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INFORMATICS

On the Set of Stable Matchings in a Bipartite Graph

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Abstract—The topic of stable matchings (marriages) in bipartite graphs gained popularity beginning from the appearance of the classical Gale and Shapley work. In this paper, a detailed review of selected and other related statements in this field that describe structured, polyhedral, and algorithmic properties of such objects and their sets accompanied by short proofs is given.

Keywords: stable matching, poset of rotations, stable matching of minimum cost, median stable matching

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1. INTRODUCTION

Beginning from the classical work by Gale and Shapley [1], problems on stable matchings attracted the attention of numerous researchers, and a lot of interesting results, both theoretical and practical, have been obtained.

In the stable marriage problem (or, more precisely, on a stable matching in a bipartite graph), a bipartite graph $G = (V, E)$ is considered, in which, for every vertex v, a linear order $<_v$ is given, which determines preferences on the set of edges $\delta_G(v)$ incident to v. Let I and J be the vertex parts in G. In [1] and in other works, the vertices in I and J were interpreted as persons of different sexes (males and females, respectively), and the edges were interpreted as possible unions (marriages) between them: if, for two vertices $m \in I$ (man) and its incident edges $e = mw$ and $e' = mw'$, it holds that $e <_m e'$, then this means that m prefers the union with w to the union with w'. The interpretation for $w \in J$ (woman) with respect to the order $<_w$ is similar.

A *matching* in G is a subset of edges $M \subseteq E$ in which no two edges have a common vertex; thus, we may assume that M describes a set of monogamous marriages. The matching M is called *stable* if, for any edge $e = mw \in E - M$, at least one vertex m or w has an incident edge in M (which determines the selected partner) and this edge is preferred to e for this vertex. A stable matching exists in any bipartite graph $G = (V, E)$ with arbitrary total orders $<_v$ on $\delta_G(v)$, $v \in V$, which was originally proved by Gale and Shapley [1] for complete bipartite graphs with parts of equal size and later generalized by other researchers to arbitrary bipartite graphs.

Later, the development in the field of stability concerning graphs went in several directions. In one of them, the concept of stability was extended to matchings in arbitrary graphs (here, the fundamental works on stable roommates were published by Irving [2] and Tan [3]). In another direction, stability issues were studied when considering variants of weight functions on the edges and vertices of a graph (we note, as one of the most general, the stable allocation problem, which was introduced and studied by Baïou and Balinski [4]).

A number of deep results were obtained within the framework of the theory of stable matchings for bipartite graphs, and the purpose of this paper is to review selected and the most significant, in the author's opinion, known results in this theory. They are to a large extent interconnected and cover combinatorial, polyhedral, algorithmic, and other aspects. A number of problems covered in this review describe structural properties of the set $\mathcal{M}(G)$ of stable matchings in $(G, <)$.

The paper is organized as follows.

Section 2 contains a description of the basic results and constructions. It begins with a reminder of the classical results going back to Gale and Shapley's work on the existence of a stable matching and the deferred acceptance method for constructing it. Then, it is explained that in the set $\mathcal{M}(G)$ of stable matchings of an arbitrary bipartite graph G all matchings cover the same set of vertices (Proposition 2), and operations on pairs of stable matchings are described that make it possible to define a partial order $\prec$ on the set $\mathcal{M}(G)$ that represents it as a distributive lattice (Proposition 3).

Section 3 is devoted to a discussion of the key concept of rotation in stable matching. It was introduced by Irving [2] in connection with the stable roommates problem on an arbitrary graph, but it turned out to be very useful for studying the structure of the set of stable matchings in a bipartite graph G. By rotation in the graph G with a stable matching M, we mean a cycle in G alternating with respect to M and its complement. Among the main well-known results discussed in this section are the following: (a) a transformation along a rotation transforms a stable matching into a stable one (Proposition 4); (b) rotation transformations correspond to immediate precedence relations in the lattice $(\mathcal{M}(G),\prec)$ (Proposition 6); (c) in all maximal chains of the lattice $(\mathcal{M}(G),\prec)$, the set of rotations denoted by $(\mathcal{R}_G)$ is the same, (Proposition 7).

The discussion of issues related to rotations is continued in Section 4. Here a partial order $\lessdot$ on the set $\mathcal{R}_G$ is defined, and an alternative representation of the set of stable matchings $\mathcal{M}(G)$ found by Irving and Leather [5] is described; namely, the stable matchings in G are in one-to-one correspondence with the ideals (or antichains) of the poset $(\mathcal{R}_G,\lessdot)$ (Proposition 8).

Section 5 considers a stronger, "cost", version of the stable matching problem. Here, the set of edges $G=(V,E,<)$ of the bipartite graph is equipped with a real weight function $c: E \to \mathbb{R}$, and it is required to find a stable matching M that minimizes the total weight $\sum(c(e)\colon e\in E)$. In the special case when the weight $c(e)$ of the edge $e=mw$ is equal to the sum of its ranks in the orders $<_m$ and $<_w$, we obtain the problem of egalitarian stable matching (in which the parts I and J are "equalized") formulated in [6]. For arbitrary weights c, the minimization (or maximization) problem is solved by an efficient combinatorial algorithm developed in [7]. This is a consequence of the following facts: firstly, stable matchings are represented by ideals of the rotation poset $(\mathcal{R}_G,\lessdot)$, in which the number of elements $|\mathcal{R}_G|$ does not exceed the number of edges $|E|$; secondly, the weight of a stable matching is expressed, up to a constant, by the weight of the corresponding ideal (with appropriate rotation weights assigned); and thirdly, the problem of finding the ideal (or "closed set") of minimum weight in an arbitrary finite poset reduces to the problem of the minimum cut of a directed graph, as shown by Picard [8].

Section 6 is devoted to polyhedral aspects. Here, the polyhedron of stable matchings $\mathcal{P}_{\rm st}(G)$ is described in terms of a system of linear inequalities (close to the one proposed by Vande Vate [9]). Also, following the polyhedral construction of Teo and Sethuraman [10], it is shown that, for an arbitrary set of stable matchings $M_1,\dots,M_\ell$ and any $k\le\ell$, we again obtain a stable matching by choosing in these matchings for each covered vertex in the part I an edge that has the order k. In particular, when ℓ is odd and $k=(\ell+1)/2$, the so-called median stable matching is determined for the given $M_1,\dots,M_\ell$.

In the final Section 7, the result on the intractability ($\#P$-completeness) of the problem of determining the number of stable matchings in a bipartite graph obtained in [7] is formulated. This answers Knuth's question [11] about the algorithmic complexity of computing such a number.

For the completeness of our description, we, as a rule, accompany the presented statements with proofs, often alternative to and shorter than the original proofs in the authors' works. Note that wherever we refer to [5], [7] and some others, the corresponding results in these papers were obtained for the case of a complete bipartite graph $K_{n,n}$ (with arbitrary orders $<$), but these results are also valid for an arbitrary bipartite graph G.

2. DEFINITIONS AND BASIC FACTS

Throughout this paper, we consider a bipartite graph $G=(V,E)$ with a partition of the set of vertices V into subsets I and J (where each edge connects vertices from different subsets). Sometimes we will consider the graph G to be directed with edges from I to J. The edge connecting vertices $m\in I$ (man) and $w\in J$ (woman) is denoted by mw. The set of edges incident to a vertex $v\in V$ is denoted by $\delta(v)=\delta_G(v)$. Each vertex v is equipped with a total order $<_v$ that defines preferences on the set $\delta(v)$. Namely, if $e<_v e'$ for edges $e,e'\in\delta(v)$, then the edge e is considered to be preferred to the edge e' for v; sometimes it will also be convenient to say that e is located earlier than e' or to the left of it (and e' is located later than e or to the right of it). If v is clear from the context then we may write $e < e'$. We usually include the order $<$ and the partition of V into the parts I, J in the graph description using the notation $G = (V, E, <)$ or $G = (I \sqcup J, E, <)$.

A matching M in G is said to be *stable* if it does not have blocking edges. An edge $e = mw \in E - M$ is said to be blocking for M if the vertex m is either not covered by M or there is an edge e' in M this incident to m and is less preferred: $e <_m e'$; the same holds for the vertex w.

Below in this section, we present two groups of basic properties of stable matchings in G.

I. The theory of stable matchings is based on the efficient algorithm for constructing a stable matching that was proposed by Gale and Shapley [1] (for the case of a complete bipartite graph with parts I, J of the same size n: $G \simeq K_{n,n}$). A recursive version of this algorithm, which is often called the deferred acceptance algorithm (DAA) was proposed in the work [6] by McVitie and Wilson (in this paper, an algorithm for successive construction of the set of all stable matchings for $G \simeq K_{n,n}$ is also described). A short nonconstructive proof of the existence of a stable matching in the bipartite case can be found in [12], Section 18.5g.

The roles of the parts I and J in the DAA are different: one part makes proposals, and the other part accepts or rejects the proposals. (Recall that at each step, the DAA arbitrarily selects a man $m \in I$ not included in the current marriage that proposes to a woman $w \in J$ according to the best edge mw from the set $\delta(m)$ not yet used by this man; this proposal is immediately rejected if w already has a better proposal from $m' \in I$ (i.e., if $m'w <_w mw$), and it is accepted otherwise while rejecting the previous proposal if any.) The results are naturally generalized to the case of an arbitrary bipartite graph G. This leads to the following important property.

Proposition 1 (see [1, 6]). *In the case when I makes proposals and J accepts or rejects them, the DAA finds (in linear time $O(|V| + |E|)$) the canonical stable matching, in which the selection of edges for the vertices in I is the best one, and for the vertices in J it is the worst one over all stable matchings.*

We denote this matching by $M^{\min} = M^{\min}(G)$. By symmetry, if the DAA is used in the version when J makes proposals, then the algorithm builds a stable matching for which the selection of edges is the best for J and the worst for I; we denote this matching by $M^{\max} = M^{\max}(G)$. To clarify the meaning of the terms *best–worst,* consider any other stable matching L in G. If the edges $e \in M^{\min}$ and $e' \in L$ have a common vertex $m \in I$, then $e \le_m e'$; and if they have a common vertex $w \in J$, then $e' \le_w e$. Here we rely on the invariance of the set of vertices covered by the stable matching (which is trivial in the case of a complete bipartite graph $K_{n,n}$, in which the stable matching is perfect, i.e., covers all the vertices). More precisely, the following property proved in [13] holds.

Proposition 2. *In any bipartite graph $G = (V, E, <)$, all stable matchings cover the same set of vertices.*

To prove this property, we will use the following definitions and notation, which also will be useful below.

A *path* in a directed graph is a sequence $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$, where e_i is the edge connecting the vertices v_{i-1} and v_i. For the path P, we may also use the short notation in terms of the vertices $v_0 v_1 \cdots v_k$ and in terms of the edges $e_1 e_2 \cdots e_k$. An edge e_i in P is called *forward* if $e_i = v_{i-1}v_i$ and *backward* if $e_i = v_i v_{i-1}$. (We denote the edge outgoing from u and incoming to v by uv instead of (u, v).) A path is said to be *directed* if all its edges are forward, and it is called *simple* if all its vertices are different. A path from a vertex u to a vertex v may be called a *u–v path*. For a subset of edges $A \subseteq E$, the set of vertices covered by A is denoted by V_A. For subsets X, Y of the set S, we write $X - Y$ instead of $X \backslash Y$ $(= \{s \in X : s \notin Y\})$ and denote by $X \Delta Y$ the symmetric difference $(X - Y) \cup (Y - X)$.

Let M, L be two stable matchings in the bipartite graph G. Consider the subgraph $\Delta_{M,L}$ in G induced by the set of edges $M \Delta L$. Note that

$$\text{if } e, e', e'' \text{ are different edges in } \Delta_{M,L} \text{ such that } e, e' \text{ have a common vertex } u \text{ and } e', e'' \text{ have a common vertex } v, u \text{ if } e >_u e', \text{ then } e' >_v e''. \tag{2.1}$$

Indeed, let, for definiteness, $e' \in M$. Then $e, e'' \in L$. If the inequality $e' <_v e''$ were true, then the edge e' would be blocking for L.

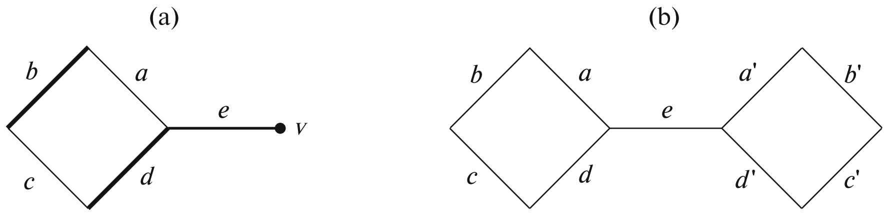

Fig. 1. Two examples.

Now, proceed to the proof of Proposition 2. Assume that $V_M \neq V_L$. Then, at least one of the components in $\Delta_{M,L}$ is a simple path $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$. In this path, the edges from M and L alternate, and the stability of M, L implies that $k \geq 2$. Without loss of generality, we assume that $e_1 \in M$. Then $v_0 \notin V_L$ and, since the edge e_1 is not a blocking one for L, then the inequality $e_1 >_{v_1} e_2$ must hold.

Therefore, by successively applying (2.1) to the triples of adjacent edges in P, we conclude that $e_{k-1} >_{v_{k-1}} e_k$. Then, if k is odd, then the edge e_k is a blocking one for L (since $e_{k-1} \in L$, $e_k \notin L$, and $v_k \notin V_L$); and if k is even, then the edge e_k is a blocking one for M (since $e_{k-1} \in M$, $e_k \notin M$, and $v_k \notin V_M$); this is a contradiction.

We denote the set of vertices in G covered by a stable matching by $\tilde{V}$. It is clear that when vertices in $V - \tilde{V}$ are removed, each stable matching in G remains stable for the resulting graph $\tilde{G}$. However, new stable matchings can appear in $\tilde{G}$ (and, therefore, the inclusion $\mathcal{M}(G) \subseteq \mathcal{M}(\tilde{G})$ can be strict), as is seen from the simple example in the fragment panel in Fig. 1. Here, the preference relation is as follows: $a < b$, $b < c$, $c < d$, and $d < e < a$; and it can be verified that there is only one stable matching, namely, $M = \{b, d\}$. Then, the vertex v is not covered by M; however, if it is removed (together with the edge e), then the new stable matching $\{a, c\}$ emerges.

Thus, when we investigate the set of stable matchings for G, we generally cannot exclude uncovered vertices from consideration and leave only the subgraph $\tilde{G} = (\tilde{V} = (\tilde{I} \sqcup \tilde{J}), \tilde{E})$ (where the parts $\tilde{I} := I \cap \tilde{V}$ and $\tilde{J} := J \cap \tilde{V}$ have the same size and all stable matchings are perfect). Due to similar reasons, if there are no uncovered vertices, the removal of an edge not included in any stable matching can entail the emergence of a new stable matching. The right fragment in Fig. 1 shows an extension of the previous graph with similar preference relation for the new edges, i.e., $a' < b'$, $b' < c'$, $c' < d'$, and $d' < e < a'$. Here, there are three stable matchings $\{a, c, b', d'\}$, $\{b, d, a', c'\}$, and $\{b, d, b', d'\}$. However, if we remove the uncovered edge e, the forth stable matching $\{a, c, a', c'\}$ emerges, for which there earlier was the blocking edge e.)

II. Next, consider an ordered pair (M, L) of stable matchings in G. Proposition 2 implies that the graph $\Delta_{M,L}$ induced by$M \Delta L$ decomposes into a set $\mathcal{C} = \mathcal{C}(M, L)$ of nonintersecting cycles. Assuming that G is directed from I to J, we also assume that each cycle $C = (v_0, e_1, v_1, \ldots, e_k, v_k = v_0) \in \mathcal{C}$ is directed in accordance with the direction of the edges in L, i.e. the forward edges in C belong to L, and the backward edges belong to M. Due to (2.1), all preferences along C "have the same direction;" more precisely,

$$\begin{gathered}\text{for a cycle } C = e_1 e_2 \cdots e_k \in \mathcal{C}(M, L) \text{ (using the notation } C \text{ in terms of edges),}\\ \text{if } e_i < e_{i+1} \text{ holds for some } i, \text{ then it holds for}\\ i = 1, \ldots, k \text{ (letting } e_{k+1} := e_1).\end{gathered} \tag{2.2}$$

In this case, we say that C is an increasing (or right) cycle with respect to M. Otherwise, C is called a decreasing (or left) cycle. Denote by $\mathcal{C}^+(M, L)$ and $\mathcal{C}^-(M, L)$ the sets of right and left cycles for (M, L), respectively. If the matching M' is obtained from M by replacing its edges along the cycle C, then we write $M \xrightarrow{C} M'$ or $M' = \mathrm{Repl}(M, C)$; similarly, we write $L \xrightarrow{C} L'$ or $L' = \mathrm{Repl}(L, C)$ for the replacement of the edges in L along C. Note that if C is a right cycle with respect to M, then the matching M' is less preferred than M for all vertices in $I \cap C$ (men) and more preferred for all vertices in $J \cap C$ (women); for

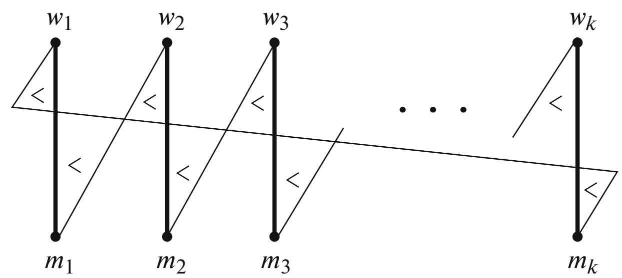

Fig. 2. Increasing cycle for (M, L). The matching M is shown in bold and L in thin lines. The vertices m_i belong to the part I.

L and L', the behavior is opposite. (In other words, when making a replacement along an increasing cycle, we sort of move away from $M^{\min}$ and approach $M^{\max}$.) An increasing cycle is shown in Fig. 2.

We could expect that the matching $M' = \mathrm{Repl}(M, C)$ is stable. However, this is not always the case. For example, consider the graph shown in the right fragment of Fig. 1 (with the preferences indicated above) and consider the stable matchings $M = \{b, d, a', c'\}$ and $L = \{a, c, b', d'\}$. Then, the edges a, b, c, d form a cycle C in $\Delta_{M,L}$; however, the matching $M' = \{a, c, a', c'\}$ obtained by replacing the edges in M along C is not stable.

Nevertheless, the stability is preserved if the replacement is simultaneously made in all cycles in $\mathscr{C}^+(M, L)$ or in $\mathscr{C}^-(M, L)$. In these cases, we write $M' = \mathrm{Repl}(M, \mathscr{C}^+(M, L))$ and $M' = \mathrm{Repl}(M, \mathscr{C}^-(M, L))$, respectively.

Lemma 1 (see [11]). *If M, L are stable matchings, then $M' = \mathrm{Repl}(M, \mathscr{C}^+(M, L))$ is also a stable matching. The same holds for* $\mathrm{Repl}(M, \mathscr{C}^-(M, L))$.

Proof. It is clear that M' is a matching. Assume that M' includes a blocking edge $e = mw$ (where $m \in I$). It is easy to verify that this is possible only if one of the vertices m, w belongs to a right cycle and the other one belongs to a left cycle in $\mathscr{C}(M, L)$. Without loss of generality, we may assume that m belongs to a cycle $C \in \mathscr{C}^+(M, L)$, and w belongs to a cycle $C' \in \mathscr{C}^-(M, L)$. Let the vertex m be incident to edges $a \in M$ and $b \in L$ in C, and w be incident to the edges $c \in M$ and $d \in L$ in C'. Then, $a <_m b$ and $c <_w d$ (taking into account that $m \in I$ and $w \in J$). Under the transformation $M \mapsto M'$, the matching M' includes the edges b and c, and it should hold that $e <_m b$ and $e <_w c$ (since e blocks M' by assumption). However, then we obtain $e < d$ (due to $c < d$), and, therefore, e blocks L, which is a contradiction.

This lemma enables us to define a lattice on the stable matchings in G. Note that, for $M, L \in \mathscr{M}(G)$, the replacement in M of the edges along all increasing cycles for M is equivalent to the replacement in L of the edges along all increasing cycles for L, i.e., $\mathrm{Repl}(M, \mathscr{C}^+(M, L)) = \mathrm{Repl}(L, \mathscr{C}^+(L, M))$. Similarly, $\mathrm{Repl}(M, \mathscr{C}^-(M, L)) = \mathrm{Repl}(L, \mathscr{C}^-(L, M))$. Due to Proposition 1, $\mathscr{C}(M, L)$ does not contain decreasing cycles in the case $M = M^{\min}$ and does not contain increasing cycles in the case $M = M^{\max}$.

Proposition 3 (see [11]). *For different $M, L \in \mathscr{M}(G)$, set $M \prec L$ if $e \le e'$ for each pair of edges $e \in M$ and $e' \in L$ that are incident to a vertex in $\tilde{I}$ ($= I \cap \tilde{V}$) (and set the converse relation for the vertices in $\tilde{J}$). Then $\prec$ determines a distributive lattice on $\mathscr{M}(G)$ with the minimal element $M^{\min}(G)$ and the maximal element $M^{\max}(G)$, in which the meet $M \wedge L$ for $M, L \in \mathscr{M}(G)$ is* $\mathrm{Repl}(M, \mathscr{C}^-(M, L)) = \mathrm{Repl}(L, \mathscr{C}^-(L, M))$*, and the join $M \vee L$ is* $\mathrm{Repl}(M, \mathscr{C}(M, L)) = \mathrm{Repl}(L, \mathscr{C}^+(L, M))$.

3. ROTATIONS

Since the set $\mathscr{M}(G)$ is a lattice, we can, using two or more stable matchings, construct new matchings by making reassignments in the corresponding cycles for pairs of matchings as indicated in Lemma 1.

Another method related to the concept of rotation makes it possible to generate new stable matchings using single elements in $\mathcal{M}(G)$. This concept was introduced by Irving and Leather in [5] for the stable marriage problem on the basis of the concept all-or-nothing cycle from Irving's paper [2] devoted to the stable roommates problem. Later, the concept of rotation was extended to other stability problems, such as the stable b-matching problem, allocations, and others (e.g., see [14]. We also point out the recent paper [15], in which a relationship between the convex hull of stable allocations in the absence of constraints on the edges and the order polytope of the poset of rotations is established). Note that even though the case of the complete bipartite graph $G \simeq K_{n,n}$ was considered in [5], the constructs and results are fairly easily extended to the arbitrary case; hence we will consider, as above, an arbitrary bipartite graph $G = (V = I \sqcup J, E, <)$.

Let M be a stable matching in G.

Definition 1. The edge $a = mw \in E - M$ (where $m \in I$) is said to be feasible if M contains an edge e incident to m that is preferred to a (i.e., $e <_m a$) and the vertex w is either uncovered or M includes an edge e' incident to w and this edge is less preferred than a (i.e. $a <_w e'$). If the set of feasible edges $\delta(m)$ for the vertex $m \in I$ is nonempty, then the first of them (the most preferred) is called *active*. We denote the set of active edges for M by A. The subgraph of G induced by the set of edges $M \cup A$ is denoted by $\Gamma = \Gamma(M)$ and is called the *active graph*.

Taking into account the fact that not more than one active edge outgoes from any vertex in I (while several active edges can be incoming into a vertex in J), any cycle in Γ must be alternating with respect to M and A. Moreover, each component K in Γ is either a tree or contains exactly one cycle. (Indeed, otherwise K would contain a pair of cycles and a simple u–v path P (all vertices in which, except for u, v do not belong to cycles). The end edges of this path must belong to A and at least on of the vertices u, v must belong to I. This vertex has two incident edges in A (one on the path P and the other on the cycle), which is a contradiction. If there is a cycle, we call the component K *cycle-containing*, the cycle itself we denote by C_K and call it a *rotation* for M. (Note that in [5] the rotation is meant to be not the cycle itself but rather its intersection with M.) The edges in C_K belonging to M are called *matching edges*, and the other edges are said to be *active*. When edges are replaced along the rotation C, there appears a matching that is less preferred for the vertices in $I \cap C$ compared with M and is more preferred to the vertices in $J \cap C$. (For the replacement operation along C, the term *rotation elimination* is used in [5].) If C is a cycle in $\Gamma(M)$, we may also say that the matching M admits the rotation C.

Example. Figure 3 shows three stable matchings M_1, M_2, M_3 (drawn in bold) in the graph $G = (V, E, <)$ as in Fig. 1b. Here, the part I consists of the vertices 1, 2, 3, 4, the part J, of the vertices x, y, v, w, and the preferences are defined as before, namely,

$$a <_y b, \quad b <_1 c, \quad c <_x d, \quad d <_2 e <_2 a,$$

$$a' <_3 b', \quad b' <_w c', \quad c' <_4 d', \quad d' <_v e <_v a'.$$

For the matching M_1, all nonmatching edges a, c, e, b', d' are feasible; however, only c, e, b', d' of them are active (since $e <_2 a$). Therefore, the subgraph $\Gamma(M_1)$ is generated by $M_1 \cup \{c, e, b', d'\}$ and has exactly one component, which is cycle-containing with the cycle $C = a'b'c'd'$. The replacement along the rotation C yields the matching M_2 (which is stable according to Proposition 4 below). The feasible edges for M_2 in $E - M_2$ are only a and c (while e, a', c' are infeasible, because $d' <_v e, a'$ and $b' <_w c'$, taking into account that $b', d' \in M_2$ and $v, w \in J$). Therefore, the active edges are a and c, and the graph $\Gamma(M_2)$ consists of three components generated by $\{a, b, c, d\}$, $\{b'\}$, and $\{d'\}$. The first of them forms the cycle $D = abcd$. Finally, the replacement along the rotation D gives the stable matching M_3. None of the edges in $E - M_3$ is feasible for it, and $\Gamma(M_3)$ is formed simply by the edges in M_3. It is seen that $M_1 = M^{\min}$ and $M_3 = M^{\max}$.

The following two propositions demonstrate important properties of $\Gamma(M)$.

Proposition 4 (see [5])**.** *For any cycle (rotation) C in $\Gamma(M)$, the matching $M' = \mathrm{Repl}(M, C)$ is stable.*

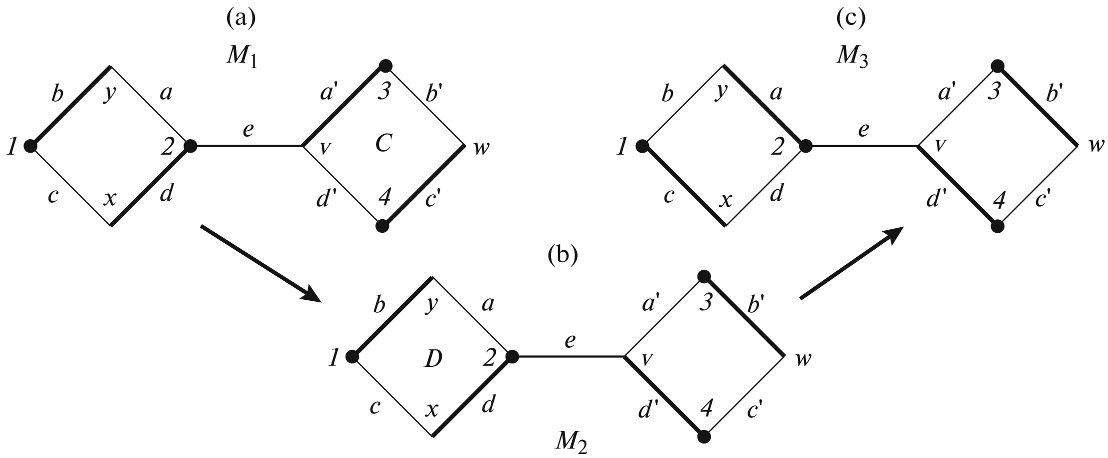

Fig. 3. Graph with three stable matchings shown in bold: $M_1 = M^{\min}$ (a), M_2 (b), and $M_3 = M^{\max}$ (c).

Proof. Both M and M' cover the same set of vertices $\tilde{V}$. Suppose (for a contradiction) that M' admits a blocking edge $e = mw \in E - M'$. Note that $e \notin M$ (taking into account that the edges in $C \cap M$ cannot be blocking for M'). Two cases are possible.

Case 1: $w \in \tilde{V}$. Then, the vertex w is incident to the edges $a \in M'$ and $b \in M$, and we have $e <_w a$ (due to blocking) and $a \leq_w b$ (since in the case $a \neq b$, the edge a lies in C and is feasible). Therefore, $e <_w b$. Then, $m \in \tilde{V}$ (otherwise, e would be blocking for M).

Let the vertex m be incident to the edges $c \in M'$ and $d \in M$. Then, $e <_m c$, and it should be $d <_m e$ (otherwise, e would be blocking for M, taking into account that $e <_w b$). Therefore, e is feasible for M and is preferred to the edge c, which belongs to A; this is a contradiction.

Case 2: $w \notin \tilde{V}$. Then $m \in \tilde{V}$. Let m be incident to the edges $c \in M'$ and $d \in M$. Since e is a blocking edge for M' but not for M, we have $d <_m e <_m c$. But then e is a feasible edge and preferred to the active edge c, which is a contradiction.

Speaking about alternating (with respect to M and A) paths and cycles in $\Gamma(M)$ below, we assume that the direction in them is chosen so that the edges in A be forward and the edges in M be backward.

Proposition 5. *Let e be an edge in M belonging to a tree component K in $\Gamma(M)$. Then, for any stable matching $L \in \mathcal{M}(G)$, either $e \in L$ or e belongs to a decreasing cycle in $\mathcal{C}(M, L)$.*

Proof. Suppose (for a contradiction) that the edge $e = mw \in M$ in the *tree component* K belongs to an increasing cycle $C \in \mathcal{C}^+(M, L)$ for some $L \in \mathcal{M}(G)$. Let C have the form $e_1 e_2 \cdots e_k$ (using the notation in terms of edges and taking into account the direction of C), and let $e := e_1 \in M$. Take a maximal alternating path $P = p_1 p_2 \cdots p_r$ (in terms of edges) in $\Gamma(M)$ that begins with $p_1 = e$. We may assume that none of the edges in $P \cap M$, except for p_1, belong to the increasing cycle $\mathcal{C}(M, L)$ (otherwise, we take it as the edge e). The edges e_1 and $e_2 = mw'$ are incident to the vertex $m \in I$ and, since C is an increasing cycle, we have $e_1 <_m e_2$ and $e_2 <_{w'} e_3$. In addition, $e_1, e_3 \in M$ and $e_2 \in L$. Therefore, e_2 is a feasible edge for M, and $\delta(m)$ contains an active edge $a = mv$, for which $e_1 <_m a \leq_m e_2$. This a coincides with the edge p_2 in P. In addition, $a \neq e_2$ (otherwise, $e_3 \in P$ and $e_3 \in C \cap M$ in contrast to the condition on $p_1 = e$).

We claim that $a = p_2$ is a blocking edge for L. This follows from the fact that $a <_m e_2 \in L$ if $r = 2$ (in this case, the end vertex v in P is uncovered), and if $r \geq 3$ and $p_3 \in L$ (since $p_3 \in M$ and $a <_v p_3$). Finally, let $p_3 \notin L$. Then p_3 belongs to the decreasing cycle $C' \in \mathcal{C}^-(M, L)$. For the edge b in C' that is incident to v and different from p_3, it holds that $b \in L$ and $p_3 <_v b$ (taking into account that $v \in J$ and the fact that C' is a decreasing cycle). Then, $a <_v p_3 <_v b$ and $a <_m e_2 \in L$ imply that a blocks L, which is a contradiction.

It is clear that the matching $M' = \mathrm{Repl}(M,C)$ obtained by the replacement along the rotation C in $\Gamma(M)$ satisfies the relation $M \prec M'$, where $\prec$ is a partial order in the lattice $\mathcal{M}(G)$ (see Proposition 3). It was proved in [5] (for $G \simeq K_{n,n}$) that, beginning with $M^{\min}$ and using rotations, all $\mathcal{M}(G)$ can be generated.

Definition 2. A *route* is defined as a sequence of stable matchings $\tau = (M_0, M_1, \dots, M_N)$ in G such that $M_0 = M^{\min}$, $M_N = M^{\max}$, and each M_i $(1 \le i \le N)$ is obtained from M_{i-1} by a replacement along one rotation C_i in $\Gamma(M_{i-1})$. The sequence of rotations $C_1, \dots, C_N$ is denoted by $\mathcal{R}(\tau)$.

Proposition 6. *The routes cover the entire lattice* $(\mathcal{M}(G), \prec)$.

Proof. It suffices to prove that, if matchings $M, L \in \mathcal{M}$ satisfy $M \prec L$, then there exists a rotation C for M such that $M' = \mathrm{Repl}(M,C)$ satisfies $M' \preceq L$. To build such a rotation, we take a vertex $m \in \tilde{I}$ for which the incident edges $e = mw \in M$ and $f = mv \in L$ are different. Then $e <_m f$ (due to $M \prec L$), and the edge f is feasible for M (since there is an edge $b \in M$ incident to v for which $f <_v b$ due to $M \prec L$ and $w \in \tilde{J}$). Therefore, $\delta(m)$ contains an active edge $a = mw'$, and it holds that $a \le_m f$. Note that $w' \in \tilde{V}$ (otherwise, it would be $a \notin L$, in which case $a \ne f$, and a blocks L).

Therefore, there is an edge $e' = m'w' \in M$ and an edge $f' \in L$ incident to m'. We claim that $f' \ne e'$. Indeed, in the case $f' = e'$, we would have $a <_{w'} f'$ and $a <_m f$, and, therefore, a would block L.

Now, from $f' \ne e'$ we obtain $e' <_{m'} f'$, and, similarly, $e <_m f$ in the initial situation. Continuing the construction, we obtain an "unbounded" alternating path P in $\Gamma(M)$ with the sequence of edges $e = e_1 = m_1w_1$, $a = a_1 = m_2w_1$, $e' = e_2 = m_2w_2$, $a_2 = m_2w_3, \dots$; and for each i there is an edge $f_i \in L$ that is incident to m_i and different from e_i. Then $e_i <_{m_i} a_i \le_{m_i} f_i$, and P contains the desired rotation C.

The *rank* of a matching M is defined as the sum $\rho(M)$ of ordinal numbers of its edges mw in the set $\delta(m)$ (ordered according to $<_m$), where $m \in I$. It is clear that $\rho(M) \le |E|$ and, if $M \prec L$, then $\rho(M) < \rho(L)$. Therefore,

$$\text{any route } \tau \text{ in } G = (V, E, <) \text{ contains not more than } |E| \text{ matchings;} \quad \text{therefore, } |\mathcal{R}(\tau)| < |E|. \tag{3.1}$$

Propositions 4–6 imply the following result.

Corollary 1. (i) The graph $\Gamma(M^{\max}(G))$ is a forest (and there are no rotations for $M^{\max}(G)$).

(ii) The edges of the minimal stable matching $M = M^{\min}(G)$ belonging to tree components of the active graph $\Gamma(M)$ belong to all stable matchings in $\mathcal{M}(G)$.

(iii) G has a single stable matching if and only if $\Gamma(M^{\min})$ is a forest.

The following proposition established in [5], despite the relative simplicity of its proof, is the most important result concerning rotations.

Proposition 7. *The set of rotations* $\mathcal{R}(\tau)$ *is the same for all routes* τ *in* $(\mathcal{M}(G), \prec)$.

Proof. For $M \in \mathcal{M}(G)$, we define $\mathcal{T}(M)$ to be the set of route segments (subroutes) beginning in M and ending in $M^{\max}$. For such a subroute $\tau \in \mathcal{T}(M)$, the corresponding set of rotations is denoted by $\mathcal{R}(\tau)$; we call the matching M *singular* if there are subroutes τ and τ' with $\mathcal{R}(\tau) \ne \mathcal{R}(\tau')$ in $\mathcal{T}(M)$. We want to prove that $M^{\min}$ is not singular.

Suppose that this is not the case, and consider a singular matching M of the maximum height, i.e., a matching such that no matching L is singular if $M \prec L$. Let $\mathcal{C}(M)$ be the set of rotations in $\Gamma(M)$. Then, for any subroute $\tau = (M = M_0, M_1, M_2, \dots, M_k = M^{\max})$, the matching M_1 is obtained from M by a replacement along a rotation $C \in \mathcal{C}(M)$. Due to the maximality of M, there exist rotations $C', C'' \in \mathcal{C}(M)$ such that the matchings $M' = \mathrm{Repl}(M,C')$ and $M'' = \mathrm{Repl}(M,C'')$ are nonsingular and the sets of rotations $\mathcal{R}(\tau')$ and $\mathcal{R}(\tau'')$ are different for the subroutes $\tau', \tau'' \in \mathcal{T}(M)$ such that τ' passes M' and τ'' passes M''.

Note that, since the rotations C' and C'' do not intersect, C' is a rotation in $\Gamma(M'')$, and C'' is a rotation in $\Gamma(M')$. Therefore, $\mathcal{T}(M)$ contains subroutes τ', τ'' such that τ' begins with $M, M', \mathrm{Repl}(M',C'')$, and τ''

begins with $M, M'', \mathrm{Repl}(M'', C')$, and the subroutes τ' and τ'' coincide after the matching $\mathrm{Repl}(M', C'') = \mathrm{Repl}(M'', C')$. However, $\mathcal{R}(\tau') = \mathcal{R}(\tau'')$, which is a contradiction.

This set of rotations $\mathcal{R}(\tau)$, which is independent of the routes $\tau \in \mathcal{T}(M^{\min})$, is denoted by $\mathcal{R}_G$. Due to (3.1), we obtain the following result.

Corollary 2. The number of rotations in $\mathcal{R}_G$ does not exceed $|E|$.

There is one more useful property:

$$\begin{gathered}\text{the set of matching edges in two different rotations do not overlap}\\ \text{(and, therefore, the total number of edges in rotations does not exceed } 2|E|\text{).}\end{gathered} \tag{3.2}$$

Indeed, if $e = mw$ is a matching edge in a rotation C, where $m \in I$, then under the transformation in C, the active edge $a = mv$ in C becomes the new matching edge in $\delta(m)$, and it holds that $e <_m a$. In the further steps along the route, the matching edges in $\delta(m)$ can move only to the "right"" (for similar reasons), and there is no return to the edge e.

In conclusion of this section, we discuss the components of the graph of the union of stable matchings Σ_G. To this end, denote by M^0 the set of edges of the minimal matching $M^{\min}$ belonging to the tree components of $\Gamma(M^{\min})$, and denote by U_G the subgraph of G that is the union of all rotations in $\mathcal{R}_G$. As indicated in Corollary 1(ii), M^0 belongs to all matchings in $\mathcal{M}(G)$ (since the active graph $\Gamma(M^{\min})$ has no decreasing cycles and taking into account Proposition 5); hence, we can conclude that each edge in M^0 forms a component in Σ_G. At the same time, obviously, each rotation completely lies in Σ_G and, therefore, $U_G \subseteq \Sigma_G$. Moreover, since each stable matching is obtained from $M^{\min}$ by a sequence of rotational transformations, we have the following result.

Corollary 3. Σ_G is the union $U_G = \cup\{C \in \mathcal{R}_G\}$ and $M^{\min}$. Each edge in M^0 and each component in U_G is a component in Σ_G.

Thus, Σ_G is constructed efficiently (in time $O(|V||E|)$ due to the reasoning in the next section). Furthermore, each edge in M^0 and each component U_G is a component in Σ_G. In light of this, we may ask if there are other components in Σ_G? If yes, then such components can be only edges from $M^{\min}$ that belong to cycle-containing components K of the graph $\Gamma(M^{\min})$ but do not belong to the union of rotations U_K. The existence of such components is an open question for the author at the moment.

4. IDEALS OF THE POSET OF ROTATIONS AND STABLE MATCHINGS

As has been mentioned above, the natural partial order $\prec$ on stable matchings turns the set $\mathcal{M}(G)$ into a distributive lattice (see Proposition 3). It is known that any distributive lattice is isomorphic to the lattice of ideals of a poset (and conversely). Irving and Leather [5] provided an explicit construct for the lattice $(\mathcal{M}(G), \prec)$.(Even though [5] and the subsequent work [7] deal with a complete bipartite graph $K_{n,n}$, the results obtained in these works can be easily extended to an arbitrary bipartite graph G.) More precisely, based on the key fact that the set of rotations associated with routes is invariant (see Proposition 7) and a method of refining the structure of the poset on the set of rotations $\mathcal{R}_G$, a correspondence between stable matchings and ideals in such a poset was demonstrated. This representation for $\mathcal{M}(G)$ has important applications; in particular, it makes it possible to efficiently solve linear optimization problems for $\mathcal{M}(G)$ and to prove that the problem of calculating the number $|\mathcal{M}(G)|$ is intractable, which will be discussed in Sections 6 and 7.

We begin with constructing of this poset on $\mathcal{R}_G$. Recall that, for the route $\tau = (M_0, M_1, \ldots, M_N)$, we denote by $\mathcal{R}(\tau)$ the sequence of rotations $C_1, \ldots, C_N$, where M_i is obtained from M_{i-1} by the replacement along C_i.

Definition 3. For rotations $C, D \in \mathcal{R}_G$, we say that C precedes D and denote this relation as $C \lessdot D$, if, for each route τ in $(\mathcal{M}(G), \prec)$, the rotation C occurs in the sequence $\mathcal{R}(\tau)$ earlier than D. In other words, the transformation of a matching in D cannot occur earlier than the transformation in C.

This binary relation is antisymmetric, transitive, and it induces a partial order on $\mathcal{R}_G$.

To explain the relationship with matchings, consider $M \in \mathcal{M}(G)$. Let $\tau = (M_0, M_1, \dots, M_N)$ be a route containing M, say, $M = M_i$, and let $\mathcal{R}(\tau) = (C_1, \dots, C_N)$. Then, M is obtained from $M_0 = M^{\min}$ by the sequence of transformations with respect to rotations $C_1, \dots, C_i$. Proposition 7 easily implies that

$$\begin{gathered}\text{the (unordered) set } \mathcal{R}_M := \{C_1, \dots, C_i\} \text{ is independent of the route } \tau, \\ \text{passing } M\text{; and similarly for the set } \mathcal{R}_M^+ := \{C_{i+1}, \dots, C_N\}, \\ \text{and for the set } \mathcal{R}_{M,M'} := \{C_{i+1}, \dots, C_j\}, \text{ where } M' = M_j \text{ for } j \geq i.\end{gathered} \tag{4.1}$$

According to the definition of $\lessdot$, the relation $C_j \lessdot C_k$ can hold only if $j < k$. Moreover, $\mathcal{R}_M$ forms a closed set or ideal in the poset $(\mathcal{R}_G, \lessdot)$ (i.e., $C \lessdot D$ and $D \in \mathcal{R}_M$ imply $C \in \mathcal{R}_M$). This gives an injective map β of the set $\mathcal{M}(G)$ to the set of ideals in $(\mathcal{R}_G, \lessdot)$. Actually, β is a bijection as is shown in [5], Theorem 4.1.

Proposition 8. *The map $M \overset{\beta}{\mapsto} \mathcal{R}_M$ establishes an isomorphism between the lattice of stable matchings $(\mathcal{M}(G), \prec)$ and the lattice $\mathcal{I}(G)$ of ideals in $(\mathcal{R}_G, \lessdot)$.*

Proof. Consider stable matchings M and L, and let E' and E'' be the sets of edges belonging to the cycles in $\mathcal{C}^+(M, L)$ and $\mathcal{C}^-(M, L)$, respectively. It is clear that the subgraphs generated by these sets do not intersect. It follows from the proof of Proposition 6 that E' is the union of the set of rotations R' required for obtaining $M \vee L$ from M (or, equivalently, L from $M \wedge L$), and E'' is the union of the set of rotations R'' required for obtaining $M \vee L$ from L (equivalently, M from $M \wedge L$). Therefore, the rotations R' and R'' "commute": in order to obtain the matching $M \vee L$ from $M \wedge L$, we may first apply R' and then R'' or conversely. This yields $\beta(M) \cap \beta(L) = \beta(M \wedge L)$ and $\beta(M) \cup \beta(L) = \beta(M \vee L)$ (i.e., β determines a homomorphism of lattices). Now, the proposition easily follows from the fact that, for rotations $C, D \in \mathcal{R}_G$, the relation $C \lessdot D$ does not hold if and only if there is a stable matching M such that $D \in \mathcal{R}_M \not\ni C$.

As a consequence, $\mathcal{M}(G)$ is bijective to the set $\mathcal{A}(G)$ of antichains in $(\mathcal{R}_G, \lessdot)$. (Recall that an *antichain* in the poset $(P, <)$ is a set A of pairwise incomparable elements. It determines the ideal $\{p \in P : p \leq a \, \exists a \in A\}$ and is determined by this ideal, being the set of its maximal elements.)

We complete this section by explaining how the relation $\lessdot$ can be efficiently checked. Note that if C, D are rotations for the same stable matching M (and, therefore, transformations in C and D may be carried out in an arbitrary order), then they do not satisfy this relation; in this case, C and D lie in different components of $\Gamma(M)$, and, therefore, $V_C \cap V_D = \varnothing$. On the other hand, if C and D have a common vertex v, then they are comparable with respect to $\lessdot$.

Indeed, the matching edges incident to v are different for C and D, the order of C and D in any route is the same, and it is determined by the order $<_v$ for these edges (see the explanation of property (3.2)). Generally, the set of rotations $\mathcal{R}^v$ containing a fixed vertex v is ordered with respect to $\lessdot$, say, $\mathcal{R}^v = (C_1 \lessdot C_2 \lessdot \dots \lessdot C_k)$, and this order is easily determined, since it agrees with the order in $<_v$ if $v \in I$ and with the reverse order if $v \in J$.

There is one more local reason for which two rotations C, D must be comparable with respect to $\lessdot$. Namely, assume that

$$\begin{gathered}C \text{ contains a matching edge } e = mw \text{ and an active edge } vw, \\ \text{and } D \text{ contains a matching edge } e' = m'w', \text{ such that } b = m'w \text{ is an edge of } G \\ \text{not contained in any rotation and the first edge in } \delta(m') \text{ after } e', \\ \text{and it holds that } a <_w b <_w e.\end{gathered} \tag{4.2}$$

Then $C \lessdot D$. Indeed, since $a <_w b <_w e$, the edge b becomes infeasible after the transformation in C. On the other hand, if D preceded C in some route, then b would be active in $\delta(m')$ and included in the rotation.

The example in Section 3 is precisely this: here, after the replacement along the rotation C, the edge $e = 2v$ becomes infeasible due to $d' <_v e <_v a'$ and $d, d' \in M_2$ (in this case, e is the first edge in $\delta(2)$ after

d). In this example, the poset consists of two rotations C, D satisfying $C \lessdot D$, and there are exactly three ideals $\varnothing$, $\{C\}$, and $\{C, D\}$ corresponding to the stable matchings $M_1 = M^{\min}$, M_2, and $M_3 = M^{\max}$ (while $\{D\}$ is not an ideal).

Note that, in order to find the set of rotations $\mathcal{R}_G$, it suffices to determine $M^{\min}$ and then construct an arbitrary route by making transformations on rotations in the current matchings. (This is done using a natural algorithm in time $O(|V||E|)$. In [16], an algorithm for building $\mathcal{R}_G$ in time $O(|V|^2)$ was proposed.)

In turn, the following important fact was established in [7], Section 5: for an efficient description of the order $\lessdot$, it is sufficient to consider the intersections of rotations and take into account (4.2). Namely, let us form the directed graph $H = (\mathcal{R}_G, \mathcal{E})$ whose edges are the pairs of rotations (C, D) such that either $V_C \cap V_D \neq \varnothing$ and $C \lessdot D$ or (4.2) holds (this graph can be easily constructed in time $O(|V||E|)$; with some ingenuity the time ca be reduced to $O(|E|)$). It can be shown that

the graph H contains an edge (C, D) if and only if there exists a stable matching M such that M admits the rotation C, but not D, and after the transformation of M with respect to C the resulting stable matching M' already admits D. (4.3)

The graph H is acyclic and, if a vertex D in H is reachable by a directed path from C, then $C \lessdot D$. The converse is true.

Proposition 9. *If $C \lessdot D$, then there is a directed path from C to D in H. Therefore, the reachability relation for the vertices of H coincides with $\lessdot$.*

Proof. Let $\mathcal{I}'$ be the set of ideals in the poset for H; then $\mathcal{I}' \supseteq \mathcal{I}(G)$. We should prove that $\mathcal{I}' \supseteq \mathcal{I}(G)$. Suppose that this is not the case, i.e., there exists an $X \in \mathcal{I}'$, that is not an ideal in $(\mathcal{R}_G, \lessdot)$. Let, in addition, X be chosen so that the number of elements $|X|$ be maximal. We also choose an ideal $R \in \mathcal{I}(G)$ such that $R \subset X$ and $|R|$ is maximal under this property. By Proposition 8, $R = \mathcal{R}_M$ for some sable matching M. Take a feasible rotation C for M. Due to the maximality of R, the element C must be outside of X. Then $R' := R \cup \{C\}$ is an ideal in $(\mathcal{R}_G, \lessdot)$ (corresponding to the matching M' obtained from M by the replacement along C). We also have $X' := X \cup \{C\} \in \mathcal{I}'$.

Due to the maximality of X, we have $X' \in \mathcal{I}(G)$. Since $X' \supset R'$ and both X', R' are ideals in $(\mathcal{R}_G, \lessdot)$, the set $X' - R'$ contains an element (rotation) D such that $R'' := R' \cup \{D\} \in \mathcal{I}(G)$. At the same time, $X'' := R \cup \{D\}$ is not an ideal in $(\mathcal{R}_G, \lessdot)$; otherwise, due to $R \subset X'' \subseteq X$, we would obtain a contradiction with the maximality of R (if $X'' \neq X$) or with the condition imposed on X (if $X'' = X$).

Thus, we have arrived at a situation in which $R, R', R'' \in \mathcal{I}(G)$, $R' = R \cup \{C\}$, $R'' = R' \cup \{D\}$, but $R \cup \{D\} \notin \mathcal{I}(G)$. Then the rotation D is infeasible for the matching M, but it becomes feasible immediately after the replacement along the rotation C in M. This exactly implies that the graph H contains the edge (C, D); cf. (4.3). However, such an edge leads from $\mathcal{R}_G - X$ to X, contrary to the fact that X belongs to $\mathcal{I}'$.

It can be shown that the edges (C, D) of the graph H, as in (4.3), determine the immediate precedence relations in the poset $(\mathcal{R}_G, \lessdot)$, namely, the validity of $C \lessdot D$ and the absence of any rotation C' between C and D (i.e., $C \lessdot C' \lessdot D$).

Remark 1. Let us consider stable matchings M and M' in a bipartite graph G such that $M \prec M'$, and let $[M, M']$ denote the set ("interval") of stable matchings L for which $M \preceq L \preceq M'$. (As special cases, we may consider $M = M^{\min}$ or $M' = M^{\max}$.) We would like to know if it is possible to "clear" G by removing a part of its edges so that the set of stable matchings for the resulting graph G' exactly coincides with the interval $[M, M']$ for G. To attempt answering this question, consider an arbitrary route τ passing both M, M' and consider the set of rotations used on the route segment from M to M', i.e., $\mathcal{R}_{M,M'}$, see (4.1). (The route τ and the set $\mathcal{R}_{M,M'}$ are constructed efficiently; cf. the proof of Proposition 6.) Then, the interval $[M, M']$ consists exactly of those $L \in \mathcal{M}(G)$ that are obtained from M by applying a sequence of rotations from the set $\mathcal{R}_{M,M'}$. Therefore, the graph G' to be found must include M and the union of all rotations from $\mathcal{R}_{M,M'}$. However, it also must respect the structure of the graph H (see (4.3)) to avoid the appearance of "superfluous" matchings that use rotations from $\mathcal{R}_{M,M'}$. Therefore, we must add edges as in (4.2) (of the form $m'w$) to guarantee the preservation of the relation $C \lessdot D$ mentioned in (4.2) for the corresponding rotations C, D in $\mathcal{R}_{M,M'}$. A conjecture: a subgraph G' does exist, and it is obtained from G by removing the other edges.

5. OPTIMAL STABLE MATCHINGS

In this section, we consider the situation when the bipartite graph $G = (V, E, <)$ is equipped with a real *weight* (or *cost*) function of edges $c : E \to \mathbb{R}$. We are interested in the *problem on stable matching of minimum weight:*

$$\text{find a stable matching } M \in \mathcal{M}(G) \text{ minimizing the total weight } (M) := \sum_{e \in M} c(e). \tag{5.1}$$

(Since all stable matchings in G have the same size, adding a constant to the function c does not actually affect the problem; therefore, we may assume that c is a nonnegative function. If c is replaced by $-c$, we obtain the problem of maximizing the weight $c(M)$.)

In an important special case of this problem known as the *optimal stable matching* problem (see [6]), the weight $c(e)$ of an edge $e = mw$ is defined as

$$c(e) := r_I(e) + r_J(e), \tag{5.2}$$

where $r_I(e)$ is the ordinal number of the edge e in the ordered (according to $<_m$) list $\delta(m)$ and, similarly, $r_J(e)$ is the number of the list $\delta(w)$. (This setting was widely discussed in the literature, in particular, in [11], [17]. In it, the matching to be found is the most favorable in terms of total (or average) preferences for all "persons" (while the deferred acceptance algorithm gives the best solution only for one set among I and J).) In [7], the problem with a weight function as in (5.2) is also called the *egalitarian stable matching* problem.

Problem (5.1) can be formulated as a linear program with a $(0, \pm 1)$ matrix of size $(|V| + |E|) \times |E|$ (this will be discussed in the next section); therefore, it can be solved using a universal strongly polynomial algorithm (by a strengthened version of the ellipsoid method [18]). However, the representation of $\mathcal{M}(G)$ as the set of ideals in the poset of rotations $(\mathcal{R}_G, \lessdot)$ described in the previous section makes it possible to solve problem (5.1) much more efficiently and simpler. This method was proposed by Irving, Leather, and Gusfield in [7], and below we describe this method as applied to an arbitrary bipartite graph G).

For the given weight function c and an arbitrary rotation $R = e_1a_1e_2a_2 \cdots e_ka_k$ in $\mathcal{R}_G$, where e_i are matching edges and a_i are active edges, we define the weight of R by

$$c^R := \sum (c(a_i) - c(e_i) : i = 1, \dots, k). \tag{5.3}$$

When the transformation with respect to the rotation R is carried out, the weights of the active edges are added to the weight of the current matching, and the weights of the matching edges in R are subtracted from it. Therefore, the weight of any stable matching M is represented by

$$c(M) = c(M^{\min}) + \sum (c^R : R \in \mathcal{R}_M),$$

where $\mathcal{R}_M$ is the ideal in $(\mathcal{R}_G, \lessdot)$ corresponding to M (i.e., the set of rotations occurring in the segment of (any) route from $M^{\min}$ to M).

Thus, (5.1) is equivalent to the problem of finding an ideal of minimum weight. In a general case, such a problem looks as follows:

$$\begin{gathered}\text{for a directed graph } Q = (V_Q, E_Q) \text{ and a function } \zeta : V_Q \to \mathbb{R} \text{ find a closed set} \\ X \subseteq V_Q \text{ of minimum weight } \zeta(X) := \sum (\zeta(v) : v \in X).\end{gathered} \tag{5.4}$$

(Recall that a set X is called *closed* if there are no edges going from $V_Q - X$ to X. In particular, closed sets are $\varnothing$ and V_Q. Without loss of generality, we may assume that the graph Q is acyclic, since a directed cycle cannot be divide by a closed set, and it can be contracted into a single vertex of the total weight.)

The solution of problem (5.4) proposed by Picard [8] is based on the reduction to the minimum cut problem in the directed graph $\hat{Q} = (\hat{V}, \hat{E})$ with the edge capacities $h(e)$, $e \in \hat{E}$, defined as follows.

Set $V^+ := \{v \in V_Q : \zeta(v) > 0\}$ and $V^- := \{v \in V_Q : \zeta(v) < 0\}$. The graph $\hat{Q}$ is obtained from Q by adding two vertices—"source" s and "sink" t and the set of edges E^+ going from s into v for all $v \in V^+$ and the set of edges E^- going from u into t for all $u \in V^-$. The capacities of these edges $e \in \hat{E}$ are given by

$$h(e) := \begin{cases} \zeta(v) & \text{if} \quad e = sv \in E^+, \\ |\zeta(u)| & \text{if} \quad e = ut \in E^-, \\ \infty & \text{if} \quad e \in E_Q. \end{cases}$$

For a set $S \subseteq \hat{V}$ such that $s \in X \not\ni t$, denote by $\delta(S)$ the set of edges in $\hat{Q}$ going from S to $\hat{V} - S$ (it is called an *$s-t$ cut*); the quantity $h(\delta(S)) := \sum (h(e) : e \in \delta(S))$ is considered to be the capacity of this cut.

Lemma 2 [8]. *A subset $X \subseteq V_Q$ is a closed set of minimum weight in (Q, ζ) if and only if $\delta((V_Q - X) \cup \{s\})$ is an $s-t$ cut of minimum capacity in $(\hat{Q}, h)$.*

Proof. Note that $X \subseteq V_Q$ is a closed set if and only if the cut $E' = \delta((V_Q - X) \cup \{s\})$ does not contain edges of infinite capacity or, equivalently, if E' is included in $E^+ \cup E^-$. For such a cut consisting of edges sv for $v \in X$ and edges ut for $u \in V_Q - X$, the capacity is

$$\begin{aligned} h(E') &= \zeta(X \cap V^+) + \sum (|\zeta(u)| : u \in (V_Q - X) \cap V^-) \\ &= \zeta(X \cap V^+) + \zeta(X \cap V^-) - \zeta(V^-) = \zeta(X) - \zeta(V^-). \end{aligned}$$

Therefore, the weight of a closed set differs from the capacity of the corresponding cut by a constant $-\zeta(V^-)$, whence we obtain the desired assertion.

Thus, problem (5.1) is reduced to the problem of minimum two-terminal cut in the network with $N = O(|E|)$ vertices and $A = O(|E|)$ edges. (Taking into account that, instead of the entire poset $(\mathcal{R}_G, \lessdot)$, it suffices to consider the graph H generating it, which has $O(|E|)$ edges, see Proposition 9. Using fast algorithms for the maximum flow and minimum cut problems (e.g., see review [12], Section 10.8), it is possible to obtain the time bound $O(NA \log N) \simeq O(|V|^4 \log |V|)$. In [7], an implementation of an algorithm solving problem (5.1) in time $O(|V|^4)$ is given.)

Remark 2. In [19], the intractability of some variants of the closed set problem was proved. Using one of them and the fact that any transitively closed graph can be implemented as a poset of rotations (which will be mentioned in Section 7), the NP-hardness of the following strengthening of problem (5.1) can be proved: for a bipartite graph $G = (V, E, <)$, functions $c, g : E \to \mathbb{R}_+$ and a number $K \in \mathbb{R}_+$, find a stable matching $M \in \mathcal{M}(G)$ minimizing $c(M)$ under the condition $g(M) \geq K$. (Here $g(M)$ may be interpreted as a profit and $c(M)$ as a cost of organizing unions (or contracts) in M.)

6. POLYHEDRAL ASPECTS AND MEDIAN STABLE MATCHINGS

As has been mentioned above, for a bipartite graph $G = (V = I \sqcup J, E, <)$, there is a polyhedral characterization of the polyhedron of stable matchings $\mathcal{P}_{\text{st}}(G)$ expressed by a linear (in $|V|, |E|$) number of equalities. Here, $\mathcal{P}_{\text{st}}(G)$ is the convex hull of the set of characteristic vectors χ^M of stable matchings M in the space $\mathbb{R}^E$. The first description of $\mathcal{P}_{\text{st}}(G)$ (in the case $G \simeq K_{n,n}$) was given in the work by Vande Vate [9] (also see [20]). Below, we give a description (which is somewhat different in form but close to that in [9]) and a proof on the basis of the exposition in [12], Section 18.5g.

For the edges $e, f \in E$, we write $f \prec e$ if they have a common vertex v and it holds that $f <_v e$ (i.e., if f is preferred to e). For $e \in E$, denote by $\gamma(e)$ the set of edges f such that $f \preceq e$ (in particular, $e \in \gamma(e)$).

Recall that the polyhedron of matchings in the bipartite case is described by the system of linear inequalities

$$x(e) \geq 0, \quad e \in E; \tag{6.1}$$

$$x(\delta(v)) \leq 1, \quad v \in V. \tag{6.2}$$

In the case of stable matchings, one more type of inequalities is added:

$$x(\gamma(e)) \geq 1, \quad e \in E. \tag{6.3}$$

Proposition 10. *System* (6.1)–(6.3) *describes exactly the set of vectors* $x \in \mathbb{R}^E$ *belonging to* $\mathcal{P}_{\text{st}}(G)$.

Proof. It is easy to verify that, or any stable matching M, the vector $x = \chi^M$ satisfies (6.1)–(6.3). Therefore, it suffices to prove that, if x is a vertex of the polyhedron $\mathcal{P}'$ defined by (6.1)–(6.3), then x is an integer-valued vector.

Set $E^+ := \{e \in E : x(e) > 0\}$ and denote by V^+ the set of vertices in G covered by E^+. For $v \in V^+$, denote by e_v the best edge in $\delta(v) \cap E^+$ with respect to the order $<_v$. The following property holds:

$$\text{for } v \in V^+ \text{ and } e_v = \{v, u\} \text{ the edge } e_v \text{ is the worst in } (\delta(u) \cap E^+, <_u); \\ \text{in addition, it holds that } x(\delta(u)) = 1. \tag{6.4}$$

Indeed, by setting $e := e_v$, we have

$$1 \leq \sum (x(f) : f \preceq e) \text{ (due to (6.3))} = \sum (x(f) : f \leq_u e) \text{ (due to } e_v)$$
$$= x(\delta(u)) - \sum (x(f) : f >_u e) \leq 1 - \sum (x(f) : f >_u e) \text{ (by applying (6.2) to } u).$$

Here, all inequalities must turn into equalities. This gives $\sum (x(f) : f >_u e) = 0$ and $x(\delta(u)) = 1$, which implies (6.4).

Form the sets $M := \{e \in E^+ : e = e_v \text{ for } v \in I\}$ and $L := \{e \in E^+ : e = e_v \text{ for } v \in J\}$. For any vertex $v \in I \cap V^+$, the best edge in $\delta(v) \cap E^+$ belongs to M, and the worst edge and only this edge belongs to L (due to (6.4)); for the vertices in $J \cap V^+$, the behavior is converse. This implies that both M and L are matchings. Each edge $e \in M \cap L$ forms a component in the subgraph (V^+, E^+); this gives $x(e) = 1$ (due to (6.2) and (6.3)). In particular, x is integer if $M = L$.

Now, let $M \neq L$. Obviously, for any edge e in $M' := M - L$ or in $L' := L - M$, it holds that $0 < x(e) < 1$. Therefore, we can choose a sufficiently small $\varepsilon > 0$ such that the vectors $x' := x + \varepsilon\chi^{M'} - \varepsilon\chi^{L'}$ satisfy (6.1) and (6.2). Let us check that both x' and x'' also satisfy (6.3).

To this end, consider an edge $e = mw$ with $x(e) < 1$ and suppose that $x'(\gamma(e)) < x(\gamma(e))$. This is possible only if $a \leq_w e <_w b$, where $\{a\} = L \cap \delta(w)$ and $\{b\} = M \cap \delta(w)$. In this case, it must be either (i) $m \notin V^+$, or (ii) $m \in V^+$, and e is preferred to any edge in $M \cap \delta(m)$ (otherwise, such an edge would made a positive contribution into $x'(\gamma(e))$, whence $x'(\gamma(e)) = x(\gamma(e))$). However, in these cases, the equality $x(\delta(w) = 1$ (due to (6.4)) and the inequality $x(b) > 0$ imply

$$x(\gamma(e)) = \sum (x(f) : f \leq_w e) \leq x(\delta(w)) - x(b) < 1,$$

which is impossible. Similarly, (6.3) holds for x''.

Thus, $x', x'' \in \mathcal{P}'$. But $x' \neq x''$ and $(x' + x'')/2 = x$. This contradicts the fact that x is a vertex in $\mathcal{P}'$.

There is one more useful property proved in [21]; we give a somewhat different but equivalent formulation.

Lemma 3. *Let* $x \in \mathcal{P}_{\text{st}}(G)$, *and let* $e = mw$ *be an edge in* G *for which* $x(e) > 0$. *Then* $x(\delta(m)) = x(\delta(w)) = x(\gamma(e)) = 1$.

Proof. Represent x as $\alpha_1\chi^{M_1} + \dots + \alpha_k\chi^{M_k}$, where M_i is a stable matching, $\alpha_i > 0$, and $\alpha_1 + \dots + \alpha_k = 1$. Define $x_i := \chi^{M_i}$. Since $x(e) > 0$, the edge e belongs to some M_i. Then $x_i(\delta(m)) = x_i(\delta(w)) = x_i(\gamma(e)) = 1$. Similar equalities also hold for any matching M_j not containing e. Indeed, since M_i and M_j cover the same set of vertices (due to Proposition 2), M_j contains an edge e' incident to m and an edge e'' incident to w. Moreover, by considering the pair M_i, M_j and using (2.1), we obtain either $e' <_m e <_w e''$ or $e' >_m e >_w e''$. In both cases exactly one edge from e', e'' belongs to $\gamma(e)$; therefore, $x_j(\gamma(e)) = 1$. Now, the assertion follows from the equality $\alpha_1 + \dots + \alpha_k = 1$.

This lemma helps us obtain an interesting result of TEo and Seturaman.

Proposition 11 [10]. *Let $M_1,\dots,M_\ell$ be stable matchings in G. For each $m \in \tilde{I}$, denote by E_m the list (with possible repetitions) of edges in $\delta(m)$ belonging to these matchings and ordered according to $<_m$; furthermore, let $e_m(i)$ denote the ith element in E_m, $i = 1,\dots,\ell$. Similarly, for each $w \in \tilde{J}$, let $e_w(j)$ be the jth element in the list E_w of edges in $\delta(w)$ belonging to these matchings and ordered according to $<_w$, $j = 1,\dots,\ell$. Then, for any $k \in \{1,\dots,\ell\}$, the set of edges $A(k) := \{e_m(k) : m \in \tilde{I}\}$ coincides with the set $B(\ell - k + 1) := \{e_w(\ell - k + 1) : w \in \tilde{J}\}$ and forms a stable matching in G.*

Proof. Set $x_i := \frac{1}{\ell}\chi^{M_i}$. Then $x = x_1 + \dots + x_\ell$ belongs to the polyhedron $\mathcal{P}_{\rm st}(G)$ (it is a so-called "fractional stable matching").

We first assume for simplicity that all edges in $M_1,\dots,M_\ell$ are different. Then, for $m \in \tilde{I}$, the list E_m consists of ℓ different edges, and the edge $e_m(k)$ is the kth element in E_m. Lemma 3 applied to x and $e = e_m(k)$ implies that the set $\gamma(e)$ consists of ℓ elements. Then e is exactly the $(\ell - k + 1)$th element in the list E_w (which, by assumption, consists of ℓ different edges) and, therefore, $e_m(k) = e_w(\ell - k + 1)$.

Thus, $A(k) = B(\ell - k + 1)$, which implies that $A(k)$ is a matching in G. To prove the stability of $A(k)$, consider an arbitrary edge $e = mw \notin A(k)$ in G. We should verify that e does not block $A(k)$ or, equivalently, that $A(k) \cap \gamma(e) \neq \varnothing$.

At least one of the endpoints of e (say, m) belongs to $\tilde{V}$ (and is covered by $A(k)$); otherwise, we would have $x(\gamma(e)) = 0$ for x and e contrary to (6.3). If $w \notin \tilde{V}$, then $x(\delta(w)) = 0$, and the inequality $x(\gamma(e)) \geq 1$ implies that, for all edges $f \in E_m$ (including $f = e_m(k)$), it holds that $f <_m e$. This gives the desired $A(k) \cap \gamma(e) \neq \varnothing$.

Now, let $w \in \tilde{V}$. Due to $x(\gamma(e)) \geq 1$, the number of edges f in $E_m \cup E_w$ for which $f \preceq e$ is not less than ℓ. Then, at least one of the inequalities $e_m(k) \leq_m e$ and $e_w(\ell - k + 1) \leq_w e$ must hold, which again implies $A(k) \cap \gamma(e) \neq \varnothing$.

If there are common edges in $M_1,\dots,M_\ell$, we can consider the multigraph obtained from G by replacing each edge $e = mw$ with $x(e) > 0$ by $\ell x(e) =: r$ parallel edges $e^1,\dots,e^r$. In this case, the extension of the order $<_m$ to these edges is assigned to be opposite to the extension of the order $<_w$, say, $e^1 <_m \dots <_m e^r$ and $e^1 >_w \dots >_w e^r$. The desired assertion for this general case is obtained by repeating (with minor refinements) the reasoning for the case on nonintersecting matchings considered above.

Corollary 4. If ℓ is odd, then for $k := (\ell + 1)/2$, the set consisting of the kth elements in the ordered lists of edges E_v incident to v and belonging to $M_1,\dots,M_\ell$ for all vertices $v \in \tilde{V}$ is a stable matching.

Such a matching is called the *median matching* for $M_1,\dots,M_\ell$. In [10], the question was raised about the possibility of efficiently finding a median stable matching among all stable matchings in G (or an "almost median matching" when the number of stable matchings is even). It seems to me that this is hardly possible, because the problem of calculating the number $|\mathcal{M}(G)|$ is intractable, which will be discussed below.

7. COUNTING THE NUMBER OF STABLE MATCHINGS

Knuth in [11] provided examples in which the number of stable matchings in a bipartite graph is exponentially large compared with the graph size, and he posed the question about the complexity of finding the exact number of these matchings. An answer was given by Irving and Leather in [5], who proved that this problem is intractable (by considering the graphs $K_{n,n}$). Below, we outline the idea of their proof.

Recall some concepts (their exact definitions can be found in [22] or [23], Section 7.3). Without going into full logical rigor, one can understand that the description of one or another enumeration problem $\mathcal{P}$ consists of an infinite family of finite sets $\mathcal{S}$, and for each $S \in \mathcal{S}$, there is a family $\mathcal{F}(\mathcal{S})$ of subsets in S ("objects"). In problem $\mathcal{P}$ it is required to find the number $|\mathcal{F}(S)|$ for a given $S \in \mathcal{S}$. The problem $\mathcal{P}$ is said to be a *$\#P$-problem* (or *KP-problem* in terminology of some authors), if the recognition of an object takes polynomial time, i.e., if there is an algorithm that, for any $S \in \mathcal{S}$ and $X \subseteq S$, determines in a polynomial time of $|S|$ if the given set X belongs to the family $\mathcal{F}(S)$. The $\#P$-problem $\mathcal{P} = \{\mathcal{F}(S), S \in \mathcal{S}\}$ is said to be *$\#P$-complete* (or universal in the class $\#P$) if any other $\#P$-problem $\mathcal{P}' = \{\mathcal{F}'(S'), S' \in \mathcal{S}'\}$ is reduced to the former problem in a polynomial time (i.e., there is a map $\omega : \mathcal{S}' \to \mathcal{S}$ such that for each $S' \in \mathcal{S}$ the number $|\mathcal{F}'(S')|$ is determined from $|\mathcal{F}(\omega(S'))|$ in a polynomial time of $|S'|$).

In the problem of our interest, the role of the family $\mathcal{S}$ is played by the collections of edge sets E of bipartite graphs $G = (V, E, <)$, and the role of the family $\mathcal{F}(S)$ $(S \in \mathcal{S})$ is played by the corresponding set of stable matchings in G. The problem of finding $|\mathcal{M}(G)|$ is indeed a $\#P$-problem, since, for any subset $M \subseteq E$, one can find out if M is a stable matching in time $O(|E|)$.

Note that the $\#P$-analog of any NP-complete problem is intractable (since in the latter problem it is required "only" to determine whether the corresponding family of objects $\mathcal{F}(S)$ is not empty (e.g., does the given graph contain at least one Hamiltonian cycle?), while in the former one it is necessary to find the number of objects). However there are P-problems whose enumeration analogs are $\#P$-complete. The problem of determining $|\mathcal{M}(G)|$ is just one of this sort. This is a consequence of the following two results.

Proposition 12. *The problem of finding the number of antichains in a finite poset is $\#P$-complete.*

Proposition 13. *Let $(P, <')$ be a poset on n elements. There exists a bipartite graph $G = (V, E, <)$ such that its rotation poset $(\mathcal{R}_G, \lessdot)$ is isomorphic to $(P, <')$; and it can be constructed in a polynomial time of n. Therefore (due to Proposition 8), the number $|\mathcal{M}(G)|$ of stable matchings in G equals the number of antichains (or the number of ideals) in the poset $(P, <')$.*

Proposition 12 was proved by Provan and Ball in [24]. Proposition 13 was proved in [5], Section 5 by explicitly constructing the desired graph G for the given poset $(P, <')$.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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