Journal of Algebra 578 (2021) 303–318

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Basic quadratic identities on quantum minors

Vladimir I. Danilov, Alexander V. Karzanov *

Central Institute of Economics and Mathematics of RAS, 47, Nakhimovskii Prospect, 117418 Moscow, Russia

A R T I C L E   I N F O

Article history:
Received 31 January 2020
Available online 18 March 2021
Communicated by David Hernandez

MSC:
16T99
05C75
05E99

Keywords:
Quantum matrix
Plűcker and Dodgson relations
Quasi-commuting minors
Cauchon graph
Path matrix

A B S T R A C T

This paper continues an earlier research of the authors on universal quadratic identities (QIs) on minors of quantum matrices. We demonstrate situations when the universal QIs are provided, in a sense, by the ones of four special types (Plűcker, co-Plűcker, Dodgson identities and quasi-commutation relations on flag and co-flag interval minors).

1. Introduction

Let $\mathcal{A}$ be a $\mathbb{K}$-algebra over a field $\mathbb{K}$ and let $q \in \mathbb{K}^*$. We deal with an $m \times n$ matrix X whose entries x_{ij} belong to $\mathcal{A}$ and satisfy the following "quasi-commutation" relations (originally appeared in Manin's work [9]): for $i < \ell \leq m$ and $j < k \leq n$,

$$x_{ij}x_{ik} = qx_{ik}x_{ij}, \qquad x_{ij}x_{\ell j} = qx_{\ell j}x_{ij}, \tag{1.1}$$

* Corresponding author.
E-mail addresses: danilov@cemi.rssi.ru (V.I. Danilov), akarzanov7@gmail.com (A.V. Karzanov).

https://doi.org/10.1016/j.jalgebra.2021.02.027

$$x_{ik}x_{\ell j} = x_{\ell j}x_{ik} \quad \text{and} \quad x_{ij}x_{\ell k} - x_{\ell k}x_{ij} = (q - q^{-1})x_{ik}x_{\ell j}.$$

We call such an X a *fine q-matrix* over $\mathcal{A}$ and are interested in relations in the corresponding *quantized coordinate ring* (the algebra of polynomials in the x_{ij} respecting the relations in $\mathcal{A}$), which are viewed as quadratic identities on *q-minors* of X. Let us start with some terminology and notation.

• For a positive integer n', the set $\{1, 2, \dots, n'\}$ is denoted by $[n']$. Let $\mathcal{E}^{n,m}$ denote the set of ordered pairs (I, J) such that $I \subseteq [m]$, $J \subseteq [n]$ and $|I| = |J|$; we will refer to such a pair as a *cortege* and may denote it as $(I|J)$. The submatrix of X whose rows and columns are indexed by elements of I and J, respectively, is denoted by $X(I|J)$. For $(I, J) \in \mathcal{E}^{m,n}$, where $I = \{i_1 < i_2 < \cdots < i_k\}$ and $J = \{j_1 < j_2 < \cdots < j_k\}$, the *$q$-determinant* (called the *q-minor*, the *quantum minor*) of $X(I|J)$ is defined as

$$\Delta_{X,q}(I|J) := \sum\nolimits_{\sigma \in S_k} (-q)^{\ell(\sigma)} \prod\nolimits_{d=1}^{k} x_{i_d j_{\sigma(d)}}, \tag{1.2}$$

where the factors in $\prod$ are ordered from left to right by increasing d, and $\ell(\sigma)$ denotes the *length* (number of inversions) of a permutation σ. The terms X and/or q in $\Delta_{X,q}(I|J)$ may be omitted when they are clear from the context. By definition $\Delta(\emptyset|\emptyset)$ is the unit of $\mathcal{A}$.

• A quantum quadratic identity (QI) of our interest is viewed as

$$\sum (\mathrm{sign}_i q^{\delta_i} \Delta_q(I_i|J_i)\, \Delta_q(I'_i|J'_i) \colon i = 1, \dots, N) = 0, \tag{1.3}$$

where for each i, $\delta_i \in \mathbb{Z}$, $\mathrm{sign}_i \in \{+, -\}$, and $(I_i|J_i)$, $(I'_i|J'_i) \in \mathcal{E}^{m,n}$. Note that any pair $(I|J), (I'|J')$ may be repeated in (1.3) many times. We restrict ourselves to merely *homogeneous* QIs, which means that in expression (1.3),

(1.4) each of the sets $I_i \cup I'_i$, $I_i \cap I'_i$, $J_i \cup J'_i$, $J_i \cap J'_i$ is invariant of i.

When, in addition, (1.3) is valid for all appropriate $\mathcal{A}, q, X$ (with m, n fixed), we say that (1.3) is *universal*.

In fact, there are plenty of universal QIs. For example, representative classes involving quantum flag minors were demonstrated by Lakshmibai and Reshetikhin [6] and Taft and Towber [11]. Extending earlier results, the authors obtained in [4] necessary and sufficient conditions characterizing all universal QIs. These conditions are given in combinatorial terms and admit an efficient verification.

Four special cases of universal QIs play a central role in this paper. They are exposed in (I)–(IV) below; for details, see [4, Sects. 6,8].

In what follows, for integers $1 \le a \le b \le n'$, we call the set $\{a, a+1, \dots, b\}$ an *interval* in $[n']$ and denote it as $[a..b]$ (in particular, $[1..n'] = [n']$). For disjoint subsets A and $\{a, \dots, b\}$, we may abbreviate $A \cup \{a, \dots, b\}$ as $Aa \dots b$. Also for $(I|J) \in \mathcal{E}^{m,n}$,

$\Delta(I|J) = \Delta_{X,q}(I|J)$ is called a *flag* (*co-flag*) q-minor if $J = [k]$ (resp. $I = [k]$), where $k := |I| = |J|$.

(I) *Plűcker-type relations with triples.* Let $A \subset [m]$, $B \subset [n]$, $\{i, j, k\} \subseteq [m] - A$, $\ell \in [n] - B$, and let $|A| + 1 = |B|$ and $i < j < k$. There are several universal QIs on such elements (see a discussion in [4, Sect. 6.4]). One of them is viewed as

$$\Delta(Aj|B)\Delta(Aik|B\ell) = \Delta(Aij|B\ell)\Delta(Ak|B) + \Delta(Ajk|B\ell)\Delta(Ai|B). \tag{1.5}$$

In the flag case (when $B = [|B|]$ and $\ell = |B| + 1$) this turns into a quantum analog of the classical Plűcker relation with a triple $i < j < k$.

(II) *Co-Plűcker-type relations with triples.* They are "symmetric" to those in (I). Namely, we deal with $A \subset [m]$, $B \subset [n]$, $\ell \in [m] - A$ and $\{i, j, k\} \subseteq [n] - B$ such that $|A| = |B| + 1$ and $i < j < k$. Then there holds:

$$\Delta(A|Bj)\Delta(A\ell|Bik) = \Delta(A\ell|Bij)\Delta(A|Bk) + \Delta(A\ell|Bjk)\Delta(A|Bi). \tag{1.6}$$

(III) *Dodgson-type relations.* Let $i, k \in [m]$ and $j, \ell \in [n]$ satisfy $k - i = \ell - j \geq 0$. Form the intervals $A := [i + 1..k - 1]$ and $B := [j + 1..\ell - 1]$. The universal QI which is a quantum analog of the classical Dodgson relation is viewed as (cf. [4, Sect. 6.5])

$$\Delta(Ai|Bj)\Delta(Ak|B\ell) = \Delta(Aik|Bj\ell)\Delta(A|B) + q\Delta(Ai|B\ell)\Delta(Ak|Bj). \tag{1.7}$$

In particular, when $A = B = \emptyset$, we obtain the expression $\Delta(ik|j\ell) = \Delta(i|j)\Delta(k|\ell) - q\Delta(i|\ell)\Delta(k|j)$ (with $k = i + 1$ and $\ell = j + 1$), taking into account that $\Delta(\emptyset|\emptyset) = 1$. This matches formula (1.2) for the q-minor of a 2×2 submatrix.

(IV) *Quasi-commutation relations on interval q-minors.* The simplest possible kind of universal QIs involves two corteges $(I|J), (I'|J') \in \mathcal{E}^{m,n}$ and is viewed as

$$\Delta(I|J)\Delta(I'|J') = q^c\Delta(I'|J')\Delta(I|J) \tag{1.8}$$

for some $c \in \mathbb{Z}$. When q-minors $\Delta(I|J)$ and $\Delta(I'|J')$ satisfy (1.8), they are called *quasi-commuting*. (For example, three relations in (1.1) are such.) Leclerc and Zelevinsky [7] characterized such minors in the *flag* case, by showing that $\Delta(I|[|I|])$ and $\Delta(I'|[|I'|])$ quasi-commute if and only if the subsets I, I' of $[m]$ are *weakly separated* (for a definition, see [7]). In a general case, a characterization of quasi-commuting q-minors is given in Scott [10] (see also [4, Sect. 8.3] for additional aspects).

For purposes of this paper, it suffices to consider only *interval q-minors*, i.e., assume that all I, J, I', J' are intervals. Let for definiteness $|I| \geq |I'|$ and define

$$\begin{aligned} \alpha &:= |\{i' \in I' \colon i' < \min(I)\}|, \qquad & \beta &:= |\{i' \in I' \colon i' > \max(I)\}|, \\ \gamma &:= |\{j' \in J' \colon j' < \min(J)\}|, \qquad & \delta &:= |\{j' \in J' \colon j' > \max(J)\}|. \end{aligned} \tag{1.9}$$

Then the facts that I, J, I', J' are intervals and that $|I| \ge |I'|$ imply $\alpha\beta = \gamma\delta = 0$.

Specializing Proposition 8.2 from [4] to our case, we obtain that

(1.10) for $|I| \ge |I'|$, interval q-minors $\Delta(I|J)$ and $\Delta(I'|J')$ quasi-commute (universally) if and only if $\alpha\gamma = \beta\delta = 0$; in this case, c as in (1.8) is equal to $\beta + \delta - \alpha - \gamma$.

In fact, we will use (1.10) only when $\Delta(I|J)$ is a flag or co-flag interval q-minor, and similarly for $\Delta(I'|J')$ (including mixed cases with one flag and one co-flag q-minors).

In this paper we explore the issue when the special quadratic identities exhibited in (I)–(IV) determine all other universal QIs. More precisely, let $\mathcal{P} = \mathcal{P}_{m,n}$, $\mathcal{P}^* = \mathcal{P}^*_{m,n}$, and $\mathcal{D} = \mathcal{D}_{m,n}$ denote the sets of relations as in (1.5), (1.6), and (1.7), respectively (concerning the corresponding objects in (I)–(III)). Also let $\mathcal{Q} = \mathcal{Q}_{m,n}$ denote the set of quasi-commuting relations in (IV) concerning the flag and co-flag interval cases.

Definitions. For $\mathcal{A}$, q, m, n as above, $f : \mathcal{E}^{m,n} \to \mathcal{A}$ is called a *QI-function* if its values satisfy the quadratic relations similar to those in the universal QIs on q-minors (i.e., when we formally replace $\Delta(I|J)$ by $f(I|J)$ in these relations). When $f : \mathcal{E}^{m,n} \to \mathcal{A}$ is assumed to satisfy the relations as in $\mathcal{P}$, $\mathcal{P}^*$ and $\mathcal{D}$, we say that f is an *RQI-function* (abbreviating “a function obeying *restricted quadratic identities*”).

Note that if $f : \mathcal{E}^{m,n} \to \mathcal{A}$ satisfies a quadratic relation Q, and a is an element of the center of $\mathcal{A}$ (i.e. $ax = xa$ for any $x \in \mathcal{A}$), then af satisfies Q as well. Hence if f is a QI- or RQI-function, then so is af. Due to this, in what follows we will default assume that any function f on $\mathcal{E}^{m,n}$ we deal with is *normalized*, i.e., satisfies $f(\emptyset|\emptyset) = 1$ (which is consistent with $\Delta(\emptyset|\emptyset) = 1$).

Our goal is to prove two results on QI-functions. Let us say that a cortege $(I|J) \in \mathcal{E}^{m,n}$ is a *double interval* if both I, J are intervals. A double interval $(I|J)$ is called *pressed* if at least one of I, J is an initial interval, i.e., either $I = [|I|]$ or $J = [|J|]$ or both (yielding a flag or co-flag case); the set of these is denoted as $Pint = Pint_{m,n}$.

Theorem 1.1. *Let RQI-functions $f, g : \mathcal{E}^{m,n} \to \mathcal{A} - \{0\}$ coincide on $Pint_{m,n}$. Let, in addition, for any $(I|J) \in \mathcal{E}^{m,n}$, the element $f(I|J)$ be not a zero divisor in $\mathcal{A}$. Then f and g coincide on the entire $\mathcal{E}^{m,n}$.*

It follows that any QI-function is uniquely determined by its values on $Pint$ and relations as in $\mathcal{P}$, $\mathcal{P}^*$ and $\mathcal{D}$.

The second theorem describes a situation when taking values on $Pint$ arbitrarily within a representative part of $\mathcal{A}$, one can extend these values to a QI-function (so one may say that, $Pint$ plays a role of “basis” for QI-functions, in a sense).

Theorem 1.2. *Let $f_0 : Pint \to \mathcal{A}^*$ (where $\mathcal{A}^*$ is the set of invertible elements of $\mathcal{A}$). Suppose that f_0 satisfies the quasi-commutation relations (as in (1.8) in (IV)) on Pint. Then f_0 is extendable to a QI-function f on $\mathcal{E}^{m,n}$.*

It should be noted that Theorems 1.1 and 1.2 can be regarded as quantum analogs of corresponding results in [5] devoted to universal quadratic identities on minors of matrices over a commutative semiring (e.g. over $\mathbb{R}_{>0}$ or over the tropical semiring $(\mathbb{R}, +, \max)$); see Theorem 7.1 there.

This paper is organized as follows. Section 2 contains a proof of Theorem 1.1. Section 3 reviews a construction, due to Casteels [2], used in our approach to proving the second theorem. According to this construction (of which idea goes back to Cauchon diagrams in [3]), the minors of a generic q-matrix can be expressed as the ones of the so-called *path matrix* of a special planar graph $G_{m,n}$, viewed as an extended square grid of size $m \times n$. There is a one-to-one correspondence between the pressed interval corteges in $\mathcal{E}^{m,n}$ and the inner vertices of $G_{m,n}$. This enables us to assign each generator involved in the construction of entries of the path matrix (formed in Lindstrőm's style via path systems, or "flows", in $G_{m,n}$) as the ratio of two values of f_0; this is just where we use that f_0 takes values in $\mathcal{A}^*$. Relying on this construction, we prove Theorem 1.2 in Section 4; here the crucial step is to show that the quasi-commutation relations on the values of f_0 imply the relations on generators needed to obtain a correct path matrix. Finally, in Section 5 we describe a situation when a function f_0 on $Pint_{m,n}$ exposed in Theorem 1.2 has a unique extension to $\mathcal{E}^{m,n}$ that is a QI-function, or, roughly speaking, when the values on $Pint$ and relations as in $\mathcal{P}, \mathcal{P}^*, \mathcal{D}$ and $\mathcal{Q}$ determine a QI-function on $\mathcal{E}^{m,n}$, thus yielding all other universal QIs.

2. Proof of Theorem 1.1

Let $f, g : \mathcal{E}^{m,n} \to \mathcal{A}$ be as in the hypotheses of this theorem. To show that $f(I|J) = g(I|J)$ holds everywhere, we consider three possible cases for $(I|J) \in \mathcal{E}^{m,n}$. In the first and second cases, we use induction on the value

$$\sigma(I, J) := \max(I) - \min(I) + \max(J) - \min(J).$$

Case 1. Let $(I|J)$ be such that: (i) $f(I'|J') = g(I'|J')$ holds for all $(I'|J') \in \mathcal{E}^{m,n}$ with $\sigma(I', J') < \sigma(I, J)$; and (ii) I is not an interval.

Define $i := \min(I)$, $k := \max(I)$ and $A := I - \{i, k\}$. Take $\ell \in J$ and let $B := J - \ell$. Since I is not an interval, there is $j \in [m]$ such that $i < j < k$ and $j \notin I$. Then $j \notin A$ and $(Aik|B\ell) = (I|J)$. Applying to f and g Plűcker-type relations as in (1.5), we have

$$f(Aj|B)f(Aik|B\ell) = f(Aij|B\ell)f(Ak|B) + f(Ajk|B\ell)f(Ai|B), \quad \text{and} \tag{2.1}$$

$$g(Aj|B)g(Aik|B\ell) = g(Aij|B\ell)g(Ak|B) + g(Ajk|B\ell)g(Ai|B). \tag{2.2}$$

The choice of i, j, k, ℓ provides that in these relations, the number $\sigma(A', B')$ for each of the five corteges $(A'|B')$ different from $(Aik|B\ell)$ $(= (I|J))$ is strictly less than $\sigma(I|J)$. So f and g coincide on these $(A'|B')$, by condition (i) on $(I|J)$. Subtracting (2.2) from (2.1), we obtain

$$f(Aj|B)\,(f(I|J) - g(I|J)) = 0.$$

This implies $f(I|J) = g(I|J)$ (since $f(Aj|B) \neq 0$ and $f(Aj|B)$ is not a zero divisor, by the hypotheses of the theorem).

Case 2. Let $(I|J)$ be subject to condition (i) from the previous case and suppose that J is not an interval. Then taking $i := \min(J)$, $k := \max(J)$, $B := J - \{i, k\}$, $\ell \in I$, $A := I - \ell$, applying to f, g the corresponding co-Plűcker-type relations as in (1.6), and arguing as above, we again obtain $f(I|J) = g(I|J)$.

Thus, it remains to examine double intervals $(I|J)$. We rely on the equalities $f(I|J) = g(I|J)$ when $(I|J)$ is pressed (belongs to $Pint$), and use induction on the value

$$\eta(I, J) := \max(I) + \min(I) + \max(J) + \min(J).$$

Case 3. Let $(I|J) \in \mathcal{E}^{m,n}$ be a non-pressed double interval. Define $i := \min(I) - 1$, $k := \max(I)$, $j := \min(J) - 1$, $\ell := \max(J)$, $A := I - k$, $B := J - \ell$. Then $i, j \geq 1$ (since $(I|J)$ is non-pressed). Also $(I|J) = (Ak|B\ell)$. Suppose, by induction, that $f(I'|J') = g(I'|J')$ holds for all double intervals $(I'|J') \in \mathcal{E}^{m,n}$ such that $\eta(I', J') < \eta(I, J)$.

Applying to f and g Dodgson-type relations as in (1.7), we have

$$f(Ai|Bj)f(Ak|B\ell) = f(Aik|Bj\ell)f(A|B) + qf(Ai|B\ell)f(Ak|Bj), \quad \text{and} \tag{2.3}$$

$$g(Ai|Bj)g(Ak|B\ell) = g(Aik|Bj\ell)g(A|B) + qg(Ai|B\ell)g(Ak|Bj). \tag{2.4}$$

One can see that for all corteges $(A'|B')$ occurring in these relations, except for $(Ak|B\ell)$, the value $\eta(A', B')$ is strictly less than $\eta(I, J)$. Therefore, subtracting (2.4) from (2.3) and using induction on η, we obtain

$$f(Ai|Bj)\,(f(Ak|B\ell) - g(Ak|B\ell)) = 0,$$

whence $f(I|J) = g(I|J)$, as required.

This completes the proof of the theorem. □

3. Flows in a planar grid

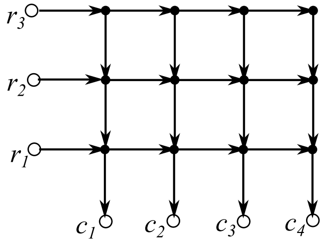
The proof of Theorem 1.2 essentially relies on a construction of quantum minors via certain path systems ("flows") in a special planar graph. This construction is due to Casteels [2] and it was based on ideas in Cauchon [3] and Lindstrőm [8]. Below we review details of the method needed to us, mostly following terminology, notation and conventions used for the corresponding special case in [4].

Extended grids. Let $m, n \in \mathbb{Z}_{>0}$. We construct a certain planar directed graph, called an *extended* $m \times n$ *grid* and denoted as $G_{m,n} = G = (V, E)$, as follows.

(G1) The vertex set V is formed by the points (i, j) in the plane $\mathbb{R}^2$ such that $i \in \{0\} \cup [m]$, $j \in \{0\} \cup [n]$ and $(i, j) \neq (0, 0)$. Hereinafter, it is convenient to us to assume that the first coordinate i of a point (i, j) in the plane is the *vertical* one.
(G2) The edge set E consists of edges of two types: "horizontal" edges, or *H-edges*, and "vertical" edges, or *V-edges*.
(G3) The H-edges are directed from left to right and go from $(i, j-1)$ to (i, j) for all $i = 1, \dots, m$ and $j = 1, \dots, n$.
(G4) The V-edges are directed downwards and go from (i, j) to $(i-1, j)$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$.

Two subsets of vertices in G are distinguished: the set $R = \{r_1, \dots, r_m\}$ of *sources*, where $r_i := (i, 0)$, and the set $C = \{c_1, \dots, c_n\}$ of *sinks*, where $c_j := (0, j)$. The other vertices are called *inner* and the set of these (i.e., $[m] \times [n]$) is denoted by $W = W_G$.

The picture illustrates the extended grid $G_{3,4}$.



Each inner vertex $v \in W$ of $G = G_{m,n}$ is regarded as a *generator*. This gives rise to assigning the *weight* $w(e)$ to each edge $e = (u, v) \in E$ (going from a vertex u to a vertex v) in a way similar to that introduced for Cauchon graphs in [2], namely:

(3.1) (i) $w(e) := v$ if e is an H-edge with $u \in R$;
(ii) $w(e) := u^{-1}v$ if e is an H-edge and $u, v \in W$;
(iii) $w(e) := 1$ if e is a V-edge.

This in turn gives rise to defining the weight $w(P)$ of a directed path $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ (where e_i is the edge from v_{i-1} to v_i) to be the ordered (from left to right) product, namely:

$$w(P) := w(e_1)w(e_2) \cdots w(e_k). \tag{3.2}$$

Then $w(P)$ forms a Laurent monomial in elements of W. Note that when P begins in R and ends in C, its weight can also be expressed in the following useful form: if $u_1, v_1, u_2, v_2, \dots, u_{d-1}, v_{d-1}, u_d$ is the sequence of vertices where P makes turns (from "east" to "south" at each u_i, and from "south" to "east" at each v_i), then, due to the "telescopic effect" caused by (3.1)(ii), there holds

$$w(P) = u_1 v_1^{-1} u_2 v_2^{-1} \cdots u_{d-1} v_{d-1}^{-1} u_d. \tag{3.3}$$

We assume that the elements of W obey quasi-commutation laws which look somewhat simpler than those in (1.1); namely, for distinct inner vertices $u = (i, j)$ and $v = (i', j')$,

(3.4) (i) if $i = i'$ and $j < j'$, then $uv = qvu$;
(ii) if $i > i'$ and $j = j'$, then $vu = quv$;
(iii) otherwise $uv = vu$,

where, as before, $q \in \mathbb{K}^*$. (Note that G has a horizontal (directed) path from u to v in (i), and a vertical path from u to v in (ii).)

Path matrix and flows. To be consistent with the vertex notation in extended grids, we visualize matrices in the Cartesian form: for an $m \times n$ matrix $A = (a_{ij})$, the row indexes $i = 1, \dots, m$ are assumed to grow upwards, and the column indexes $j = 1, \dots, n$ from left to right.

Given an extended $m \times n$ grid $G = G_{m,n} = (V, E)$ with the corresponding partition (R, C, W) of V as above, we form the *path matrix* Path $=$ Path$_G$ of G in a spirit of [2]; namely, Path is the $m \times n$ matrix whose entries are defined by

$$\mathrm{Path}(i|j) := \sum\nolimits_{P \in \Phi_G(i|j)} w(P), \qquad (i, j) \in [m] \times [n], \tag{3.5}$$

where $\Phi_G(i|j)$ is the set of (directed) paths from the source r_i to the sink c_j in G. Thus, the entries of Path$_G$ belong to the $\mathbb{K}$-algebra $\mathcal{L}_G$ of Laurent polynomials generated by the set W if inner vertices of G subject to (3.4).

Definition. Let $(I|J) \in \mathcal{E}^{m,n}$. Borrowing terminology from [5], by an $(I|J)$-*flow* we mean a set ϕ of *pairwise disjoint* directed paths from the source set $R_I := \{r_i : i \in I\}$ to the sink set $C_J := \{c_j : j \in J\}$ in G.

The set of $(I|J)$-flows ϕ in G is denoted by $\Phi(I|J) = \Phi_G(I|J)$. We order the paths forming ϕ by increasing the indexes of sources: if I consists of $i(1) < i(2) < \cdots < i(k)$ and J consists of $j(1) < j(2) < \cdots < j(k)$ and if P_ℓ denotes the path in ϕ beginning at $r_{i(\ell)}$, then P_ℓ is just ℓ-th path in ϕ, $\ell = 1, \dots, k$. Note that the planarity of G and the fact that the paths in ϕ are pairwise disjoint imply that each P_ℓ ends at the sink $c_{j(\ell)}$.

Similar to the assignment of weights for path systems in [2], we define the weight of $\phi = (P_1, P_2, \dots, P_k)$ to be the ordered product

$$w(\phi) := w(P_1)w(P_2)\cdots w(P_k). \tag{3.6}$$

Using a version of Lindström Lemma, Casteels showed a correspondence between path systems and q-minors of path matrices.

Proposition 3.1 *([2]). For the extended grid $G = G_{m,n}$ and any $(I|J) \in \mathcal{E}^{m,n}$,*

$$\Delta(I|J)_{\mathrm{Path}_G,q} = \sum\nolimits_{\phi\in\Phi_G(I|J)} w(\phi). \tag{3.7}$$

(This is generalized to a larger set of graphs and their path matrices in [4, Theorem 3.1].)

The next property, surprisingly provided by (3.4), is of most importance to us.

Proposition 3.2 *([2]). The entries of* Path_G *obey Manin's relations (similar to those in (1.1)).*

It follows that the q-minors of Path_G satisfy all universal QIs, and therefore, the function $g : \mathcal{E}^{m,n} \to \mathcal{L}_G$ defined by $g(I|J) := \mathrm{Path}_G(I|J)$ is a QI-function.

4. Proof of Theorem 1.2

Let $f_0 : Pint_{m,n} \to \mathcal{A}^*$ be a function as in the hypotheses of this theorem. Our goal is to extend f_0 to a QI-function f on $\mathcal{E}^{m,n}$. The idea of our construction is prompted by Propositions 3.1 and 3.2; namely, we are going to obtain the desired f as the function of q-minors of an appropriate path matrix Path_G for the extended $m \times n$ grid $G = G_{m,n}$.

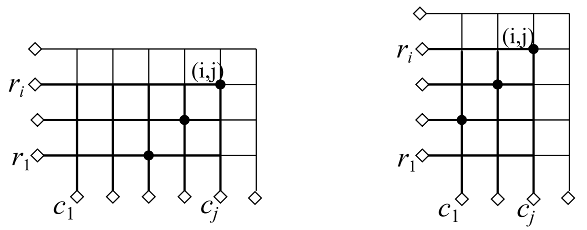
For this purpose, we first have to determine the "generators" in W in terms of values of f_0 (so as to provide that these values are consistent with the corresponding pressed interval q-minors of the path matrix), and second, using the quasi-commutation relations (as in (1.8)) on the values of f_0, to verify validity of relations (3.4) on the generators. Then $Path_G$ will be indeed a fine q-matrix and its q-minors will give the desired QI-function f.

(It should be emphasized that we may speak of a vertex of G in two ways: either as a point in $\mathbb{R}^2$, or as a generator of the corresponding algebra. In the former case, we use the coordinate notation (i, j) (where $i \in \{0\} \cup [m]$ and $j \in \{0\} \cup [n]$). And in the latter case, we use notation $w(i, j)$, referring to it as the *weight* of (i, j).)

To express the elements of W via values of f_0, we associate each pair $(i, j) \in [m] \times [n]$ with the pressed interval cortege $\pi(i, j) = (I|J)$, where

(4.1) $I := [i - k + 1..i]$ and $J := [j - k + 1..j]$, where $k := \min\{i, j\}$.

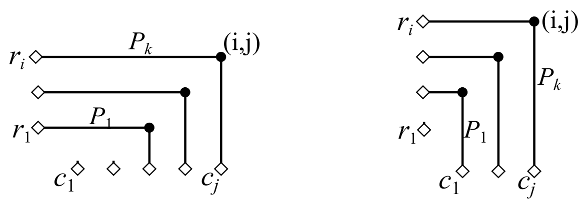
In other words, if $i \le j$ (i.e., (i,j) lies "south-east" from the "diagonal" $\{\alpha,\alpha\}$ in $\mathbb{R}^2$), then $(I|J)$ is the co-flag interval cortege with $I=[i]$ and $\max(J)=j$, and if $i \ge j$ (i.e., (i,j) is "north-west" from the diagonal), then $(I|J)$ is the flag interval cortege with $\max(I)=i$ and $J=[j]$. Also it is useful to associate to (i,j): the (almost rectangular) subgrid induced by the vertices in $(\{0\}\cup[i])\times(\{0\}\cup[j])-\{(0,0)\}$, and the *diagonal* $D(i|j)$ formed by the vertices $(i,j),(i-1,j-1),\dots,(i-k+1,j-k+1)$. See the picture where the left (right) fragment illustrates the case $i<j$ (resp. $i>j$), the subgrids are indicated by thick lines, and the diagonals $D(i,j)$ by bold circles.



An important feature of a pressed interval cortege $(I|J)\in\mathcal{E}^{m,n}$ (which is easy to see) is that

(4.2) $\Phi(I|J)$ consists of a unique flow ϕ and this flow is formed by paths $P_1,\dots,P_k$, where for $i:=\max(I)$, $j:=\max(J)$, $k:=\min\{i,j\}$, and $\ell=1,\dots,k$, the path P_ℓ begins at $r_{i-k+\ell}$, ends at $c_{j-k+\ell}$ and makes exactly one turn, namely, the east to south turn at the vertex $(i-k+\ell,j-k+\ell)$ of the diagonal $D(i|j)$.

We denote this flow $(P_1,\dots,P_k)$ as $\phi(i|j)$; it is illustrated in the picture (for both cases $i<j$ and $i>j$ from the previous picture).



Therefore, for each $(i,j)\in[m]\times[n]$, taking the cortege $(I|J)=\pi(i,j)$ and the flow $\phi(i|j)=(P_1,\dots,P_k)$ with $k=\min\{i,j\}$ and using expressions (3.3) and (3.6) for them, we obtain that

$$\sum\nolimits_{\phi\in\Phi_G(I|J)} w(\phi)=w(\phi(i|j))=w(i-k+1,j-k+1)\cdots w(i-1,j-1)w(i,j). \tag{4.3}$$

Now imposing the conditions

$$w(\phi(i|j)):=f_0(I|J)\quad\text{for all }\ (I|J)=\pi(i,j)\in Pint_{m,n}, \tag{4.4}$$

we come to the rule of defining appropriate weights of inner vertices of G. Namely, relying on (4.3), we define $w(i,j)$ for each $(i,j) \in [m] \times [n]$ by

$$w(i,j) := \begin{cases} f_0(\{i\}|\{j\}) & \text{if } \min\{i,j\} = 1, \\ (f_0(\pi(i-1,j-1)))^{-1} f_0(\pi(i,j)) & \text{otherwise.} \end{cases} \tag{4.5}$$

Such a $w(i,j)$ is well-defined since $f_0(\pi(i-1,j-1))$ is invertible.

The crucial step in our proof is to show that these weights satisfy the relations as in (3.4), i.e., for (i,j) and (i',j'),

(4.6) (i) if $i = i'$ and $j < j'$, then $w(i,j)w(i',j') = qw(i',j')w(i,j)$;
(ii) if $i > i'$ and $j = j'$, then $w(i',j')w(i,j) = qw(i,j)w(i',j')$;
(iii) otherwise $w(i,j)w(i',j') = w(i',j')w(i,j)$.

This would provide that $Path_G$ is indeed a fine q-matrix, due to (3.7) and Proposition 3.2, and setting $f(I|J) := \Delta(I|J)_{Path_G}$ for all $(I|J) \in \mathcal{E}^{m,n}$, we would obtain the desired function, thus completing the proof of the theorem.

First of all we have to explain that

(4.7) f_0 satisfies the quasi-commutation relation for any two pressed interval corteges $(I|J), (I'|J') \in Pint$, i.e., $f_0(I|J)f_0(I'|J') = q^c f_0(I'|J')f_0(I|J)$ holds for some $c \in \mathbb{Z}$.

This is equivalent to saying that such corteges determine a universal QI of the form (1.8) on associated q-minors. To see the latter, assume that $|I| \ge |I'|$ and define $\alpha, \beta, \gamma, \delta$ as in (1.9). One can check that: $\gamma = \delta = 0$ if both interval corteges are flag ones; $\alpha = \beta = 0$ if they are co-flag ones; and either $\beta = \gamma = 0$ or $\alpha = \delta = 0$ (or both) if one of these is a flag, and the other a co-flag interval cortege. So in all cases, we have $\alpha\gamma = \beta\delta = 0$, and (4.7) follows from (1.10).

Next we start proving (4.6). Given $(i,j), (i',j') \in [m] \times [n]$, let $(I|J) := \pi(i,j)$ and $(I'|J') := \pi(i',j')$, and define

$$A := f_0(I|J), \quad B := f_0(I-i|J-j), \quad C := f_0(I'|J'), \quad D := f_0(I'-i'|J'-j'),$$

letting by definition $B := 1$ ($D := 1$) if $|I| = 1$ (resp. $|I'| = 1$). (Here for an element $p \in P$, we write $P - p$ for $P - \{p\}$.)

Then $w(i,j)$ is rewritten as $B^{-1}A$, and $w(i',j')$ as $D^{-1}C$ (by (4.5)), and our goal is to show that

$$B^{-1}AD^{-1}C = q^d D^{-1}CB^{-1}A, \tag{4.8}$$

where d is as required in (4.6) (i.e., equal to 1, -1, 0 in cases (i), (ii), (iii), respectively).

Define c_1, c_2, c_3, c_4 from the quasi-commutation relations (as in (1.8))

$$AC = q^{c_1}CA, \quad AD = q^{c_2}DA, \quad BC = q^{c_3}CB, \quad BD = q^{c_4}DB. \tag{4.9}$$

One can see that

$$d = c_1 - c_2 - c_3 + c_4. \tag{4.10}$$

Indeed, in order to transform the string $B^{-1}AD^{-1}C$ into $D^{-1}CB^{-1}A$, one should swap each of A, B^{-1} with each of C, D^{-1}. The second equality in (4.9) implies $AD^{-1} = q^{-c_2}D^{-1}A$, and for similar reasons, $B^{-1}C = q^{-c_3}CB^{-1}$ and $B^{-1}D^{-1} = q^{c_4}D^{-1}B^{-1}$.

Now we are ready to examine possible combinations for (i, j) and (i', j') and compute d in these cases by using (4.10). We will denote the intervals $I - i,\ J - j,\ I' - i',\ J' - j'$ in question by $\widetilde{I},\ \widetilde{J},\ \widetilde{I}',\ \widetilde{J}'$, respectively. Also for an ordered pair $((P|Q), (P'|Q'))$ of double intervals in $\mathcal{E}^{m,n}$ (where $|P'| = |Q'|$ may exceed $|P| = |Q|$), we define

$$\begin{aligned} \alpha(P, P') &:= \min\{|\{p' \in P' : p' < \min(P)\}|,\ |\{p \in P : p > \max(P')\}|\}; \\ \beta(P, P') &:= \min\{|\{p' \in P' : p' > \max(P)\}|,\ |\{p \in P : p < \min(P')\}|\}, \end{aligned} \tag{4.11}$$

and define $\gamma(Q, Q')$ and $\delta(Q, Q')$ in a similar way (this matches the definition of $\alpha, \beta, \gamma, \delta$ in (1.9) when $|P| \ge |P'|$). Using (1.10), we observe that the sum $\beta(I, I') + \delta(J, J') - \alpha(I, I') - \gamma(J, J')$ is equal to c_1, and similarly for the pairs concerning c_2, c_3, c_4.

In our analysis we also will use the values

$$\varphi := (\beta(I, I') - \alpha(I, I')) - (\beta(I, \widetilde{I}') - \alpha(I, \widetilde{I}')) - (\beta(\widetilde{I}, I') - \alpha(\widetilde{I}, I')) + (\beta(\widetilde{I}, \widetilde{I}') - \alpha(\widetilde{I}, \widetilde{I}'));$$
$$\psi := (\delta(J, J') - \gamma(J, J')) - (\delta(J, \widetilde{J}') - \gamma(J, \widetilde{J}')) - (\delta(\widetilde{J}, J') - \gamma(\widetilde{J}, J')) + (\delta(\widetilde{J}, \widetilde{J}') - \gamma(\widetilde{J}, \widetilde{J}')).$$

In view of (4.10) and (4.11),

$$\varphi + \psi = c_1 - c_2 - c_3 + c_4 = d. \tag{4.12}$$

The lemmas below compute φ using (4.11). Let $r := \min(I)$ $(= \min(\widetilde{I}))$ and $r' := \min(I')$ $(= \min(\widetilde{I}'))$.

Lemma 4.1. *Suppose that $|I| \ne |I'|$ and $i \ne i'$. Then $\varphi = 0$.*

Proof. Assume that $|I| > |I'|$. Then $|I| > |\widetilde{I}| \ge |I'| > |\widetilde{I}'|$. Consider possible cases.

Case 1: $r \le r'$ and $i' < i$. Then $I', \widetilde{I}' \subseteq I, \widetilde{I}$. Therefore, both α and β are zero everywhere, implying $\varphi = 0$.

Case 2: $I \cap I' = \emptyset$. If $i' < r$, then β is zero. Also $\alpha(I, I') = |I'| = \alpha(\widetilde{I}, I')$ and $\alpha(I, \widetilde{I}') = |\widetilde{I}'| = \alpha(\widetilde{I}, \widetilde{I}')$.

And if $i < r'$, then α is zero. Also $\beta(I, I') = |I'| = \beta(\widetilde{I}, I')$ and $\beta(I, \widetilde{I}') = |\widetilde{I}'| = \beta(\widetilde{I}, \widetilde{I}')$. So in both situations, $\varphi = 0$.

Case 3: $r' < r \le i' < i$. Then β is zero. Also $\alpha(P, P') = r - r'$ holds for all $P \in \{I, \widetilde{I}\}$ and $P' \in \{I', \widetilde{I}'\}$, implying $\varphi = 0$.

Case 4: $r < r' \le i < i'$. Then α is zero, and

$$\beta(I, I') = i' - i = \beta(\widetilde{I}, \widetilde{I}'), \quad \beta(I, \widetilde{I}') = i' - 1 - i \quad \text{and} \quad \beta(\widetilde{I}, I') = i' - (i - 1),$$

again implying $\varphi = 0$.

When $|I| < |I'|$, the argument follows by swapping $I, \widetilde{I}$ by $I', \widetilde{I}'$. □

Lemma 4.2. *Let* $|I| = |I'|$. (a) *If* $i < i'$ *then* $\varphi = 1$. (b) *If* $i > i'$ *then* $\varphi = -1$. (c) *If* $i = i'$ *then* $\varphi = 0$.

Proof. We have $|I'|, |\widetilde{I}'| \le |I|$ and $|\widetilde{I}'| = |\widetilde{I}|$ but $|I'| = |\widetilde{I}| + 1$. Let $i > i'$. Then, using (4.11)), one can check that β is zero. Also if $I \cap I' = \emptyset$, then

$$\alpha(I, I') = |I|, \quad \alpha(I, \widetilde{I}') = |\widetilde{I}'| = \alpha(\widetilde{I}, \widetilde{I}'), \quad \alpha(\widetilde{I}, I') = |\widetilde{I}| = |I| - 1.$$

And if $I \cap I' \ne \emptyset$, then

$$\alpha(I, I') = \alpha(I, \widetilde{I}') = \alpha(\widetilde{I}, \widetilde{I}') = r - r' = i - i' \quad \text{and} \quad \alpha(\widetilde{I}, I') = |\widetilde{I} - I'| = (i - 1) - i'.$$

Therefore, in both situations

$$\varphi = -\alpha(I, I') + \alpha(I, \widetilde{I}') + \alpha(\widetilde{I}, I') - \alpha(\widetilde{I}, \widetilde{I}') = \alpha(\widetilde{I}, I') - \alpha(I, I') = -1,$$

as required in (b).

Case (a) reduces to (b). And if $i = i'$ then $r = r'$, implying that both α, β are zero (since for any two intervals among $I, \widetilde{I}, I', \widetilde{I}'$, one is included in the other). □

Lemma 4.3. *Let* $i = i'$. (a) *If* $|I| > |I'|$ *then* $\varphi = -1$. (b) *If* $|I| < |I'|$ *then* $\varphi = 1$.

Proof. Let $|I| > |I'|$. Then $I', \widetilde{I} \subset I$ and $\widetilde{I}' \subset \widetilde{I}$. Hence α and β are zero on each of $(I|I'), (I|\widetilde{I}'), (\widetilde{I}|\widetilde{I}')$. Also $|\widetilde{I}| \ge |I'|$ and $r < r'$ imply $\alpha(\widetilde{I}, I') = 0$ and $\beta(\widetilde{I}, I') = i' - (i - 1) = 1$ (since $\max(\widetilde{I}) = i - 1$). This gives $\varphi = -\beta(\widetilde{I}, I') = -1$.

Case (b) reduces to (a). □

Replacing i, i' by j, j', and I, I' by J, J' in Lemmas 4.1–4.3, we obtain the corresponding statements concerning ψ.

(4.13) (i) If $|J|=|J'|$ and $j<j'$, or if $|J|<|J'|$ and $j=j'$, then $\psi=1$.
(ii) Symmetrically, if $|J|=|J'|$ and $j>j'$, or if $|J|>|J'|$ and $j=j'$, then $\psi=-1$.
(iii) Otherwise $\psi=0$.

Now we finish the proof with showing (4.6) in the corresponding three cases.

Case A: $i=i'$ and $j<j'$. First suppose that $i\le j$. Then both $(I|J)$ and $(I'|J')$ are co-flag corteges, and $|I|=|I'|=i$. We have $\varphi=0$ (by Lemma 4.2(c)) and $\psi=1$ (by (4.13)(i)).

Next suppose that $j<i<j'$. Then $(I|J)$ is flag, $(I'|J')$ is co-flag, and $|I|=j<i=|I'|$. This gives $\varphi=1$ (by Lemma 4.3(b)) and $\psi=0$ (by (4.13)(iii)).

Finally, suppose that $j'\le i$. Then both $(I|J),(I'|J')$ are flag, and $|I|=j<j'=|I'|$. This gives $\varphi=1$ (by Lemma 4.3(b)) and $\psi=0$ (by (4.13)(iii)).

Thus, in all situations, $d=\varphi+\psi=1$, as required in (4.6)(i).

Case B: $i<i'$ and $j=j'$. This is symmetric to the previous case, yielding $d=1$. This matches assertion (ii) in (4.6) (since replacing $i<i'$ by $i>i'$ changes $d=1$ to $d=-1$).

Case C: $i\ne i$ and $j\ne j'$. When $\varphi=\psi=0$, (4.6)(iii) is immediate. The situation with $\varphi\ne 0$ arises only when $|I|=|I'|$; then (a) $i<i'$ implies $\varphi=1$, and (b) $i>i'$ implies $\varphi=-1$ (see Lemma 4.2). Similarly, $\psi\ne 0$ happens only if $|J|=|J'|$; then (c) $j<j'$ implies $\psi=1$, and (d) $j>j'$ implies $\psi=-1$ (by (4.13)(i),(ii))

In subcase (a), $i<i'$ and $|I|=|I'|=:k$ imply $i'>k$ (in view of $i\ge|I|$). Therefore, $j'=k$ must hold (i.e., $(I'|J')$ is flag). Then $j\ne j'$ implies $j>j'$, and we obtain $\psi=-1$, by (4.13)(ii).

In subcase (b), $i>i'$ and $|I|=|I'|=:k$ imply $i>k$. Therefore, $j=k$. Then $j'>j$, yielding $\psi=1$, by (4.13)(i).

So in both (a) and (b), we obtain $\varphi+\psi=0$. In their turn, subcases (c) and (d) are symmetric to (a) and (b), respectively. Thus, in all situations, $d=0$ takes place, as required in (4.6)(iii).

This completes the proof of Theorem 1.2.

5. Uniqueness

Let $f_0: Pint_{m,n}\to\mathcal{A}^*$ be a function in the hypotheses of Theorem 1.2, i.e., f_0 satisfies quasi-commutation relations for all pairs of pressed interval corteges in $\mathcal{E}^{m,n}$ (cf. (4.7)). A priori, f_0 may have many extensions to $\mathcal{E}^{m,n}$ that are QI-functions. One of them is the function f whose values $f(I|J)$ are q-minors $\Delta(I|J)$ of the corresponding path matrix constructed in the proof in Sect. 4.

In light of Theorems 1.1 and 1.2, it is tempting to ask when f_0 has a unique QI-extension. Since any QI-extension is an RQI-function (i.e., satisfies the corresponding relations of Plűcker, co-Plűcker and Dodgson types) and in view of Theorem 1.1, we may address an equivalent question: when an RQI-extension g of f_0 is a QI-function (and therefore $g = f$). We give sufficient conditions below (which is, in fact, a corollary of Theorems 1.1 and 1.2).

To this aim, let us associate to each $(I|J) \in Pint_{m,n}$ an indeterminate $y_{I|J}$ and form the $\mathbb{K}$-algebra $\mathcal{L}_Y$ of quantized Laurent polynomials generated by these $y_{I|J}$ (where the quantization is agreeable with that for f_0). The values of f_0 are said to be *algebraically independent* if the map $y_{I|J} \mapsto f_0(I|J)$, $(I|J) \in Pint_{m,n}$, gives an isomorphism between $\mathcal{L}_Y$ and the $\mathbb{K}$-subalgebra $\mathcal{A}^{f_0}$ of $\mathcal{A}$ generated by these values.

Corollary 5.1. *Let f_0 and f be as above. Let the following additional conditions hold:*

(i) *the values of f_0 are algebraically independent;*
(ii) *if an element $a \in \mathcal{A}^{f_0}$ is a zero divisor in $\mathcal{A}$, then a is a zero divisor in $\mathcal{A}^{f_0}$.*

Suppose that g is an RQI-function on $\mathcal{E}^{m,n}$ coinciding with f_0 on $Pint_{m,n}$. Then g is a QI-function (and therefore $g = f$).

Proof (a sketch). Considering the construction of q-minors of the path matrix related to f_0 (cf. (3.5), (3.7), (4.3)–(4.5)), one can deduce that for each cortege $(I|J) \in \mathcal{E}^{m,n}$, $y_{I|J}$ is a nonzero polynomial in $\mathcal{L}_Y$. Then condition (i) implies that $f(I|J)$ is a nonzero element of $\mathcal{A}^{f_0}$. Furthermore, since $\mathcal{L}_Y$ is free of zero divisors (by a known fact on Laurent polynomials; see, e.g. [1], ch. II, \$11.4, Prop. 8), so is $\mathcal{A}^{f_0}$. Therefore, by condition (ii), $f(I|J)$ is not a zero divisor in $\mathcal{A}$. Now applying Theorem 1.1, we obtain $g = f$, as required. □

Acknowledgment

We thank the anonymous referees for pointing out to us inaccuracies and typos in the original version of this paper.

References

[1] N. Bourbaki, Algebre, Hermann, Paris, 2007.
[2] K. Casteels, A graph theoretic method for determining generating sets of prime ideals in quantum matrices, J. Algebra 330 (2011) 188–205.
[3] G. Cauchon, Spectre premier de $O_q(M_n(k))$: image canonique et séparation normale, J. Algebra 260 (2) (2003) 519–569.
[4] V. Danilov, A. Karzanov, On universal quadratic identities for minors of quantum matrices, J. Algebra 488 (2017) 145–200.
[5] V. Danilov, A. Karzanov, G. Koshevoy, Planar flows and quadratic relations over semirings, J. Algebraic Comb. 36 (2012) 441–474.
[6] V. Lakshmibai, N. Reshetikhin, Quantum flag and Schubert schemes, Contemp. Math. 134 (1992) 145–181.
[7] B. Leclerc, A. Zelevinsky, Quasicommuting families of quantum Plűcker coordinates, Amer. Math. Soc. Transl. Ser. 2 181 (1998) 85–108.

[8] B. Lindstrőm, On the vector representations of induced matroids, Bull. Lond. Math. Soc. 5 (1973) 85–90.
[9] Yu.I. Manin, Quantum Groups and Non-commutative Geometry, vol. 49, Centre de Recherches Mathematiques Montreal, 1988.
[10] J.S. Scott, Quasi-commuting families of quantum minors, J. Algebra 290 (1) (2005) 204–220.
[11] E. Taft, J. Towber, Quantum deformation of flag schemes and Grassmann schemes. I. q-deformation of the shape-algebra for GL(n), J. Algebra 142 (1991) 1–36.