Journal of Combinatorial Theory, Series A 180 (2021) 105392

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

The weak separation in higher dimensions

Vladimir I. Danilov [a], Alexander V. Karzanov [a,*], Gleb A. Koshevoy [b]

[a] *Central Institute of Economics and Mathematics of the RAS, 47, Nakhimovskii Prospect, 117418 Moscow, Russia*
[b] *Institute for Information Transmission Problems of the RAS, 19, Bol'shoi Karetnyi per., 127051 Moscow, Russia*

A R T I C L E I N F O

Article history:
Received 8 January 2020
Received in revised form 11 December 2020
Accepted 14 December 2020
Available online xxxx

Keywords:
Weakly separated sets
Cyclic zonotope
Fine zonotopal tiling
Cubillage
Higher Bruhat order

A B S T R A C T

For an odd integer $r > 0$ and an integer $n > r$, we introduce a notion of *weakly r-separated* collections of subsets of $[n] = \{1, 2, \dots, n\}$. When $r = 1$, this corresponds to the concept of weak separation introduced by Leclerc and Zelevinsky. In this paper, extending results due to Leclerc-Zelevinsky, we develop a geometric approach to establish a number of nice combinatorial properties of maximal weakly r-separated collections (such as an exact upper bound on the maximal size of weakly r-separated collections, mutations rules, relations to the so-called weak membranes in zonotopes of dimension $r+2$, and etc.) A possible analog with r even is briefly discussed as well.

1. Introduction

Let n be a positive integer and let $[n]$ denote the set $\{1, 2, \dots, n\}$. For subsets $X, Y \subseteq [n]$, we write $X < Y$ if the maximal element $\max(X)$ of X is smaller than the minimal

* Corresponding author.
E-mail addresses: danilov@cemi.rssi.ru (V.I. Danilov), akarzanov7@gmail.com (A.V. Karzanov), koshevoyga@gmail.com (G.A. Koshevoy).

https://doi.org/10.1016/j.jcta.2020.105392

element $\min(Y)$ of Y, letting $\max(\emptyset) := 0$ and $\min(\emptyset) := n+1$. An *interval* in $[n]$ is a subset of the form $\{a, a+1, \dots, b\}$ in it, denoted as $[a..b]$ (so $[n] = [1..n]$).

The well-known concept of *strongly separated* sets introduced by Leclerc and Zelevinsky [10] is extended as follows.

Definition. For $r \in \mathbb{Z}_{\geq 0}$, sets $A, B \subseteq [n]$ are called (strongly) *r-separated* if there is no sequence $i_1 < i_2 < \dots < i_{r+2}$ of elements of $[n]$ such that the elements with odd indices (namely, $i_1, i_3, \dots$) belong to one of $A - B$ and $B - A$, while the elements with even indices ($i_2, i_4, \dots$) belong to the other of these two sets (where $A' - B'$ denotes the set difference $\{i\colon A' \ni i \notin B'\}$). Accordingly, a *set-system* $\mathcal{S} \subseteq 2^{[n]}$ (a collection of subsets of $[n]$) is called r-separated if any two members of $\mathcal{S}$ are such.

Equivalently, $A, B \subseteq [n]$ are r-separated if there are intervals $I_1 < I_2 < \dots < I_{r'}$ in $[n]$ with $0 \leq r' \leq r+1$ such that one of $A - B$ and $B - A$ is included in $I_1 \cup I_3 \cup \dots$, and the other in $I_2 \cup I_4 \cup \dots$. If, in addition, (a) r' is minimal, and (b) $|I_1| + \dots + |I_{r'}|$ is minimal subject to (a), we say that $(I_1, \dots, I_{r'})$ is the *interval cortege* associated with A, B.

In particular, A, B are 0-separated if $A \subseteq B$ or $B \subseteq A$, and 1-separated if either $\max(A - B) < \min(B - A)$ or $\max(B - A) < \min(A - B)$. The 1-separation relation is just what is called the strong separation one in [10]. The case $r = 2$ was studied by Galashin [6]. A study for a general r is conducted in Galashin and Postnikov [7].

When A, B are r-separated but not $(r-1)$-separated, they are called $(r+1)$-*intertwined*. In other words, the interval cortege associated with A, B consists of $r+1$ intervals. When A, B are such that $\min(A - B) < \min(B - A)$ and $\max(A - B) > \max(B - A)$, we say that A *surrounds* B.

For example, $A = \{1, 2, 5, 6, 7, 10, 11\}$ and $B = \{1, 3, 4, 6, 9, 11\}$ are 5-intertwined (with the interval cortege $(\{2\}, [3..4], [5..7], \{9\}, \{10\})$) and A surrounds B.

Another kind of set separation introduced by Leclerc and Zelevinsky is known under the name of *weak separation* (which appeared in [10] in connection with the problem of characterizing quasi-commuting flag minors of a quantum matrix; for a discussion on this and wider relations between the weak separation and quantum minors, see also [1, Sect. 8]). We generalize that notion to "higher dimensions" in the following way (where the term "higher dimensions" is justified by appealing to a geometric interpretation explained later).

Definition. Let r be odd. Sets $A, B \subseteq [n]$ are called *weakly r-separated* if they are r'-intertwined with $r' \leq r+2$, and if $r' = r+2$ takes place, then either (a) A surrounds B and $|A| \leq |B|$, or (b) B surrounds A and $|B| \leq |A|$. Accordingly, a set-system $\mathcal{W} \subseteq 2^{[n]}$ is called weakly r-separated if any two members of $\mathcal{W}$ are such.

In other words, A and B are weakly r-separated if they are either (strongly) r-separated or $(r+2)$-intertwined, and in the latter case, for the interval cortege

$(I_1, \ldots, I_{r+2})$ associated with A, B, if the cardinalities of A and B are different, say, $|A| < |B|$, then $I_1 \cup I_3 \cup \ldots \cup I_{r+2}$ contains $A - B$ (while $I_2 \cup I_4 \cup \ldots \cup I_{r+1}$ contains $B - A$). For example, $\{1, 2, 6\}$ and $\{2, 3, 4, 5\}$ are weakly 1-separated, whereas $\{1, 2, 5, 6, 7\}$ and $\{1, 3, 4, 5\}$ are 3-intertwined but not weakly 1-separated.

When $r = 1$, the above definition turns into the classical notion of weak separation due to Leclerc and Zelevinsky [10] (where sets $A, B \subseteq [n]$ are called weakly separated if either (a) $|A| \le |B|$ and there is a partition of $A - B$ into subsets A', A'' (admitting empty ones) such that $A' < B - A < A''$, or (b) $|B| \le |A|$ and there is a partition of $B - A$ into subsets B', B'' such that $B' < A - B < B''$, or both).

In this paper we generalize, to an arbitrary odd $r \ge 1$, two results on weakly separated collections obtained in [10]. One of those (Theorem 1.2 in [10]) says that

(1.1) the maximal possible sizes (numbers of members) of strongly and weakly separated collections in $2^{[n]}$ are the same and equal to $\frac{1}{2}n(n+1) + 1$ $(= \binom{n}{2} + \binom{n}{1} + \binom{n}{0})$.

To formulate a generalization of (1.1), let $r < n$ and denote the maximal possible size $|\mathcal{S}|$ of an r-separated collection $\mathcal{S}$ in $2^{[n]}$ by $s_{n,r}$. Also when r is odd, denote the maximal possible size of a weakly r-separated collection $\mathcal{W} \subseteq 2^{[n]}$ by $w_{n,r}$. Extending results in [10] (for $r = 1$) and [6] (for $r = 2$), it is shown in [7] that

$$s_{n,r} = \binom{n}{\le r+1} \qquad \left(= \binom{n}{r+1} + \binom{n}{r} + \cdots + \binom{n}{0}\right). \tag{1.2}$$

We prove the following

Theorem 1.1. *Let r be odd. Then $w_{n,r} = s_{n,r}$.*

Example. In case $(n, r) = (4, 1)$, one can easily construct a maximal by size weakly separated collection $\mathcal{W}$ which is not strongly separated. It has $s_{4,1} = \binom{4}{2} + \binom{4}{1} + \binom{4}{0} = 11$ sets of which eight are the intervals containing 1 or/and 4 plus the empty interval $\emptyset$ (each of them is necessary since it is separated from any set in [4]). The other three sets in $\mathcal{W}$ are 14, 23, 24 (where we write $a \cdots b$ for $\{a, \ldots, b\}$). Here 23 and 14 are weakly but not strongly separated. Also one can check that any of the five sets in $2^{[4]} - \mathcal{W}$ (namely, 2, 3, 13, 124, 134) is not weakly separated from some of $\{14, 23, 24\}$; so $\mathcal{W}$ is maximal indeed.

Another impressive result in [10] says that a weakly separated collection can be transformed into another one by making a *flip* (a sort of mutation) "in the presence of four witnesses". This relies on the following property (Theorem 1.7 in [10]):

(1.3) let $\mathcal{W} \subset 2^{[n]}$ be weakly separated, and suppose that there are elements $i < j < k$ of $[n]$ and a set $X \subseteq [n] - \{i, j, k\}$ such that $\mathcal{W}$ contains four sets ("witnesses") Xi, Xk, Xij, Xjk and a set $U \in \{Xj, Xik\}$; then the collection obtained from $\mathcal{W}$ by replacing U by the other member of $\{Xj, Xik\}$ is again weakly separated.

Hereinafter, for disjoint subsets A and $\{a,\dots,b\}$ of $[n]$, we write $Aa\dots b$ for $A\cup\{a,\dots,b\}$. Also for $a\in A$, we will abbreviate $A-\{a\}$ as $A-a$.

We generalize (1.3) as follows.

Theorem 1.2. *For an odd r, let $r' := (r+1)/2$. Let $P = \{p_1,\dots,p_{r'}\}$ and $Q = \{q_0,\dots,q_{r'}\}$ consist of elements of $[n]$ such that $q_0 < p_1 < q_1 < p_2 < \dots < p_{r'} < q_{r'}$, and let $X \subseteq [n]-(P\cup Q)$. Define the sets of "upper" and "lower"* neighbors *(or "witnesses") of P,Q to be*

$$\mathcal{N}^{\uparrow}(P,Q) := \{Pq\colon q\in Q\}\cup\{(P-p)q\colon p\in P,\ q\in Q\}; \qquad \text{and} \tag{1.4}$$

$$\mathcal{N}^{\downarrow}(P,Q) := \{Q-q\colon q\in Q\}\cup\{(Q-q)p\colon p\in P,\ q\in Q\}. \tag{1.5}$$

Suppose that a weakly r-separated collection $\mathcal{W}\subset 2^{[n]}$ contains the set $X\cup P$ (resp. $X\cup Q$) and the sets $X\cup S$ for all $S\in \mathcal{N}^{\downarrow}(P,Q)$ (resp. $S\in \mathcal{N}^{\uparrow}(P,Q)$). Then the collection obtained from $\mathcal{W}$ by replacing $X\cup P$ by $X\cup Q$ (resp. by replacing $X\cup Q$ by $X\cup P$) is weakly r-separated as well.

(Note that since Q surrounds P but $|Q|>|P|$, the sets P and Q are not weakly r-separated. Also $|P\cup Q| = r+2$ implies that any two sets in $\{P,Q\}\cup\mathcal{N}^{\uparrow}(P,Q)\cup\mathcal{N}^{\downarrow}(P,Q)$ except for P,Q are weakly r-separated. If $r=1$ then, denoting q_0,p_1,q_1 as i,j,k, respectively, we obtain $\mathcal{N}^{\uparrow}(P,Q)=\mathcal{N}^{\downarrow}(P,Q)=\{i,k,ij,jk\}$, and the theorem turns into (1.3). When $r>1$, the sets $\mathcal{N}^{\uparrow}(P,Q)$ and $\mathcal{N}^{\downarrow}(P,Q)$ become different.)

In general, for two weakly r-separated collections $\mathcal{W}$ and $\mathcal{W}'$, if there are P,Q,X as above such that $\mathcal{W}'=(\mathcal{W}-\{X\cup P\})\cup\{X\cup Q\}$ and $\mathcal{W}=(\mathcal{W}'-\{X\cup Q\})\cup\{X\cup P\}$, then we say that $\mathcal{W}'$ is obtained from $\mathcal{W}$ by a *raising* (combinatorial) *flip*, while $\mathcal{W}$ is obtained from $\mathcal{W}'$ by a *lowering flip*.

Our method of proof of the above theorems and subsequent results essentially use a geometric approach and some facts on fine zonotopal tilings, or *cubillages*, on a *cyclic zonotope* in a space $\mathbb{R}^d$. (The term "cubillage" was introduced by Kapranov and Voevodsky in paper [9] containing, in particular, a geometric interpretation of higher Bruhat orders.)

An important fact is that the maximal by size (strongly) $(d-1)$-separated collections $\mathcal{S}$ in $2^{[n]}$ one-to-one correspond to the cubillages Q in a cyclic zonotope $Z(n,d)$ (generated by a cyclic configuration of n vectors in $\mathbb{R}^d$); moreover, the set of vertices of Q "encodes" $\mathcal{S}$. (When $d=2$, a cubillage becomes a rhombus tiling on a planar n-zonogon; a bijection between these tilings and the maximal strongly separated collections in $2^{[n]}$ is due to [10, Theorem 1.6] (where the language of pseudo-line arrangements, dual to rhombus tilings, is used). For $d=3$, a bijection between the corresponding cubillages and maximal 2-separated sets was originally established in [6]. For a general d, the corresponding bijection was shown by Galashin and Postnikov [7].)

Another important fact, inspired by results in the classical work due to Manin and Schechtman [11] on higher Bruhat orders and their geometric counterparts in [9,12], is that any cubillage on $Z(n, d-1)$ can be lifted as a certain $(d-1)$-dimensional subcomplex, that we call an *s-membrane*, in some cubillage on $Z(n, d)$. For more explanations and other relevant facts, see [5].

We further develop the theory of cubillages by constructing a certain *fragmentation* $Q^{\equiv}$ of a cubillage Q on $Z(n, d)$, introducing a class of $(d-1)$-dimensional subcomplexes in $Q^{\equiv}$, called *w-membranes*, and showing (in Theorem 6.4) that when d is odd, the vertex set of any w-membrane forms a maximal by size weakly $(d-2)$-separated collection in $2^{[n]}$. It turns out that the collections of this sort (over all cubillages on $Z(n, d)$), called *representable* ones, constitute a poset, with a unique minimal element and a unique maximal element, in which neighboring collections are linked by flips; this is obtained as a consequence of Theorems 1.2 and 6.4.

This paper is organized as follows. Sect. 2 contains basic definitions and reviews needed facts on cyclic zonotopes and cubillages. Sect. 3 recalls the construction of s-membranes in cubillages and describes their properties important to us. Here we also introduce the so-called *bead-thread* relation on vertices of a cubillage, which is used in the proof of Theorem 1.1. Sect. 4 proves Theorem 1.1, and Sect. 5 proves Theorem 1.2.

Sect. 6 introduces the notions of cubillage fragmentation and w-membranes. It proves the above-mentioned results on w-membranes in a cubillage on $Z(n, d)$ and on the poset of representable $(d-2)$-separated collections in $2^{[n]}$ (Theorem 6.4 and Corollary 6.5). Here we also raise a conjecture on the representability of all maximal by size weakly r-separated set-systems, and briefly discuss, in Remark 1, the phenomenon of violation of purity for the weak r-separation (i.e., the situation when a maximal by inclusion weakly r-separated collection is not maximal by size).

The paper finishes with two appendixes. Appendix A contains proofs of two propositions stated in Sect. 6 (of which one is of a rather fundamental character). In Appendix B we discuss a possible analog of the weak r-separation when r is even, outline some constructions and results on this way and raise two more conjectures.

Note that, in order to avoid a possible mess, we throughout prefer to use one symbol (namely, r) for the parameter of weak separation, and the other (namely, d) for the dimension of related geometric constructions. (Usually, but not always, $r = d - 2$.)

2. Preliminaries

This section contains additional definitions, notation and conventions that will be needed later on. Also we review some known properties of cubillages.

• Let n, d be integers with $n \geq d > 1$. By a *cyclic configuration* of size n in $\mathbb{R}^d$ we mean an ordered set Ξ of n vectors $\xi_i = (\xi_i(1), \dots, \xi_i(d)) \in \mathbb{R}^d$, $i = 1, \dots, n$, satisfying:

(2.1) (a) $\xi_i(1) = 1$ for each i, and
(b) for the $d \times n$ matrix A formed by $\xi_1, \dots, \xi_n$ as columns (in this order), any flag minor of A is positive.

A typical (and commonly used) sample of such configurations Ξ is generated by the Veronese curve; namely, take reals $t_1 < t_2 < \cdots < t_n$ and assign $\xi_i := \xi(t_i)$, where $\xi(t) = (1, t, t^2, \ldots, t^{d-1})$.

The *zonotope* $Z = Z(\Xi)$ generated by Ξ is the Minkowski sum of line segments $[0, \xi_i]$, $i = 1, \ldots, n$. A *fine zonotopal tiling* is a subdivision Q of Z into d-dimensional parallelotopes such that any two of them either are disjoint or share a face, and each face of the boundary of Z is contained in some of these parallelotopes. For brevity, we refer to these parallelotopes as *cubes*, and to Q as a *cubillage* (following [9]). (Two examples for $(n, d) = (4, 2)$ are illustrated in Fig. 1.)

• When n, d are fixed, the choice of one or another cyclic configuration Ξ (subject to (2.1)) does not matter in essence, and for this reason, we unify notation $Z(n, d)$ for $Z(\Xi)$, referring to it as the *cyclic zonotope* for (n, d).

• Let π denote the projection $\mathbb{R}^d \to \mathbb{R}^{d-1}$ given by $(x(1), \ldots, x(d)) \mapsto (x(1), \ldots, x(d-1))$. Due to (2.1), the vectors $\pi(\xi_1), \ldots, \pi(\xi_n)$ form a cyclic configuration as well, and we may say that π projects $Z(n, d)$ onto the zonotope $Z(n, d-1)$.

• Each subset $X \subseteq [n]$ naturally corresponds to the point $\sum_{i \in X} \xi_i$ in $Z(n, d)$, and the cardinality $|X|$ is called the *height* of this subset/point. (W.l.o.g., we usually assume that all combinations of vectors ξ_i with coefficients 0,1 are different.)

• Depending on the context, we may think of a cubillage Q on $Z(n, d)$ in two ways: either as a set of d-dimensional cubes (and write $C \in Q$ for a cube C in Q) or as the corresponding polyhedral complex. The 0-, 1-, and $(d-1)$-dimensional faces of Q are called *vertices*, *edges*, and *facets*, respectively. By the above-mentioned subset-to-point correspondence, each vertex is identified with a subset of $[n]$. In turn, each edge e is a parallel translation of some segment $[0, \xi_i]$; we say that e has *color* i, or is an i-*edge*. When needed, e is regarded as a directed edge (according to the direction of ξ_i).

• Let $V(Q)$ denote the set of vertices of a cubillage Q. Galashin and Postnikov [7] established a relationship between fine zonotopal tilings and alternating oriented matroids; as a consequence, the following one-to-one correspondence takes place:

(2.2) for any cubillage Q on $Z(n, d)$, the set $V(Q)$ of its vertices (regarded as subsets of $[n]$) constitutes a maximal by size $(d-1)$-separated collection in $2^{[n]}$; conversely, for any maximal by size $(d-1)$-separated collection $\mathcal{S} \subseteq 2^{[n]}$, there exists a cubillage Q on $Z(n, d)$ with $V(Q) = \mathcal{S}$.

• When a face C of Q has $X \subseteq [n]$ as the minimum height vertex, and $T \subseteq [n]$ as the set of edge colors in C, we say that C has *root* X and *type* T, and may write $C = (X \mid T)$. One easily shows that $X \cap T = \emptyset$. Another appealing fact is that for any cubillage Q, the types of all (d-dimensional) cubes in it are different and form the set $\binom{[n]}{d}$ of d-element subsets of $[n]$ (so Q has exactly $\binom{n}{d}$ cubes). See, e.g., [12] or [5].

• For a closed subset U of points in $Z = Z(n, d)$, let U^{fr} (U^{rear}) denote the subset of U "seen" in the direction of the last, d-th, coordinate vector e_d (resp. $-e_d$), i.e., formed by the points $x \in U$ such that there is no $y \in U$ with $\pi(y) = \pi(x)$ and $y(d) < x(d)$ (resp. $y(d) > x(d)$). It is called the *front* (resp. *rear*) *side* of U. Also we call $U^{\rm fr} \cap U^{\rm rear}$ the *rim* of U and denote it as $U^{\rm rim}$ (this term is justified when U is a ball in $\mathbb{R}^3$.)

In particular, $Z^{\rm fr}$, $Z^{\rm rear}$, and $Z^{\rm rim}$ denote the front side, the rear side, and the rim, respectively, of the zonotope Z.

• When a set $X \subseteq [n]$ is the union of k intervals and k is as small as possible, we say that X is a k-*interval*. Note that its complementary set $[n] - X$ is a k'-interval with $k' \in \{k-1, k, k+1\}$. In the next section we will use the following known characterization of the sets of vertices in the front and rear sides of a zonotope of an odd dimension (this can be easily shown by induction on n using the "n-pie contraction technique" as in [5]).

(2.3) Let d be odd. Then for $Z = Z(n, d)$,
  (i) $V(Z^{\rm fr})$ is formed by all k-intervals of $[n]$ with $k \le (d-1)/2$;
  (ii) $V(Z^{\rm rear})$ is formed by the subsets of $[n]$ complementary to those in (i); specifically, it consists of all k-intervals with $k < (d-1)/2$, all $(d-1)/2$-intervals containing at least one of the elements 1 and n, and all $(d+1)/2$-intervals containing both 1 and n.

This implies that: $V(Z^{\rm rim})$ consists of the k-intervals with $k < (d-1)/2$ and the $(d-1)/2$-intervals containing at least one of 1 and n; the set $V(Z^{\rm fr}) - V(Z^{\rm rim})$ of *inner* vertices in $Z^{\rm fr}$ consists of the $(d-1)/2$-intervals containing none of 1 and n; and $V(Z^{\rm rear}) - V(Z^{\rm rim})$ consists of the $(d+1)/2$-intervals containing both 1 and n.

• Consider a cube $C = (X \,|\, T)$ and let $T = (p_1 < p_2 < \cdots < p_d)$. This cube has $2d$ facets $F_1, \dots, F_d, G_1, \dots, G_d$, where

(2.4) $F_i = F_i(C)$ is viewed as $(X \,|\, T - p_i)$, and $G_j = G_j(C)$ as $(Xp_j \,|\, T - p_j)$.

3. S-membranes and bead-threads

In this section we recall the definition of s-membranes, associate with a cubillage a certain path structure, and review additional basic properties.

Definition. Let Q be a cubillage on $Z(n, d)$. An *s-membrane* in Q is a (closed) subcomplex M of Q such that M (regarded as a subset of $\mathbb{R}^d$) is *bijectively* projected by π to $Z(n, d-1)$. (So π gives a homeomorphism between M and $Z(n, d-1)$.)

Then each facet of Q occurring in M is projected to a cube of dimension $d-1$ in $Z(n, d-1)$ and these cubes constitute a cubillage on $Z(n, d-1)$, denoted as $\pi(M)$. In view of (2.2) and (1.2) (applied to $\pi(Q)$), we obtain that

(3.1) each s-membrane M in a cubillage Q on $Z(n, d)$ has $s_{n,d-2}$ vertices, and the vertex set of M (regarded as a collection in $2^{[n]}$) is $(d-2)$-separated.

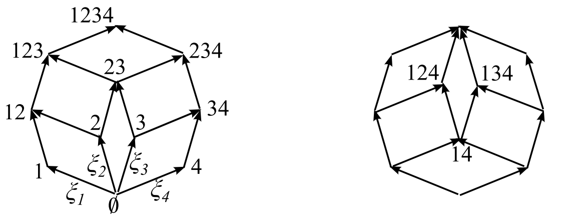

Fig. 1. Left: standard tiling; right: anti-standard tiling.

Two s-membranes are of an especial interest. These are the front side Z^{fr} and the rear side Z^{rear} of $Z = Z(n, d)$ (in these cases the choice of a cubillage on Z is not important). Following terminology in [4,5], their projections $\pi(Z^{\text{fr}})$ and $\pi(Z^{\text{rear}})$ (regarded as complexes) are called the *standard* and *anti-standard* cubillages on $Z(n, d-1)$, respectively. Such cubillages in dimension 2 (viz. rhombus tilings) with $n = 4$ are drawn in Fig. 1.

Next we distinguish certain vertices in cubes. When $n = d$, the zonotope turns into the cube $C = (\emptyset|[d])$, and there holds:

(3.2) the front side C^{fr} of $C = (\emptyset|[d])$ has a unique *inner* vertex (i.e., a vertex not contained in C^{rim}), namely, $t_C := \{i \in [n] \colon d-i \text{ odd}\}$; symmetrically, the rear side C^{rear} of C has a unique inner vertex, namely, $h_C := \{i \in [n] \colon d-i \text{ even}\}$.

(When d is odd, (3.2) can be obtained from (2.3). A direct proof of (3.2) for an arbitrary d is as follows (a sketch). The facets of C are $F_i := (\emptyset|[d]-i)$ and $G_i := (i|[d]-i)$, $i = 1, \dots, d$ (cf. (2.4)). A facet F_i is contained in C^{fr} (C^{rear}) if, when looking at the direction e_d, C lies "behind" (resp. "before") the hyperplane containing F_i, or, equivalently, $\det(A_i) > 0$ (resp. $\det(A_i) < 0$), cf. (2.1)(b), where A_i is the matrix with the columns $\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots \xi_d, \xi_i$ (in this order). It follows that $F_i \subset C^{\text{fr}}$ if and only if $d-i$ is even. By "central symmetry", $G_i \subset C^{\text{fr}}$ if and only if $d-i$ is odd.

Now consider a vertex $X \subseteq [d]$ of C. If X (resp. $[d] - X$) has consecutive elements $i-1$ and i, then $X \in G_{i-1}$ and simultaneously $X \in G_i$ (resp. $X \in F_{i-1}$ and $X \in F_i$). This implies that X is in both C^{fr} and C^{rear}, i.e., $X \in C^{\text{rim}}$. The remaining vertices of C are just t_C and h_C as in (3.2); one can see that the former (latter) is contained in all facets F_j and G_i with $d-j$ even and $d-i$ odd (resp. $d-j$ odd and $d-i$ even). So t_C lies in C^{fr}, and h_C in C^{rear}; moreover, both are not in C^{rim} (since C is full-dimensional).)

When n is arbitrary and Q is a cubillage on $Z = Z(n, d)$, we distinguish vertices t_C and h_C of a cube $C \in Q$ in a similar way; namely (cf. (3.2)):

(3.3) if $C = (X\,|\,T)$ and $T = (p_1 < \ldots < p_d)$, then $t_C := X \cup \{p_i \colon d-i \text{ odd}\}$ is the unique inner vertex in C^{fr}, and $h_C := X \cup \{p_i \colon d-i \text{ even}\}$ is the unique inner vertex in C^{rear}.

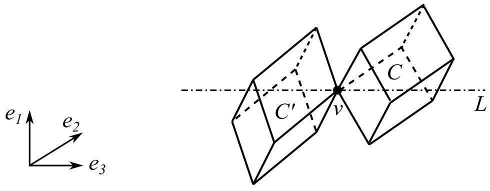

Fig. 2. $v = t_C = h_{C'}$.

Also for each vertex v of Q, unless v is in Z^{rear}, there is a unique cube $C \in Q$ such that $t_C = v$, and symmetrically, unless v is in Z^{fr}, there is a unique cube $C' \in Q$ such that $h_{C'} = v$. (To see this, consider the line L going through v and parallel to e_d. Since e_d and any $(d-1)$ vectors in Ξ are linearly independent, L intersects interiors of cubes in a vicinity of v, namely, C and C'. See Fig. 2, where $d = 3$.)

Therefore, by drawing for each cube $C \in Q$, the edge-arrow from t_C to h_C, we obtain a directed graph whose connectivity components are directed paths beginning at $Z^{\text{fr}} - Z^{\text{rim}}$ and ending at $Z^{\text{rear}} - Z^{\text{rim}}$. We call these paths *bead-threads* in Q. It is convenient to add to this graph the elements of $V(Z^{\text{rim}})$ as isolated vertices, forming *degenerate* bead-threads, each going from a vertex to itself. Let B_Q be the resulting directed graph. Then

(3.4) B_Q contains all vertices of Q, and each component of B_Q is a bead-thread going from Z^{fr} to Z^{rear}.

Note that along every bead-thread, the heights $|X|$ of vertices X are monotone increasing when d is odd, and constant when d is even.

4. Proof of Theorem 1.1

Let r be odd and $n > r$. We have to show that

(4.1) if $\mathcal{W}$ is a weakly r-separated collection in $2^{[n]}$, then $|\mathcal{W}| \le \binom{n}{\le r+1}$.

This is valid when $r = 1$ (cf. (1.1)) and is trivial when $n = r + 1$. So one may assume that $3 \le r \le n-2$. We prove (4.1) by induction, assuming that the corresponding inequality holds for $\mathcal{W}', n', r'$ when $n' \le n$, $r' \le r$, and $(n', r') \neq (n, r)$.

Define the following subcollections in $\mathcal{W}$:

$$\mathcal{W}^- := \{A \subseteq [n-1] : \{A, An\} \cap \mathcal{W} \neq \emptyset\}, \qquad \text{and}$$
$$\mathcal{T} := \{A \subseteq [n-1] : \{A, An\} \subseteq \mathcal{W}\}.$$

Observe that

(4.2) any $A, B \in \mathcal{W}^-$ are weakly r-separated.

Indeed, this is trivial when $A, B \in \mathcal{W}$ or $An, Bn \in \mathcal{W}$. So one may assume that $A \in \mathcal{W}$ and $B' := Bn \in \mathcal{W}$, and that A, B' are $(r+2)$-intertwined (for if A, B' are r'-intertwined with $r' \le r+1$, then so is for A, B, and we are done). Since $\max(B' - A) = n > \max(A - B')$ and $r+2$ is odd, B' surrounds A. Therefore, $\min(B' - A) < \min(A - B')$ and $|B'| \le |A|$. Then $|B| < |A|$ and $\min(B - A) = \min(B' - A) < \min(A - B)$, implying that A, B are weakly r-separated, as required.

By induction, $|\mathcal{W}^-| \le \binom{n-1}{\le r+1}$. Also one can see that $|\mathcal{W}| = |\mathcal{W}^-| + |\mathcal{T}|$. Therefore, using the identity $\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$ for any $j \le n-1$, in order to obtain the inequality in (4.1), it suffices to show that

$$|\mathcal{T}| \le \binom{n-1}{\le r}. \tag{4.3}$$

For $i = 0, 1, \dots, n-1$, define $\mathcal{T}^i := \{A \in \mathcal{T} : |A| = i\}$. We will rely on two claims.

Claim 1. *For each i, the collection $\mathcal{T}^i$ is $(r-1)$-separated; moreover, $\mathcal{T}^i$ is weakly $(r-2)$-separated.*

Proof. Let $A, B \in \mathcal{T}^i$. Take the interval cortege $(I_1, \dots, I_{r'})$ for A, B, and let for definiteness $I_{r'}$ concerns A (i.e., $I_{r'} \cap (A - B) \ne \emptyset$). Then $(I_1, \dots, I_{r'}, I_{r'+1} := \{n\})$ is the interval cortege for A and $B' := Bn$. Since $|A| = |B| < |B'|$ and $\max(A - B') < \max(B' - A) = n$ and since A, B' are weakly r-separated, $r'+1$ must be strictly less than $r+2$. Then $r' \le r$, implying that A, B are $(r-1)$-separated. Since $|A| = |B|$ and r is odd, we also can conclude that A, B are weakly $(r-2)$-separated. $\square$

Now consider the zonotope $Z = Z(n-1, r)$. For $j = 0, 1, \dots, n-1$, define $\mathcal{S}^j$ (resp. $\mathcal{A}^j$) to be the set of vertices X of $Z^{\rm fr}$ (resp. $Z^{\rm rear}$) with $|X| = j$. We extend each $\mathcal{T}^i$ to the collection

$$\mathcal{D}^i := \mathcal{T}^i \cup (\mathcal{S}^{i+1} \cup \ldots \cup \mathcal{S}^{n-1}) \cup (\mathcal{A}^0 \cup \mathcal{A}^1 \cup \ldots \cup \mathcal{A}^{i-1}). \tag{4.4}$$

Claim 2. *$\mathcal{D}^i$ is weakly $(r-2)$-separated.*

Proof. The vertex sets of $Z^{\rm fr}$ and $\pi(Z^{\rm fr})$ are essentially the same (regarding a vertex as a subset of $[n-1]$), and similarly for $Z^{\rm rear}$ and $\pi(Z^{\rm rear})$. Since $\pi(Z^{\rm fr})$ and $\pi(Z^{\rm rear})$ are cubillages on $Z(n-1, r-1)$ (namely, the "standard" and "anti-standard" ones, respectively), (2.2) implies that both collections $V(Z^{\rm fr}) = \mathcal{S}^0 \cup \ldots \cup \mathcal{S}^{n-1}$ and $V(Z^{\rm rear}) = \mathcal{A}^0 \cup \ldots \cup \mathcal{A}^{n-1}$ are $(r-2)$-separated, and therefore, each of them is weakly $(r-2)$-separated as well.

Next, by (2.3)(i), each vertex X of $Z^{\rm fr}$ is a k-interval, where $k \le (r-1)/2$. Such an X and any subset $Y \subseteq [n-1]$ are k'-intertwined with $k' \le 2k+1$. Then $k' \le r$, and this holds with equality when X and Y are r-intertwined and Y surrounds X. It follows that X is weakly $(r-2)$-separated from any $Y \subseteq [n-1]$ with $|Y| \le |X|$ (in particular, if $X \in \mathcal{S}^j$ and $j \ge i$, then X is weakly $(r-2)$-separated from each member of $\mathcal{T}^i \cup \mathcal{A}^0 \cup \ldots \cup \mathcal{A}^{i-1}$).

Symmetrically, by (2.3)(ii), each vertex X of $Z^{\rm rear}$ is the complement to $[n-1]$ of a k-interval with $k \le (r-1)/2$, implying that X is weakly $(r-2)$-separated from any $Y \subseteq [n-1]$ with $|Y| \ge |X|$.

Now the result is provided by Claim 1 and the inequalities $|X| > |A| > |X'|$ for any $X \in \mathcal{S}^{i+1} \cup \ldots \cup \mathcal{S}^{n-1}$, $A \in \mathcal{T}^i$, and $X' \in \mathcal{A}^0 \cup \ldots \cup \mathcal{A}^{i-1}$. $\square$

By induction, $|\mathcal{D}^i| \le \binom{n-1}{\le r-1}$. Then, using (1.2) and (3.1) (relative to $n-1$ and $r-2$), we have

$$|\mathcal{D}^i| \le \binom{n-1}{\le r-1} = s_{n-1,r-2} = |V(Z^{\rm fr})|. \tag{4.5}$$

Let $\mathcal{S}' := \mathcal{S}^0 \cup \mathcal{S}^1 \cup \ldots \cup \mathcal{S}^i$ and $\mathcal{A}' := \mathcal{A}^0 \cup \mathcal{A}^1 \cup \ldots \cup \mathcal{A}^{i-1}$. Since $\mathcal{S}^{i+1} \cup \ldots \cup \mathcal{S}^{n-1} = V(Z^{\rm fr}) - \mathcal{S}'$, we obtain from (4.4) and (4.5) that

$$|\mathcal{T}^i| = |\mathcal{D}^i| - (|V(Z^{\rm fr}) - \mathcal{S}'|) - |\mathcal{A}'| \le |\mathcal{S}'| - |\mathcal{A}'|. \tag{4.6}$$

We now finish the proof by using bead-thread techniques (as in Sect. 3). Fix an arbitrary cubillage Q on $Z = Z(n-1,r)$. Let $\mathcal{R}^i$ be the set of vertices X of Q with $|X| = i$, and let $\mathcal{B}$ be the set of paths (bead-threads) in the graph B_Q beginning at $Z^{\rm fr}$ and ending at $Z^{\rm rear}$. Since r is odd, each edge (X,Y) of B_Q is “ascending” (satisfies $|Y| > |X|$). This implies that each path $P \in \mathcal{B}$ beginning at $\mathcal{S}'$ must meet (once) either $\mathcal{R}^i$ or $\mathcal{A}'$, and conversely, each path meeting $\mathcal{R}^i \cup \mathcal{A}'$ begins at $\mathcal{S}'$. Therefore, $|\mathcal{R}^i| = |\mathcal{S}'| - |\mathcal{A}'|$, and this together with (4.6) implies

$$|\mathcal{T}^i| \le |\mathcal{R}^i|.$$

Summing up these inequalities for $i = 0, 1, \ldots, n-1$, we have

$$|\mathcal{T}| = \sum\nolimits_i |\mathcal{T}^i| \le \sum\nolimits_i |\mathcal{R}^i| = |V_Q| = s_{n-1,r-1} = \binom{n-1}{\le r},$$

yielding (4.3) and completing the proof of Theorem 1.1. $\square$ $\square$

5. Proof of Theorem 1.2

Let r, r', $P = \{p_1, \ldots, p_{r'}\}$, $Q = \{q_0, \ldots, q_{r'}\}$ and X be as in the hypotheses of Theorem 1.2 (where r is odd and $r' = (r+1)/2$).

In what follows, for sets $A, B \subset [n]$, when $A \cap B = \emptyset$, we abbreviate $A \cup B$ as AB. When A, B are not weakly r-separated, we say that the pair $\{A, B\}$ is *bad*.

Note that XP and XQ are $(r+2)$-intertwined; XQ surrounds XP; $|XQ| > |XP|$; and $\{XP, XQ\}$ is the unique bad pair in the collection $\{XS \colon S \in \{P,Q\} \cup \mathcal{N}^{\uparrow}(P,Q) \cup \mathcal{N}^{\downarrow}(P,Q)\}$. The theorem is reduced to the following assertion.

(5.1) Let $Y \subset [n]$ be different from XP and XQ. Then:
 (i) if $\{Y, XP\}$ is bad, then there exists $S \in \mathcal{N}^{\uparrow}(P,Q)$ such that $\{Y, XS\}$ is bad;
 (ii) if $\{Y, XQ\}$ is bad, then there exists $S \in \mathcal{N}^{\downarrow}(P,Q)$ such that $\{Y, XS\}$ is bad.

(Indeed, to obtain Theorem 1.2 from (5.1), suppose that a weakly r-separated collection $\mathcal{W} \subset 2^{[n]}$ includes $\{XS\colon S \in \{P\} \cup \mathcal{N}^{\downarrow}(P,Q)\}$ (resp. $\{XS\colon S \in \{Q\} \cup \mathcal{N}^{\uparrow}(P,Q)\}$). Let $Y \in \mathcal{W} - \{XP, XQ\}$. Then (ii) (resp. (i)) in (5.1) implies that Y and XQ (resp. Y and XP) are weakly r-separated, and the theorem follows.)

We first prove assertion (i) in (5.1) (obtaining (ii) by symmetry, as we explain in the end of the proof). Suppose, for a contradiction, that

(5.2) there is $Y \subset [n]$ different from XQ such that $\{Y, XP\}$ is bad but none of the pairs $\{Y, XS\}$ with $S \in \mathcal{N}^{\uparrow}(P,Q)$ is bad.

This will impose sharp restrictions on Y and will eventually lead us to the conclusion that Y is impossible. W.l.o.g., one may assume that $Y \cap X = \emptyset$.

In what follows, the interval cortege for sets $A, B \subset [n]$ is denoted by $\mathcal{I}(A,B)$, and when it is not confusing, we refer to the intervals in it concerning $A - B$ $(B - A)$ as *A-bricks* (resp. *B-bricks*). For brevity we will write $\mathcal{N}^{\uparrow}$ for $\mathcal{N}^{\uparrow}(P,Q)$, and $\mathcal{I}$ for $\mathcal{I}(Y, XP)$. Also we refer to an element $p \in P$ $(q \in Q)$ as *refined* if it forms the single-element XP-brick $\{p\}$ (resp. the single-element Y-brick $\{q\}$) in $\mathcal{I}$.

The core of the proof consists in the next lemma.

Lemma 5.1. *Let Y be as in (5.2). Then at least one of the following holds:*
 ($*$) *all elements of P are refined;*
 ($**$) *all elements of Q are refined.*

This lemma will be proved later, and now assuming its validity, we show (5.1)(i) as follows. Note that $Y \cap P \neq \emptyset$ is possible (whereas $Y \cap X = \emptyset$, as assumed above).

Let a and b denote the numbers of Y- and XP-bricks in $\mathcal{I}$, respectively. Then $a + b = |\mathcal{I}| \geq r + 2 = 2r' + 1$ and $|a - b| \leq 1$. We assume that the intervals in $\mathcal{I}$ are viewed as $\ldots < A_{i-1} < B_i < A_i < B_{i+1} \ldots$, where $A_{i'}$ $(B_{i'})$ stands for a Y-brick (resp. XP-brick). The first (last) Y-brick is denoted by A^m (resp. A^M), and the first (last) XP-brick by B^m (resp. B^M). Also for a set $C \subset [n]$ and a singleton $c \in [n]$, we write $c < C$ $(c > C)$ if $c < \min(C)$ (resp. $c > \max(C)$).

We first assume that ($**$) from Lemma 5.1 is valid. Then $Y \supseteq Q$ and $a \geq |Q| = r' + 1$. Consider two possible cases for a.

<u>*Case I*</u>: $a \geq r' + 2$. Then $b \geq r' + 1$ and $|\mathcal{I}| \geq 2r' + 3$. If $q_0 < B^m$, then $q_0 \in A^m$, implying $A^m = \{q_0\}$ (since q_0 is refined). Taking $S := Pq_0 \in \mathcal{N}^{\uparrow}(P,Q)$, we obtain $|\mathcal{I}(Y, XS)| = |\mathcal{I}| - 1 \geq 2r' + 2 > r + 2$ (since the Y-brick $\{q_0\}$ disappears, while the other bricks of $\mathcal{I}$ preserve). Hence $\{Y, XS\}$ is bad. Similarly, if $B^M < q_{r'}$, then

$A^M = \{q_{r'}\}$, and taking $S := Pq_{r'}$, we again obtain $|\mathcal{I}(Y, XS)| \ge 2r' + 2$, whence $\{Y, XS\}$ is bad.

So we may assume that $B^m < q_0$ and $q_{r'} < B^M$. Then $b \ge |Q| + 1 = r' + 2$ and $|\mathcal{I}| = a + b \ge 2r' + 4$. Taking $S := Pq_0$, we obtain $|\mathcal{I}(Y, XS)| = |\mathcal{I}| - 2 \ge 2r' + 2$ (since the Y-brick $\{q_0\}$ disappears and the XP-bricks preceding and succeeding $\{q_0\}$ merge). Thus, in all cases, $\{Y, XS\}$ is bad; a contradiction.

<u>*Case II*</u>: $a = r' + 1$. Then $A^m = \{q_0\}, \{q_1\}, \dots, \{q_{r'}\} = A^M$ are exactly the Y-bricks of $\mathcal{I}$. If $B^m < q_0$ and $B^M < q_{r'}$, then $b = a = r' + 1$ and $|\mathcal{I}| = 2r' + 2$. Taking $S := Pq_{r'}$, we obtain $|\mathcal{I}(Y, XS)| = |\mathcal{I}| - 1 = 2r' + 1$. Also XS surrounds Y (since B^M becomes the last interval in $\mathcal{I}(Y, XS)$). Hence $|Y - XS| = r' < r' + 1 \le |XS - Y|$, implying that $\{Y, XS\}$ is bad.

Similarly, if $q_0 < B^m$ and $q_{r'} < B^M$, then $S := Pq_0$ gives $|\mathcal{I}(Y, XS)| = |\mathcal{I}| - 1 = 2r' + 1$, and XS surrounds Y as well. And if $B^m < q_0$ and $q_{r'} < B^M$, then $b = r' + 2$, and for $S := Pq_0$, we obtain $|\mathcal{I}(Y, XS)| = |\mathcal{I}| - 2 = 2r' + 1$. Again, XS surrounds Y, whence $\{Y, XS\}$ is bad.

So it remains to consider the situation when $q_0 < B^m$ and $B^M < q_{r'}$. Then $b = r'$ and Y surrounds XP. Since $\{Y, XP\}$ is bad and $Y - XP = Q$, we have $r' + 1 = |Y - XP| > |XP - Y| \ge r'$. It follows that $|XP - Y| = r'$. This implies that each XP-brick is a singleton. Also in case $Y \cap P = \emptyset$, the XP-bricks of $\mathcal{I}$ are exactly $\{p_1\}, \dots, \{p_{r'}\}$. But then $X = \emptyset$ and $Y = Q = XQ$, contradicting the condition $Y \ne XQ$ in (5.1).

Therefore, Y must contain an element p_i for some i. Then $p_i \notin B_i$ (in view of $|B_i| = 1$ and $p_i \notin XP - Y$). So one of two situations takes place: $q_{i-1} < p_i < B_i < q_i$, or $q_{i-1} < B_i < p_i < q_i$. Define $S := (P - p_i)q_{i-1}$ in the former case, and $S := (P - p_i)q_i$ in the latter case. The transformation $XP \mapsto XS$ replaces the Y-brick $\{q_{i-1}\}$ or $\{q_i\}$ by $\{p_i\}$. We obtain: $|\mathcal{I}(Y, XS)| = |\mathcal{I}| = 2r' + 1$, Y surrounds XS, and $|Y| > |XS|$. Hence $\{Y, XS\}$ is bad; a contradiction.

Next we assume that $(*)$ from Lemma 5.1 is valid. Then $b \ge r'$ and each $p_i \in P$ forms the XP-brick $\{p_i\}$ in $\mathcal{I}$ (admitting the possibility of other XP-bricks); in particular, $Y \cap P = \emptyset$. Consider two possibilities for b.

<u>*Case III*</u>: $b = r'$. Then $\{p_1\}, \dots, \{p_{r'}\}$ are exactly the XP-bricks of $\mathcal{I}$, $X = \emptyset$, and $a = r' + 1$ (in view of $|\mathcal{I}| \ge 2r' + 1$). So Y surrounds $P = XP$, and $|Y| > |P|$. Assuming, w.l.o.g., that $(**)$ from Lemma 5.1 is not valid, there is $i \in \{0, \dots, r'\}$ such that $\{q_i\}$ is not a Y-brick of $\mathcal{I}$. Then either (a) q_i lies in some Y-brick A_j with $|A_j| \ge 2$, or (b) q_i lies in no interval of $\mathcal{I}$.

In case (a), take $S := Pq_i$. Note that $|Y| \ge r' + 2 = |P| + 2$ (in view of $a = r' + 1$ and $|A_j| \ge 2$). If $q_i \in Y$, then the transformation $P \mapsto S$ replaces A_j by a (possibly smaller but nonempty) Y-brick in $\mathcal{I}(Y, S)$, while preserving the other intervals of $\mathcal{I}$. It follows that $|\mathcal{I}(Y, S)| = |\mathcal{I}| = 2r' + 1$, Y surrounds S, and $|Y| > |P| + 1 = |S|$; so $\{Y, S\}$ is bad. And if $q_i \notin Y$, then, obviously, $\min(A_j) < q_i < \max(A_j)$. This gives $|\mathcal{I}(Y, S)| = |\mathcal{I}| + 2$ (since $P \mapsto S$ replaces A_j by the S-brick $\{q_i\}$ and two Y-bricks, one containing $\min(A_j)$ and the other containing $\max(A_j)$); so $\{Y, S\}$ is bad again.

In case (b), it is clear that $q_i \notin Y$. Then one of four subcases takes place: (b1) $p_i < q_i < A_i$; (b2) $A_i < q_i < p_{i+1}$; (b3) $i = 0$ and $q_0 < A^m$; and (b4) $i = r'$ and $A^M < q_{r'}$. In subcases (b3) and (b4), taking $S := Pq_i$, we have $|\mathcal{I}(Y,S)| = |\mathcal{I}| + 1$ (since $\{q_i\}$ becomes a new brick), whence $\{Y,S\}$ is bad. In subcases (b1) and (b2), for $S' := (P - p_i)q_i$ and $S' := (P - p_{i+1})q_i$, respectively, the transformation $P \mapsto S'$ replaces the P-brick $\{p_i\}$ or $\{p_{i+1}\}$ by the S'-brick $\{q_i\}$, and the badness of $\{Y,P\}$ implies that of $\{Y,S'\}$.

<u>*Case IV*</u>: $b \geq r' + 1$. Assuming, as before, that we are not in (∗∗) from Lemma 5.1, there is i such that $\{q_i\}$ is not a Y-brick of $\mathcal{I}$. Take $S := Pq_i$. We can observe that in all possible cases for q_i (as exposed in Case III above), the transformation $XP \mapsto XS$ leads to the following: $|XS| > |XP|$, each Y-brick of $\mathcal{I}$ either preserves or is replaced by a (nonempty) Y-brick of $\mathcal{I}(Y,XS) =: \mathcal{I}'$, and similarly for the XP-bricks of $\mathcal{I}$. Then $|\mathcal{I}'| \geq |\mathcal{I}| \geq 2r' + 1$. Moreover, in case $|\mathcal{I}'| = 2r' + 1$, the number of XS-bricks (Y-bricks) of $\mathcal{I}'$ and the number of XP-bricks (resp. Y-bricks) of $\mathcal{I}$ are the same, which is equal to $r' + 1 = b$ (resp. $r' = a$). So $b > a$, XP surrounds Y, and $|XP| > |Y|$ (since $\{Y,XP\}$ is bad). Now the badness of $\{Y,XS\}$ follows from $|XS| > |XP|$, yielding a contradiction.

Thus, assertion (i) in (5.1) is proven (subject to Lemma 5.1).

It remains to show (ii). We reduce it to the previous case, using the following observation. For $A \subseteq [n]$, let $\overline{A}$ denote the complementary set $[n] - A$. One can see that $\mathcal{I}(A,B) = \mathcal{I}(\overline{A},\overline{B})$ and that the bricks for $A - B$ coincide with those for $\overline{B} - \overline{A}$. It follows that if A,B are $(r+2)$-intertwined (where r is odd, as before) and A surrounds B, then $\overline{A},\overline{B}$ are $(r+2)$-intertwined, $\overline{B}$ surrounds $\overline{A}$, and $|A| - |B| = |\overline{B}| - |\overline{A}|$. Therefore, if A,B are weakly r-separated then so are $\overline{A},\overline{B}$.

Now for Y,P,Q,X as above and $U := XQ$, consider $Y' := \overline{Y}$, $X' := \overline{XPQ}$ ($= [n] - (X \cup P \cup Q)$) and $U' := \overline{XQ}$. Suppose that $\{Y,U\}$ is bad. Then $\{Y',U'\}$ is bad as well. Note also that $U' = X'P$. By the theorem applied to Y',P,Q,X' and U', there exists $S' \in \mathcal{N}^{\uparrow}(P,Q)$ such that $\{Y',X'S'\}$ is bad. Then $\{Y,\overline{X'S'}\}$ is bad as well. Take $S := (P \cup Q) - S'$. One can see that $S \in \mathcal{N}^{\downarrow}(P,Q)$ and $\overline{X'S'} = XS$. Therefore, $\{Y,XS\}$ is bad, as required.

This completes the proof of (5.1) (yielding Theorem 1.2), modulo Lemma 5.1.

Proof of Lemma 5.1. Suppose that there are simultaneously $p \in P$ and $q \in Q$ that are not refined. Form $S' := P - p$, $S'' := Pq$ and $S := (P - p)q$ (note that S'' and S are in $\mathcal{N}^{\uparrow} = \mathcal{N}^{\uparrow}(P,Q)$, whereas S' is not). Let $\mathcal{I} := \mathcal{I}(Y,XP)$, $\mathcal{I}' := \mathcal{I}(Y,XS')$ and $\mathcal{I}'' := \mathcal{I}(Y,XS'')$. We write A_i (B_i) for Y-bricks (resp. XP-bricks) in $\mathcal{I}$ and assume that they follow in $\mathcal{I}$ in the order $\ldots < A_{i-1} < B_i < A_i < B_{i+1} \ldots$.

For $A,B,A',B' \subseteq [n]$, let us say that the ordered pairs (A,B) and (A',B') *have the same type* if $|\mathcal{I}(A,B)| = |\mathcal{I}(A',B')|$ and the first interval of $\mathcal{I}(A,B)$ concerns $A - B$ if and only if the first interval of $\mathcal{I}(A',B')$ concerns $A' - B'$ (implying that a similar property holds for the last intervals of $\mathcal{I}(A,B)$ and $\mathcal{I}'(A',B')$).

We examine four possible cases for p, q and establish important interrelations for $\mathcal{I}, \mathcal{I}', \mathcal{I}''$ which will be used later.

Case 1: p lies in an interval C of $\mathcal{I}$.

(1a) Suppose that C is an XP-brick B_i. Since p is not refined, $|B_i| \geq 2$. If $p \notin Y$, then the transformation $XP \mapsto XS'$ replaces B_i by a (nonempty) XS'-brick B'_i in $\mathcal{I}$ with $B'_i \subseteq B_i$, while the other intervals in $\mathcal{I}$ and $\mathcal{I}'$ coincide. Therefore, $\mathcal{I}'$ and $\mathcal{I}$ have the same type.

And if $p \in Y$, then $\min(B_i) < p < \max(B_i)$, and $XP \mapsto XS'$ replaces B_i by three bricks, say, $B' < A' < B''$, where A' is the single-element Y-brick $\{p\}$, and B', B'' are XS'-bricks (with $\min(B') = \min(B_i)$ and $\max(B'') = \max(B_i)$). Then $|\mathcal{I}'| = |\mathcal{I}| + 2$.

(1b) Now suppose that C is a Y-brick A_i. This is possible only if $p \in Y$ and $\min(A_i) < p < \max(A_i)$. Then $p \in Y - XS'$, and $XP \mapsto XS'$ preserves A_i (as well as the other intervals of $\mathcal{I}$), whence $\mathcal{I}' = \mathcal{I}$.

Case 2: q lies in an interval C of $\mathcal{I}$.

(2a) Suppose that $C = B_i$. This is possible only if $q \notin Y$ and $\min(B_i) < q < \max(B_i)$ (since $q \notin XP$). Then $XP \mapsto XS''$ preserves B_i, yielding $\mathcal{I}'' = \mathcal{I}$.

(2b) Now suppose that $C = A_i$. Since q is not refined, $|A_i| \geq 2$. If $q \in Y$, then $XP \mapsto XS''$ replaces A_i by a (nonempty) Y-brick A'_i with $A'_i \subseteq A_i$. Therefore, $\mathcal{I}''$ and $\mathcal{I}$ have the same type. And if $q \notin Y$, then $XP \mapsto XS''$ replaces A_i by three bricks $A' < B' < A''$, where B' is the XS''-brick $\{q\}$, and A', A'' are Y-bricks (with $\min(A') = \min(A_i)$ and $\max(A'') = \max(A_i)$), whence $|\mathcal{I}''| = |\mathcal{I}| + 2$.

Case 3: p belongs to no interval of $\mathcal{I}$. Then $p \in Y$.

(3a) Suppose that $A_i < p < B_{i+1}$ or $B_i < p < A_i$ for some i. Then $p \in Y - XS'$, and $XP \mapsto XS'$ extends A_i (making a Y-brick with the beginning or end at p). Hence $\mathcal{I}'$ and $\mathcal{I}$ have the same type.

(3b) Suppose that $p < C$, where C is the first interval of $\mathcal{I}$. If C is an XP-brick, then $XP \mapsto XS'$ produces a new Y-brick, namely, $\{p\}$, and preserves the other intervals of $\mathcal{I}$, whence $|\mathcal{I}'| = |\mathcal{I}| + 1$. And if C is a Y-brick, then $XP \mapsto XS'$ extends C (making a Y-brick with the beginning p), whence $\mathcal{I}'$ and $\mathcal{I}$ have the same type.

(3c) Similarly, if $p > D$, where D is the last interval of $\mathcal{I}$, then either $|\mathcal{I}'| = |\mathcal{I}| + 1$, or $\mathcal{I}'$ and $\mathcal{I}$ have the same type (when D is extended to a Y-brick with the end p).

Case 4: q belongs to no interval of $\mathcal{I}$. Then $q \notin Y$.

(4a) Suppose that $A_{i-1} < q < B_i$ or $B_i < q < A_i$ for some i. Then $XP \mapsto XS''$ extends B_i (making a XS''-brick with the beginning or end at q). Hence $\mathcal{I}'', \mathcal{I}$ have the same type.

(4b) Suppose that $q < C$, where C is the first interval of $\mathcal{I}$. If C is an XP-brick, then $XP \mapsto XS''$ extends C (making an XS''-brick with the beginning q), whence $\mathcal{I}'', \mathcal{I}$ have the same type. And if C is a Y-brick, then $XP \mapsto XS''$ preserves C and produces the new brick $\{q\}$ (concerning XS''), whence $|\mathcal{I}''| = |\mathcal{I}| + 1$.

(4c) Similarly, if $q > D$, where D is the last interval of $\mathcal{I}$, then either $\mathcal{I}''$ and $\mathcal{I}$ have the same type, or $|\mathcal{I}''| = |\mathcal{I}| + 1$.

Now we finish proving the lemma as follows. Analyzing Cases 1,3 above, we observe that $|\mathcal{I}'| \geq |\mathcal{I}|$ is valid throughout, and if this holds with equality, then $\mathcal{I}'$ and $\mathcal{I}$ have the same type. For $\mathcal{I}''$ and $\mathcal{I}$, the behavior is similar (in Cases 2,4).

If $|\mathcal{I}''| > |\mathcal{I}|$ happens, then Y and XS'' form a bad pair (since they are $|\mathcal{I}''|$-intertwined with $|\mathcal{I}''| > r+2$ and taking into account that $S'' = Pq \in \mathcal{N}^{\uparrow}$). This contradicts (5.2).

Now let $|\mathcal{I}''| = |\mathcal{I}|$ (then $\mathcal{I}'', \mathcal{I}$ have the same type). We consider the neighbor $S = (P-p)q \in \mathcal{N}^{\uparrow}$ and assert that $\{Y, XS\}$ is bad, thus coming to a contradiction again.

To show this, let $\widetilde{\mathcal{I}} := \mathcal{I}(Y, XS)$. Suppose that $q \in Y$. Setting $Y^- := Y - q$, we have $Y^- - XP = Y - XS''$ and $XP - Y^- = XS'' - Y$, implying that $\mathcal{I}^- := \mathcal{I}(Y^-, XP)$ coincides with $\mathcal{I}''$. Hence $\mathcal{I}^-$ and $\mathcal{I}$ have the same type. Moreover, under the correspondence of intervals in these corteges (exposed in (2b)), each Y^--brick of $\mathcal{I}^-$ is included in the corresponding Y-brick of $\mathcal{I}$, and each XP-brick of $\mathcal{I}^-$ includes the corresponding XP-brick of $\mathcal{I}$. In particular, p is not refined w.r.t. $\mathcal{I}^-$. So we can apply to X, P, Y^-, p the analysis as in Cases 1 and 3 and conclude that under the transformation $XP \mapsto XS'$, the cortege $\mathcal{I}^-$ turns into $\widehat{\mathcal{I}} := \mathcal{I}(Y^-, XS')$ such that either $|\widehat{\mathcal{I}}| > |\mathcal{I}^-|$, or $\widehat{\mathcal{I}}$ and $\mathcal{I}^-$ have the same type. But $Y = Y^- q$ and $S = S'q$ imply $\widehat{\mathcal{I}} = \widetilde{\mathcal{I}}$. Now the badness of $\{Y, XS\}$ is immediate when $|\widehat{\mathcal{I}}| > |\mathcal{I}^-|\ (= |\mathcal{I}|)$, and follows from the badness of $\{Y, XP\}$ when $|\widehat{\mathcal{I}}| = |\mathcal{I}^-|$ (since $\widehat{\mathcal{I}}$ and $\mathcal{I}$ have the same type and $|Y^-| - |XS'| = |Y| - |XS| = |Y| - |XP|$).

Finally, let $q \notin Y$. Then (in view of $|\mathcal{I}''| = |\mathcal{I}|$) we are in one of the following subcases: (2a) with $\min(B_i) < q < \max(B_i)$ for some i; or (4a) with $A_{i-1} < q < B_i$ or $B_i < q < A_i$ for some i; or (4b) with $q < B^m < A^m$; or (4c) with $q > B^M > A^M$ (where, as before, A^m and A^M (resp. B^m and B^M) are the first and last Y-bricks (resp. XP-bricks) in $\mathcal{I}$, respectively). By the explanations above, in all of these situations, $XP \mapsto XS''$ leads to increasing at most one brick concerning X and preserving the other intervals of $\mathcal{I}$. This implies that p is not refined w.r.t. $\mathcal{I}''$, and we can apply to X, S'', Y, p the reasoning as in Cases 1 and 3 and conclude that $XS'' \mapsto XS$ turns $\mathcal{I}''$ into $\widetilde{\mathcal{I}}$ so that either $|\widetilde{\mathcal{I}}| > |\mathcal{I}''|$ $(= |\mathcal{I}|)$, or $\widetilde{\mathcal{I}}$ and $\mathcal{I}''$ have the same type. Then the badness of $\{Y, XS\}$ follows.

This completes the proof of the lemma. □

6. Weakly *r*-separated collections generated by cubillages

In Sects. 2, 3 we outlined an interrelation between (strongly) ∗-separated collections on the one hand, and cubillages and s-membranes on the other hand (see (2.2) and (3.1)). This section is devoted to geometric aspects of the weak r-separation when r is odd. Being motivated by geometric constructions for maximal weakly 1-separated collections elaborated in [3,4], we explain how to construct maximal by size weakly r-separated collections by use of the so-called *w-membranes*; these are analogs of s-membranes in certain *fragmentations* of cubillages.

In the subsections below we introduce the notions of fragmentation and w-membrane, demonstrate their properties (extending results from [4, Sect. 6]) and finish with a theorem saying that the vertex set of any $(r+1)$-dimensional w-membrane gives rise to a maximal by size weakly r-separated collection (for corresponding n). Note that in Sects. 6.1–6.3 the dimension d of a zonotope/cubillage in question is assumed to be arbitrary (not necessarily odd).

6.1. Fragmentation

Let Q be a cubillage on $Z(n,d)$. For $\ell = 0,1,\dots,n$, we denote the "horizontal" hyperplane at "height" ℓ in $\mathbb{R}^d$ by H_ℓ, i.e., $H_\ell := \{x = (x(1),\dots,x(d)) \in \mathbb{R}^d :\ x(1) = \ell\}$. The *fragmentation* of Q is meant to be the complex $Q^{\equiv}$ obtained by cutting Q by $H_1,\dots,H_{n-1}$.

Such hyperplanes subdivide each cube $C = (X\,|\,T)$ of Q into d pieces $C^{\equiv}_1,\dots,C^{\equiv}_d$, where $C^{\equiv}_h$ is the (closed) portion of C between $H_{|X|+h-1}$ and $H_{|X|+h}$. We say that $C^{\equiv}_h$ is h-th *fragment* of C and, depending on the context, may also think of $Q^{\equiv}$ as the set of fragments over all cubes of Q. Let $S_h(C)$ denote h-th horizontal *section* $C \cap H_{|X|+h}$ of C (where $0 \le h \le d$); this is the convex hull of the set of vertices

$$(X\,|\,\tbinom{T}{h}) \qquad (= \{X \cup A :\ A \subset T,\ |A| = h\}). \tag{6.1}$$

(Such an $S_h(C)$ is called a *hyper-simplex*, in terminology of [8]. It turns into a usual simplex when $h = 1$ or $d-1$.) Observe that for $h = 1,\dots,d$,

(6.2) the h-th fragment $C^{\equiv}_h$ of C is the convex hull of the set of vertices $(X\,|\,\binom{T}{h-1})$ and $(X\,|\,\binom{T}{h})$; it has two "horizontal" facets, namely, $S_{h-1}(C)$ and $S_h(C)$, and $2d$ other facets (conditionally called "vertical" ones), namely, the portions of $F_i(C)$ and $G_i(C)$ between $H_{|X|+h-1}$ and $H_{|X|+h}$ for $i = 1,\dots,d$, denoted as $F_{h,i}(C)$ and $G_{h,i}(C)$, respectively.

Here $F_i(C)$ and $G_i(C)$ are the facets of $C = (X|T)$ defined in (2.4), letting $T = (p_1 < p_2 < \dots < p_d)$. We call $S_{h-1}(C)$ and $S_h(C)$ the *lower* and *upper* facets of the fragment $C^{\equiv}_h$, respectively. Note that $S_0(C)$ and $S_d(C)$ degenerate to the single points X and XT, respectively. The vertical facets $F_{d,i}(C)$ and $G_{1,i}(C)$ (for all i) degenerate as well.

The horizontal facets are "not fully seen" under the projection π. To visualize all facets of fragments of $Q^{\equiv}$, it is convenient to look at them as though "from the front and slightly from below", i.e., by use of the projection $\pi^\epsilon : \mathbb{R}^d \to \mathbb{R}^{d-1}$ defined by

$$x = (x(1),\dots,x(d)) \mapsto (x(1) - \epsilon x(d), x(2),\dots,x(d-1)) =: \pi^\epsilon(x) \tag{6.3}$$

for a sufficiently small $\epsilon > 0$. (Compare π^ϵ with π.) Fig. 3 illustrates the case $d = 3$; here the fragments of a cube $C = (X\,|\,T)$ with $T = (i < j < k)$ are drawn.

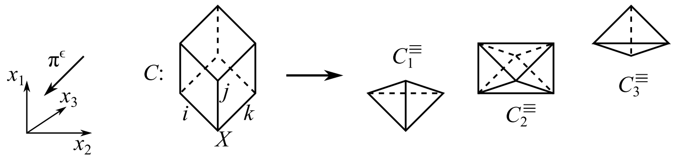

Fig. 3. The fragmentation of cube $C = (X \mid T)$.

Using this projection, we obtain slightly slanting front and rear sides of objects in $Q^{\equiv}$. More precisely, for a closed set U of points in $Z = Z(n, d)$, let $U^{\epsilon,\text{fr}}$ ($U^{\epsilon,\text{rear}}$) be the subset of U "seen" in the direction $e_d + \epsilon e_1$ (resp. $-e_d - \epsilon e_1$), where e_i is i-th coordinate vector, i.e., formed by the points $x \in (\pi^\epsilon)^{-1}(x') \cap U$ with $x(d)$ minimum (resp. maximum) over all $x' \in \pi^\epsilon(U)$. We call it the *ϵ-front* (resp. *ϵ-rear*) *side* of U.

Obviously, $Z^{\epsilon,\text{fr}} = Z^{\text{fr}}$ and $Z^{\epsilon,\text{rear}} = Z^{\text{rear}}$. Also for a cube $C = (X|T)$ in Z, $C^{\epsilon,\text{fr}} = C^{\text{fr}}$ and $C^{\epsilon,\text{rear}} = C^{\text{rear}}$. As to fragments of C, their ϵ-front and ϵ-rear sides are viewed as follows:

(6.4) for $h = 1, \dots, d$, $C_h^{\epsilon,\text{fr}}$ is the union of C_h^{fr} and the lower facet $S_{h-1}(C)$ (degenerating to the point X when $h = 1$); in turn, $C_h^{\epsilon,\text{rear}}$ is formed by the union of C_h^{rear} and the upper facet $S_h(C)$ (degenerating to the point XT when $h = d$).

So $C_h^{\epsilon,\text{fr}} \cup C_h^{\epsilon,\text{rear}}$ is just the full boundary of $C_h^{\equiv}$.

6.2. W-membranes

Membranes of this sort represent certain $(d-1)$-dimensional subcomplexes of $Q^{\equiv}$. To introduce them, we consider small deformations of cyclic zonotopes in $\mathbb{R}^{d-1}$ using the projection π^ϵ. More precisely, given a cyclic configuration $\Xi = (\xi_1, \dots, \xi_n)$ as in (2.1), define

$$\psi_i := \pi(\xi_i) \quad \text{and} \quad \psi_i^\epsilon := \pi^\epsilon(\xi_i), \quad i = 1, \dots, n.$$

Then $\Psi = (\psi_1, \dots, \psi_n)$ obeys (2.1) (with $d-1$ instead of d), and when ϵ is small enough, $\Psi^\epsilon = (\psi_1^\epsilon, \dots, \psi_n^\epsilon)$ obeys the condition (2.1)(b), though slightly violates (2.1)(a); yet we keep the term "cyclic configuration" for Ψ^ϵ as well. Consider the zonotope in $\mathbb{R}^{d-1}$ generated by Ψ^ϵ, denoted as $Z^\epsilon(n, d-1)$ (when it is not confusing).

Definition. A *w-membrane* of a cubillage Q on $Z(n, d)$ is a (closed) subcomplex M of the fragmentation $Q^{\equiv}$ such that M (regarded as a subset of $\mathbb{R}^d$) is bijectively projected by π^ϵ onto $Z^\epsilon(n, d-1)$.

(Cf. [5, Appendix C].) A w-membrane M has facets (of dimension $d-1$) of two sorts, called *H-tiles* and *V-tiles*. Each H-tile is a horizontal facet of some fragment (viz. the section $S_h(C)$ of a cube C in Q at height $h \in [d-1]$). And V-tiles are vertical facets of some fragments $C_h^{\equiv}$ (see (6.2)).

6.3. *Acyclicity and the lattice structure of w-membranes*

Let $\mathbf{C}(n,d)$ denote the set of all cubes in $Z(n,d)$ (occurring in all cubillages there). For $C, C' \in \mathbf{C}(n,d)$, we say that C *immediately precedes* C' if C^{rear} and $(C')^{\text{fr}}$ share a facet (of dimension $d-1$). Generalizing the known acyclicity property for cubes in a cubillage, one can show the following

Proposition 6.1. *The directed graph $\Gamma_{n,d}$ whose vertices are the cubes in $\mathbf{C}(n,d)$ and whose edges are the pairs (C, C') of cubes such that C immediately precedes C' is acyclic.*

(As a consequence, the transitive closure of this “immediately preceding” relation forms a partial order on $\mathbf{C}(n,d)$.) Proposition 6.1 enables us to further construct a partial order on the set of fragments for a cubillage Q, which in turn is used to show that the set of w-membranes in $Q^{\equiv}$ forms a distributive lattice.

More precisely, given a cubillage Q on $Z(n,d)$, consider fragments $\Delta = C_i^{\equiv}$ and $\Delta' = (C')_j^{\equiv}$ of $Q^{\equiv}$. Let us say that Δ *immediately precedes* Δ' if the ϵ-rear side of Δ and the ϵ-front side of Δ' share a facet. In other words, either $C \neq C'$ and $\Delta^{\text{rear}} \cap (\Delta')^{\text{fr}}$ is a V-tile, or $C = C'$ and $j = i+1$. The following is important for us.

Proposition 6.2. *The directed graph $\Gamma_{Q^{\equiv}}$ whose vertices are the fragments in $Q^{\equiv}$ and whose edges are the pairs (Δ, Δ') of fragments such that Δ immediately precedes Δ' is acyclic.*

Proofs of Propositions 6.1 and 6.2 will be given in Appendix A.

From Proposition 6.2 it follows that the transitive closure of the immediately preceding relation on the fragments of $Q^{\equiv}$ forms a partial order; denote it as $(Q^{\equiv}, \prec)$.

Let us associate with a w-membrane M of Q the (closed) region $\Omega(M)$ of $Z = Z(n,d)$ between Z^{fr} and M, and let $Q^{\equiv}(M)$ be the set of fragments in $Q^{\equiv}$ lying in $\Omega(M)$. The constructions of π^{ϵ} and M imply that M is the ϵ-rear side of $\Omega(M)$ (while Z^{fr} is its ϵ-front side). This leads to the following property: for fragments Δ, Δ' of $Q^{\equiv}$, if Δ immediately precedes Δ' and if $\Delta' \in Q^{\equiv}(M)$, then $\Delta \in Q^{\equiv}(M)$ as well (since the common facet of Δ, Δ' lies in $\Omega(M)$ and belongs to the ϵ-rear side of Δ). Then a similar property for fragments Δ, Δ' with $\Delta \prec \Delta'$ is valid as well. Hence $Q^{\equiv}(M)$ is an ideal of $(Q^{\equiv}, \prec)$. A converse property is also true: any ideal I of $(Q^{\equiv}, \prec)$ is expressed as $Q^{\equiv}(M)$ for some w-membrane M of Q (this M is the ϵ-rear side of the minimal region of Z containing Z^{fr} and I). Therefore (cf. [5, Appendix C]),

(6.5) the set $\mathcal{M}^{\text{w}}(Q)$ of w-membranes of a cubillage Q on $Z = Z(n,d)$ is a distributive lattice in which for $M, M' \in \mathcal{M}^{\text{w}}(Q)$, the w-membranes $M \wedge M'$ and $M \vee M'$

satisfy $Q^{\equiv}(M \wedge M') = Q^{\equiv}(M) \cap Q^{\equiv}(M')$ and $Q^{\equiv}(M \vee M') = Q^{\equiv}(M) \cup Q^{\equiv}(M')$; the minimal and maximal elements of this lattice are $Z^{\rm fr}$ and $Z^{\rm rear}$, respectively.

Suppose that $M \in \mathcal{M}^{\rm w}(Q)$ is different from $Z^{\rm fr}$. Then $Q^{\equiv}(M) \neq \emptyset$. Take a maximal (relative to the order $\prec$ in $Q^{\equiv}$) fragment Δ in $Q^{\equiv}(M)$. Then $\Delta^{\epsilon,\rm rear}$ is entirely contained in M. Indeed, if a facet $F \in \Delta^{\epsilon,\rm rear}$ lies in $Z^{\rm rear}$, then F is automatically in M. And if F is not in $Z^{\rm rear}$, then F is shared by $\Delta^{\epsilon,\rm rear}$ and $(\Delta')^{\epsilon,\rm fr}$ for another fragment Δ'. Hence Δ immediately precedes Δ', implying that Δ' lies in the region between M and $Z^{\rm rear}$. Then F is in M, as required.

For Δ as above, the set $Q^{\equiv}(M) - \{\Delta\}$ is again an ideal of $(Q^{\equiv}, \prec)$, and therefore it is expressed as $Q^{\equiv}(M')$ for some w-membrane M'. Moreover, M' is obtained from M by replacing the disk $\Delta^{\epsilon,\rm rear}$ by $\Delta^{\epsilon,\rm fr}$. We call the transformation $M \mapsto M'$ the (geometric) *lowering flip* on M *using* Δ, and call the reverse transformation $M' \mapsto M$ the (geometric) *raising flip* on M' *using* Δ. As a result, we obtain the following nice property.

Corollary 6.3. *Let M be a w-membrane of a cubillage Q. Then there exists a sequence of w-membranes $M_0, M_1, \dots, M_k \in \mathcal{M}^{\rm w}(Q)$ such that $M_0 = Z^{\rm fr}$, $M_k = M$, and for $i = 1, \dots, k$, M_i is obtained from M_{i-1} by the (geometric) raising flip using some fragment in $Q^{\equiv}$.*

6.4. Weakly r-separated collections via w-membranes

Now we throughout assume that r is odd and $d = r + 2$. Consider a cubillage Q on $Z = Z(n, d)$. Based on Theorems 1.1, 1.2 and Corollary 6.3, we establish the main result of Sect. 6.

Theorem 6.4. *For any w-membrane M of a cubillage Q on $Z(n, d)$, the set $V(M)$ of vertices of M (regarded as subsets of $[n]$) constitutes a maximal by size weakly r-separated collection in $2^{[n]}$ (where, as before, r is odd and $d = r+2$). In particular, all w-membranes in Q have the same number of vertices, namely, $w_{n,d-2}$ ($= s_{n,d-2}$).*

Example. Fig. 4 illustrates a w-membrane M for $(n, d) = (4, 3)$ for which $V(M)$ is the weakly separated collection $\mathcal{W}$ exposed in Example in Sect. 1; here there are ten V-tiles and two H-tiles, which are shadowed. For simplicity we do not indicate a cubillage on $Z(4, 3)$ whose fragmentation contains this membrane.

Proof. Let $M \in \mathcal{M}^{\rm w}(Q)$ and consider a sequence $Z^{\rm fr} = M_0, M_1, \dots, M_k = M$ as in Corollary 6.3. Let $\Delta_1, \dots, \Delta_k$ be the fragments of Q such that M_i is obtained from M_{i-1} by the raising flip using Δ_i. The collection $V(Z^{\rm fr})$ is weakly r-separated (as it is strongly r-separated, cf. (3.1)), and our aim is to show that if $V(M_{i-1})$ is weakly r-separated, then so is $V(M_i)$, and $|V(M_i)| = |V(M_{i-1})|$ is valid.

To show this, consider w-membranes M, M' of Q such that M' is obtained from M by the raising flip using a fragment $\Delta \in Q^{\equiv}$. Let $\Delta = C_h^{\equiv}$ for a cube $C = (X \,|\, T)$ with

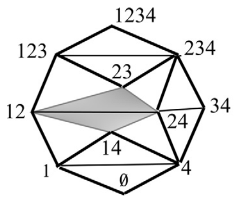

Fig. 4. An example of w-membrane for $d = 3$ and $n = 4$.

$T = (p_1 < \ldots < p_d)$), and $h \in [d]$. By explanations in Sect. 3, $C^{\,\rm fr}$ and $C^{\,\rm rear}$ differ by exactly two vertices; namely, $V(C^{\,\rm fr}) = V(C^{\,\rm rim}) \cup \{t_C\}$ and $V(C^{\,\rm rear}) = V(C^{\,\rm rim}) \cup \{h_C\}$, where $t_C = Xp_2p_4 \ldots p_{d-1}$ and $h_C = Xp_1p_3 \ldots p_d$ (cf. (3.3)). Define R to be the set of vertices of $C^{\,\rm rim}$ occurring in Δ, and let $r' := (d-1)/2$. We consider three cases.

Case 1: $h \le r'$. Since the vertices of Δ are formed by the sections $S_{h-1}(C)$ and $S_h(C)$,

$$V(\Delta) = (X \mid \tbinom{T}{h-1}) \cup (X \mid \tbinom{T}{h}) \quad \text{and} \quad R \subseteq V(\Delta^{\,\rm fr}) \cup V(\Delta^{\,\rm rear})$$

(cf. (6.1)). Also $V(\Delta^{\,\rm fr}) \subseteq V(\Delta^{\epsilon,\rm fr})$ and $V(\Delta^{\,\rm rear}) \subseteq V(\Delta^{\epsilon,\rm rear})$. If $h < r'$, then all vertices of Δ belong to $C^{\,\rm rim}$; this implies $V(\Delta^{\epsilon,\rm fr}) = R = V(\Delta^{\epsilon,\rm rear})$. And if $h = r'$, then the only vertex of Δ not in R is t_C. Since $t_C \in V(C^{\,\rm fr})$, t_C belongs to $\Delta^{\epsilon,\rm fr}$. But t_C also lies in the upper facet $S_{r'}(C)$ (in view of $|p_2p_4 \ldots p_{d-1}| = r'$), and this facet is included in $\Delta^{\epsilon,\rm rear}$. Hence $t_C \in \Delta^{\epsilon,\rm fr} \cap \Delta^{\epsilon,\rm rear}$, implying $V(\Delta^{\epsilon,\rm fr}) = V(\Delta^{\epsilon,\rm rear})$.

Case 2: $h \ge r' + 2$. This is "symmetric" to the previous case. If $h > r' + 2$, then all vertices of Δ belong to $C^{\,\rm rim}$, implying $V(\Delta^{\epsilon,\rm fr}) = R = V(\Delta^{\epsilon,\rm rear})$. And if $h = r' + 2$, then $\Delta^{\epsilon,\rm fr}$ includes the lower facet $S_{r'+1}(C)$, which in turn contains the vertex h_C (since $|p_1p_3 \ldots p_d| = r' + 1$). Also $h_C \in V(C^{\,\rm rear})$ implies $h_C \in V(\Delta^{\epsilon,\rm rear})$, and we again obtain $V(\Delta^{\epsilon,\rm fr}) = V(\Delta^{\epsilon,\rm rear})$.

Thus, in both cases the raising flip $M \mapsto M'$ using Δ does not change the vertex set of the current w-membrane.

Case 3: $h = r' + 1$. This case is most important. Now the lower facet $S_{h-1=r'}(C)$ of Δ contains t_C, while the upper facet $S_{h=r'+1}(C)$ contains h_C. Hence $t_C \in V(\Delta^{\epsilon,\rm fr})$ and $h_C \in V(\Delta^{\epsilon,\rm rear})$. On the other hand, neither t_C belongs to $\Delta^{\epsilon,\rm rear}$ ($= \Delta^{\,\rm rear} \cup S_{r'+1}(C)$), nor h_C belongs to $\Delta^{\epsilon,\rm fr}$ ($= \Delta^{\,\rm fr} \cup S_{r'}(C)$).

It follows that $V(\Delta^{\epsilon,\rm rear}) = (V(\Delta^{\epsilon,\rm fr}) - \{t_C\}) \cup \{h_C\}$, and therefore the raising flip $M \mapsto M'$ using Δ replaces t_C by h_C, while preserving the other vertices of the w-membrane. Since the vertices of Δ are of the form XS with S running over the r'- and $r'+1$-element subsets of $\{p_1, \ldots, p_d\}$, this vertex set includes $\{XS \colon S \in \mathcal{N}^{\downarrow}(\widetilde{P}, \widetilde{Q})\}$ (which is contained in R and in M), where $\widetilde{P} = p_2p_4 \ldots p_{d-1}$ and $\widetilde{Q} = p_1p_3 \ldots p_d$ (i.e., $t_C = X\widetilde{P}$ and $h_C = X\widetilde{Q}$).

Now applying Theorem 1.2 to $\mathcal{W} := V(M)$, $X, \widetilde{P}, \widetilde{Q}$ and $\mathcal{N}^{\downarrow}(\widetilde{P}, \widetilde{Q})$, we conclude that $\mathcal{W}(M')$ is weakly r-separated, as required.

This completes the proof of the theorem. $\square$

It should be noted that any w-membrane in a cubillage on $Z(n,3)$ can be expressed as a *quasi-combined tiling* in the planar zonogon $Z(n,2)$, and in this particular case, the statement of Theorem 6.4 with $r=1$ is equivalent to Corollary 6.5 in [4].

Next, in light of the above discussion, given an odd r and $n>r$, we can specify three classes $\mathbf{W}_{n,r}$, $\mathbf{W}^{=}_{n,r}$ and $\mathbf{W}^{*}_{n,r}$ of weakly r-separated collections $\mathcal{W}$ in $2^{[n]}$ such that $\mathcal{W}$ is maximal by inclusion, maximal by size, and representable, respectively. (Recall that $\mathcal{W}$ is called *representable* if it can be represented as the vertex set of a w-membrane in a cubillage on $Z(n,r+2)$; in particular, $\mathcal{W}$ is maximal by size.) We have the following hierarchy:

$$\mathbf{W}_{n,r} \supseteq \mathbf{W}^{=}_{n,r} \supseteq \mathbf{W}^{*}_{n,r}\,.$$

Theorem 6.4 together with (6.5) implies the following nice property of $\mathbf{W}^{*}_{n,r}$.

Corollary 6.5. $\mathbf{W}^{*}_{n,r}$ *is a poset with the unique minimal element* $V(Z^{\,\mathrm{fr}})$ *and the unique maximal element* $V(Z^{\,\mathrm{rear}})$ *in which any two neighboring elements are linked by a (raising or lowering) combinatorial flip, where* $Z := Z(n,r+2)$.

Indeed, for a cubillage Q on Z, let $\mathbf{W}(Q)$ be the set of collections $\mathcal{W}\subseteq 2^{[n]}$ such that $\mathcal{W}=V(M)$ for some w-membrane M in (the fragmentation of) Q. Typically, the set $\mathcal{M}^{\mathrm{w}}(Q)$ of w-membranes of Q is larger that $\mathbf{W}(Q)$ since no geometric raising flip on $\mathcal{M}^{\mathrm{w}}(Q)$ occurring in Cases 1 and 2 of the proof of Theorem 6.4 changes the vertex set of the membrane. On the other hand, each flip $M\mapsto M'$ in Case 3 of the proof induces the combinatorial raising flip $V(M)\mapsto V(M')$ on $\mathbf{W}(Q)$, which replaces the set (vertex) t_C by h_C for some cube $C\in Q$. The fact that $|t_C|<|h_C|$ implies that the directed graph $\Gamma(Q)$ on $\mathbf{W}(Q)$ whose edges correspond to such raising flips is acyclic. Also $\Gamma(Q)$ is connected and has one minimal vertex (namely, $V(Z^{\,\mathrm{fr}})$) and one maximal vertex ($V(Z^{\,\mathrm{rear}})$); this follows from similar properties of $\Gamma_{Q^{\equiv}}$ (defined in Proposition 6.2). Combining the graphs $\Gamma(Q)$ over all cubillages Q on Z, we obtain an acyclic graph on $\mathbf{W}^{*}_{n,r}$, giving rise to the desired poset.

A natural question is whether any two members of the set $\mathbf{W}^{=}_{n,r}$ can be connected by a sequence of flips. This is strengthened in the following

Conjecture 1. Let r be odd. Then any maximal by size weakly r-separated collection in $2^{[n]}$ is representable.

Its validity together with Theorem 6.4 would imply $\mathbf{W}^{*}_{n,r}=\mathbf{W}^{=}_{n,r}$. This has been proved for $r=1$ (cf. Theorems 3.4, 3.5 in [3] and Theorem 6.8 in [4]).

We finish the main content of this paper with one more aspect, as follows.

Remark 1. For a symmetric binary relation R on a set N, let G be the graph whose vertices are the elements of N and whose edges are the pairs $\{u,v\}$ of distinct vertices subject to uRv. Let $\mathcal{C}$ be the set of *cliques* in G (i.e., inclusion-wise maximal subsets of vertices of which any two are connected by edge in G). Then $\mathcal{C}$ is said to be *pure* if all cliques of G have the same size.

Recall that for an odd r, $\mathbf{W}_{n,r}$ denotes the set of all maximal by inclusion weakly r-separated collections in $2^{[n]}$. It was shown in [2] that $\mathbf{W}_{n,1}$ is pure for any n (which affirmatively answers Leclerc-Zelevinsky's conjecture on maximal weakly separated set-systems in [10]). In other words, $\mathbf{W}_{n,1} = \mathbf{W}^{=}_{n,1}$ $(= \mathbf{W}^{*}_{n,1})$. A reasonable question is: whether $\mathbf{W}_{n,r}$ is pure when $r \geq 3$? It is not difficult to show that this is so if $n - r \leq 2$ (see [5]). On the other hand, it turns out that already $\mathbf{W}_{6,3}$ is not pure. Here a counterexample to the purity can be constructed as follows.

The set-system $2^{[6]}$ consists of 64 sets, and a direct enumeration shows that exactly 52 of them are formed by (a) intervals in the six-element set $[6]$, and (b) 2-intervals containing at least one of the elements 1 and 6; let $\mathcal{S}$ denote the set of these. One easily shows that each member of $\mathcal{S}$ is weakly 3-separated from any subset of $[6]$. So there are $2^6 - 52 = 12$ other subsets of $[6]$; these are:

(6.6) **24**, 245, 25, 235, **35**, 135, 1356, 136, **1346**, 146, 1246, 246.

(Recall that $a \cdots b$ stands for $\{a, \dots, b\}$.) Let $\mathcal{A}$ be the collection formed by the members of the sequence in (6.6) indicated in bold, i.e., $\mathcal{A} = \{24, 35, 1346\}$; these sets are weakly 3-separated from each other. Then $\mathcal{S} \cup \mathcal{A}$ consists of $52 + 3 = 55$ sets, whereas the number $s_{6,3} = w_{6,3}$ is equal to $\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 57$. So $\mathcal{S} \cup \mathcal{A}$ is weakly 3-separated. Moreover, it is maximal by inclusion, since any element of (6.6) not in $\mathcal{A}$ is not weakly 3-separated from some element of $\mathcal{A}$ (which can be verified directly). For example, 245, 25, 235 are not weakly 3-separated from 1346. Thus, $\mathbf{W}_{6,3}$ is not pure, yielding $\mathbf{W}_{6,3} \neq \mathbf{W}^{=}_{6,3}$.

Acknowledgments

We thank the anonymous referees for meticulously reading the original version of this paper and many remarks and suggestions. The third author is supported in part by grant RSF 16-11-10075.

Appendix A. Proofs of two propositions on acyclicity

In Sect. 6.3 we stated two propositions on acyclicity for cubes and their fragments. Their proofs can be found in our recent paper [5] (in Appendixes C and D there), but in order to make our description more self-contained, below we give proofs, using some stylistic modifications and improvements.

Proof of Proposition 6.1. Let C immediately precede C', and let the cubes C, C' and the facet $F := C^{\text{rear}} \cap (C')^{\text{fr}}$ be of the form $(X \,|\, T)$, $(X' \,|\, T')$ and $(\widetilde{X} \,|\, \widetilde{T})$, respectively.

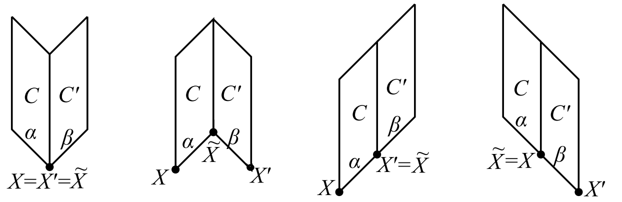

Fig. 5. Cases (i), (ii), (iii), (iv) (from left to right).

Then $T = \widetilde{T}\alpha$ and $T' = \widetilde{T}\beta$ for some $\alpha, \beta \in [n]$. Four cases are possible (as illustrated in Fig. 5):

(i) $X = X' = \widetilde{X}$;
(ii) $X, X', \widetilde{X}$ are different (then $\widetilde{X} = X\alpha = X'\beta$);
(iii) $X \neq X' = \widetilde{X}$ (then $\widetilde{X} = X\alpha$);
(iv) $X' \neq X = \widetilde{X}$ (then $\widetilde{X} = X'\beta$).

Let us associate with a cube $C'' = (X'' \,|\, T'')$ a label $\omega(C'') \in \{0, 1, 2\}$ by the following rule:

$(*)$ $\omega(C'') = 0$ if $n \notin X'', T''$; $\omega(C'') = 1$ if $n \in T''$; $\omega(C'') = 2$ if $n \in X''$.

The following observation is the key.

Claim. *For C, C' as above, $\omega(C) \leq \omega(C')$.*

Proof of the Claim. We may assume that $\omega(C) \neq \omega(C')$. Then $n \in T \cup X \cup T' \cup X'$ but n belongs to neither $\widetilde{T}$ nor $X \cap X'$. This implies that either $\alpha = n$ or $\beta = n$ (in view of $\widetilde{T} = T - \alpha = T' - \beta$). We use the following characterization of facets (in notation as in (2.4)) of the front and rear sides of a cube, which is shown by arguing as in Sect. 3 (when proving (3.2)):

(A.1) for a cube $\widehat{C} = (\widehat{X}|\widehat{T})$ with $\widehat{T} = (p_1 < \cdots < p_d)$, a facet $F_i(\widehat{C})$ is in $\widehat{C}^{\rm fr}$ if and only if $d - i$ is even, whereas $G_i(\widehat{C})$ is in $\widehat{C}^{\rm fr}$ if and only if $d - i$ is odd.

Using this for C and F as above and considering the inclusion $F \subset C^{\rm rear}$, one can conclude that if $\alpha = n$, then the root $\widetilde{X}$ of F and the root X of C are different (taking into account that n is the maximal element in T). In turn, $F \subset (C')^{\rm fr}$ implies that if $\beta = n$, then $\widetilde{X} = X'$. This leads to the following:

(A.2) $\alpha = n$ is possible only in cases (ii) and (iii), whereas $\beta = n$ is possible only in cases (i) and (iii).

In particular, case (iv) is impossible (when $\omega(C) \neq \omega(C')$). As to the other three cases, we obtain from (A.2) that

(a) in case (i), $\omega(C) = 0$ and $\omega(C') = 1$ (since $n = \beta \in T'$);
(b) in case (ii), $\omega(C) = 1$ (since $n = \alpha \in T$) and $\omega(C') = 2$ (since $\widetilde{X} = X\alpha = X'\beta$ implies $\alpha \in X'$);
(c) in case (iii), if $\alpha = n$ then $\omega(C) = 1$ and $\omega(C') = 2$ (since $X' = X\alpha$), and if $\beta = n$ then $\omega(C) = 0$ and $\omega(C') = 1$.

Thus, $\omega(C) \le \omega(C')$ holds in all cases, as required. □

Now we finish the proof of the proposition by induction on n. This is trivial when $n = d$, so assume that $n > d$ and that the assertion is valid for (n', d') with $n' < n$.

Suppose, for a contradiction, that $\Gamma_{n,d}$ has a directed cycle $\mathcal{C} = (C_0, C_1, \ldots, C_k = C_0)$ (where each C_i immediately precedes C_{i+1}). Then the Claim implies that $\omega(C_i)$ is the same number q for all i. Consider three cases (where $C_i = (X_i \,|\, T_i)$).

Case 1: $q = 0$. Then $\mathcal{C}$ is a directed cycle in $\Gamma_{n-1,d}$, contrary to the inductive assumption.

Case 2: $q = 2$. Define $X'_i := X_i - n$ and $C'_i := (X'_i \,|\, T_i)$, $i = 0, \ldots, k$. Then each C'_i is a cube in $Z(n-1, d)$, and the sequence $C'_0, C'_1, \ldots, C'_k$ forms a directed cycle in $\Gamma_{n-1,d}$; a contradiction.

Case 3: $q = 1$. Define $T'_i := T_i - n$ and $C'_i := (X_i \,|\, T'_i)$, $i = 0, \ldots, k$. Then each C'_i can be regarded as a cube in $Z(n-1, d-1)$ (in view of $|T'_i| = d - 1$). Considering (A.1) and using the fact that n is the maximal element in T_i, one can conclude that if $(Y \,|\, U)$ is a facet with $n \in U$ in $C_i^{\rm fr}$, then $(Y \,|\, U \cap [n-1])$ is a facet in $(C'_i)^{\rm rear}$, and similarly for facets in $C_i^{\rm rear}$ and $(C'_i)^{\rm fr}$. Then the fact that $C_i^{\rm rear} \cap C_{i+1}^{\rm fr}$ is a facet (having n in its type) implies that $(C'_i)^{\rm fr} \cap (C'_{i+1})^{\rm rear}$ is a facet as well. This means that C'_{i+1} immediately precedes C'_i. Therefore, the sequence $C'_k, C'_{k-1}, \ldots, C'_1, C'_0$ forms a directed cycle in $\Gamma_{n-1,d-1}$; a contradiction.

This completes the proof of the proposition. □

Proof of Proposition 6.2. For a fragment $\Delta = C_h^{\equiv}$ of a cube $C = (X \,|\, T)$, denote $|X| + h - 1/2$ by $\ell(\Delta)$, called the *mid-level* of Δ.

Suppose that there exist fragments $\Delta_0, \Delta_1, \ldots, \Delta_k = \Delta_0$ forming a directed cycle in $\Gamma_{Q^{\equiv}}$. Consider two consecutive fragments $\Delta = \Delta_{i-1}$ and $\Delta' = \Delta_i$. Then the sides $\Delta^{\epsilon,\rm rear}$ and $\Delta^{\epsilon,\rm fr}$ share a facet F, and either (a) F is a vertical facet of both (in terminology of (6.2)), or (b) F is the upper facet of Δ and the lower facet of Δ'. Obviously, $\ell(\Delta') = \ell(\Delta)$ in case (a), and $\ell(\Delta') = \ell(\Delta) + 1$ in case (b). This implies

$$\ell(\Delta_0) \le \ell(\Delta_1) \le \cdots \le \ell(\Delta_{k-1}) \le \ell(\Delta_0).$$

Then all fragments Δ_i have the same mid-level, and therefore each pair of consecutive fragments shares a vertical facet. But this means that the sequence of cubes containing these fragments forms a cycle in the graph $\Gamma_{n,d}$, contrary to Proposition 6.1. □

Appendix B. A concept of weak r-separation when r is even

Up to now, we have dealt with the weak r-separation when r is odd. In this section we attempt to introduce and explore an analogous concept when r is even.

For $A', B' \subseteq [n]$, we say that A' *surrounds* B' *from the right* if $\max(A' - B') > \max(B' - A')$.

Definition. For an *even* integer $r > 0$ and an integer $n > r$, sets $A, B \subseteq [n]$ are called *weakly r-separated* if they are $\widetilde{r}$-intertwined with $\widetilde{r} \le r+2$, and in case $\widetilde{r} = r+2$, either (a) A surrounds B from the right and $|A| \le |B|$, or (b) B surrounds A from the right and $|B| \le |A|$. Accordingly, a set-system $\mathcal{W} \subseteq 2^{[n]}$ is called weakly r-separated if any two members of $\mathcal{W}$ are such.

(Note that this matches the definition for r odd in the Introduction.)

Remark 2. In contrast to the odd case, the size $|\mathcal{W}|$ of a weakly r-separated collection $\mathcal{W} \subseteq 2^{[n]}$ with r even can exceed the value $s_{n,r}$ (defined in (1.2)). The simplest example is given by $n = r+2$ and $\mathcal{W} = 2^{[n]}$. Indeed, in this case $s_{n,r}$ amounts to $\sum(\binom{r+2}{i} : i = 0, \ldots, r+1) = 2^{r+2} - 1$, which is less than $|\mathcal{W}| = 2^{r+2}$. Observe that $\mathcal{W}$ has only one pair $\{A, B\}$ of $(r+2)$-intertwined sets, namely, $A = \{2, 4, \ldots, r\}$ and $B = \{1, 3, \ldots, r-1\}$. These A, B are weakly r-separated since $|A| = |B|$. (By the way, one can see that this $\mathcal{W}$ represents the vertex set of a w-membrane in the fragmentation of the cube $C = (\emptyset \,|\, [n])$, forming the trivial cubillage on $Z(n, n)$.)

Thus, if one wished to get rid of exceeding $s_{n,r}$, one should impose additional restrictions on $\mathcal{W}$. The above example prompts an idea on this way.

Definition. Let us say that a pair $\{A, B\}$ of subsets of $[n]$ is a *double r-comb* if A, B are $(r+2)$-intertwined and $|A \triangle B| = r+2$, i.e., $A \triangle B$ consists of elements $a_1 < a_2 < \cdots < a_{r+2}$ of $[n]$, one of $A - B$ and $B - A$ is formed by $a_1, a_3, \ldots, a_{r+1}$, and the other by $a_2, a_4, \ldots, a_{r+2}$ (where $A \triangle B$ denotes the symmetric difference $(A - B) \cup (B - A)$).

A result involving a double r-comb and related neighbors is presented in the theorem below. This is in the spirit of Theorem 1.2 (concerning an odd r), but now the situation becomes more intricate.

More precisely, let $r' := r/2 + 1$, where r is even as before. Let $P = \{p_1, \ldots, p_{r'}\}$ and $Q = \{q_1, \ldots, q_{r'}\}$ consist of elements of $[n]$ such that $p_1 < q_1 < \ldots < p_{r'} < q_{r'}$. Define the sets $\mathcal{N}^{\uparrow}(P, Q)$ and $\mathcal{N}^{\downarrow}(P, Q)$ of neighbors of P, Q in the same way as in (1.4) and (1.5) (where now P, Q satisfy $|P| = |Q| = r'$). Let $X \subseteq [n] - (P \cup Q)$ and let $Y \subseteq [n]$ be different from XP and XQ. We assert the following (cf. (5.1)).

Theorem B.1. *Let r, n, r', P, Q, X, Y be as above.*

1. *Suppose that Y and XS are weakly r-separated for any $S \in \mathcal{N}^{\uparrow}(P,Q)$, but Y and XP are not. If, in addition, Y and XP are $(r+2)$-intertwined, then*

(B.1) *Y is equal to $XQ \cup \{a\}$ for some element $a \in [n] - XPQ$ such that $a > p_1$.*

2. *Suppose that Y and XS are weakly r-separated for any $S \in \mathcal{N}^{\downarrow}(P,Q)$, but Y and XQ are not. If, in addition, Y and XQ are $(r+2)$-intertwined, then*

(B.2) *Y is equal to $XP - b$ for some $b \in X$ such that $b > p_1$.*

(One can check that Y as in (B.1) (resp. (B.2)) is indeed weakly r-separated from any member of $\mathcal{N}^{\uparrow}(P,Q)$ (resp. $\mathcal{N}^{\downarrow}(P,Q)$) but not from XP (resp. XQ), and the essence of the theorem is that there is no other $Y \neq XP, XQ$ with such a property.)

Proof. Let us prove assertion 1.

Analyzing the proof of Lemma 5.1, one can realize that it remains valid for corresponding P, Q, X, Y when r is even as well. We further rely on this lemma, borrowing, with a due care, terminology and notation from Section 5. In particular: when sets A, B are not weakly r-separated, the pair $\{A, B\}$ is called *bad*; $\mathcal{I}$ abbreviates the interval cortege $\mathcal{I}(Y, XP)$; the intervals in $\mathcal{I}$ are viewed as $\ldots < A_{i-1} < B_i < A_i < B_{i+1} \ldots$, where $A_{i'}$ ($B_{i'}$) stands for a Y-brick (resp. XP-brick). Also we may assume that $Y \cap X = \emptyset$ (though Y and P need not be disjoint).

Since Y and XP are required be $(r+2)$-intertwined, $\mathcal{I}$ consists of r' Y-bricks and r' XP-bricks. So we may assume that $\mathcal{I}$ is viewed as either

(V1) $B_1 < A_1 < B_2 < A_2 < \ldots < B_{r'} < A_{r'}$, or
(V2) $A_1 < B_2 < A_2 < \ldots < B_{r'} < A_{r'} < B_{r'+1}$.

Next we consider two possible cases.

<u>*Case A*</u>: $(**)$ from Lemma 5.1 is valid. Then $A_i = \{q_i\}$ for each i. Hence $Y - XP = Q$ and $|Y - XP| = r' \le |XP - Y|$. Suppose that (V1) takes place. Then Y-surrounds XP from the right. This contradicts the condition that $\{Y, XP\}$ is bad.

And if (V2) takes place, then $p_1 \notin XP - Y$ (since $p_1 < q_1$ and $\{q_1\} = A_1$). Hence $p_1 \in Y$. For $S := (P - p_1)q_1$, the transformation $XP \mapsto XS$ replaces the first brick $\{q_1\}$ of $\mathcal{I}$ by $\{p_1\}$, forming the first Y-brick of $\mathcal{I}(Y, XS)$. Then $|\mathcal{I}(Y, XS)| = r' + 2$, $|XS| = |XP|$, XS surrounds Y from the right, and therefore the badness of $\{Y, XP\}$ implies that of $\{Y, XS\}$; a contradiction.

<u>*Case B*</u>: $(*)$ from Lemma 5.1 is valid. Then $Y \cap P = \emptyset$, and the fact that $\mathcal{I}$ has exactly r' XP-bricks implies that $X = \emptyset$; so we may ignore X in what follows. If (V2) takes place, then XP surrounds Y from the right. Since each P-brick B_i is a singleton, $|P| = r' \le |Y|$, contradicting the condition that $\{Y, P\}$ is bad.

Now let (V1) take place. Then $B_i = \{p_i\}$ for each i. Suppose that there is $q_i \in Q$ such that $q_i \notin Y$. Then either (a) q_i lies in some Y-brick A_j, or (b) no brick of $\mathcal{I}$ contains q_i. In case (a), we have $i = j$ (in view of $p_i < q_i < p_{i+1}$ and $p_i < A_i < p_{i+1}$, letting $p_{r'+1} := n+1$). Moreover, $\min(A_i) < q_i < \max(A_i)$ (since both $\min(A_i)$ and $\max(A_i)$ are in Y). Taking $S := Pq_i$, we obtain $|\mathcal{I}(Y,S)| = |\mathcal{I}|+2$ (since $P \mapsto S$ replaces A_i by the S-brick $\{q_i\}$ and two Y-bricks). Hence $\{Y,S\}$ is bad; a contradiction.

In case (b), three subcases are possible: either (b1) $p_i < q_i < A_i$; or (b2) $A_i < q_i < p_{i+1}$, or (b3) $i = r'$ and $A_{r'} < q_{r'}$. In these subcases, taking as S the neighbors $(P-p_i)q_i$, $(P-p_{i+1})q_i$, and $Pq_{r'}$, respectively, one can see that $\{Y,S\}$ is bad.

Thus, $Q \subseteq Y$. Note that any Y-brick A_i contains at most one element of Q (for if A_i would contain q_{j-1} and q_j say, then A_i should contain p_j as well, which is impossible). It follows that each A_i contains exactly one element of Q, namely, q_i. Since $\{Y,P\}$ is bad and Y surrounds P from the right, there must be $|Y| > |P| = r'$. So at least one Y-brick A_i has size ≥ 2. For such an A_i, taking $S := Pq_i$, one can see that $|\mathcal{I}(Y,S)| = |\mathcal{I}|$ and that Y surrounds S from the right. Then $|Y| \leq |S|$ (otherwise $\{Y,S\}$ is bad). This together with $|Y| > r'$ and $|S| = |P|+1 = r'+1$ gives $|Y| = r'+1$. The latter means that there is exactly one brick A_i of size ≥ 2; moreover, $|A_i| = 2$. Then $A_i = \{q_i, a\}$, where a is as required in (B.1), yielding assertion 1 of the theorem.

Assertion 2 of the theorem can be shown by symmetry and we leave details to the reader. □

Remark 3. Some neighbors of P,Q arising in connection with Theorem B.1 play an especial role. More precisely, let $Y = XQ \cup \{a\}$ be as in (B.1); then $p_i < a < p_{i+1}$ for some $i \in [r']$ (letting $p_{r'+1} := n+1$). One can check that in the upper neighbor collection $\{S \subset P \cup Q:\ S \neq P,Q,\ r' \leq |S| \leq r'+1\}$ (which includes $\mathcal{N}^{\uparrow}(P,Q)$) there is *exactly one* set S such that $\{Y, XS\}$ is a double r-comb; this is $S = Pq_i$. (Then $XS - Y = \{p_1,\dots,p_{r'}\}$ and $Y - XS = \{q_1,\dots,q_{i-1},a,q_{i+1},\dots,q_{r'}\}$.) Symmetrically, for $Y = XP - b$ as in (B.2), in the lower neighbor collection $\{S \subset P \cup Q:\ S \neq P,Q,\ r'-1 \leq |S| \leq r'\}$ (which includes $\mathcal{N}^{\downarrow}(P,Q)$) there is exactly one S such that $\{Y,XS\}$ is a double r-comb. Namely, if $p_i < b < p_{i+1}$ (letting $p_{r'+1} := n+1$), then $S = Q - q_i$. (In this case, $Y - XS = \{p_1,\dots,p_{r'}\}$ and $XS - Y = \{q_1,\dots,q_{i-1},b,q_{i+1},\dots,q_{r'}\}$.)

The rest of this section is devoted to a geometric construction representing a class of r-separated collections. This relies on Theorem B.1 and is in the spirit of the construction from Sect. 6.4 (with r odd), though looks a bit more intricate. We will use terminology and notation from Sect. 6.

As before, let r be even and $r' = r/2+1$. For $d := r+2$, consider a cubillage Q on the zonotope $Z = Z(n,d)$ and its fragmentation $Q^{\equiv}$. For each cube $C = (X|T) \in Q$, we distinguish two "central" fragments $C^{\equiv}_{r'}$ and $C^{\equiv}_{r'+1}$. They share the middle horizontal section $S_{d/2}(C)$ ($= C \cap H_{|X|+r'}$), which contains the specified vertices $t_C = XP$ and $h_C = XQ$, where $T = (p_1 < q_1 < \ldots < p_{r'} < q_{r'})$, $P = \{p_1,\dots,p_{r'}\}$ and $Q = \{q_1,\dots,q_{r'}\}$ (so $\{t_C,h_C\}$ forms a double r-comb).

Definitions. For a cube $C \in Q$, the set $C^{\equiv}_{r'} \cup C^{\equiv}_{r'+1}$ is called the *doubled fragment*, or the *center*, of C and denoted by $C^{\equiv}_{\rm cup}$; the remaining fragments $C^{\equiv}_{h}$ of C ($h \neq r', r'+1$) are called *ordinary* ones. By the *enlarged fragmentation* of Q we mean the complex generated by the centers and ordinary fragments of all cubes of Q, denoted as $Q^{\equiv}_{\rm en}$, i.e., it is obtained from $Q^{\equiv}$ by merging the pieces $C^{\equiv}_{r'}$ and $C^{\equiv}_{r'+1}$ into one piece $C^{\equiv}_{\rm cup}$ for each cube $C \in Q$. Depending on the context, we may also think of $Q^{\equiv}_{\rm en}$ as the collection of doubled and ordinary fragments over all cubes of Q.

This gives rise to an important subclass of w-membranes. More precisely, when a w-membrane M of Q is a subcomplex (of dimension $d-1$) of $Q^{\equiv}_{\rm en}$, we say that M is an *e-membrane*. It is not difficult to show that a w-membrane M of this sort is characterized by the property that no facet of M is the middle section of a cube of Q, or, equivalently, for each cube $C \in Q$, M contains at most one vertex among t_C, h_C.

Like s- and w-membranes, the set of e-membranes of Q forms a distribute lattice. This is based on the following

Proposition B.2. *The directed graph $\Gamma_{Q^{\equiv}_{\rm en}}$ whose vertices are the fragments in $Q^{\equiv}_{\rm en}$ and whose edges are the pairs (Δ, Δ') of fragments such that Δ immediately precedes Δ' (in the sense that $\Delta^{\epsilon,\rm rear}$ and $(\Delta')^{\epsilon,\rm fr}$ share a facet) is acyclic.* □

Proof. This is similar to the proof of Proposition 6.2 and is briefly as follows. Suppose that fragments $\Delta_0, \Delta_1, \dots, \Delta_k = \Delta_0$ of $Q^{\equiv}_{\rm en}$ form a directed cycle in $\Gamma_{Q^{\equiv}_{\rm en}}$. For each i, let C_i be the cube of Q containing Δ_i. If $C_i = C_{i+1}$, then the height of Δ_{i+1} is greater than that of Δ_i. Therefore, a maximal subsequence S of different cubes among $C_0, C_1, \dots, C_{k-1}$ consists of more than one element. Moreover, consecutive cubes in S share a (vertical) facet, whence S determines a directed cycle in $\Gamma_{n,d}$, contradicting Proposition 6.1. □

Thus, the transitive closure of the above relation on the fragments of $Q^{\equiv}_{\rm en}$ forms a partial order, denoted as $\prec_{\rm en}$. As a consequence (cf. (6.5) and Corollary 6.3):

(B.3) For an e-membrane M of Q, let $Q^{\equiv}_{\rm en}(M)$ be the collection of fragments of $Q^{\equiv}_{\rm en}$ lying between $Z^{\rm fr}$ and M. Then the set $\mathcal{M}^{\rm e}(Q)$ of e-membranes of a cubillage Q on $Z(n,d)$ is a distributive lattice, with the minimal element $Z^{\rm fr}$ and the maximal element $Z^{\rm rear}$, in which for $M, M' \in \mathcal{M}^{\rm e}(Q)$, the e-membranes $M \wedge M'$ and $M \vee M'$ satisfy $Q^{\equiv}_{\rm en}(M \wedge M') = Q^{\equiv}_{\rm en}(M) \cap Q^{\equiv}_{\rm en}(M')$ and $Q^{\equiv}_{\rm en}(M \vee M') = Q^{\equiv}_{\rm en}(M) \cup Q^{\equiv}_{\rm en}(M')$.

(B.4) Let M be an e-membrane of Q. Then there exists a sequence of e-membranes $M_0, M_1, \dots, M_k \in \mathcal{M}^{\rm e}(Q)$ such that $M_0 = Z^{\rm fr}$, $M_k = M$, and for $i = 1, \dots, k$, M_{i-1} is obtained from M_i by the lowering flip using some maximal (w.r.t. $\prec_{\rm en}$) fragment Δ in $Q^{\equiv}_{\rm en}(M_i)$ (in the sense that $\Delta^{\epsilon,\rm rear} \subset M_i$, and M_{i-1} is obtained from M_i by replacing the disk $\Delta^{\epsilon,\rm rear}$ by $\Delta^{\epsilon,\rm fr}$).

Based on the above properties, we obtain a geometric result which can be viewed, to some extent, as a counterpart of Theorem 6.4 (concerning the odd case).

Theorem B.3. *Let r be even and $d := r + 2$. Suppose that a cubillage Q on $Z = Z(n, d)$ possesses the property that*

(P) *no e-membrane of Q has a pair of vertices forming a double r-comb.*

Then for any e-membrane M of Q,

(i) *the set $V(M)$ of vertices of M (regarded as subsets of $[n]$) is weakly r-separated;*
(ii) $|V(M)| = s_{n,r}$.

Proof. We argue as in the proof of Theorem 6.4. For $M \in \mathcal{M}^{\rm e}(Q)$, consider a sequence $Z^{\rm fr} = M_0, M_1, \ldots, M_k = M$ of e-membranes of Q as in (B.4). Since $Z^{\rm fr}$ satisfies (i), (ii), it suffices to prove the following assertion.

(B.5) For $M, M' \in \mathcal{M}^{\rm e}(Q)$, let M' be obtained from M by the raising flip using a (doubled or ordinary) fragment Δ of $Q^{\equiv}_{\rm en}$, and suppose that M satisfies (i), (ii). Then M' satisfies (i), (ii) as well.

To show this, assume that Δ belongs to a cube $C = (X|T) \in Q$. When Δ is ordinary, i.e., $\Delta = C^{\equiv}_{h}$ with $h \le r' - 1$ or $h \ge r' + 2$ (where $r' = r/2 + 1$), then $V(M') = V(M)$, and we are done (cf. the explanations in Cases 1 and 2 of the proof of Theorem 6.4).

So let Δ be the center $C^{\equiv}_{\rm cup}$ of C and let $T = (p_1 < q_1 < \ldots < p_{r'} < q_{r'})$. The raising flip using Δ replaces in M the side $\Delta^{\epsilon,{\rm fr}}$ by $\Delta^{\epsilon,{\rm rear}}$. One can see that the vertex t_C of C is in $\Delta^{\epsilon,{\rm fr}}$ by not in $\Delta^{\epsilon,{\rm rear}}$, while h_C is in $\Delta^{\epsilon,{\rm rear}}$ by not in $\Delta^{\epsilon,{\rm fr}}$, and that the other vertices of Δ and Δ' coincide. Therefore, the flip replaces $t_C = XP$ by $h_C = XQ$, yielding $V(M') = (V(M) - \{t_C\}) \cup \{h_C\}$, where $P = \{p_1, \ldots, p_{r'}\}$ and $Q = \{q_1, \ldots, q_{r'}\}$.

We have $|V(M')| = |V(M)|$; so M' satisfies (ii). Suppose, for a contradiction, that (i) is false for M', i.e., there are two vertices of M' that are not weakly r-separated from each other. Then one of them is XQ, and the other, Y say, belongs to M and differs from XP. By (i) for M, the vertex Y is weakly r-separated from XS for each neighbor $S \in \mathcal{N}^{\downarrow}(P, Q)$. (Note that XS lies in $\Delta^{\epsilon,{\rm fr}}$ even if $|S| = r'$.) So we can apply assertion 2 of Theorem B.1 and conclude that Y is viewed as in (B.2). But then, as mentioned in Remark 3, there is $S \in \mathcal{N}^{\downarrow}(P, Q)$ such that $\{Y, XS\}$ is a double r-comb; namely, $Y = XP - b$ and $S = Q - q_i$, where $p_i < b < p_{i+1}$. Hence M contains a double r-comb, contrary to condition (P).

Thus, (B.5) is valid, and the theorem follows. □

Remark 4. By the construction of an e-membrane M of a cubillage Q, M has no double r-comb of the form $\{t_C, h_C\}$ for a cube C of Q. However, a priori it is not clear whether

M is free of double r-combs at all. We conjecture that this is so for any e-membrane, i.e., property (P) holds for any cubillage Q. Its validity would give a strengthening of Theorem B.3. We state it as follows:

Conjecture 2. For r even, the vertex set $V(M)$ of any e-membrane M of an arbitrary cubillage Q on $Z(n, r+2)$ gives a weakly r-separated collection.

(Note that such a $V(M)$ is automatically of size $s_{n,r}$, by explanations above.) It is tempting to conjecture a sharper property (which is just a direct analog of Theorem 6.4), by claiming that $V(M)$ is weakly r-separated for any *w-membrane* M of a cubillage Q on $Z(n, r+2)$ (where $|V(M)|$ may exceed $s_{n,r}$), but we do not go so far at the moment.

We finish the paper with a counterpart of Conjecture 1 from Sect. 6.4:

Conjecture 3. For r even, the maximal size of a weakly r-separated collection $W \subseteq 2^{[n]}$ without double r-combs is equal to $s_{n,r}$ and such a W with $|W| = s_{n,r}$ is representable, in the sense that there exists a cubillage Q on $Z(n, r+2)$ and an e-membrane M of Q such that $V(M) = W$.

References

[1] V.I. Danilov, A.V. Karzanov, On universal quadratic identities for minors of quantum matrices, J. Algebra 488 (2017) 145–200.

[2] V.I. Danilov, A.V. Karzanov, G.A. Koshevoy, On maximal weakly separated set-systems, J. Algebraic Comb. 32 (2010) 497–531.

[3] V.I. Danilov, A.V. Karzanov, G.A. Koshevoy, Combined tilings and the purity phenomenon on separated set-systems, Sel. Math. New Ser. 23 (2017) 1175–1203.

[4] V.I. Danilov, A.V. Karzanov, G.A. Koshevoy, On interrelations between strongly, weakly and chord separated set-systems (a geometric approach), arXiv:1805.09595 [math.CO], 2018.

[5] V.I. Danilov, A.V. Karzanov, G.A. Koshevoy, Cubillages of cyclic zonotopes, Usp. Mat. Nauk 74 (6) (2019) 55–118 (in Russian), English translation in Russ. Math. Surv. 74 (6) (2019) 1013–1074.

[6] P. Galashin, Plabic graphs and zonotopal tilings, Proc. Lond. Math. Soc. 117 (4) (2018) 661–681.

[7] P. Galashin, A. Postnikov, Purity and separation for oriented matroids, arXiv:1708.01329 [math.CO], 2017.

[8] I.M. Gel'fand, V.V. Serganova, Combinatorial geometries and torus strata on homogeneous compact manifolds, Usp. Mat. Nauk 42 (2) (1987) 107–134 (in Russian), English translation in Russ. Math. Surv. 42 (2) (1987) 133–168.

[9] M.M. Kapranov, V.A. Voevodsky, Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher Bruhat orders, Cah. Topol. Géom. Différ. Catég. 32 (1) (1991) 11–28.

[10] B. Leclerc, A. Zelevinsky, Quasicommuting families of quantum Plücker coordinates, Amer. Math. Soc. Trans., Ser. 2 181 (1998) 85–108.

[11] Yu. Manin, V. Schechtman, Arrangements of hyperplanes, higher braid groups and higher Bruhat orders, in: Algebraic Number Theory – in Honour of K. Iwasawa, in: Advance Studies in Pure Math., vol. 17, Academic Press, NY, 1989, pp. 289–308.

[12] G. Ziegler, Higher Bruhat orders and cyclic hyperplane arrangements, Topology 32 (1993) 259–279.