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Corrigendum

Erratum to “B_2-crystals: Axioms, structure, models” [J. Comb. Theory, Ser. A 116 (2009) 265–289]

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A B S T R A C T

In this erratum we explain how to implement two axioms stated in our paper of 2009 so as to get a purely “local” characterization for finite B_2-crystals, which was declared but not clarified at some moments there. Also we correct some inaccuracies in that paper.

1. Local axioms

Crystal graphs of type B_2 constitute the simplest case of doubly-laced Kashiwara's crystals in representation theory, and our paper [1] was devoted to a combinatorial study of this class of 2-edge-colored directed graphs. For these crystals, there had been

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known several classical models (in particular, those mentioned in Sect. 1 of [1]) or global characterizations (such as [2, Proposition 3.2.3]), and one of the main purposes of [1] was to find a characterization of B_2-crystals in "local terms". The latter is understood as a way of defining this class of crystal graphs via a finite list of all possible structures of small neighborhoods of vertices (in the sense that the radius of a neighborhood is bounded by a constant). Earlier a characterization of local type was devised for regular simply-laced crystals due to Stembridge [3].

Theorem 4 in [1] characterizes the set of so-called S-graphs by axioms (B0)–(B4), (B′3), (B′4) and (BA) (where the last axiom reduces to a series of other ones), which are then shown in the paper to be precisely the set of B_2-crystals, in both finite and infinite cases. It is seen that a majority of these axioms are obviously local, but there are two axioms, namely, (B1) and (B2), for which a possibility to be implemented by using merely local terms is not clarified in [1].

Next we explain how to fulfill this task in the assumption that the input graphs are finite and acyclic (i.e., having no directed cycles).

More precisely, we deal with a finite acyclic directed graph $G = (V, E)$ in which the edge set E is partitioned into two subsets E_1 and E_2, consisting of edges of *color 1* and *color 2*, also called *1-edges* and *2-edges*, respectively. Also it is usually assumed that G is weakly connected (i.e., the underlying undirected graph is connected). According to Axiom (B0), for $i = 1, 2$, each vertex $v \in V$ has at most one entering and at most one leaving edge of color i. Therefore, the subgraph (V, E_i) consists of disjoint directed paths covering all vertices of G, called *i-strings*. (Here if a vertex v has no incident i-edge, then v itself forms a (degenerate) i-string; e.g. the vertices 2 and 5 in Figure 5(e) of [1] form 1-strings.) Besides, some vertices and edges are distinguished as *central* ones. Then Axioms (B1) and (B2) read as follows.

(B1) Each 1-string has exactly one central element, which is either a vertex or an edge.

This partitions V into three subsets, consisting of central, left and right vertices, where a vertex is called *left* (resp. *right*) if it lies in its *1-string* before (resp. after) the central element. (Note that when (u, v) is a central edge, its beginning vertex u is regarded as left, while the end vertex v as right vertex.) Accordingly, a non-central 1-edge (u, v) is called *left* (*right*) if u is left (resp. v is right).

(B2) Each 2-string P contains exactly one central vertex v. Moreover, all vertices of P lying in this string *before* v are right, whereas all vertices lying *after* v are left.

Formally speaking, Axioms (B1) and (B2) are not local. In order to obtain their local implementations, we assume that an input graph $G = (V, E)$ as before is equipped with *labels* $\ell(v)$ on the vertices $v \in V$ which take values in the 3-element set $\{L, C, R\}$. We impose the following local requirements on (G, ℓ).

(B1(i)) For each 1-edge (u, v), the pair $(\ell(u), \ell(v))$ is equal to one of (L, L), (L, C), (L, R), (C, R), (R, R).

(B1(ii)) If a vertex v has no entering 1-edge, then $\ell(v) \neq R$ (i.e., $\ell(v) \in \{L, C\}$); and if a vertex v has no leaving 1-edge, then $\ell(v) \neq L$.

(B2(i)) For each 2-edge (u, v), the pair $(\ell(u), \ell(v))$ is equal to one of (R, R), (R, C), (C, L), (L, L).

(B2(ii)) If a vertex v has no entering 2-edge, then $\ell(v) \neq L$; and if a vertex v has no leaving 2-edge, then $\ell(v) \neq R$.

In particular, (B1(ii)) implies that if v has neither entering nor leaving 1-edge, then $\ell(v) = C$, and similarly for (B2(ii)).

The vertices labeled L, C, R are naturally interpreted as left, central and right ones, respectively. Accordingly, a 1-edge (u, v) is regarded as left if $(\ell(u), \ell(v)) \in \{(L, L), (L, C)\}$, central if $(\ell(u), \ell(v)) = (L, R)$, and right if $(\ell(u), \ell(v)) \in \{(C, R), (R, R)\}$. As to a 2-edge (u, v), it is regarded as left if $(\ell(u), \ell(v)) \in \{(C, L), (L, L)\}$, and right if $(\ell(u), \ell(v)) \in \{(R, R), (R, C)\}$.

As an easy consequence of the above assignments, we conclude with the following

Proposition. *For finite acyclic graphs satisfying (B0), Axioms (B1)–(B2) are equivalent to imposing (B1(i),(ii)),(B2(i),(ii)).*

Thus, when dealing with finite acyclic graphs, we obtain a "purely local" axiomatics for finite B_2-crystals, as required. Note that the formal requirement that an input graph G is acyclic can be realized by imposing additional variables and constraints (where, however, each variable has size $O(\log |V|)$ in binary notation). Namely, endow each vertex v with an integer $\pi(v)$ (a "potential") and impose the condition: $\pi(u) < \pi(v)$ for each edge (u, v) of G. Clearly G is acyclic if and only if a feasible π does exist.

However, the above labeling method does not work when an input graph is infinite (since in this case Axioms (B1(i),(ii)),(B2(i),(ii)) do not forbid the existence of infinite monochromatic strings without central elements). Therefore, our "local characterization" is applicable only to finite B_2-crystals.

We finish this section with one more useful observation.

Remark. In fact, [1] exhibits one more system of axioms for B_2-crystals, which is "almost local" (using variables of size $O(\log |V|)$). It is provided by the so-called *worm model* developed in Section 4 of [1]. In this model, each vertex v of a graph is endowed with a six-tuple $\tau(v)$ of integers (x, x', x'', y, y', y'') satisfying conditions (A),(B),(C) on page 278 (due to this, one can associate v with a connected figure formed by one or two line segments in a rectangle of the plane). Conditions in (i)–(vi) (on pages 278–279) prescribe how a six-tuple $\tau(v)$ can change under the action of crystal operators (thus defining the edges of colors 1 and 2 entering and leaving v).

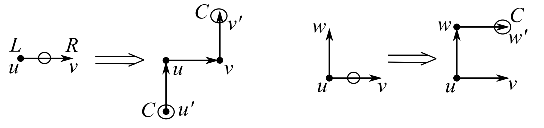

Fig. 1. In Corollary 2 (left); in Corollary 3 (right).

2. Other corrections

In this section we correct three inaccuracies from [1]; they were pointed out by Shunsuke Tsuchioka in [4].

The first one concerns the remark on page 273. More precisely, [1, Remark 2] gives a commentary on axiom (B3) and Corollary 2 and has incorrect sentences in the second paragraph of this remark, which can be disproved by elementary examples. It should be noted that this remark does not affect the main content of the paper; moreover, there is no statement in the remaining part of the paper where details from this remark are quoted or used. Due to this, the second paragraph in Remark 2 should be deleted, which causes no flaw in the whole content.

The second and third inaccuracies involve Corollaries 2 and 3 on page 273 in [1]. They read as follows.

Corollary 2. *Let (u, v) be a central 1-edge. Then there exist a 2-edge (u', u) and a 2-edge (v, v'). Moreover, both vertices u', v' are central.*

Corollary 3. *Let (u, v) be a central 1-edge. Let (u, w) be a 2-edge. Then there exists a 1-edge (w, w'), and the vertex w' is central.*

These corollaries are illustrated in Fig. 1.

Both corollaries are correct but their proofs given in [1] contain gaps. Correct proofs essentially use Axiom (B4) (and its dual (B′4)), and accordingly, these corollaries should be placed after this axiom. Below we give correct proofs.

Proof of Corollary 2. Since the edge (u, v) is central, the vertex u is left and the vertex v is right. Hence (cf. (B2(ii))), u has an entering 2-edge, (u', u) say, and v has a leaving 2-edge, (v, v') say. The former edge is left (since $\ell(u) = L$) and the latter is right (since $\ell(v) = R$); that is, $\ell(u') \in \{L, C\}$ and $\ell(v') \in \{C, R\}$ (by (B2(i))). Suppose that $\ell(u') = L$. Then u' has a leaving 1-edge, (u', u'') say (by (B1(ii))). The edge (u', u'') cannot be left; for otherwise u', u'', u, v would give a commutative square (by Axiom (B3)), whence the 1-edge (u, v) should be left, not central. Therefore, (u', u'') is central. Applying Axiom (B4) to the edges $(u', u''), (u', u), (u, v), (v, v')$, we obtain that v' is the beginning vertex of a central 1-edge. Therefore, $\ell(v') = L$. But we know that $\ell(v') \in \{C, R\}$; a contradiction. It follows that $\ell(u') = C$, as required.

The assertion that v' is central is symmetric.

Proof of Corollary 3. Since u is left, w is left as well. So a 1-edge (w, w') does exist. This edge cannot be central; for otherwise the vertex u would be central, by Corollary 2, contradicting the condition that (u, v) is central. Suppose that w' is not central. Then w' is left, and therefore, there exists a 2-edge (w'', w'). This edge is left and we can apply (B3)(ii) to the edges (w'', w') and (w, w'), obtaining a commutative square containing the vertices w, w', w''. This square must contain the vertices u, v as well, implying $v = w''$ and leading to a contradiction to the fact that the edge (u, v) is central.

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