as $\Pi_{\mathcal{H}}$. We say that an s – t path P in G is an \mathcal{H} -geodesic (resp. an H -geodesic for $H \in \mathcal{H}$) if P is shortest w.r.t. ℓ and $st \in \Pi_{\mathcal{H}}$ (resp. $st \in \Pi_H$). For vertices $x, y, z \in V$, define the values

$$\begin{aligned} \varepsilon(x|yz) &:= d(xy) + d(xz) - d(yz); \quad \text{and} \\ \Delta(xy) &:= \min\{d(sx) + d(xy) + d(yt) - d(st) : st \in \Pi_{\mathcal{H}}\} \end{aligned} \quad (2.1)$$

(which are efficiently computed when needed). For brevity we write $\Delta(x)$ for $\Delta(xx)$. Clearly each $\Delta(xy)$ is nonnegative and even, and we say that xy (resp. x) is *tight* if $\Delta(xy) = 0$ (resp. $\Delta(x) = 0$). In operations (OP2)–(OP4) we decrease the current length ℓ , trying to make the values Δ as small as possible while preserving the cyclical evenness and the original distances d on $\Pi_{\mathcal{H}}$.

(OP2) Suppose that there exists (and is chosen) a non-tight vertex x (this is possible only if x is inner). Then we decrease the length of *each* edge incident to x by the minimum of these lengths and $\Delta(x)/2$ (this minimum is a positive integer).

As a result, at least one of the following takes place: (i) $\ell(e)$ becomes 0 for some edge e incident to x , or (ii) $\Delta(x)$ becomes 0. In case (i), we contract e (by applying (OP1)), and if $\Delta(x)$ is still nonzero, repeat (OP2) with the same x .

(OP3) Suppose that there exists an edge $e = xy \in E$ with $\ell(e) \geq 2$ and $\Delta(xy) > 0$. Then we reduce the length $\ell(e)$ to the minimal nonnegative integer α so that α and $d(xy)$ have the same parity and $d(sx) + \alpha + d(yt) \geq d(st)$ for all $st \in \Pi_{\mathcal{H}}$.

(OP4) Suppose that there are two different vertices x, y in an *inner* face F which are not adjacent in G and such that $\Delta(xy) > 0$. Then we connect x, y by edge e , inserting it inside F (thus subdividing F into two inner faces), and assign the length $\ell(e)$ in the same way as in (OP3).

Clearly (OP3) and (OP4) preserve both the cyclical evenness of lengths and the distances on $\Pi_{\mathcal{H}}$, and we can see that

(2.2) if the new value $\Delta(xy)$ is still nonzero, then $\ell(e) = 1$

(taking into account that e should be contracted when $\ell(e) = 0$). Note that if none of (OP3) and (OP4) is applicable, then each edge $e = xy$ satisfies $\ell(e) = d(xy)$. Moreover, the following useful property takes place:

(C2): for each face F and vertices $x, y \in V_F$, the pair xy is tight.

Indeed, this is trivial when F is a hole. So suppose that F is inner and xy is non-tight. By (OP4), this is possible only if xy is an edge. By (2.2), $\ell(xy) = 1$. Take a vertex $z \in V_F$ different from x, y (existing by (C1)). Since $d(zx) + d(zy) + \ell(xy)$ is even and $\ell(xy) = 1$, either $d(zx) = d(zy) + \ell(xy) \geq 1 + 1 = 2$ or $d(zy) = d(zx) + \ell(xy) \geq 2$. For definiteness, assume the former. Then zx is tight (since $\Delta(zx) > 0$ would imply that z, x are connected by edge e with $\ell(e) = d(zx) = 1$, by (2.2), contradicting $d(zx) \geq 2$). This implies that there exists an \mathcal{H} -geodesic passing z, x , and hence an \mathcal{H} -geodesic passing x, y . Then $\Delta(xy) = 0$; a contradiction.

The final operation is intended for getting rid of “redundant” edges.

- (OP5) Suppose that some face F contains a *dominating* edge $e = xy$, which means that $d(xy) = \ell(P)$, where P is the x - y path in $\text{bd}(F)$ not containing e (so $\text{bd}(F) = P \cup \{e\}$). Then we delete e from G (thus merging F with the other face containing e and preserving the distance d).

Let ℓ, d, Δ be the corresponding functions obtained upon termination of the preprocessing stage. Then (C1) and (C2) hold, and

- (C3): no face of G has a dominating edge.

The preprocessing stage takes $O(|V|^2)$ applications of (OP2)–(OP5). This follows from observations that if, at some step, a pair $x, y \in V$ becomes tight, then it remains tight upon termination of the process (unless x, y merge). Also if a dominating edge is deleted by (OP5), then this edge could never appear again, and (“symmetrically”) if an edge xy is created by (OP4) (from a non-tight pair), then e becomes tight and could not be handled again (unless it is deleted by (OP5) or (OP2)).

3 Reduction I

In this section we further simplify (G, ℓ, \mathcal{H}) by using the algorithm from [2] which finds a packing of cuts realizing the corresponding distances in the two-hole case.

For $X \subset V$, define $\delta X = \delta_G X$ to be the set of edges of G connecting X and $V - X$, referring to it as the *cut* generated by X (or by $V - X$), and define $\rho X = \rho_{\mathcal{H}} X$ to be the set of pairs $st \in \Pi_{\mathcal{H}}$ separated by X , i.e., $s \neq t$ and $|\{s, t\} \cap X| = 1$. The cut δX is associated with the cut-metric corresponding to the partition $\{X, V - X\}$.

Let $\chi^{E'}$ denote the incidence vector of a subset $E' \subseteq E$, i.e., $\chi^{E'}(e) = 1$ if $e \in E'$, and 0 if $e \in E - E'$.

Definition. Let \mathcal{C} be a collection of cuts δX in G equipped with weights $\lambda(X) \in \mathbb{Z}_{\geq 0}$. We call (\mathcal{C}, λ) *reducible* (for G, ℓ, \mathcal{H}) if the function $\ell' := \ell - \sum (\lambda(X) \chi^{\delta X} : \delta X \in \mathcal{C})$ (of *reduced lengths*) is nonnegative and the distance function $d' := \text{dist}_{G, \ell'}$ satisfies

$$d'(st) = d(st) - \sum (\lambda(X) : \delta X \in \mathcal{C}, st \in \rho X) \quad \text{for each } st \in \Pi_{\mathcal{H}}. \quad (3.1)$$

We also say that the lengths ℓ' are obtained by a *good reduction* using (\mathcal{C}, λ) .

An advantage from such a reduction is clear: once we succeeded to find a reducible (\mathcal{C}, λ) , it remains to solve PMP with (G, ℓ', \mathcal{H}) . Indeed, ℓ' is cyclically even (since any cycle and cut have an even number of edges in common), and taking an integer solution to PMP with (G, ℓ', \mathcal{H}) and adding to it the weighted cut metrics associated with (\mathcal{C}, λ) , we obtain an integer solution to the original problem (since (1.1)–(1.2) for ℓ are provided by the nonnegativity ℓ' and relation (3.1)). Also $\ell \mapsto \ell'$ does not decrease the set of \mathcal{H} -geodesics. In particular, the following property (which will be used to show strongly polynomial complexity of the algorithm) holds:

(3.2) if $\varepsilon(x|st)$ for $st \in \Pi_{\mathcal{H}}$ or $\Delta(xy)$ (defined in (2.1)) is zero before a good reduction, then it remains zero after the reduction.

Any subcollection of weighted cuts in a reducible (\mathcal{C}, λ) is reducible as well (for one can see that if $\lambda_1, \lambda_2 : \mathcal{C} \rightarrow \mathbb{Z}_{\geq 0}$ and $\lambda_1 + \lambda_2 = \lambda$, then each (\mathcal{C}, λ_i) satisfies the equality as in (3.1)). Also if a cut δX (with unit weight) is reducible and if the subgraph $\langle X \rangle = \langle X \rangle_G$ of G induced by $X \subset V$ consists of k components $\langle X_1 \rangle, \dots, \langle X_k \rangle$, then each cut δX_i is reducible as well (in view of $\chi^{\delta X} = \chi^{\delta X_1} + \dots + \chi^{\delta X_k}$ and $\rho X \subseteq \rho X_1 \cup \dots \cup \rho X_k$), and similarly for the components of $\langle V - X \rangle$. So we always may deal with only those cuts δX for which both subgraphs $\langle X \rangle$ and $\langle V - X \rangle$ are connected, called *simple* cuts. The planarity of G implies that

(3.3) for a simple cut δX and any face F , $|\delta X \cap E_F| \in \{0, 2\}$.

The main part of the proof of Theorem 1.1 will consist in showing (throughout Sects. 3–7) the following assertion.

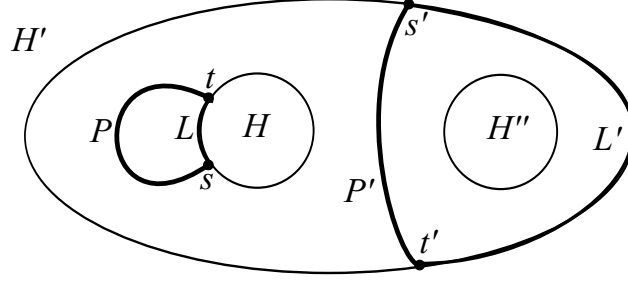
Proposition 3.1 *When $|\mathcal{H}| = 3$, one can find, in strongly polynomial time, a reducible collection of cuts with integer weights so that the reduction of (G, ℓ, \mathcal{H}) by these cuts results in a triple $(G', \ell', \mathcal{H}')$ where $|\mathcal{H}'| \leq 3$ and G' has no inner faces: $\mathcal{F}_{G'} = \mathcal{H}'$.*

As a step toward proving this assertion, in the rest of this section we eliminate one sort of reducible cuts. Unless otherwise is explicitly said, when speaking of a shortest path (or a geodesic), we mean that it is shortest w.r.t. the current length ℓ , or ℓ -shortest.

Fix a hole H and consider an H -geodesic P with ends $s, t \in V_H$. Let $\mathcal{L}_H(st)$ denote the pair of s – t paths that form the boundary of H .

Definitions. For P as above and $\mathcal{L}_H(st) = \{L, L'\}$, define $\Omega(P, L)$ to be the closed region of the sphere bounded by P and L and not containing the hole H , and define $G_{P,L} = (V_{P,L}, E_{P,L})$ to be the subgraph of G lying in $\Omega(P, L)$. We say that the pair (P, L) (and the region $\Omega(P, L)$) is of *type* i and denote $\tau(P, L) := i$ if $\Omega(P, L)$ contains exactly i holes (then $\tau(P, L) + \tau(P, L') = |\mathcal{H}| - 1$). The pair (P, L) is called *normal* if L is shortest, and *excessive* otherwise (when $\ell(L) > \text{dist}(st) = \ell(P)$). Also we write $\tau(P) := \min\{\tau(P, L), \tau(P, L')\}$ and define $\tau(H)$ to be the maximum $\tau(P)$ over all H -geodesics P , referring to $\tau(P)$ and $\tau(H)$ as the *type* of P and H , respectively.

In particular, if $|\mathcal{H}| \leq 4$ then $\tau(P), \tau(H) \leq 1$. In the picture below, $\mathcal{H} = \{H, H', H''\}$, $\tau(P, L) = 0$ and $\tau(P', L') = 1$.



An algorithm of eliminating excessive pairs of types 0,1 (Reduction I).

Suppose that a pair (P, L) as above (concerning H, s, t) has type $i \leq 1$ and is excessive. For convenience, assume that the region $\Omega(P, L)$ is bounded. Let $\alpha := (\ell(L) - d(st))/2$; then α is an integer ≥ 1 . We wish to make a good reduction by cuts so as to turn L into an H -geodesic while preserving the distance $d(st)$.

To this aim, we consider the auxiliary PMP with $(G_{P,L}, \ell_{P,L}, \mathcal{H}')$, where $\ell_{P,L}$ is the restriction of ℓ to the edge set $E_{P,L}$, and \mathcal{H}' consists of the i holes of \mathcal{H} located in $\Omega(P, L)$ plus the outer face \overline{H} of $G_{P,L}$ (with the boundary $P \cup L$).

Since $|\mathcal{H}'| = i + 1 \leq 2$, we can apply the strongly polynomial algorithm of [2] to find a packing of (simple) cuts δX with integer weights $\lambda(X) > 0$ realizing the distances on $\Pi_{\mathcal{H}'}$. From this packing we extract the set \mathcal{C} of those cuts that meet $\text{bd}(\overline{H})$ (twice). Since P is shortest, each cut in \mathcal{C} meets P at most once. Then \mathcal{C} is partitioned into the set \mathcal{C}' of cuts δX with $|\delta X \cap L| = 2$ (and $\delta X \cap P = \emptyset$) and the rest (formed by the cuts δX with $|\delta X \cap L| = |\delta X \cap P| = 1$).

One can see that $\sum(\lambda(X) : \delta X \in \mathcal{C}') = \alpha$ (in view of $\ell(xy) = d(xy)$ for all edges xy in $L \cup P$). We assert that (\mathcal{C}', λ') is reducible in the whole (G, ℓ) , where $\lambda' := \lambda|_{\mathcal{C}'}$.

To show this, assume that for each $\delta X \in \mathcal{C}'$, the vertices s, t are not in X (since the generating set X of this cut can be taken up to the complement to $V_{P,L}$ and in view of $|\delta X \cap L| = 2$). Then $X \cap P = \emptyset$, and X generates the same cut in both $G_{P,L}$ and G .

Define $\ell' := \ell - \sum(\lambda(X)\chi^{\delta X} : \delta X \in \mathcal{C}')$. Let \tilde{d} be the distance within the subgraph $G_{P,L}$ with the lengths $\ell'(e)$ of edges $e \in E_{P,L}$. The fact that the path P separating $G_{P,L}$ from the rest of G is ℓ -shortest easily implies that any ℓ -shortest path in $G_{P,L}$ is ℓ -shortest in the whole G . Also (\mathcal{C}', λ') is reducible for $(G_{P,L}, \ell_{P,L}, \mathcal{H}')$, the cuts in \mathcal{C}' do not meet P , and $\sum(\lambda(X) : \delta X \in \mathcal{C}') = \alpha$. These properties imply:

$$(3.4) \text{ in } G_{P,L}, \text{ both } P, L \text{ are } \ell'\text{-shortest, and } \ell'(L) = \tilde{d}(st) = d(st) = \ell(P); \text{ and}$$

$$(3.5) \text{ any } pq \in \Pi_{\mathcal{H}'} \text{ satisfies } \tilde{d}(pq) = d(pq) - \sum(\lambda(X) : \delta X \in \mathcal{C}', pq \in \rho_{\mathcal{H}'}(X)).$$

Lemma 3.2 (\mathcal{C}', λ') is reducible for (G, ℓ, \mathcal{H}) .

Proof Consider $pq \in \Pi_{\mathcal{H}}$ and a simple p - q path Q in G . It suffices to show that

$$\ell'(Q) \geq d(pq) - \sum(\lambda(X) : \delta X \in \mathcal{C}', pq \in \rho_{\mathcal{H}}(X)). \quad (3.6)$$

Let p, q belong to $H' \in \mathcal{H}$ and let the subgraph $Q \cap P$ consist of k components Y_1, \dots, Y_k , occurring in this order in Q . We use induction on k .

Suppose that $k \geq 2$. Take vertices $u \in Y_1$ and $v \in Y_2$, and let $Q' := Q[u, v]$ and $P' := P[u, v]$. Then $\ell'(Q') \geq \ell'(P')$ (by (3.4)). Therefore, replacing in Q the piece Q' by P' , we obtain a p - q path Q'' in G with $\ell'(Q'') \leq \ell'(Q)$ and such that the number of components of $Q'' \cap P$ is less than k , and then we apply induction.

It remains to consider the cases when either (a) $Q \cap P = \emptyset$, or (b) $Q \cap P$ is nonempty and connected. In case (a), Q is entirely contained in one of the subgraphs $G_{P,L}$ and $G' := (G - G_{P,L}) \cup P$. If $Q \subset G_{P,L}$, then (3.6) follows from (3.5). And if $Q \subset G'$, then (3.6) follows from $\ell'(Q) = \ell(Q) \geq d(pq)$.

In case (b), if Q is entirely contained in $G_{P,L}$ or in G' , then we argue as in case (a). So we may assume that p is in $G_{P,L} - P$, q is in $G' - P$. Take a vertex v in $P \cap Q$, and let $Q_1 := Q[p, v]$ and $Q_2 := Q[v, q]$ (which lie in $G_{P,L}$ and G' , respectively). By planarity reasons, there is a unique hole containing both p, q , namely, H (where p occurs in L , and q in the other path in $\mathcal{L}_H(st)$). Then $\ell'(Q_1) \geq \tilde{d}(pv)$ (by (3.5)) and $\ell'(Q_2) = \ell(Q_2) \geq d(vq)$. Also for any $\delta X \in \mathcal{C}'$, neither q nor v is in X . This implies that $pq \in \rho_{\mathcal{H}}X$ if and only if $pv \in \rho_{\mathcal{H}'}X$. Hence $a := \sum(\lambda(X) : \delta X \in \mathcal{C}', pq \in \rho_{\mathcal{H}}X)$ is equal to $b := \sum(\lambda(X) : \delta X \in \mathcal{C}', pv \in \rho_{\mathcal{H}'}X)$, and we have

$$\begin{aligned} \ell'(Q) &= \ell'(Q_1) + \ell(Q_2) \geq \tilde{d}(pv) + d(vq) \\ &= d(pv) - b + d(vq) = d(pv) + d(vq) - a \geq d(pq) - a, \end{aligned}$$

implying (3.6). ■

Applying the above procedure, step by step, to the excessive pairs of types 0 and 1, we get rid of all such pairs. Each path $L \in \mathcal{L}_H(s, t)$ ($H \in \mathcal{H}$, $s, t \in V_H$) is treated at most once (in view of (3.2)), and therefore the whole process, called *Reduction I*, takes $O(|V|^2)$ iterations and is implemented in strongly polynomial time (relying on the complexity of the algorithm in [2]).

So we may further assume that

- (C4): (G, ℓ, \mathcal{H}) has no excessive pairs (P, L) with $\tau(P, L) \leq 1$; therefore, when $|\mathcal{H}| \leq 4$, for any $H \in \mathcal{H}$ and $s, t \in V_H$, at least one of the two s - t paths L, L' in $\text{bd}(H)$ is shortest: $d(st) = \min\{\ell(L), \ell(L')\}$

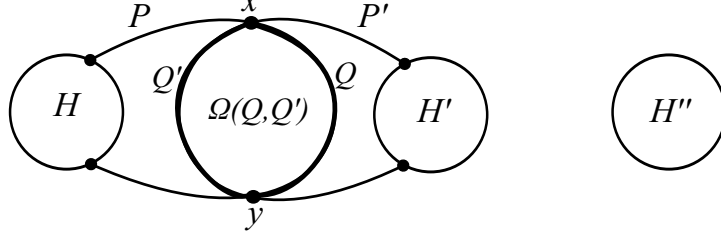
(since at least one of the pairs (P, L) and (P, L') has type ≤ 1). In other words, for each $H \in \mathcal{H}$, the cycle $\text{bd}(H)$ is isometric. One more useful observation for $|\mathcal{H}| = 3$ is:

- (3.7) if $\mathcal{H} = \{H, H', H''\}$ and P is an H -geodesic with ends s, t separating the holes H' and H'' , then both paths in $\mathcal{L}_H(st)$ are shortest.

4 Elimination of lenses without holes

Consider distinct holes $H, H' \in \mathcal{H}$, an H -geodesic P , and an H' -geodesic P' . Suppose that P, P' have common vertices x, y and let Q (resp. Q') be the subpath of P (resp. P') between x and y . Suppose that the interiors of Q and Q' are disjoint. Then the fact that both P, P' are shortest implies $P \cap Q' = P' \cap Q = \{x, y\}$.

Definition. We denote the (closed) region of the sphere bounded by $Q \cup Q'$ and containing neither H nor H' by $\Omega(Q, Q')$ and call it a *lens* for P, P' with ends x, y . If, in addition, $\Omega(Q, Q')$ contains no hole, it is called a *0-lens*. (See the picture.)



In what follows, for a vertex v and an edge e of a path P , we may liberally write $v \in P$ for $v \in V_P$ and $e \in P$ for $e \in E_P$.

It turns out that conditions (C2)–(C3) provide the following nice property.

Proposition 4.1 *There exists no 0-lens at all.*

Proof Suppose, for a contradiction, that a 0-lens $\Omega(Q, Q')$ does exist, and let $H, H', P, P', x, y, Q, Q'$ be as above. Let $G_{Q, Q'} = (V_{Q, Q'}, E_{Q, Q'})$ be the subgraph of G lying in $\Omega := \Omega(Q, Q')$. We rely on the following

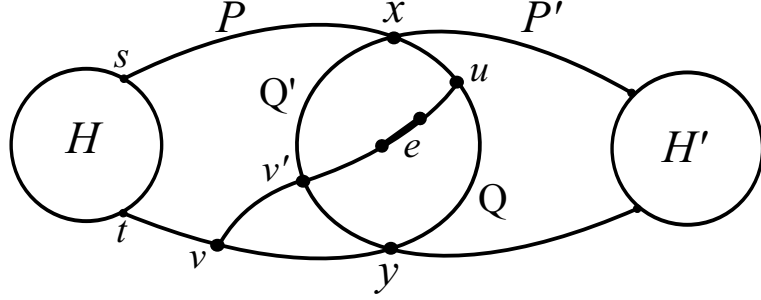
Claim *In the graph $G_{Q, Q'}$, each edge e is contained in a shortest x – y path, and similarly for any pair of vertices in a face of $G_{Q, Q'}$ in Ω .*

Proof of Claim. One may assume that e is an inner edge of $G_{Q, Q'}$ (i.e., not on $Q \cup Q'$). By (C2), there exists an \mathcal{H} -geodesic L containing e . Using the fact that Ω has no hole, one can realize that L crosses at least twice some of P, P' . Moreover, there are two vertices u, v of L such that e belongs to $L' := L[u, v]$ and at least one of the following takes place: both u, v are in P ; both u, v are in P' . Assume that u, v are chosen so that L' is minimal under this property and let for definiteness $u, v \in P$. Let $R := P[u, v]$.

Let \tilde{L} be the path obtained from P by replacing its part R by L' ; this is again a geodesic containing e . Moreover, the minimal choice of L' implies that at least one of the vertices u and v , say, u , belongs to Q , while the other, v , either (a) belongs to Q as well, or (b) is not in Ω .

In case (a), we may assume that L' is entirely contained in Ω . Then replacing in Q the part between u and v by L' , we just obtain the desired x – y path containing e .

In case (b), L' meets Q' at a vertex v' such that $L'[u, v']$ lies in Ω and contains e . Let s, t be the ends of \tilde{L} (and P); we may assume that s, u, v', v, t occur in this order in \tilde{L} . Since $u \in Q$, the subpath $\tilde{L}[s, u]$ ($= P[s, u]$) passes one of the ends x, y of the lens Ω , say, x ; see the picture below. Then the desired shortest x – y path in $G_{Q, Q'}$ containing e is obtained by concatenating $Q[x, u]$, $L'[u, v']$ and $Q'[v', y]$ (taking into account that the lengths of $Q'[x, v']$ and $\tilde{L}[x, v']$ are equal).



The assertion for a pair of vertices w, z in a face F of $G_{Q,Q'}$ in Ω is proved in a similar way. More precisely, take an \mathcal{H} -geodesic D passing w and z , existing by (C2), and let $D' := D[w, z]$. Making, if needed, appropriate exchange operations involving P, D' and/or P', D' , one can “improve” D' so as to get it entirely contained in Ω (keeping w, z). Now we argue as above, with D' in place of e . \blacksquare

By the Claim, $G_{Q,Q'}$ is the union of shortest x - y paths; therefore, one can direct the edges of $G_{Q,Q'}$ so that each shortest x - y path turns into a directed x - y path, and vice versa. It follows that each face F of the planar graph $G_{Q,Q'}$ has two vertices x_F and y_F such that $\text{bd}(F)$ is formed by two x_F - y_F paths A and B , which are extended to shortest x - y paths $C := P'_F \cdot A \cdot P''_F$ and $D := P'_F \cdot B \cdot P''_F$ (where P'_F, P''_F are shortest x - x_F and y_F - y paths in $G_{Q,Q'}$, respectively).

Suppose that A has an intermediate vertex u and B has an intermediate vertex v (assuming $F \subset \Omega$). By the Claim, u and v belong to a shortest x - y path L in $G_{Q,Q'}$; let for definiteness x, u, v, y occur in this order in L . By the planarity, $L' := L[u, v]$ must intersect either (a) the path P'_F , or (b) the path P''_F . The graph $G_{Q,Q'}$, being directed as indicated above, is acyclic. But in case (a), the subgraph $C \cup L'$ has a directed cycle, and in case (b), so does the subgraph $D \cup L'$; a contradiction.

Thus, either A or B has no intermediate vertex, i.e., has only one edge e . Since $\ell(A) = \ell(B)$, e is dominating in F . This contradicts (C3), and the result follows. \blacksquare

One consequence of the non-existence of 0-lenses that will be used later is as follows.

- (4.1) Let $(P_i, L_i), \dots, (P_k, L_k)$ be normal pairs of type 0 for a hole H , and let $\Omega_i := \Omega(P_i, L_i) \cup H$. Then for each hole $H' \neq H$, no H' -geodesic has a vertex in $\text{Int}(\Omega_1 \cup \dots \cup \Omega_k)$.

Indeed, such an H' -geodesic would create a 0-lens with some P_i .

5 Necklaces

In this section we further simplify the graph (G, ℓ) by handling one more sort of reducible cuts. In fact, our description in the previous sections was applicable to an arbitrary number of holes. This and the next sections will be devoted to the three-hole case only (though some ingredients are valid for $|\mathcal{H}| > 3$ as well).

Fix a hole H and denote the ℓ -length of $\text{bd}(H)$ by $\sigma = \sigma_H$. By (C4) and (3.7), $\text{bd}(H)$ is isometric and each H -geodesic of type 1 (if exists) connects antipodal terminals in $\text{bd}(H)$ (i.e. being at distance $\sigma/2$).

To simplify our description technically, we insert (for a while) extra terminals in the boundary $\text{bd}(H)$ to make it central symmetric. More precisely, for each $s \in V_H$, when s does not have the antipodal terminal in $\text{bd}(H)$, we add such a vertex by splitting the corresponding edge $pq \in E_H$ into two edges pt and tq whose lengths are such that $\ell(pt) + \ell(tq) = \ell(pq)$ and $d(st) = d(sp) + \ell(pt) = d(sq) + \ell(qt) = \sigma/2$ (using $d(sp) + \ell(pq) + d(qs) = \sigma$, cf. (C4)). This modification does not affect the problem, and we keep the previous notation (G, ℓ, \mathcal{H}) . (It increases the number of vertices and, possibly, creates non-tight pair of vertices involving t , violating (C2); but this will not be important for us.)

Let $s_1, s_2, \dots, s_{2n} = s_0$ be the sequence of vertices of the (modified) boundary cycle $\text{bd}(H)$ in the clockwise order around $\text{Int}(H)$. (Note that in illustrations, when H is the outer face of G , then the order of these vertices around the rest of G looks anti-clockwise.) For each s_i , its antipodal vertex s_{i+n} is also denoted as t_i (hereinafter the indices are taken modulo $2n$). A path of the form $s_i s_{i+1} \dots s_j$ is denoted by $L(s_i s_j)$; then $L(s_j s_i)$ is the path “complementary” to $L(s_i s_j)$ in $\text{bd}(H)$. When vertices $s_{i(1)}, s_{i(2)}, \dots, s_{i(k)}$ (admitting $i(j) = i(j+1)$) follow in this order cyclically, making at most one turn, we write $s_{i(1)} \rightarrow s_{i(2)} \rightarrow \dots \rightarrow s_{i(k)}$.

When $L := L(st)$ is shortest, we denote by $\mathcal{P}^0(st)$ the set of shortest s - t paths P such that $\tau(P, L) = 0$. For $P, P' \in \mathcal{P}^0(st)$, let $P \wedge P'$ (resp. $P \vee P'$) be the s - t path which together with L forms the boundary of $\Omega(P, L) \cap \Omega(P', L)$ (resp. $\Omega(P, L) \cup \Omega(P', L)$). Then $\ell(P \wedge P') + \ell(P \vee P') = \ell(P) + \ell(P')$, implying that both $P \wedge P'$, $P \vee P'$ are shortest as well. Hence $(\mathcal{P}^0(st), \wedge, \vee)$ is a lattice with the minimal element L . The maximal (most remote from L) element of $\mathcal{P}^0(st)$ is denoted by $D(st)$; then $\Omega(P, L) \subseteq \Omega(D(st), L)$ for any $P \in \mathcal{P}^0(st)$. Note that $D(st)$ can be extracted from the subgraph of shortest s - t paths in strongly polynomial time.

We are going to examine an interrelation of paths for two antipodal pairs s_i, t_i and s_j, t_j with $s_i \rightarrow s_j \rightarrow t_i \rightarrow t_j$. For brevity we write $L_i, \mathcal{P}_i^0, D_i, \Omega_i$ for $L(s_i t_i)$, $\mathcal{P}^0(s_i t_i)$, $D(s_i t_i)$, $\Omega(D(s_i t_i), L(s_i t_i))$, respectively, and similarly for j . Also we abbreviate $s := s_i$, $t := t_i$, $s' := s_j$, and $t' := t_j$.

Consider paths $M \in \mathcal{P}_i^0$ and $N \in \mathcal{P}_j^0$. Then $M \cap N \neq \emptyset$; let x and y be the first and last vertices of M occurring in $M \cap N$. Since M, N are shortest, $\ell(M[x, y]) = \ell(N[x, y])$. Two cases are possible:

Case 1 : either $x = y$ or x precedes y in N ;

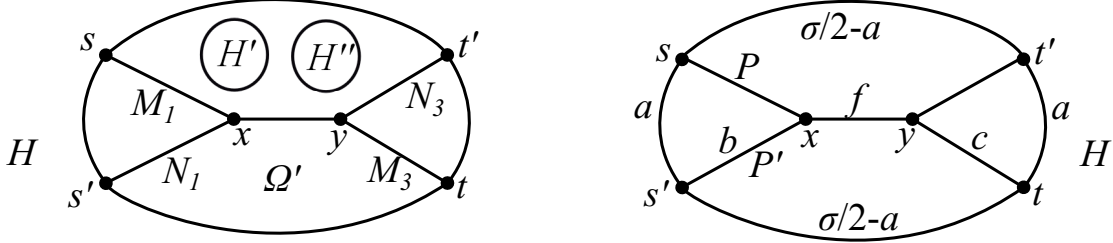
Case 2 : y precedes x in N .

We first consider Case 1. Represent M as $M_1 \cdot M_2 \cdot M_3$, and N as $N_1 \cdot N_2 \cdot N_3$, where $M_2 := M[x, y]$ and $N_2 := N[x, y]$. One can see that exchanging the parts M_2 and N_2 in M and N , we again obtain paths of type 0, one belonging to \mathcal{P}_i^0 , and the other to \mathcal{P}_j^0 . To slightly simplify our considerations, we will assume that $M_2 = N_2$, denoting

this path as \widehat{M} . (Note that if $M = D_i$ and $N = D_j$, then $M_2 = N_2$ follows from the maximality of D_i, D_j .) Form the s - t' path P and the s' - t path P' as

$$P := M_1 \cdot \widehat{M} \cdot N_3 \quad \text{and} \quad P' := N_1 \cdot \widehat{M} \cdot M_3,$$

and let $\Omega := \Omega(P, L(st'))$ and $\Omega' := \Omega(P', L(s't))$. Clearly both regions Ω', Ω contain no hole; also $\Omega \subseteq \Omega_i \cup \Omega_j$ and $\Omega' \subseteq \Omega_i \cap \Omega_j$. See the left fragment of the picture (where H is the outer face of G).



Define

$$a := d(ss') \quad (= d(tt')), \quad b := d(s'x), \quad c := d(yt), \quad \text{and} \quad f := d(xy).$$

Since the paths L_i, L_j, M, N are shortest and have the same length $\sigma/2$, we have

$$\ell(P') = b + f + c \geq d(s't) = \sigma/2 - a \quad \text{and} \quad \ell(P) = \sigma - \ell(P') \leq \sigma/2 + a. \quad (5.1)$$

We distinguish between two subcases:

Subcase 1a : the path P' is shortest: $\ell(P') = d(s't)$;

Subcase 1b : $\ell(P') > d(s't)$.

If Subcase 1b happens, we devise a certain collection of reducible cuts and make a good reduction, aiming to obtain a situation as in Subcase 1a. For this purpose, we apply the algorithm of [2] to solve the auxiliary one-hole PMP with $(G, \ell, \{H\})$, i.e., we handle the same G and ℓ but regard H', H'' as inner faces (see the right fragment of the above picture). It finds a packing \mathcal{C} of (simple) cuts δX with integer weights $\lambda(X) > 0$ realizing the distances on Π_H .

Let $\varepsilon := \ell(P') - d(s't)$. Then (cf. (5.1))

$$\ell(P') = \sigma/2 - a + \varepsilon \quad \text{and} \quad \ell(P) = \sigma - \ell(P') = \sigma/2 + a - \varepsilon. \quad (5.2)$$

Since the path M is shortest and connects antipodal terminals of H , each cut $\delta X \in \mathcal{C}$ meets M exactly once, and similarly for N . Let \mathcal{C}' be the set of cuts $\delta X \in \mathcal{C}$ meeting $L(s't)$. Then each $\delta X \in \mathcal{C}'$ meets $L(t's)$ as well, whereas each $\delta X \in \mathcal{C} - \mathcal{C}'$ meets one edge in each of $L(ss')$ and $L(tt')$. Partition $\mathcal{C} - \mathcal{C}'$ as $\mathcal{C}_1 \cup \mathcal{C}_2$, where \mathcal{C}_1 is formed by the cuts not meeting P , and \mathcal{C}_2 is the rest (consisting of the cuts δX with $|\delta X \cap M_1| = |\delta X \cap N_3| = 1$). Let h', h_1, h_2 be the sums of values $\lambda(X)$ over the cuts δX in $\mathcal{C}', \mathcal{C}_1, \mathcal{C}_2$, respectively. The cuts in \mathcal{C} (weighted by λ) must *saturate* the shortest

paths L_i, L_j, M, N (in the sense, that for each edge e in these paths, the sum of $\lambda(X)$'s over $\delta X \in \mathcal{C}$ containing e is equal to $\ell(e)$). Then they saturate P and P' . So $L(st')$ and $L(t's)$ are saturated by the cuts of \mathcal{C}' , P by the cuts of $\mathcal{C}' \cup \mathcal{C}_2$, and $L(ss') \cup L(tt')$ by the cuts of $\mathcal{C}_1 \cup \mathcal{C}_2$. Then (in view of (5.2))

$$h' = \sigma/2 - a, \quad h_2 = (\ell(P) - h')/2 = a - \varepsilon/2 \quad \text{and} \quad h_1 = a - h_2 = \varepsilon/2. \quad (5.3)$$

For each $\delta X \in \mathcal{C}_1$, since δX do not meet P , we may assume that $X \subset \Omega - P$ (where $\Omega = \Omega(P, L(st'))$). Let λ_1 be the restriction of λ to \mathcal{C}_1 . We assert the following

Lemma 5.1 $(\mathcal{C}_1, \lambda_1)$ is reducible for (G, ℓ, \mathcal{H}) ; in other words, for the reduced length $\ell' := \ell - \sum (\lambda(X) \chi^{\delta X} : \delta X \in \mathcal{C}_1)$, any p - q path Q in G with $pq \in \Pi_{\mathcal{H}}$ satisfies

$$\ell'(Q) \geq d(pq) - \sum (\lambda(X) : \delta X \in \mathcal{C}_1, pq \in \rho_{\mathcal{H}} X). \quad (5.4)$$

Proof If Q is a p - q path with $p, q \in V_H$, then (5.4) immediately follows from the reducibility of $(\mathcal{C}_1, \lambda_1)$ for $(G, \ell, \{H\})$. So assume that $pq \in \Pi_{H'} \cup \Pi_{H''}$. Then both p, q are not in $\Omega - P$.

If Q does not meet $\Omega - P$, then we have $\ell'(Q) = \ell(Q)$ (in view of $X \subset \Omega - P$ for each $\delta X \in \mathcal{C}_1$), and (5.4) is trivial. Suppose that $Q \cap (\Omega - P) \neq \emptyset$. Then Q crosses at least one of M, N . Let for definiteness Q meets $\Omega(M, L_i) - M$, and take a maximal subpath R of Q such that $R \subset \Omega(M, L_i)$ and $R \not\subset M$. Let u, v be the endvertices of R , and let $M' := M[u, v]$. The fact that M is ℓ' -shortest implies that $\ell'(M') \leq \ell'(R)$. Then, replacing in Q the part R by M' , we obtain a p - q path Q' with $\ell'(Q') \leq \ell'(Q)$. If Q' still meets $\Omega - P$, we repeat the procedure (treating the pair (Q', M) or (Q', N)), and so on. Eventually, we obtain a p - q path \tilde{Q} such that $\ell'(\tilde{Q}) \leq \ell'(Q)$ and $\tilde{Q} \cap (\Omega - P) = \emptyset$, yielding (5.4) for Q . ■

From (5.2), (5.3) it follows that for the updated length function,

(5.5) the path P' becomes shortest (of length $\sigma/2 - a$), and the lengths of P and $L(st')$ become the same.

Thus, the above procedure turns Subcase 1b into Subcase 1a, as required.

Next we consider Case 2. Let $x = v_0, v_1, \dots, v_k = y$ be the common vertices of M and N , in this order in M and, accordingly, in the reverse order in N (taking into account that M, N are shortest). For $p = 1, \dots, k$, let ω_p be the region bounded by the subpaths $M(p) := M[v_{p-1}, v_p]$ and $N(p) := N[v_p, v_{p-1}]$ and not containing H . Then either (a) ω_p lies in $\Omega(M, L_i) \cap \Omega(N, L_j)$ (in particular, $M(p)$ may coincide with the reverse path $N(p)^{-1}$ to $N(p)$), or (b) $\text{Int}(\omega_p)$ is nonempty and lies in the complement of $\Omega(M, L_i) \cup \Omega(N, L_j)$. To slightly simplify our considerations, we exclude the cases when $\text{Int}(\omega_p)$ is nonempty and contains no hole (which will lead to no loss of generality). Namely, for each p where such a situation happens, we perturb N , by replacing its part $N(p)$ by $M(p)^{-1}$. Clearly the updated s' - t path N is again shortest and of type 0.

So we will further assume that for each p where $M(p) \neq N(p)^{-1}$, the region ω_p contains one or two holes among H', H'' ; in this case we say that ω_p is *essential*. We come to four subcases of Case 2.

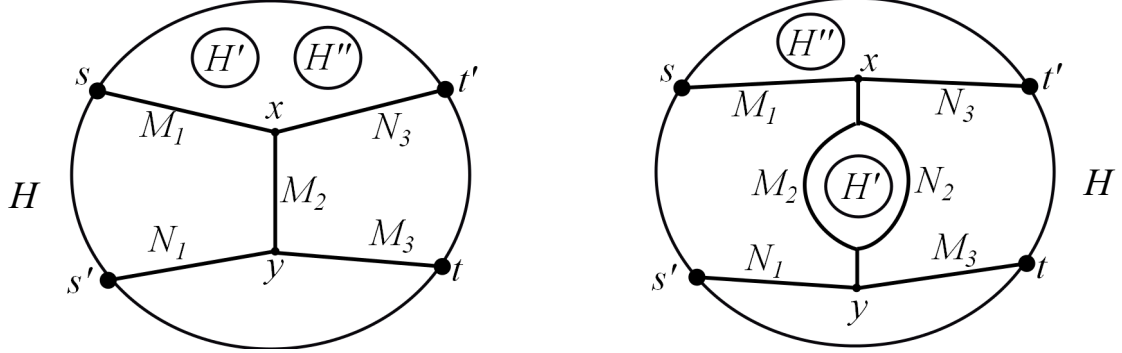
Subcase 2a : No region ω_p is essential; equivalently, $M[x, y] = N^{-1}[y, x]$.

Subcase 2b : Only one ω_p is essential and it contains exactly one hole, say, H' .

Subcase 2c : Two ω_p, ω_q are essential (each containing one hole).

Subcase 2d : One ω_p is essential and it contains both H', H'' .

We first handle (simultaneously) Subcases 2a and 2b; they are illustrated in the left and right fragments of the picture, respectively.



We are going to reduce these subcases to a situation as in Subcase 1a. As before, let $a := d(ss')$. Represent M as $M_1 \cdot M_2 \cdot M_3$, and N as $N_1 \cdot N_2 \cdot N_3$, where $M_2 := M[x, y]$ and $N_2 := N[y, x]$ (then $M_2 = N_2^{-1}$ in Subcase 2a). Form the s - t' path $P := M_1 \cdot N_3$ and the s' - t path $P' := N_1 \cdot M_3$ and define

$$f := \ell(M_2) \quad (= \ell(N_2)) \quad \text{and} \quad \varepsilon := \ell(P') - d(s't).$$

Then $\ell(P') = d(s't) + \varepsilon = \sigma/2 - a + \varepsilon$. Since $\ell(M) = \ell(N) = \sigma/2$, we have

$$\ell(P) = \ell(M) + \ell(N) - \ell(P') - \ell(M_2) - \ell(N_2) = \sigma/2 + a - \varepsilon - 2f. \quad (5.6)$$

To make the desired reduction, we use the algorithm of [2] to solve the auxiliary one- or two-hole PMP with $(G, \ell, \tilde{\mathcal{H}})$, where $\tilde{\mathcal{H}} := \{H\}$ in Subcase 2a, and $\tilde{\mathcal{H}} := \{H, H'\}$ in Subcase 2b. Let (\mathcal{C}, λ) be an integer solution to it. We extract from \mathcal{C} the set \mathcal{C}' of cuts meeting $L(s't)$ (and its opposite path $L(t's)$) and the set \mathcal{C}'' of cuts meeting $L(ss')$ (and $L(tt')$). Then

$$h' := \sum (\lambda(X) : \delta X \in \mathcal{C}') = \sigma/2 - a \quad \text{and} \quad h'' := \sum (\lambda(X) : \delta X \in \mathcal{C}'') = a. \quad (5.7)$$

Partition \mathcal{C}'' as $\mathcal{C}_1 \cup \mathcal{C}_2$, where \mathcal{C}_1 consists of the cuts not meeting P , and accordingly define $h_i := \sum (\lambda(X) : \delta X \in \mathcal{C}_i)$, $i = 1, 2$. Each cut $\delta X \in \mathcal{C}_2$ meets P twice (since $\delta X \cap P \neq \emptyset$ implies $|\delta X \cap M_1| = |\delta X \cap N_3| = 1$). Therefore, $\ell(P) \geq h' + 2h_2$, and this holds with equality since P is saturated by (\mathcal{C}, λ) . Using (5.6) and (5.7), we have

$$2h_2 = \ell(P) - h' = (\sigma/2 + a - \varepsilon - 2f) - (\sigma/2 - a) = 2a - 2f - \varepsilon.$$

This and $h_1 + h_2 = a$ imply

$$h_1 = f + \varepsilon/2. \quad (5.8)$$

The following assertion is similar to Lemma 5.1.

Lemma 5.2 *In Subcases 2a, 2b, $(\mathcal{C}_1, \lambda_1)$ is reducible for (G, ℓ, \mathcal{H}) , where $\lambda_1 := \lambda|_{\mathcal{C}_1}$.*

Proof Let Q be a p - q path in G with $pq \in \Pi_{\mathcal{H}}$. We have to show relation (5.4) for Q and $\ell' := \ell - \sum(\lambda(X)\chi^{\delta X} : \delta X \in \mathcal{C}_1)$. This is done in a way similar to the proof of Lemma 5.1. More precisely, if $pq \in \Pi_{\tilde{\mathcal{H}}}$, then (5.4) is immediate from the reducibility of $(\mathcal{C}_1, \lambda_1)$ for $(G, \ell, \tilde{\mathcal{H}})$. And if $pq \in \Pi_{\mathcal{H}-\tilde{\mathcal{H}}}$, then both p, q are not in $\Omega - P$, where $\Omega := \Omega(P, L(st'))$. Making, if needed, the corresponding replacements in Q using pieces of M or N (like in the proof of Lemma 5.1), we obtain a p - q path Q' disjoint from $\Omega - P$ and such that $\ell'(Q') \leq \ell'(Q)$. Then $\ell'(Q') = \ell(Q')$, implying (5.4) for Q . \blacksquare

Note that since each cut $\delta X \in \mathcal{C}_1$ does not meet P and M, N are shortest, either $|\delta X \cap M_2| = 1$ or $|\delta X \cap N_1| = |\delta X \cap M_3| = 1$. Then (5.8) implies that $(\mathcal{C}_1, \lambda_1)$ saturates M_2 and uses ε units of the ℓ -length of P' . It follows that for the updated lengths,

(5.9) the path P' becomes shortest, and the lengths of M_2 and N_2 become zero.

In other words, contracting the edges with zero length, we obtain a situation as in Subcase 1a, as required. (Note that the hole H' vanishes if $|\tilde{\mathcal{H}}| = 2$.)

In Subcase 2c, we act in a similar fashion. Suppose that $H' \subset \omega_p$, $H'' \subset \omega_q$, and $q < p$. Let $z := v_q$. Then H'' is located between $M[x, z]$ and $N[z, x]$, whereas H' between $M[z, y]$ and $N[y, z]$; see the left fragment of Fig. 1.

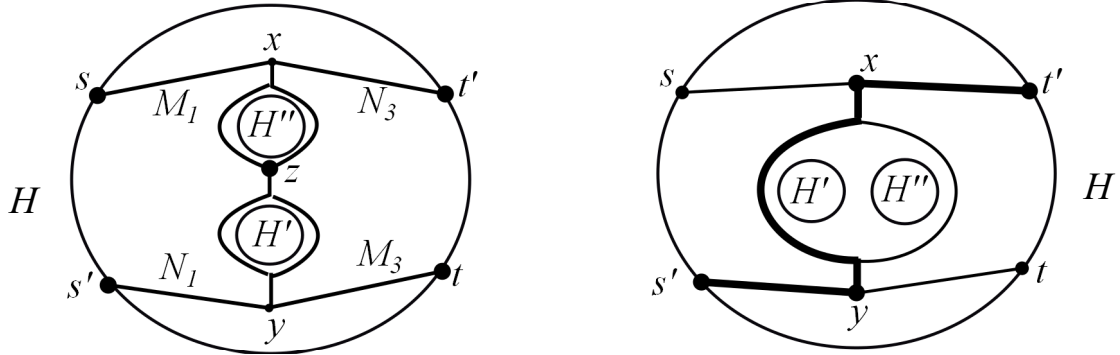


Figure 1: Subcase 2c (left) and Subcase 2d (right)

Let $P' := N_1 \cdot M_3$ and define $\varepsilon := \ell(P') - d(s't)$ and $f := \ell(M[z, y])$ ($= \ell(N[y, z])$). Find an integer solution (\mathcal{C}, λ) to the two-hole PMP with (G, ℓ, \mathcal{H}) , where $\mathcal{H} := \{H, H'\}$. Extract from \mathcal{C} the set \mathcal{C}'' of cuts meeting $L(ss')$ and partition \mathcal{C}'' as $\mathcal{C}_1 \cup \mathcal{C}_2$, where \mathcal{C}_1 is formed by the cuts not meeting $M[s, z]$. Let λ_1 be the restriction of λ to \mathcal{C}_1 . Arguing as in the previous case, one can conclude that $(\mathcal{C}_1, \lambda_1)$ is reducible for (G, ℓ, \mathcal{H}) and that $\sum(\lambda(X) : \delta X \in \mathcal{C}_1) = f + \varepsilon/2$. Then for the reduced length function, the path P' becomes shortest, and the lengths of the paths $M[z, y]$ and $N[y, z]$ become zero. Then the hole H' vanishes and we obtain PMP with two holes.

It remains to consider Subcase 2d. We distinguish between two possibilities for $s = s_i$ and $s' = s_j$.

I. Suppose that $j = i + n$; then $s = t' = x$ and $s' = t = y$. Let $z := v_{p-1}$ and $u := v_p$, i.e., the region ω_p containing H' and H'' is bounded by the paths $M[z, u]$ and $N[u, z]$. Suppose that $u \neq y$. Then we make a good reduction, aiming to decrease the length of the path $M' := M[u, y]$ ($= N^{-1}[u, y]$) to zero (and then to contract M'). For this purpose, we find a solution (\mathcal{C}, λ) to the auxiliary one-hole problem PMP with $(G, \ell, \{H\})$ and extract from \mathcal{C} the set \mathcal{C}_1 of cuts meeting M' . Since M, N are shortest paths connecting the antipodal terminals x, y and both holes H', H'' are disposed between $M[z, u]$ and $N[u, z]$, one can conclude that $(\mathcal{C}_1, \lambda_1)$ is reducible for (G, ℓ, \mathcal{H}) and saturates M' . Then, after the reduction using $(\mathcal{C}_1, \lambda_1)$ followed by the corresponding contractions, u and y become merged into one vertex.

If $x \neq z$, we handle the subpath $M[x, z]$ in a similar way.

II. Suppose that $0 < j - i < n$. As in Subcases 2a–2c, we represent M as $M_1 \cdot M_2 \cdot M_3$, and N as $N_1 \cdot N_2 \cdot N_3$, where $M_2 := M[x, y]$ and $N_2 := N[y, x]$. Besides N , consider the shortest $t' - s'$ path $N' := N_3^{-1} \cdot M_2 \cdot N_1^{-1}$ (see the right fragment in Fig. 1 where N' is drawn in bold). Then $\tau(N') = 0$, and the paths N and N' form a pair as in I. Therefore, we may assume that $N \cap N' = \{s', t'\}$. But then all subpaths M_1, M_3, N_1, N_3 must degenerate, giving $s = t'$ and $s' = t$. So this is not the case.

We apply the good reductions by cuts described above to all pairs $\{i, j\} \subset \{1, \dots, 2n\}$ for the hole H (with $\text{bd}(H)$ extended to be central symmetric), and then treat the other holes H', H'' in a similar way, referring to the whole process as *Procedure II*. Every time we take as M, N the most remote paths $D_i \in \mathcal{P}_i^0$ and $D_j \in \mathcal{P}_j^0$. Summing up the above results, we can conclude with the following:

Proposition 5.3 *Procedure II takes strongly polynomial time and terminates with (G, ℓ, \mathcal{H}) such that $|\mathcal{H}| \leq 3$, ℓ is cyclically even, and: for each $H \in \mathcal{H}$ and antipodal pairs $\{s_i, t_i\}$ and $\{s_j, t_j\}$ with $s_i \rightarrow s_j \rightarrow t_i \rightarrow t_j$ in $\text{bd}(H)$, the paths $M := D(s_i t_i)$ and $N := D(s_j t_j)$ are subject to Subcase 1a when $s_i \neq t_j$, and subject to Subcase 2d with $M \cap N = \{s_i, s_j\}$ when $s_i = t_j$.*

Removing the extra terminals that were added before the procedure, we obtain a similar result in terms of the original graph. More precisely, for $H \in \mathcal{H}$, let $s_1, s_2, \dots, s_m = s_0$ be the vertices of (the original) $\text{bd}(H)$ in the clockwise order around $\text{Int}(H)$, and define \mathcal{L}_H^{\max} to be the set of inclusion-wise *maximal* shortest paths $s_i s_{i+1} \dots s_{i'}$ in $\text{bd}(H)$ (taking indices modulo m). For such a path, we denote $s_{i'}$ by t_i . Also we denote the set of indices i for which \mathcal{L}_H^{\max} has a path starting with s_i by $I = I_H$. Like the central symmetric case, for $i \in I$, we denote $L(s_i t_i)$ by L_i and write $\mathcal{P}_i^0, D_i, \Omega_i$ for the set of shortest $s_i - t_i$ paths of type 0, the most remote path in \mathcal{P}_i^0 , and the region $\Omega(D_i, L_i)$, respectively. Then (after removing the extra terminals) the following holds:

- (5.10) for $H \in \mathcal{H}$ and $i, j \in I_H$ with $s_i \rightarrow s_j \rightarrow t_i \rightarrow t_j$: (a) if $s_i = t_j$ (and therefore $s_j = t_i$), then $D_i \cap D_j = \{s_i, s_j\}$; and (b) if $s_i \neq t_j$, then for any common vertex v of D_i and D_j , the $s_j - t_i$ path $D_j[s_j, v] \cdot D_i[v, t_i]$ is shortest.

For a fixed $H \in \mathcal{H}$ and for $i \in I_H$, let $G_i = (V_i, E_i)$ be the directed graph that is the union of shortest paths $P \in \mathcal{P}^0(s_i t_i)$, each P being directed from s_i to t_i . We will take an advantage from the following

Proposition 5.4 (i) *The directions of edges in all graphs G_i are agreeable.*

(ii) *In the graph $N_H = (W, U) := \cup(G_i : i \in I)$, for each simple directed cycle C , the region bounded by C and containing H contains no other hole.*

(iii) *N_H admits a function (cyclic potential) $\pi : W \rightarrow \{0, 1, \dots, \sigma_H - 1\}$ such that $\pi(s_1) = 0$ and for each edge $e = (u, v) \in U$, $\ell(e) \equiv \pi(v) - \pi(u) \pmod{\sigma_H}$.*

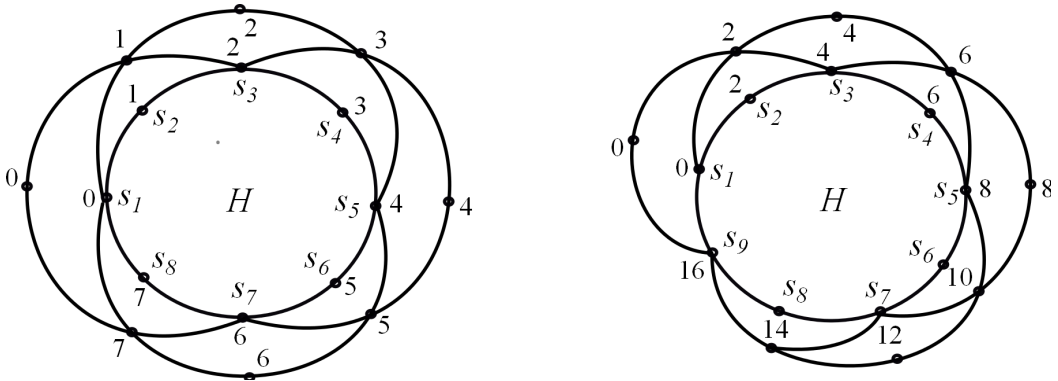
Proof For $k = 1, \dots, m = |V_H|$, define $\pi(s_k) := \ell(L(s_1 s_k))$. Then each edge $e = s_k s_{k+1}$ of $\text{bd}(H)$ satisfies $\ell(e) = \pi(s_{k+1}) - \pi(s_k)$, and each $i \in I$ satisfies $\ell(L(s_i t_i)) = \pi(t_i) - \pi(s_i)$, taking indices modulo m and taking lengths/potentials modulo $\sigma = \sigma_H$.

In order to extend π to the other vertices of N_H , we first introduce, for each $i \in I$, its own potential $\pi_i : V_i \rightarrow \mathbb{Z}$ as $\pi_i(v) := \ell(P)$, where $v \in V_i$ and P is a directed s_i - v path in G_i . Then $\pi_i(s_i) = 0$ and π_i satisfies $\ell(e) = \pi_i(v) - \pi_i(u)$ for each $e = (u, v) \in E_i$.

We assert that for any $i, j \in I$ and $v \in V_i \cap V_j$, the numbers $\pi(s_i) + \pi_i(v)$ and $\pi(s_j) + \pi_j(v)$ are the same modulo σ (whence $\pi(v) := \pi(s_i) + \pi_i(v)$ is as required).

Indeed, one may assume that $s_i \rightarrow s_j \rightarrow t_i \rightarrow t_j$. Then, unless s_i, s_j are antipodal (in which case $G_i \cap G_j$ consists of two isolated terminals $s_i = t_j$ and $s_j = t_i$ and the assertion is trivial), the region $\Omega_i \cap \Omega_j$ lies between the shortest paths $L(s_j t_i)$ and $D(s_j t_i)$ (in view of (5.10)). This implies that the subgraphs of G_i and G_j lying in $\Omega_i \cap \Omega_j$ are the same and equal to $G_i \cap G_j$. Moreover, the latter is just the union of shortest s_j - t_i paths: $G_i \cap G_j = \cup(P \in \mathcal{P}^0(s_j t_i))$ (whence the directions of edges in G_i and G_j are agreeable, yielding (i)). Since $\pi(s_i) + \pi_i(s_j) \equiv \pi(s_j) \pmod{\sigma}$ and $\pi_i(v) = \pi_i(s_j) + \pi_j(v)$, we can conclude that $\pi(s_i) + \pi_i(v) \equiv \pi(s_j) + \pi_j(v) \pmod{\sigma}$ holds for each $v \in V_i \cap V_j$, and (iii) follows. To see (ii) is easy. \blacksquare

We call the graph $N = N_H$ defined in this proposition the *necklace* for H . (Depending on the context in what follows, we may also think of the necklace as the underlying undirected graph.) Two examples are illustrated in the picture; here the potentials are indicated when all edges in the left (right) fragment have length 1 (resp. 2).



Denoting the set of all H -geodesics of type 0 by \mathcal{P}_H^0 , we can summarize the above observations and results as follows.

Corollary 5.5 *The initial problem can be reduced, in strongly polynomial time, to PMP with (G, ℓ, \mathcal{H}) satisfying (C1)–(C4) and the next property:*

(C5) *for each $H \in \mathcal{H}$, the subgraph $\cup(P \in \mathcal{P}_H^0)$ of G can be directed so that any simple directed cycle separates H from $\mathcal{H} - \{H\}$ and has ℓ -length exactly σ_H .*

(This can be concluded from the existence of potential π as above, which implies that any directed cycle in N_H has length $k\sigma_H$ for some integer k and that each inner face in N_H is bounded by two directed paths of the same length.)

We denote the set of such cycles in N_H embracing H by \mathcal{C}_H . For $C \in \mathcal{C}_H$, let $\Omega(C)$ denote the closed region bounded by C and containing H . The correspondence $C \mapsto \Omega(C)$ leads to representing \mathcal{C}_H as the distributive lattice with operations \wedge, \vee defined by the relations: for $C, C' \in \mathcal{C}_H$, $\Omega(C \wedge C') = \Omega(C) \cap \Omega(C')$ and $\Omega(C \vee C') = \Omega(C) \cup \Omega(C')$. Then $\text{bd}(H)$ is the minimal element of \mathcal{C}_H , and we denote the maximal element in it as D_H ; so

$$\ell(D_H) = \sigma_H \quad \text{and} \quad \Omega(C) \subseteq \Omega(D_H) \quad \text{for all } C \in \mathcal{C}_H.$$

Note that (4.1) implies the following property:

(5.11) for distinct $H, H' \in \mathcal{H}$, no H' -geodesic meets $\text{Int}(\Omega(D_H))$.

We finish this section with one important special case.

Definition. The necklace N_H is called *trivial* if $N_H = \text{bd}(H)$.

Proposition 5.6 *If $\tau(H) = 0$ then N_H is trivial.*

(Note that the converse need not hold.)

Proof Suppose that this is not so. Then $D_H \neq \text{bd}(H)$ and $\Omega(D_H)$ contains a face $F \neq H$ of G (which lies in a face of N_H but may not coincide with the latter). We can choose two vertices u, v in $\text{bd}(F)$ not contained in a directed path of N_H (taking into account that F has no dominating edge, by (C3)). Then at least one of these vertices, v say, is not in D_H . By (C2), u and v belong to an \mathcal{H} -geodesic Q . By (5.11), Q cannot be an H' -geodesic for $H' \neq H$. So Q is an H -geodesic. Moreover, $\tau(Q) = 0$. Then Q turns into a directed path in N_H containing u and v ; a contradiction. \blacksquare

6 Elimination of inner edges

In this section we demonstrate one more sort of reducible cuts, aiming to obtain the following result (as a weakened version of Proposition 3.1).

Proposition 6.1 *When $|\mathcal{H}| = 3$, one can find, in strongly polynomial time, a reducible collection of cuts with integer weights so that the reduction by these cuts results in a triple $(G', \ell', \mathcal{H}')$ with $|\mathcal{H}'| \leq 3$ and G' having no inner edges (i.e., covered by the boundaries of holes).*

Let (G, ℓ, \mathcal{H}) be as in Corollary 5.5. In what follows, until otherwise is explicitly said, we assume that

(6.1) G has an inner edge or an inner face F with $|V_F| \geq 4$.

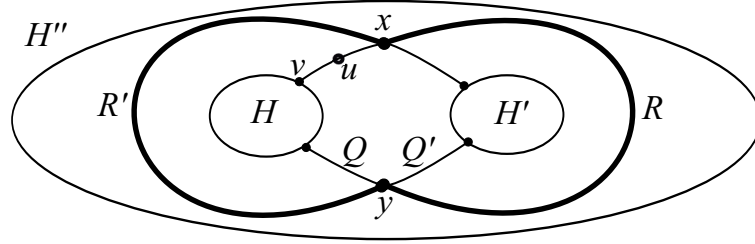
Lemma 6.2 *Suppose that for some $H \in \mathcal{H}$, the necklace N_H is nontrivial. Then the other two holes are of type 0 (and their necklaces are trivial, by Proposition 5.6).*

Proof Let v be a vertex in $\text{bd}(H) - D_H$ (existing since $\text{bd}(H) \neq D_H$). One can see that $\Omega(D_H)$ contains an inner face F of G and a vertex $u \neq v$ such that

(6.2) both u, v belong to F but not connected by a directed path in N_H .

By (C2), u and v belong to an \mathcal{H} -geodesic Q . By (5.11), Q is an H -geodesic, and (6.2) implies that $\tau(Q) = 1$, i.e., Q separates the holes H' and H'' , where $\mathcal{H} = \{H, H', H''\}$. One may assume that the u - v part of Q lies in $\Omega(D_H)$.

Now suppose that $\tau(H') = 1$ as well. Take an H' -geodesic Q' separating H and H'' . Then Q and Q' “cross” each other; let x and y be the first and last vertices of Q' occurring in Q , respectively. Let R and R' be the x - y parts of Q and Q' , respectively; see the picture where R, R' are drawn in bold.



Exchange in Q, Q' the pieces R, R' , forming H -path \tilde{Q} and H' -path \tilde{Q}' , respectively. In view of $\ell(\tilde{Q}) + \ell(\tilde{Q}') = \ell(Q) + \ell(Q')$, both \tilde{Q}, \tilde{Q}' are shortest. Also \tilde{Q} does not separate H' and H'' and contains both vertices v, u . So \tilde{Q} is an H -geodesic of type 0 passing v and u , which contradicts (6.2). ■

This lemma is generalized as follows.

Lemma 6.3 *Subject to (6.1), exactly one hole has type 1.*

Proof In view of lemma 6.2, we may assume that the necklaces of all holes are trivial. From (6.1) it follows that there are two vertices u, v contained in an inner face F but not in the boundary of one hole. By (C2), for some $H \in \mathcal{H}$, there is an H -geodesic Q passing u, v . Since N_H is trivial, $\tau(Q) = \tau(H) = 1$. Suppose that there is another hole H' of type 1. Choose an H' -geodesic Q' of type 1. Then (like in the proof of Lemma 6.2) there are vertices $x, y \in V_Q \cap V_{Q'}$ such that exchanging in Q, Q' the pieces $R := Q[x, y]$ and $R' := Q'[x, y]$, we obtain an H -geodesic \tilde{Q} and an H' -geodesic \tilde{Q}' , both of type 0. Since N_H and $N_{H'}$ are trivial, \tilde{Q} lies in $\text{bd}(H)$ and \tilde{Q}' lies in $\text{bd}(H')$. It follows that $x, y \in V_H \cap V_{H'}$, R is an H -geodesic, and R' is an H' -geodesic, both of type 1. Now $\ell(R) = \sigma_H/2 = \ell(Q)$ implies $R = Q$. Then $Q \subset \text{bd}(H')$, contradicting the choice of u, v . ■

Lemma 6.4 *For any two holes $H, H' \in \mathcal{H}$, $\text{bd}(H) \cap \text{bd}(H')$ is connected (possibly empty).*

Proof Suppose this is not so. Then there are two paths $L \subset \text{bd}(H)$ and $L' \subset \text{bd}(H')$ that have the same ends, x, y say, but no common intermediate vertices, and such that one of the two regions of the plane bounded by $L \cup L'$, Ω say, contains at least one inner face and no hole (in view of $|\mathcal{H}| = 3$).

Each pair of vertices in a face within Ω belongs to an \mathcal{H} -geodesic lying in Ω and having both ends in one of L, L' . It follows that the subgraph G' of G lying in Ω is included in $N_H \cup N_{H'}$. In view of Lemma 6.2, exactly one of the necklaces of H, H' , say, N_H , is nontrivial. Therefore, $G' \subset N_H$. This implies $\ell(L) = \ell(L')$. But then L belongs to $N_{H'}$; a contradiction. \blacksquare

Consider the auxiliary graph Γ whose vertices are the inner faces of G and whose edges are the pairs of inner faces sharing an edge. For a component Γ' of Γ , let $\Omega_{\Gamma'}$ be the union of faces that are the vertices of Γ' . Lemma 6.4 implies that Γ has at most two components, and for each component Γ' , the region $\Omega_{\Gamma'}$ is surrounded by three paths in the boundaries of holes. One more important fact is as follows.

Lemma 6.5 *Let an inner face F share an edge e with the hole H of type 1. Suppose that at least one is true: F has an inner edge, or $|V_F| \geq 4$ (cf. (6.1)). Then there is an endvertex x of e and a vertex $z \in V_F$ such that both x, z belong to an H -geodesic P of type 1 and satisfy $d(zx) + d(xy) > d(zy)$, where y is the other endvertex of e .*

Proof Let x_0, x_1, \dots, x_k be the sequence of vertices in $\text{bd}(F)$. One may assume that e connects x_0, x_k and is directed from x_0 to x_k in N_H . Consider two cases.

Case 1: $F \not\subset \Omega(D_H)$. Then there exists x_i which is not in $\Omega(D_H)$. (Since $V_F \subset \Omega(D_H)$ together with $F \not\subset \Omega(D_H)$ would imply $V_F \subset D_H$, and therefore for some j , the sequence $x_j x_{j-1} \dots x_0 x_k x_{k-1} \dots x_{j+1}$ forms a directed path Q in D_H . Moreover, (C5) and $\ell(x_j x_{j+1}) = d(x_j x_{j+1})$ imply that Q is shortest. Then the edge $x_j x_{j+1}$ is dominating, contrary to (C3).) Choose $x_i \notin \Omega(D_H)$.

If there is no hole H' such that $x_0, x_i \in V_{H'}$, we assign $x := x_0, y := x_k, z := x_i$. Then an \mathcal{H} -geodesic P containing x and z is an H -geodesic of type 1 (since $x_i \notin \Omega(D_H)$ and any H' -geodesic for $H' \neq H$ lies in $\text{bd}(H')$, by Lemma 6.3). Also the inequality $d(zx) + d(xy) \geq d(zy)$ is strict. For otherwise, taking an \mathcal{H} -geodesic Q passing y, z and replacing its part $Q[y, z]$ by the concatenation of e and a shortest x - z path, we would obtain a geodesic Q' passing y, x, z . Then Q' is an H -geodesic of type 1 connecting y with its antipodal terminal, s say. But the part of Q' between x and s is an H -geodesic of type 1 as well, whence x is antipodal to s ; a contradiction.

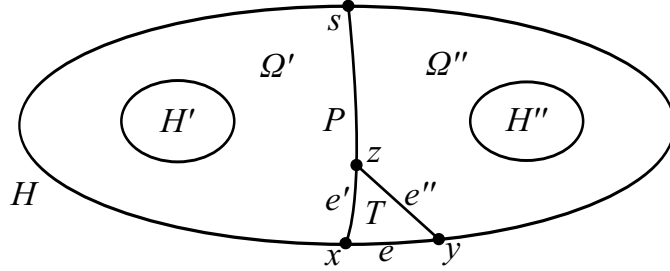
If there is no hole H'' such that $x_k, x_i \in V_{H''}$, we argue in a similar way and assign $x := x_k, y := x_0, z := x_i$.

Now let $x_0, x_i \in V_{H'}$ and $x_k, x_i \in V_{H''}$ for $H', H'' \in \mathcal{H} - \{H\}$. In case $H' = H''$, the shortest path x_0, e, x_k would lie in $\text{bd}(H')$, which is impossible since e is shared by H and F . So $H' \neq H''$. Then concatenating the edge e with one x_0 - x_i path in $\text{bd}(H')$ and one x_i - x_k path in $\text{bd}(H'')$, we obtain a simple cycle surrounding a region of the

plane that contains F but no hole. This easily implies that if (a) $i \geq 2$, then x_1 belongs to neither $\text{bd}(H'')$ nor D_H , and if (b) $i \leq k - 2$ then x_{k-1} belongs to neither $\text{bd}(H')$ nor D_H . Since $|V_F| = k \geq 4$ (by the hypotheses of the lemma), at least one of (a),(b) takes place. So we can choose either x_1 or x_{k-1} and argue as above for x_i .

Case 2: $F \subset \Omega(D_H)$. Let F' be the face of N_H containing e and different from H ; then $F \subseteq F'$ (possibly $F = F'$). Let $\text{bd}(F')$ be formed by two directed $x'-y'$ paths Q and R , and let e belong to Q . Since e is non-dominating, at least one of $x' \neq x_0$ and $y' \neq x_k$ is true. Let for definiteness $x' \neq x_0$. Then $x_0 \notin D_H$. We assign $x := x_0$, $y := x_k$, and assign z to be a vertex $x_i \in V_F$ not in Q . Let P be an \mathcal{H} -geodesic passing x, z . Then P is an H -geodesic (in view of (4.1) and $x \in \Omega(D_H) - D_H$). Also $\tau(P) = 1$ (since the choice of x_0 and x_i ensures that the pair x, z does not belong to a directed path in N_H). The required inequality $d(xz) + d(xy) > d(yz)$ is shown as in Case 1. ■

Consider F, e, x, y, z, P as in this lemma and let s be the endvertex of P different from x . It is convenient for us to add to G (for a while) extra edges $e' = xz$ and $e'' = yz$ with the lengths $\ell(e') := d(xz)$ and $\ell(e'') := d(yz)$, placing them in the face F (unless such edges already exist); this does not affect the problem. Accordingly replace in P the $x-z$ part by the edge e' . The updated path P divides $\mathbb{R}^2 - \text{Int}(H)$ into two closed regions Ω' and Ω'' containing H' and H'' , respectively. Let for definiteness y lies in Ω'' ; then the face (triangle) T bounded by e, e', e'' lies in Ω'' as well; see the picture.



Let P' be the $y-s$ path that is the concatenation of e'' and $P[z, s]$. We wish to construct a collection of reducible cuts containing both edges e, e' and traversing Ω' . To this aim, we extract from G the subgraph $G' = (V', E')$ lying in $\Omega' \cup T$ and, using the algorithm of [2], solve the auxiliary two-hole PMP with $G', \ell|_{E'}$ and $\mathcal{H}' := \{H', \tilde{H}\}$, where \tilde{H} is the face of G' containing H . This gives a packing \mathcal{C} of cuts δX in G' with integer weights $\lambda(X) > 0$ that realize the distances on $\Pi_{\mathcal{H}'}$.

Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be the collections of cuts $\delta X \in \mathcal{C}$ containing $\{e, e'\}$, $\{e, e''\}$ and $\{e', e''\}$, respectively. Define $a_i := \sum(\lambda(X) : \delta X \in \mathcal{C}_i)$, $i = 1, 2, 3$. The edges of T must be saturated by (\mathcal{C}, λ) , i.e., the following equalities hold:

$$d(xy) = a_1 + a_2, \quad d(xz) = a_1 + a_3, \quad d(yz) = a_2 + a_3.$$

Hence $2a_1 = d(xy) + d(xz) - d(yz) > 0$, in view of Lemma 6.5. For each $\delta X \in \mathcal{C}_1$, we may assume that $x \in X$. Since δX can meet any \tilde{H} -geodesic at most once, e' is the unique common edge of δX and P . Then X does not meet P' and therefore δX is a cut in the whole G .

Lemma 6.6 *Let $\ell' := \ell - \sum(\lambda(X)\chi^{\delta X} : \delta X \in \mathcal{C}_1)$ and $\lambda_1 := \lambda|_{\mathcal{C}_1}$. Then $(\mathcal{C}_1, \lambda_1)$ is reducible for (G, ℓ, \mathcal{H}) , i.e., any p - q path Q in G with $pq \in \Pi_{\mathcal{H}}$ satisfies (5.4).*

Proof We denote the distance in $(G', \ell'_{|E'})$ by d' . For Q as above, let R_1, \dots, R_k be the components of $Q \cap P$, occurring in this order in Q . We use induction on k . If Q is entirely contained in one of Ω', Ω'' (in particular, if $k = 0$), then (5.4) is easy (taking into account that $(\mathcal{C}_1, \lambda_1)$ is reducible for $(G', \ell'_{|E'}, \mathcal{H}')$, whence $d'(e) = \ell'(e) = a_2$, $d'(e') = \ell'(e') = a_3$, $d'(e'') = d(e'') = a_2 + a_3$).

In case $k \geq 2$, choose a vertex u in R_1 and a vertex v in R_2 , and let $\tilde{Q} := Q[u, v]$ and $\tilde{P} := P[u, v]$. The reducibility of $(\mathcal{C}_1, \lambda_1)$ for $(G', \ell'_{|E'}, \mathcal{H}')$ implies $d'(uv) = \ell'(\tilde{P})$. In its turn, $\ell'(\tilde{Q}) \geq d'(uv)$ when \tilde{Q} lies in Ω' . And when \tilde{Q} lies in Ω'' , we have $\ell'(\tilde{Q}) = \ell(\tilde{Q}) \geq d(uv)$ if $x \neq u, v$, and $\ell'(\tilde{Q}) = \ell(\tilde{Q}) - a_1 \geq d'(uv)$ otherwise (since in the latter case \tilde{P} contains the edge e' and \tilde{Q} must contain either e or e'). Hence $\ell'(\tilde{Q}) \geq \ell'(\tilde{P})$ always hold, and (5.4) follows by induction (by replacing \tilde{Q} by \tilde{P} in Q).

It remains to consider the situation when $k = 1$ and Q meets both $\Omega' - P$ and $\Omega'' - P$. Let $v \in V_Q \cap V_P$, $Q' := Q[p, v]$ and $Q'' := Q[v, q]$. One may assume that $Q' \subset \Omega'$ (and $Q'' \subset \Omega''$); then p is in $\Omega' - P$ and q is in $\Omega'' - P$. This implies $pq \in \Pi_H$.

Take in (G, ℓ) a shortest p - v path A and a shortest v - q path B . One may assume that $A \subset \Omega'$ and $B \subset \Omega''$. Since $pv \in \Pi_{\tilde{H}}$, A is shortest in $(G', \ell'_{|E'})$, whence $\ell'(Q') \geq \ell'(A)$. Also one can see that $\ell'(Q'') \geq \ell'(B)$ (considering both cases $v = x$ and $v \neq x$). So we can replace Q by the concatenation L of A and B . Then $\ell(L) = d(pv) + d(vq) \geq d(pq)$. Since each cut in \mathcal{C}_1 meets A at most once,

$$\ell'(A) = d(pv) - a, \quad (6.3)$$

where $a := \sum(\lambda(X) : \delta X \in \mathcal{C}_1, pv \in \rho_{\mathcal{H}'}X)$.

If $v \neq x$ then, obviously, no cut in \mathcal{C}_1 meets B , whence $\ell'(B) = \ell(B) = d(vq)$ holds, and (6.3) implies (5.4) (with L in place of Q).

Now suppose that $v = x$. Let M', M'' be the x - s paths in $\text{bd}(H)$ lying in Ω' and Ω'' , respectively. Since both M', M'' are geodesics (of length $\sigma_H/2$), we may assume that A, B are parts of $(M')^{-1}$ and M'' , respectively. Then B (and therefore L) contains the edge e , whence

$$\ell'(B) = d(xq) - a_1. \quad (6.4)$$

Note that (6.3) and (6.4) immediately imply (5.4) if $a = 0$. So assume $a > 0$.

Let r be the last vertex of M' such that $\ell(M'[r, s]) \geq a_1$. The path $R := M'[r, s]$ is “opposite” (in a sense) to the edge e' in the cycle $P \cup M'$, and under the reduction by $(\mathcal{C}_1, \lambda_1)$, the lengths of all edges of R , except the first one in case $\ell(R) > a_1$, reduce to zero. Then $a > 0$ implies that the terminal p lies in R and is different from r . Hence the ℓ -length of $R[p, s]$ is equal to $a_1 - a$ (taking into account that each cut of \mathcal{C}_1 meeting $M'[x, p]$ ($= A^{-1}$) should meet the part $R[r, p]$ of R , and the “contribution” from these cuts is just a). It follows that $\ell(A) = \sigma_H/2 - a_1 + a$. This together with $\ell(B) \geq \ell(e) \geq a_1$ gives

$$\ell(L) = \ell(A) + \ell(B) \geq \sigma_H/2 + a.$$

Then the p - q path $L' \in \mathcal{L}_H(pq)$ different from L satisfies $\ell(L') = \sigma_H - \ell(L) \leq \sigma_H/2 - a$. This implies $d(pq) = \ell(L') \leq \sigma_H/2 - a$ and $d(px) + d(xq) = \ell(L) \geq d(pq) + 2a$. Now using (6.3) and (6.4), we have

$$\ell'(L) = \ell'(A) + \ell'(B) = (d(px) - a) + (d(xq) - a_1) \geq d(pq) + a - a_1. \quad (6.5)$$

But $a_1 - a$ is equal to the sum of values $\lambda(X)$ over the cuts $\delta X \in \mathcal{C}_1$ separating p and q . So (6.5) implies the required relation (5.4). \blacksquare

Reduction III: Implementation and convergency. We refer to a natural procedure behind Lemmas 6.5 and 6.6 as *Reduction III*. It scans all F, e as in Lemma 6.5 and, at a current iteration, finds corresponding x, y, z, P for F, e , compute $(\mathcal{C}_1, \lambda_1)$ and reduce ℓ to ℓ' as in Lemma 6.6, after which the extra edges e', e'' (if exist) are deleted. Note that

$$(6.6) \quad d(yz) < d(xy) + d(xz) \text{ turns into } d'(yz) = d'(xy) + d'(xz), \text{ where } d' := d_{G, \ell'}.$$

Then we update $\ell := \ell'$, and so on until F, e as in Lemma 6.5 no longer exist.

The process finishes in $O(|V|^3)$ iterations. To see this, consider a current iteration and use notation as above. Assume that $\ell'(e) = a_2 > 0$ (otherwise e is contracted). Take an \mathcal{H} -geodesic C in (G, ℓ) passing y, z and let C' be formed from C by replacing $C[y, z]$ by the concatenation of e and an ℓ -shortest x - z path. By (6.6), C' is ℓ' -shortest but not ℓ -shortest.

Next we argue as follows. If C' is a \tilde{H} -geodesic for ℓ' and $\tilde{H} \in \{H', H''\}$, then the trivial necklace for (\tilde{H}, ℓ) captures C' and becomes nontrivial (since $e \notin E_{H'}$). Now let C' be an H -geodesic for ℓ' . Then $\tau(C') = 0$ (for otherwise C' connects antipodal terminals y and s in $\text{bd}(H)$, and similarly for its part $C'[x, s]$; this is impossible since $\ell'(e) = a_2 > 0$). In this case, N_H captures a new geodesic (in view of (6.6)).

Thus, in all cases at least one of $N_H, N_{H'}, N_{H''}$ grows. This is possible only if for some $v \in V$ and $st \in \Pi_{\mathcal{H}}$, the excess $\varepsilon(v|st)$ changes from a positive value to zero.

This yields the desired complexity and completes the proof of Proposition 6.1. Moreover, we have shown that upon termination of Reduction III,

$$(6.7) \quad G \text{ has at most two inner faces, and each inner face } F \text{ has exactly three edges and shares one edge with each hole.}$$

7 Final reductions and the proof of Theorem 1.1

To finish the proof of the main theorem, it remains to consider $G, \ell, \mathcal{H} = \{H_1, H_2, H_3\}$ satisfying (6.7). An example with two inner faces F, F' is illustrated in the left fragment of Fig. 2.

Let $V_F = \{v_1, v_2, v_3\}$ and let $v_{i-1}v_{i+1}$ be the common edge of F and H_i (taking indices modulo 3). We replace the “triangle” F by a star. More precisely, insert in $\text{Int}(F)$ one extra vertex s , delete the edges of $\text{bd}(F)$ and connect s with each v_i by an edge with the length a_i such that

$$a_{i-1} + a_{i+1} = d(v_{i-1}v_{i+1}) \quad (= \ell(v_{i-1}v_{i+1})), \quad i = 1, 2, 3;$$

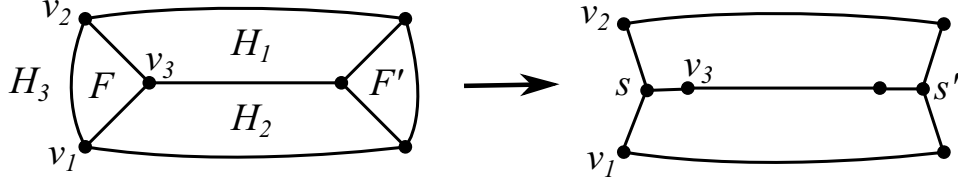


Figure 2: Transformation to the situation without inner faces

these lengths are unique, nonnegative and integral. Act similarly for the other inner face F' (if exists). This transformation results in a graph with three holes and without inner faces; see the right fragment of Fig. 2. Also the lengths of edges in this graph are cyclically even and preserve the distances between the old vertices. Then a solution for the new triple (keeping notation $(G, \ell, \mathcal{H} = \{H_1, H_2, H_3\})$ for it) determines a solution for the old one, yielding Proposition 3.1.

Thus, we come to the case when the current graph G is formed by three openly disjoint paths P_1, P_2, P_3 connecting two vertices s and s' , and each hole H_i is bounded by $P_{i-1} \cup P_{i+1}$. Assume that $\ell(P_1) \leq \ell(P_2) \leq \ell(P_3)$. Note that if $\ell(P_1) < \ell(P_3)$ then P_1 is an H_1 -geodesic of type 1 and the pair (P_1, P_3) is excessive w.r.t. H_1 (causing Reduction I from Sect. 3). So we may assume that P_1, P_2, P_3 have the same length b .

It is convenient to slightly modify (G, ℓ) to make it “path-invariant” and “mirror-reflective”. More precisely, subdividing some edges of G (which adds at most $5|V|$ new vertices), we can update G so that for any vertex x of a path P_i , each other path P_j has vertex y with $d(sy) = d(sx)$ and vertex z with $d(sz) = b - d(sx)$. Let $s = x_0^i, x_1^i, \dots, x_k^i = s'$ be the sequence of vertices in P_i , $i = 1, 2, 3$, and define

$$\lambda_r := \ell(x_{r-1}^i x_r^i) \quad (= d(sx_r^i) - d(sx_{r-1}^i)), \quad r = 1, \dots, k,$$

which does not depend on i and satisfies $\lambda_r = \lambda_{k-r+1}$.

Now we represent d as the sum of weighted (2,3)-metrics and, possibly, one cut-metric as follows:

(7.1) for $r = 1, 2, \dots, \lfloor k/2 \rfloor$, define m_r to be the (2,3)-metric on V determined by the partition $(S_r^1, S_r^2; T_r^1, T_r^2, T_r^3)$, where

$$\begin{aligned} S_r^1 &:= \{x_p^i : i = 1, 2, 3, p = 0, 1, \dots, r-1\}; \\ S_r^2 &:= \{x_p^i : i = 1, 2, 3, p = k-r+1, \dots, k\}; \\ T_r^j &:= \{x_p^j : p = r, \dots, k-r\}, \quad j = 1, 2, 3. \end{aligned}$$

(Recall that the (2,3)-metric determined by a partition $(S^1, S^2; T^1, T^2, T^3)$ of V is the metric induced by the map $\gamma : V \rightarrow V(K_{2,3})$ with $S^i = \gamma^{-1}(s_i)$ and $T^j = \gamma^{-1}(t_j)$, where $\{s_1, s_2\}$ and $\{t_1, t_2, t_3\}$ are the color classes of $K_{2,3}$.) Also when k is odd, we assign as $m_{\lfloor k/2 \rfloor}$ the cut metric on V associated with the cut δX for $X := \{x_p^i : i = 1, 2, 3, p = 0, 1, \dots, \lfloor k/2 \rfloor\}$. Each metric m_r is endowed with the weight λ_r .

A routine verification shows that

$$d = \lambda_1 m_1 + \lambda_2 m_2 + \dots + \lambda_{\lfloor k/2 \rfloor} m_{\lfloor k/2 \rfloor},$$

yielding a solution to PMP for (G, ℓ, \mathcal{H}) . This completes the proof of Theorem 1.1.

In conclusion it should be noted that for PMP with four holes ($|\mathcal{H}| = 4$) and cyclically even lengths of edges in a planar graph, it is open at present whether there exists an efficient (weakly or strongly polynomial time) algorithm of finding an integer solution (consisting of cut-, (2,3)-, and 4f-metrics with integer weights, as required in (1.5)). At the same time, the non-constructive proof of the existence theorem (1.5) given in [4] shows that using merely cut-metrics one can properly reduce the problem so as to obtain a planar graph having at most four faces (after which the reduced problem becomes rather simple). But how to find such cut-metrics (with integer weights) efficiently? (Note that using uncrossing techniques, one can show that there exists a solution with $O(|V|^9)$ different cut-metrics.)

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