Polyhedra Related to Undirected Multicommodity Flows

A. V. Karzanov
Institute for System Studies
Prospect 60 Let Oktyabrya, 9
117312 Moscow, USSR

Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Alexander Schrijver

ABSTRACT

Let $G=(VG, EG)$ and $H=(VH, EH)$ be two undirected graphs, and $VH \subseteq VG$. We associate with G and H the (unbounded) polyhedron $P(G,H)$ in $\mathbf{Q}^{EG}$ which consists of all nonnegative rational-valued functions (vectors) l on EG such that, for each edge st in H, the distance between s and t in the graph G whose edges $e \in EG$ have the lengths $l(e)$ is no less than 1. Let $\nu(H)$ be the least positive integer k such that each vertex of $P(G,H)$ is $1/k$-integral for any G with $VG \supseteq VH$ [$\nu(H)=\infty$ if such a k does not exist]. In other terms, $\nu(H)$ is the least positive integer k such that each problem dual to a maximum undirected multicommodity flow problem with the "commodity graph" H has an optimal solution that is $1/k$-integral. We prove that $\nu(H)$ can be only 1, 2, 4, or ∞, and moreover, for each $k=1,2,4,\infty$, we describe the class of H's with $\nu(H)=k$. Also results concerning extreme rays of cones related to feasibility multicommodity flow problems are presented.

1. INTRODUCTION

Suppose that $\mathscr{K}$ is a class of *optimization* problems:

(O): given $S \subseteq \mathbf{Q}^n$ and $g: S \to \mathbf{Q}$, find an $x \in S$ with $g(x)$ maximum (or minimum),

or a class of *feasibility* problems:

(F): given $S \subseteq \mathbf{Q}^n$, find some element in S.

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[$\mathbf{Q}$ is the set of rationals.] We define:

(i) where $x \in \mathbf{Q}^n$, $\varphi(x)$ to be the least positive integer k such that x is $1/k$-integral, i.e., each component of the vector kx is integer-valued;

(ii) where $P \in \mathscr{K}$, $\varphi(P)$ to be the least positive integer k such that P has an optimal (respectively, feasible) solution that is $1/k$-integral; if P has no optimal (feasible), solution we put $\varphi(P) := 0$;

(iii) $\varphi(\mathscr{K})$ to be the least positive integer k (if it exists) such that each problem P in $\mathscr{K}$ with $\varphi(P) > 0$ has an optimal (feasible) solution that is $1/k$-integral; if such a k does not exist, then $\varphi(\mathscr{K}) := \infty$.

We call $\varphi(x)$ [$\varphi(P)$, $\varphi(\mathscr{K})$] the *fractionality* of a vector x [a problem P, a class $\mathscr{K}$]; if $\varphi(\mathscr{K}) = \infty$, we say that $\mathscr{K}$ has *unbounded* fractionality.

In the present work we mainly study the fractionality of classes of problems dual to maximum undirected multicommodity flow problems.

We start with some definitions and conventions. Throughout the paper, by a *function* we shall mean a function taking values in the set $\mathbf{Q}_+$ of nonnegative rationals. By a *graph* we mean a finite undirected graph without loops and multiple edges; an edge with ends x and y may be denoted by xy. If K is a graph, then VK denotes its vertex set and EK its edge set. A *chain*, or an *st-chain*, of a graph is a subgraph L in it such that $VL = \{s = v_0, v_1, \dots, v_m = t\}$ (v_i are distinct) and $EL = \{v_i v_{i+1} : i = 0, \dots, m-1\}$; we may denote L also as $v_0 v_1 \dots v_m$.

Let G and H be two graphs with $VH \subseteq VG$, and c be a function on EG; we refer to c, H, and VH as *capacities* of edges of G, a (flow) *scheme*, and a set of *terminals*, respectively. It is more convenient for our purposes to use the "edge-chain" formulation for multicommodity flow problems, which is equivalent to the usual "edge-vertex" one (see [5]). For $s, t \in V$, let $\mathscr{L}(G, st)$ denote the set of st-chains in G. Let $\mathscr{L} = \mathscr{L}(G, H)$ be $\bigcup(\mathscr{L}(G, st) : st \in EH)$. A *multicommodity flow*, or a *multiflow*, for G and H is a function f on $\mathscr{L}$. f is called *c-admissible* if

$$\zeta^f(e) := \sum (f(L) : L \in \mathscr{L},\ e \in EL) \leqslant c(e) \qquad \text{for all} \quad e \in EG.$$

The value $v(f) := \Sigma(f(L) : L \in \mathscr{L})$ is called the *total value* of f.

The *maximum multiflow problem*, denoted by $M(G, c, H)$, is: given G, c, and H as above, find a c-admissible multiflow f for G and H with $v(f)$ maximum; this maximum is denoted by $v(G, c, H)$.

The problem $M^*(G, c, H)$ dual (in the linear-programming sense) to $M(G, c, H)$ is: find a nonnegative function l on EG minimizing $cl :=$

$\Sigma(c(e)l(e): e \in EG)$ subject to

$$l(EL) \geqslant 1 \qquad \text{for each} \quad L \in \mathscr{L}(G,H). \tag{1}$$

[For $h: X \to \mathbf{Q}$ and $X' \subseteq X$, $h(X')$ denotes $\Sigma(h(e): e \in X')$.] Denote by $M(G,H)$ the set of problems $M(G,c,H)$ with fixed G and H and an arbitrary nonnegative *integer-valued* function $c: EG \to \mathbf{Z}_+$; let $M(H) := \bigcup(M(G,H): G$ is a graph with $VG \supseteq VH)$. Similarly define the sets of dual problems $M^*(G,H)$ and $M^*(H)$. We denote $\varphi(M^*(H))$ also as $\nu(H)$ and call it the *fractionality* of H (with respect to the dual maximum multiflow problems).

Throughout the paper we shall assume that $EH \neq \varnothing$ and H has no isolated vertex, i.e., each vertex of H has at least one incident edge [clearly, removing an isolated vertex (if any) from H does not change any problem $M(G,c,H)$ or its dual].

Definition. We say that a scheme H has *property* (P) if $A \cap B = B \cap C = C \cap A$ holds for any three distinct pairwise intersecting anticliques A, B, and C in H.

[An *anticlique* of a graph is a maximal (with respect to inclusion) independent set of its vertices.] For example, if H consists of two disjoint complete graphs H_1 and H_2, then the set of anticliques of H is $\{\{s,t\}: s \in VH_1,\ t \in VH_2\}$, and H has property (P).

The following two theorems are central in the paper.

Theorem 1. *If a scheme H has property* (P), *then $\nu(H)$ is* 1, 2, *or* 4.

Theorem 2. *If a scheme H does not have property* (P), *then $\nu(H) = \infty$.*

Theorems 1 and 2 will be proved in Sections 3 and 4. They can be reformulated in polyhedral terms. A polyhedron P (possibly unbounded) in $\mathbf{Q}^n$ is said to be $1/k$*-integral*, where k is a positive integer, if each of its facets contains a $1/k$-integral point. For G and H as above, let $P(G,H)$ denote the (unbounded) polyhedron in $\mathbf{Q}^{EG}$ formed by the nonnegative functions (vectors) l on EG satisfying (1); we refer to $P(G,H)$ as a *dual flow polyhedron*. Obviously, for any $c \in \mathbf{Q}_+^{EG}$, there is a vertex l of $P(G,H)$ such that $cl \leqslant cx$ for all $x \in P(G,H)$, and, on the other hand, for an arbitrary vertex l of $P(G,H)$, there is $c \in \mathbf{Z}_+^{EG}$ such that $cl < cx$ for all $x \in P(G,H) - \{l\}$. Thus, Theorems 1 and 2 are equivalent to the following theorems.

THEOREM 1′. *If a scheme H has property* (P), *then, for any graph G with $VG \supseteq VH$, the polyhedron $P(G, H)$ is $\frac{1}{4}$-integral.*

THEOREM 2′. *If a scheme H does not have property* (P), *then, for any positive integer k, there exists a graph G with $VG \supseteq VH$ such that the polyhedron $P(G, H)$ is not $1/k$-integral.*

The values $\nu(H)$ are already known for a number of schemes H. For example, $\nu(H) = 1$ if $|EH| = 1$, by the max-flow min-cut theorem of Ford and Fulkerson [5], or if $|EH| = 2$, by the max-two-commodity-flow min-cut theorem of Hu [7]. Section 5 contains a refinement of Theorem 1 which describes completely the classes of schemes H with $\nu(H) = 1$ and $\nu(H) = 2$ (Theorem 4). Thus, Theorems 1, 2, and 4 give the values $\nu(H)$ for all schemes H.

Theorem 2 enables us to state unbounded fractionality of the class $M(H)$ of "primal" maximum multiflow problems for each H not having property (P). To this end we use the following statement, similar to one occurred in [6] for polyhedra with 0, 1 vertices (and to one known for the totally dual integral system; see [4]).

STATEMENT 1.1. *Let P be a polyhedron $\{x \in \mathbf{Q}^n : x \geq 0,\ Ax \geq b\}$, where A is a nonnegative $m \times n$ matrix and b is an integral m-component column vector, and let k be a positive integer. Let the program $D(c) := \max\{yb : y \geq 0,\ yA \leq c\}$ have a $1/k$-integral optimal solution for each n-component integral row vector $c \geq 0$ whenever $D(c)$ has an optimal solution. Then P is $1/k$-integral.*

Proof. Let x be a vertex in P, and x_i be a component of the vector x. It follows from nonnegativity of A that there exists $c = (c_1, \dots c_n) \in \mathbf{Z}^n$ so that any $c' \in \mathbf{Q}^n$ with $\|c' - c\|_\infty \leq 1$ satisfies

$$c'x = \max\{yb : y \geq 0,\ yA \leq c'\}$$

[$\|a\|_\infty$ is the norm $\max\{|a_i| : i = 1, \dots, n\}$ of a vector $a = (a_1, \dots, a_n)$]. Put $c'_i := c_i + 1$ and $c'_j := c_j$, $j \in \{1, \dots, n\} - \{i\}$. Let y and y' be $1/k$-integral optimal solutions of $D(c)$ and $D(c')$, respectively. Then

$$kx_i = kc'x - kcx = ky'b - kyb,$$

whence the value kx_i is integral. ∎

Applying Statement 1.1 to a matrix A whose rows are the characteristic vectors of the sets EL for $L \in \mathscr{L}(G, H)$ and to the all-unit vector b, we obtain

STATEMENT 1.2. *$\varphi(M(G, H)) \geqslant \varphi(M^*(G, H))$ for any G and H with $VG \supseteq VH$.*

COROLLARY 1.3 (to Statement 1.2 and Theorem 2). *If a scheme H does not have the property* (P), *then $\varphi(M(H)) = \infty$.*

It should be noted that Theorem 1 gives no possibility of determining whether or not $\varphi(M(H))$ is finite for H having the property (P) [in particular, the converse to Statement 1.1 is, in general, false]. Studying fractionality of $M(H)$ for such H's has turned out to be more difficult than for $M^*(H)$, and I shall return to this subject in a forthcoming paper.

In Section 6 another popular kind of multiflow problems will be considered. This is the *feasibility multiflow problem*, denoted by $F(G, c, H, d)$: given G, c, H as above and a *demand* function d on EH, find a c-admissible multiflow f for G and H satisfying

$$v(f, st) := \sum (f(L) : L \in \mathscr{L}(G, st)) \geqslant d(st) \qquad \text{for each} \quad st \in EH. \quad (2)$$

By Farkas's lemma, we have (see [14])

STATEMENT 1.4. *The problem $F(G, c, H, d)$ is solvable (i.e., a required multiflow f exists) if and only if*

$$cl - dq \geqslant 0$$

holds for all functions l on EG and q on EH satisfying

$$l(EL) \geqslant q(st) \qquad \textit{for all} \quad st \in EH \textit{ and } L \in \mathscr{L}(G, st). \qquad (3)$$

Let $C(G, H)$ be the cone of vectors $(l, q) \in \mathbf{Q}_+^{EG} \times \mathbf{Q}_+^{EH}$ satisfying (3); we refer to $C(G, H)$ as a *dual flow cone* (slightly different cones were considered in [15, 16]). According to Statement 1.4 we may define the problem $F^*(G, c, H, d)$ dual to $F(G, c, H, d)$ as: find $(l, q) \in C(G, H)$ such that $\|(l, q)\|_\infty = 1$ and

$$cl - dq < 0. \qquad (4)$$

Thus, the alternative is: $F(G, c, H, d)$ is solvable if and only if $F^*(G, c, H, d)$

is not. Let $F^*(G, H)$ $[F^*(H)]$ be the set of problems $F^*(G, c, H, d)$ with fixed G and H $[H]$ and arbitrary c and d $[G$, c, and $d]$.

THEOREM 3.

(i) *If a scheme H contains no matching with three edges, then* $\varphi(F^*(H))$ *is* 1, 2, *or* 12.

(ii) *If a scheme H contains a matching with three edges, then* $\varphi(F^*(H)) = \infty$.

[A *matching* in a graph is a set of its edges such that no two meet the same vertex.] This theorem will be proved in Section 6; its proof uses Theorem 1 and known results for special cases of H's. In fact we give there the exact values $\varphi(F^*(H))$ for all schemes H.

One more type of multicommodity flow problems is known. This is the *minimum-cost maximum-multiflow problem*: given G, c, H as above and an *edge-cost* function a on EG, find a c-admissible multiflow F whose "total cost" $\Sigma(a(e)\zeta^f(e): e \in E)$ is minimum, subject to the value $v(f)$ being maximum.

Let $CS(H)$ be the set of such problems with fixed H and with arbitrary graph G and integer-valued c and a, and let $CS^*(H)$ be the set of problems dual (in the linear-programming sense) to the problems in $CS(H)$. According to a multiterminal version of the minimum-cost maximum-flow theorem of Ford and Fulkerson [5], $\varphi(CS(H)) = \varphi(CS^*(H)) = 1$ if H is a complete bipirtite graph, i.e., $EH = \{st: s \in S,\ t \in VH - S\}$ for some $\varnothing \neq S \subset VH$. It was proved in [9] that $CS(H) = CS^*(H) = 2$ if H is a complete graph with $r \geqslant 3$ vertices; this result is easily extended to an arbitrary complete r-partite graph H, i.e., $EH = \{st: s \in S_i,\ t \in S_j,\ i < j\}$ for some partition $\{S_1, \dots, S_r\}$ of VH. On the other hand, it was shown in [12] that $\varphi(CS(H)) = \varphi(CS^*(H)) = \infty$ for all other schemes H.

2. METRICS

In this section we establish elementary properties of solutions of the abovementioned problems as well as some facts about vertices of polyhedra $P(G, H)$ and extreme rays of cones $C(G, H)$.

Let V be a finite set with $|V| \geqslant 2$. Denote by $[V]$ the set of all unordered pairs of distinct elements of V; thus, $(V, [V])$ is the complete graph with the vertex set V. By a *metric on* V we mean a function on $[V]$ satisfying the triangle inequality $m(xy) + m(yz) \geqslant m(xz)$ for any $x, y, z \in V$; we assume by definition that $m(vv) = 0$ for $v \in V$.

Let K be a connected graph with $VK = V$. Two metrics associated with K are distinguished:

(a) the *distance* function m_l induced by a function l (of *lengths* of edges) on EK, i.e., $m_l(xy) = \min\{l(EL): L \in \mathscr{L}(K, xy)\}$ for $x, y \in V$;

(b) the metric m_K induced by K, defined to be m_l for the all-unit function l on EK.

Let U be a subset of $[V]$ and l be a function on EK. For $x, y \in V$, an xy-chain L in K is called a *geodesic*, or an *xy-geodesic*, of l if $l(EL) = m_l(xy)$, i.e., L is a shortest chain in the graph K with the lengths l of edges. An xy-geodesic is *U-geodesic* if $xy \in U$. The set of U-geodesics of l is denoted by $\Gamma(l, U)$. If l is the all-unit function, we apply the term "a U-geodesic of K" and the symbol $\Gamma(K, U)$.

We say that a function l' on EK *U-decomposes* l if there is a rational $\lambda > 0$ such that the function $l'' := l - \lambda l'$ is nonnegative and

$$m_l(st) = m_{l'}(st) + m_{l''}(st) \qquad \text{for all} \quad st \in U.$$

l is called *U-primitive* if (i) $m_l(st) > 0$ for some $st \in U$ unless $U = \varnothing$, and (ii) each l' which U-decomposes l is *proportional* to l, i.e., $l' = \lambda l$ for some $\lambda \geqslant 0$. K is called U-primitive if the all-unit function on EK is primitive. Obviously, l is $\varnothing$-primitive if and only if $|Z(l)| \geqslant |EK| - 1$, where $Z(l) := \{e \in E: l(e) = 0\}$.

One popular kind of U-primitive functions for $U \neq \varnothing$ give functions induced by certain cuts of K. More precisely, for $X \subseteq VK$, let $\delta X = \delta_K X$ be the set of edges of K with just one end in X; $\delta X \neq \varnothing$ is called a *simple cut* of K if there is no $Y \subset VK$ such that $\varnothing \neq \delta Y \subset \delta X$. Denote by $\rho X = \rho_K X$ the characteristic function (on EK) of δX. It is easy to check that, for $X \subset V$ and $U \neq \varnothing$, ρX is U-primitive if and only if $|X \cap \{s, t\}| = 1$ for some $st \in U$ and δX is a simple cut of K.

The following describes elementary properties of U-primitive functions (similar statements for metrics occur, for example, in [16, 8]).

STATEMENT 2.1. *Let l be a function on EK and $\varnothing \neq U \subseteq [V]$.*

(i) *If l is U-primitive, then $l(e) = m_l(e)$ for all $e \in EK$, and each $e \in EK$ with $l(e) > 0$ belongs to some U-geodesic of l. In particular, any U-primitive function on $[V]$ is a metric.*

(ii) *A function l' on EK U-decomposes l if and only if $Z(l) \subseteq Z(l')$ and $\Gamma(l, U) \subseteq \Gamma(l', U)$. In particular, any U-primitive function l is determined uniquely up to proportionality by the sets $Z(l)$ and $\Gamma(l, U)$, and K is*

U-primitive if and only if each function l' on EK with $\Gamma(K,U) \subseteq \Gamma(l',U)$ is constant.

(iii) *There exists a finite sequence $l^1,\dots,l^r$ of functions on EG such that each l^i is U-primitive or $\varnothing$-primitive, $l = l^1 + \cdots + l^r$, and $m_l(st) = m_{l^1}(st) + \cdots + m_{l^r}(st)$ for all $st \in U$.*

(iv) *If l is U-primitive, then the metric m_l is U-primitive (m_l is related to $[V]$).*

In [15] and [2] a number of classes of primitive graphs were found. One of them is described as follows. For K and U as above, two edges e and e' in K are said to be *vis-à-vis* if there exists a circuit C of K such that (i) $r := |VC|$ is even, (ii) e and e' are opposite edges in C (i.e., a minimal chain in C containing e and e' has $1 + r/2$ edges), and (iii) each chain in C with $r/2$ edges is a part of some geodesic in $\Gamma(K,U)$. [A *circuit* of a graph is a connected subgraph C in it each vertex of which has valency 2 in C.]

STATEMENT 2.2 [15, 2]. *K is U-primitive if, for any two edges e and e' in K, there is a sequence $e = e_0, e_1, \dots, e_n = e'$ of edges of K such that each two edges e_i and e_{i+1} are vis-à-vis.*

We list several examples of graphs K whose primitivity follows from (2.2) (a verification is left to the reader). These graphs will be used in further sections.

EXAMPLE 1. Let K be the graph drawn in Figure 1. Then K is U-primitive for $U = \{s_1s_2, s_2s_3, s_2s_1, s_4s_5, s_5s_6, s_6s_4\}$. This example was pointed out to the author by V. P. Grishuhin.

EXAMPLE 2. Where T_1 and T_2 are disjoint sets with $|T_i| \geqslant 3$, $i = 1,2$, let K be the graph with vertex set $T_1 \cup T_2 \cup \{x_{st} : s \in T_1,\ t \in T_2\} \cup \{v\}$, whose edges are sx_{st}, tx_{st}, and vx_{st} for all $s \in T_1$ and $t \in T_2$. Then K is U-primitive for $U = [T_1] \cup [T_2]$.

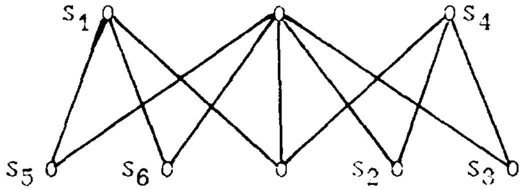

FIG. 1.

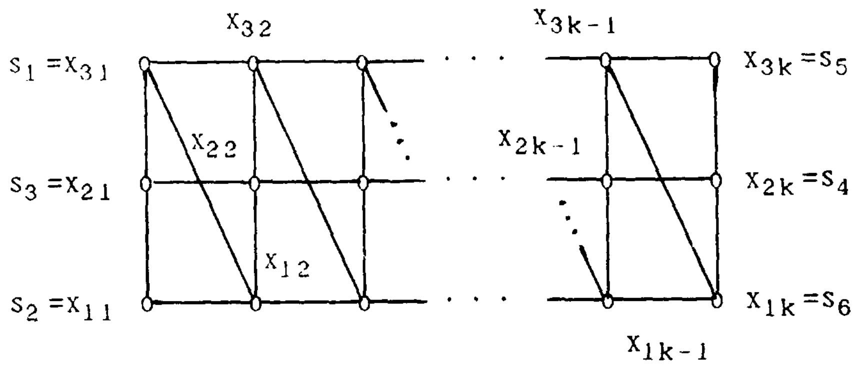

FIG. 2.

EXAMPLE 3. Where $k \geqslant 2$, let K be the graph shown in Figure 2. Then K is U-primitive for $U = \{s_1s_4, s_2s_5, s_3s_6\}$.

EXAMPLE 4. Where $p, q, r \geqslant 2$, let K be the graph with vertex set $\{x_{ijk}: i = 1,\ldots, p,\ j = 1,\ldots, q,\ k = 1,\ldots, r\}$ in which vertices x_{ijk} and $x_{i'j'k'}$ are joined by an edge if and only if either $|i - i'| + |j - j'| + |k - k'| = 1$ or $i' - i = j' - j = k' - k = 1$. Put $s_1 := x_{p11}$, $s_1' := x_{1qr}$, $s_2 := x_{1q1}$, $s_2' := x_{p1r}$, $s_3 := x_{11r}$, $s_3' := x_{pq1}$. Then K is U-primitive for $U = \{s_1s_1', s_2s_2', s_3s_3'\}$.

An important feature of the U-primitive functions is that they generate extreme rays of dual flow cones. More precisely, consider a cone $C = C(G, H)$. Where G is connected, l is a function on EG and $U \subseteq EH$, let $m = m[l, U]$ be the function on EH defined by $m(e) := m_l(e)$ for $e \in U$ and 0 for $e \in EH - U$. Clearly the cone C is polyhedral and pointed (i.e., $a \in C$ and $a \neq 0$ imply $-a \notin C$). $a \in C$ is an *extremal vector* of C if $a = a^1 + a^2$, where $a^i \in C$, implies that $a^1 = \lambda a$ for some $\lambda \geqslant 0$.

STATEMENT 2.3. *Let G be connected. The following are equivalent*:

(i) (l, q) *is an extremal vector of* $C(G, H)$;
(ii) l *is a U-primitive function on EG and* $q = m[l, U]$ *for some* $U \subseteq EH$.

Proof. (ii) $\to$ (i) is trivial. For the converse, let us show that an arbitrary vector (l, q) in C is the sum of some vectors (l', q') as in (ii). We proceed by induction on $|W(q)|$, where $W(q) := \{st \in EH: q(st) > 0\}$. If $W(q) = \varnothing$, then (l, q) is the sum of the vectors $(l_e, 0)$, $e \in EG$, and l_e, the unit basis vector of e in $\mathbf{Q}^{EG}$, is $\varnothing$-primitive. Thus, one may assume that $W(q) \neq \varnothing$.

Let b be the maximum of $\lambda \geqslant 0$ such that $m_{\lambda l}(st) \leqslant q(st)$ for all $st \in W(q)$. Put $l' := bl$, $l'' := l - l'$, $q' := m[l', W(q)]$ and $q'' := q - q'$. Clearly, l'' and q'' are nonnegative, (l', q'), $(l'', q'') \in C$, and $W(q)$ strongly includes $W(q'')$. By induction, (l'', q'') is the sum of some vectors as in (ii). By Statement 2.1(iii), the same is true for (l', q'). ■

Remark 2.4. This statement is easily generalized to an arbitrary (not necessarily connected) graph G. Let $G_1, \ldots, G_r$ be the components of G, $U_i := EH \cap [VG_i]$, and $U_0 := EH - (U_1 \cup \cdots \cup U_r)$. One can see that $C(G, H)$ is the direct product of the cones $C(G_i, H_i)$, $i = 1, \ldots, r$, and $C_{st} := \mathbf{Q}_+^{\{st\}}$, $st \in U_0$; so a vector (l, q) in $C(G, H)$ is extremal if and only if it is the direct product of an extremal vector of one of these cones and the zero vectors of the others. By analogy with the case of a connected graph, the function l for such an (l, q) is called *U-primitive* (for corresponding U).

Standard linear-programming arguments show that, on the one hand, if a problem $F(G, c, H, d)$ has no solution, then there is an extremal vector (l, q) in $C(G, H)$ such that $cl < dq$, and on the other hand, each extremal vector $(l, q) \neq 0$ with $\|(l, q)\|_\infty = 1$ is essential in the sense that, for any $\varepsilon > 0$, there exists an unsolvable problem $F(G, c, H, d)$ such that $cl < dq$ but $cl' \geqslant dq'$ for any $(l', q') \in C(G, H)$ with $\|(l', q')\|_\infty = 1$ and $\|(l, q) - (l', q')\|_\infty \geqslant \varepsilon$. Thus, by Statement 2.3 and Remark 2.4, $\varphi(F^*(G, H))$ is the least positive integer k such that $k\varphi(\xi)$ is integral for all $\xi = (l, m[l, U])$ with $\|\xi\|_\infty = 1$, where l is a U-primitive function on EG for some $U \subseteq EH$.

Now consider a problem $M(G, c, H)$ and its dual problem. Let f and l be feasible solutions of $M(G, c, H)$ and of $M^*(G, c, H)$, respectively [i.e., f is c-admissible and l satisfies (1)]. Note that (1) can be rewritten as

$$m_l(st) \geqslant 1 \qquad \text{for each} \quad st \in EH \tag{5}$$

[m_l is defined as above assuming that $m_l(st) := \infty$ if $\mathscr{L}(G, st) = \varnothing$, i.e., if s and t are in different components of G]. By the linear-programming duality theorem applied to these problems, f and l are optimal if and only if the following (complementary slackness) relations hold:

$$e \in EG,\ l(e) > 0 \text{ imply that } f \text{ saturates } e, \text{ i.e., } \zeta^f(e) = c(e); \tag{6}$$

$$L \in \mathscr{L}(G, st),\ st \in EH,\ f(L) > 0 \text{ imply that } L \text{ is a geodesic of } l \text{ having length } 1, \text{ i.e., } l(EL) = m_l(st) = 1. \tag{7}$$

We distinguish one kind of capacity functions and multiflows. Let $\Gamma \subseteq \mathscr{L}(G, H)$. For $e \in EG$, define $c(e)$ to be the number of chains in Γ

containing e, and for $L \in \mathscr{L}(G, H)$, define $f(L)$ to be 1 if $L \in \Gamma$ and 0 otherwise. These f and c are called the capacity function (for G) and the multiflow (for G and H) *generated by* Γ, respectively. Clearly f saturates each edge in G. Note also that the relation (7) can be rewritten for f and a function l on EG satisfying (5) as

$$l(EL) = 1 \qquad \text{for all} \quad L \in \Gamma. \tag{8}$$

One possible way to prove that the fractionality of a certain scheme H is more than k is as follows. Suppose we succeed in constructing a graph G with $VG \supseteq VH$, a subset $U \subseteq EH$, and a function l^0 on EG that is not $1/k'$-integral for any $k' = 1, \ldots, k$ so that

$$m_{l^0}(st) \geqslant 1 \quad \text{for all} \ st \in EH \quad \text{and} \quad m_{l^0}(st) = 1 \quad \text{for all} \ st \in U, \tag{9}$$

and the system (8) for $\Gamma := \Gamma(l^0, U)$ has a unique solution. Then l^0 is the unique solution of $M^*(G, c, H)$, where c is the capacity function for G generated by Γ, whence $\nu(H) > k$. Indeed, let f be the multiflow for G and H generated by Γ. By (9), l^0 is a feasible solution of $M^*(G, c, H)$. Since (6) and (8) hold for f and l^0, f is optimal. Now if l is an optimal solution of $M^*(G, c, H)$, then l must satisfy (8), and the result follows.

Such an approach will be applied in Section 5 to prove that $\nu(H) > 2$ for some schemes H. Note also that uniqueness of the solution of (8) is ensured whenever we take for G a U-primitive graph such that $m_G(st)$ is a constant b for $st \in U$, and for l^0 the function on EG taking identically the value $1/b$.

Now we explain a relation between vertices of a polyhedron $P(G, H)$ and metrics. We shall identify an edge in H and an edge in G if they have the same ends.

STATEMENT 2.5. *Let l be a vertex of a polyhedron $P(G, H)$. Then $l(e) = m_l(e) \leqslant 1$ for each $e \in EG$.*

Proof. Define l' by $l'(e) := \min\{1, m_l(e)\}$ for $e \in EG$, and let $l'' := 2l - l'$. Clearly $l' \leqslant l \leqslant l''$ and l' satisfies (5); hence l' and l'' are contained in $P(G, H)$. Now since l is a vertex in $P(G, H)$, we have $l = l' = l''$. ∎

COROLLARY 2.6 (to Statement 2.5). *If G is complete and l is a vertex of $P(G, H)$, then l is a metric and $l(st) = 1$ for each $st \in EH$.*

STATEMENT 2.7. *Let l be a vertex of a polyhedron $P(G, H)$, and let G' be the complete graph with $VG' = VG$. Then there exists a vertex l' in $P(G', H)$ such that l' coincides with l on EG.*

Proof. Let c be chosen so that l is the unique optimal solution of $M^*(G, c, H)$. Let c' be the extension of c by zero on $EG' - EG$, and $\tilde{l}$ be the extension of l by unity on $EG' - EG$. Then $\tilde{l} \in P(G', H)$ and $c'\tilde{l} = cl$. Take a vertex l' of $P(G', H)$ that is an optimal solution of $M^*(G', c', H)$, and let l'' be the restriction of l' to EG. Clearly $l'' \in P(G, H)$. We have $cl'' \leqslant c'l' \leqslant c'\tilde{l} = cl$, and now uniqueness of l implies $l = l'' = l'|_{EG}$. ■

It follows from Corollary 2.6 and Statement 2.7 that in order to determine $\nu(H)$ for an arbitrary scheme H it suffices to consider only the set of complete graphs G and the functions l on EG that are metrics with $l(st) = 1$ for all $st \in EH$. This fact will be used, in particular, in the proof of Theorem 1.

3. PROOF OF THEOREM 1

Let G be a graph, c be a capacity function on EG, and H, $VH \subseteq VG$, be a scheme having property (P). One must prove that $M^*(G, c, H)$ has a $\frac{1}{4}$-integral optimal solution. We shall denote VG by V.

Let f and l be optimal solutions of $M(G, c, H)$ and its dual, respectively. As it was explained in the previous section, we may assume that the graph G is complete and l is a metric on V satisfying $l(st) = 1$ for all $st \in EH$. Our end is to find a $\frac{1}{4}$-integral metric m on V satisfying

$$m(st) = 1 \qquad \text{for all} \quad st \in EH \tag{10}$$

and EH-decomposing l, i.e.,

$$x, y \in V, \quad l(xy) = 0 \quad \text{imply} \quad m(xy) = 0; \tag{11}$$

$$\Gamma(l, EH) \subseteq \Gamma(m, EH). \tag{12}$$

Then holding (6) and (7) for f and l implies holding them for f and m; thus, m will be also optimal, and Theorem 1 will be proved. The metric m is derived from l, and it belongs to a special class of metrics on V which we now introduce.

For an arbitrary scheme H', let $\mathscr{A} = \mathscr{A}(H')$ denote the set of anticliques in H', and $\mathscr{D} = \mathscr{D}(H')$ denote the set of nonempty subsets $\alpha \subset VH'$ such that $\alpha = A \cap B$ for some distinct anticliques A and B in H'. Note that if H' has property (P), the members of $\mathscr{D}$ are disjoint. We say that $s \in VH'$ is a 1-*terminal* if s is in exactly one anticlique in H', and a 2-*terminal* otherwise.

For two not necessarily distinct vertices s and t in H' such that $st \notin EH'$, we write $s \sim t$ if s and t are 2-terminals and, for each $A \in \mathscr{A}$, either $s, t \in A$ or $s, t \notin A$; and we write $s \nsim t$ otherwise (in particular, $s \nsim s$ if s is a 1-terminal). Obviously, the relation $\sim$ is transitive, and if $s \sim t$, then, for $p \in VH'$, $sp \in EH'$ implies $tp \in EH'$ (and vice versa).

For H and V as above, let $\mathscr{R} = (X_A; Y_\alpha; Z_A)$ be a family of $2|\mathscr{A}| + |\mathscr{D}|$ subsets X_A $(A \in \mathscr{A})$, Y_α $(\alpha \in \mathscr{D})$, and Z_A $(A \in \mathscr{A})$ of V (each of these subsets can be empty). We say that $\mathscr{R}$ is a *framework* (for V and H) if the sets in $\mathscr{R}$ are disjoint and the following are true:

$$\text{each 1-terminal } s \in A \in \mathscr{A} \text{ is contained in } X_A \text{ and each 2-terminal } s \in \alpha \in \mathscr{D} \text{ is contained either in } Y_\alpha \text{ or in } X_A \text{ for some } A \in \mathscr{A} \text{ such that } \alpha \subset A; \tag{13}$$

$$\text{unless } X_A \text{ is empty, it contains some } s, t \in A \text{ (possibly } s = t) \text{ such that } s \nsim t; \text{ unless } Y_\alpha \text{ is empty, it contains some } s \in \alpha \tag{14}$$

(thus, each Z_A contains no terminal). It should be noted that (14) is not essential in our proof, but it will be useful for considerations in Section 5. Let M be the set of elements of V contained in none of the X_A's, Y_α's, and Z_A's, and define $\mathscr{S}(\mathscr{R})$ to be the family of all sets in $\mathscr{R}$ and the set M. We associate with $\mathscr{R}$ the graph $\mathscr{G} = \mathscr{G}(\mathscr{R})$ with vertex set $\mathscr{S}(\mathscr{R})$ whose edges are

$X_A Z_A$	for all $A \in \mathscr{A}$,
$Y_\alpha Z_A$	for all $A \in \mathscr{A}$ and $\alpha \in \mathscr{D}$ such that $\alpha \subset A$,
$Z_A M$	for all $A \in \mathscr{A}$.

Define the metric $h = h[\mathscr{R}]$ on $\mathscr{S}(\mathscr{R})$ to be $\frac{1}{4} m_{\mathscr{G}}$. In particular, one can see that $h(X_A X_B) = h(X_A Y_\alpha) = h(Y_\alpha Y_\beta) = 1$ if $A, B \in \mathscr{A}$, $A \neq B$, $\alpha, \beta \in \mathscr{D}$, $\alpha \not\subset A$, and neither α nor β is included in any common anticlique in H. Finally, we define the metric $m = m[\mathscr{R}]$ on V *induced by* $\mathscr{R}$ as follows:

(i) $m(xy) := 0$ if x and y are in the same set in $\mathscr{S}(\mathscr{R})$;

(ii) $m(xy) := h(SS')$ if $x \in S$, $y \in S'$, where S and S' are distinct members of $\mathscr{S}(\mathscr{R})$.

STATEMENT 3.1. *$m = m[\mathscr{R}]$ satisfies* (10) *for any framework $\mathscr{R}$ for V and H.*

This follows from (13) and the constructions of $\mathscr{G}$ and $m[\mathscr{R}]$. Thus, the frameworks generate a class of $\frac{1}{4}$-integral metrics that are feasible solutions of $M^*(G, c, H)$.

Construction of the metric m. For $x \in V$, let

$$N(x) := \{z \in V: l(xz) = 0\}.$$

Define $\mathscr{R} = (X_A; Y_\alpha; Z_A)$ as

$$\begin{aligned} X_A &:= \{x \in V: l(sx) + l(xt) < \tfrac{1}{2} \text{ for some } s, t \in A,\ s \nrightarrow t\} \qquad \text{for} \quad A \in \mathscr{A}, \\ Y_\alpha &:= \bigcup(N(s): s \in \alpha) - \bigcup(X_A: A \in \mathscr{A},\ \alpha \subset A) \qquad \text{for} \quad \alpha \in \mathscr{D}, \\ Z_A &:= \{x \in V: l(sx) + l(xt) = \tfrac{1}{2} \text{ for some } s, t \in A,\ s \nrightarrow t\} \\ &\qquad - X_A - \bigcup(Y_\alpha: \alpha \in \mathscr{D},\ \alpha \subset A) \qquad \text{for} \quad A \in \mathscr{A}. \end{aligned} \tag{15}$$

Below we shall prove the following.

LEMMA 3.2. *The collection $\mathscr{R}$ defined by* (15) *is a framework for V and H.*

Let M and $\mathscr{S}(\mathscr{R})$ be defined as above for given $\mathscr{R}$. The required metric m is just $m[\mathscr{R}]$.

As an illustration, consider a scheme H consisting of two complete graphs with vertex sets T_1 and T_2, where $|T_i| \geqslant 3$, $i = 1,2$. Given T_1 and T_2, take the graph K from Example 2 in Section 2. Let G be the complete graph with $VG = VK$, c be the capacity function for G generated by $\Gamma(K, EH)$, and $l := \frac{1}{4}m_K$. Then l is an optimal solution of $M^*(G, c, H)$ (by arguments in Section 2). One can check that the sets in $\mathscr{R}$ defined by (15) for given l are $X_A = \varnothing$, $Z_A = \{x_{st}\}$ $[A = \{s, t\} \in \mathscr{A}(H)]$, and $Y_\alpha = \{s\}$ $(s \in VH)$, whence $M = \{v\}$ and $m[\mathscr{R}]$ coincides with l. Note also that l is the unique optimal solution of $M^*(G, c, H)$ because of the EH-primitivity of K, and so the fractionality of this problem is just 4.

Now we begin to prove correctness and the optimality of m. Put $\mathscr{S} := \mathscr{S}(\mathscr{R})$.

Proof of Lemma 3.2. First of all we observe from (15) that if $x \in S$ for some $S \in \mathscr{S}$, then $N(x) \subseteq S$. Let $s \in VH$. If s is a 1-terminal contained in an anticlique A, then $s \in X_A$ (since $s \nrightarrow s$ and $l(sx) + l(xs) = 0 < \frac{1}{2}$ for $x := s$). If s is a 2-terminal contained in $\alpha \in \mathscr{D}$, then, by (15), either $s \in X_A$ for some

$A \in \mathscr{A}$ such that $\alpha \subset A$, or $s \in Y_\alpha$. Thus, (13) holds. (14) is trivial. It remains to prove that the sets in $\mathscr{S}$ are disjoint. This falls into a number of claims.

CLAIM 3.3.

(i) *$s \sim t$ if and only if s and t are in the same set in $\mathscr{D}$.*

(ii) *$s \nsim t$ if and only if there is a unique anticlique containing both s and t.*

CLAIM 3.4.

(i) *$M \cap S = \varnothing$ for all $S \in \mathscr{S}(\mathscr{R}) - \{M\}$.*

(ii) *$X_A \cap Y_\alpha = X_A \cap Z_A = Z_A \cap Y_\alpha = \varnothing$ for $A \in \mathscr{A}$ and for $\alpha \in \mathscr{D}$ such that $\alpha \subset A$.*

Claim 3.3 obviously follows from property (P), and Claim 3.4 from (15).

CLAIM 3.5. *If $s, t \in A \in \mathscr{A}$, $s \nsim t$, $p \in VH$, $l(sp) < 1$, and $l(tp) < 1$, then $p \in A$.*

Indeed, $sp \notin EH$, since $l(sp) < 1$, and similarly $tp \notin EH$. Hence s, t, and p are in some anticlique B, and we have $B = A$ because $s \nsim t$.

It follows from Claim 3.5 that, firstly, each terminal in $X_A \cup Z_A$ is in A and, secondly, $X_A \cap Y_\alpha = Z_A \cap Y_\alpha = \varnothing$ if $\alpha \not\subset A$.

CLAIM 3.6. *Let A and B be distinct anticliques, and let $s, t \in A$, $s \nsim t$, $p, q \in B$, $p \nsim q$, and $x \in V$. Then:*

(i) *if $l(sx) + l(xt) < \frac{1}{2}$, then $l(px) + l(xq) > \frac{1}{2}$;*

(ii) *if $l(sx) + l(xt) = \frac{1}{2}$, then $l(px) + l(xq) \geqslant \frac{1}{2}$; moreover, if equality holds, then there are $v \in \{s, t\}$ and $w \in \{p, q\}$ such that $v, w \in A \cap B$ and $l(vx) = l(wx) = 0$.*

Indeed, $s \nsim t$, $p \nsim q$, and $A \neq B$ imply $s'p' \in U$ for some $s' \in \{s, t\}$ and $p' \in \{p, q\}$ (otherwise s, t, p, and q would be all in some anticlique C and we would have $A = C = B$, by Claim 3.3). Letting, for definiteness, $s' = s$ and $p' = p$, we have $l(sx) + l(tx) + l(px) + l(qx) \geqslant l(sp) = 1$, whence the required inequalities in (i) and (ii) follow. Next, if $l(sx) + l(tx) = l(px) + l(qx) = \frac{1}{2}$, then $l(tx) + l(qx) = l(tq) = 0$ [since $l(sx) + l(px) \geqslant 1(sp) = 1$]. Now $l(tq) = 0$ and $l(sq) = l(st) < 1$ imply $tq, sq \notin U$, whence $q \in A$, by Claim 3.3(ii). Similarly, $t \in B$.

It follows immediately from Claim 3.6 that $X_A \cap X_B = X_A \cap Z_B = \varnothing$ for distinct anticliques A and B. Thus, we obtain $X_A \cap S = \varnothing$ for all $S \in \mathscr{S} - \{X_A\}$. Next, consider a set Z_A. It was shown that $Z_A \cap X_B = Z_A \cap Y_\alpha = \varnothing$ for any $B \in \mathscr{A}$ and $\alpha \in \mathscr{D}$. This and (13) imply that Z_A contains no terminal. Suppose that $Z_A \cap Z_B \neq \varnothing$ for some $B \neq A$, and let $x \in Z_A \cap Z_B$. Then, by

Claim 3.6(ii), $N(x)$ contains some terminal s, and hence $s \in Z_A$, a contradiction. Thus, $Z_A \cap S = \varnothing$ for all $S \in \mathscr{S} - \{Z_A\}$.

CLAIM 3.7. *$Y_\alpha \cap Y_\beta = \varnothing$ for any distinct $\alpha, \beta \in \mathscr{D}$.*

Indeed, assuming that Y_α and Y_β are nonempty, consider arbitrary terminals s and t in $Y_\alpha \cap \alpha$ and $Y_\beta \cap \beta$, respectively. If $st \in EH$ then $l(st) = 1$, and if $s \nsim t$ then $l(st) \geqslant \frac{1}{2}$ (otherwise we would have $s, t \in X_A$ for the anticlique A containing s and t). In both cases we have $N(s) \cap N(t) = \varnothing$, and the result follows.

Thus, $\mathscr{S}$ consists of pairwise disjoint sets. Lemma 3.2 is proven. ∎

We continue the proof of the theorem. We have remarked above that $m = m[\mathscr{R}]$ satisfies (10) for any framework $\mathscr{R}$. Next, (11) obviously follows from the fact that, for any $S \in \mathscr{S}$, $x \in S$ implies $N(x) \subseteq S$. It remains to prove (12). We start with the following auxiliary statement.

STATEMENT 3.8. *Let s, t, p, and q be (not necessarily distinct) terminals, so that $s \nsim t$ and $pq \in EH$. Then there are $s' \in \{s, t\}$ and $p' \in \{p, q\}$ for which $s'p' \in EH$.*

Proof. Suppose that it is not so for some s, t, p, and q as in the hypotheses of Statement 3.8. Then s, t, and p (or q) are in some anticlique A (or B). Since $s \nsim t$, we have $A = B$ (by Claim 3.3), contrary to $pq \in U$. ∎

Consider an arbitrary EH-geodesic $L = x_0 x_1 \dots x_k$ of l. One must prove that L is also a geodesic of m. Let $p := x_0$ and $q := x_k$. By (10), $m(pq) = l(pq) = l(EL) = 1$. Claims 3.9–3.11 below clarify how L can pass across each of the sets in $\mathscr{S}(\mathscr{R})$. In particular, we shall show that unless $VL \cap S = \varnothing$ for some $S \in \mathscr{S}(\mathscr{R})$, the vertices x_i in L contained in S go in succession (without gaps). For $0 \leqslant i \leqslant j \leqslant k$, let $VL(x_i, x_j)$ denote the set $\{x_i, x_{i+1}, \dots, x_j\}$.

CLAIM 3.9. *If $V' := VL \cap X_A$ is nonempty for some $A \in \mathscr{A}$, then either $V' = VL(p, x_i)$ for some $i < k$ or $V' = VL(x_j, q)$ for some $j > 0$.*

Proof. Choose a vertex x in V', and let s and t be elements of A such that $s \nsim t$ and $l(sx) + l(xt) < \frac{1}{2}$. By Statement 3.8, we may assume, for definiteness, that $sp \in EH$. Then $l(sx) + l(xp) \geqslant l(sp) = 1$ and

$$l(px) + l(xq) + l(sx) + l(xt) < 1 + \tfrac{1}{2} = \tfrac{3}{2},$$

whence $l(tq) \leqslant l(tx) + l(xq) < \frac{1}{2}$ and $l(sq) \leqslant l(st) + l(tq) < 1$. Thus, $tq, sq \notin EH$, and hence $q \in A$. Two cases are possible.

(1) $t \nsim q$. For any $x' \in VL(x, q)$, we have

$$l(tx') + l(x'q) \leqslant l(tx) + l(xx') + l(x'q) = l(tx) + l(xq) < \tfrac{1}{2};$$

therefore $x' \in X_A$. Thus, $VL(x, q) \subseteq X_A$.

(2) $t \sim q$. Then $tp \in EH$ (since $qp \in EH$) and $s \nsim q$ (otherwise $s \sim q$ and $t \sim q$ would imply $s \sim t$). Following the same reasoning for t as for s above, we also obtain $VL(x, q) \subseteq X_A$.

Now choosing as x the vertex x_j in V' with the least number j and taking into account that $p \notin X_A$ (since $q \in A$ implies $p \notin A$) we obtain the claim. ■

Claim 3.10. *If $V' := VL \cap Y_\alpha$ is nonempty for some $\alpha \in \mathscr{D}$, then one of the following is valid*:

(i) $V' = VL(p, x_i) \subseteq N(p)$ *for some* $i < k$;
(ii) $V' = VL(x_j, q) \subseteq N(q)$ *for some* $j > 0$;
(iii) $V' = VL(x_i x_j) \subseteq N(s)$ *for some* $0 < i \leqslant j < k$ *and some terminal* s *in* Y_α *such that* $s \nsim p$, $s \nsim q$, *and* $l(ps) = l(qs) = \frac{1}{2}$.

Proof. Let x be an arbitrary vertex in V'. Two cases are possible.

(1) $l(px) = 0$ [the case $l(qx) = 0$ is symmetric]. Then $N(x) \subseteq Y_\alpha$ implies $VL(p, x) \subseteq Y_\alpha$.

(2) $0 < l(px) < 1$ [and hence $0 < l(qx) < 1$]. Choose a terminal s in Y_α such that $l(sx) = 0$. Then $l(ps) = l(px) < 1$ and $l(qs) = l(qx) < 1$, whence $ps, qs \notin EH$. Therefore $p, s \in A$ and $q, s \in B$ for some anticliques A and B. Now $pq \in EH$ implies $A \neq B$, $p \notin B$, and $q \notin A$, whence $s \nsim p$ and $s \nsim q$. Since $x \notin X_A$, we have $l(px) = l(px) + l(xs) \geqslant \frac{1}{2}$; similarly $l(qx) \geqslant \frac{1}{2}$. Now $l(px) + l(xq) = 1$ implies $l(px) = l(qx) = \frac{1}{2}$.

The rest of the proof is trivial. ■

Note that if case (iii) in Claim 3.10 occurs and $s, p \in A \in \mathscr{A}$, then each vertex in $VL(p, x_{i-1})$ which is contained in none of X_A and Y_β $(\beta \subset A)$ must be in Z_A, and similarly for $VL(x_{j+1}, q)$. Hence $VL \cap M = \varnothing$.

Claim 3.11. *Let $V' := VL \cap Z_A$ be nonempty for some $A \in \mathscr{A}$. Then $V' = VL(x_i, x_j)$ for some $0 < i \leqslant j < k$, and exactly one of the following is* *valid*:

(i) $VL(p, x_{i-1}) \subseteq X_A$;
(ii) $VL(p, x_{i-1}) \subseteq Y_\alpha$ *for some* $\alpha \in \mathscr{D}$, $\alpha \subset A$;
(iii) $VL(x_{j+1}, q) \subseteq X_A$;
(iv) $VL(x_{j+1}, q) \subseteq Y_A$ *for some* $A \in \mathscr{D}$, $\alpha \subset A$.

Proof. Let x be an arbitrary vertex in V'. Then there are $s, t \in A$ such that $s \nleftrightarrow t$ and $l(sx) + l(xt) = \frac{1}{2}$. By Statement 3.8, one may assume, for definiteness, that $sp \in EH$. Using similar arguments to those in the proof of Claim 3.9, we conclude that $q \in A$ and that there is $v \in \{s, t\}$ such that $v \nleftrightarrow q$ and $l(vx') + l(x'q) \leqslant \frac{1}{2}$ for any $x' \in VL(x, q)$. This implies that each vertex in $VL(x, q)$ is in one of the sets X_A, Z_A, or Y_α, $\alpha \subset A$. Now applying Claims 3.10 and 3.11, we obtain the conclusion [note that at most one of (i)–(iv) can hold because $|\{p, q\} \cap A| \leqslant 1$]. ■

One easy consequence of Claims 3.9–3.11 is that if $VL \cap M$ is nonempty, then it is $VL(x_i, x_j)$ for some $0 < i \leqslant j < k$. Claims 3.9–3.11 enable to describe all possible cases of passing L across sets in $\mathscr{S}(\mathscr{R})$. More precisely, we assert the following (a verification is straightforward and left to the reader): there are indices $0 = i(0) < i(1) \leqslant i(2) \leqslant i(3) \leqslant i(4) < i(5) = k + 1$ and a mapping $j \to S_j \in \mathscr{S}(\mathscr{R})$, $j = 0, \ldots, 4$, such that:

(i) for $j = 0, \ldots, 4$, $V_j := VL(x_{i(j)}, x_{i(j+1)-1}) \subset S_j$ [if $i' > i''$ we assume by definition $VL(x_{i'}, x_{i''}) := \varnothing$; so each of V_1, V_2, and V_3 can be empty];
(ii) S_0 is either X_A for some $A \in \mathscr{A}$ or Y_α for some $\alpha \in \mathscr{D}$;
(iii) S_4 is either X_B for some $B \in \mathscr{A}$ or Y_β for some $\beta \in \mathscr{D}$;
(iv) S_1 (S_3) is Z_A (Z_B), and if $S_0 = Y_\alpha$ ($S_4 = Y_\beta$) then $\alpha \subset A$ ($\beta \subset B$);
(v) S_2 is either M or Y_ν for some $\nu \in \mathscr{D}$, and if $S_2 = Y_\nu$ then $\nu = A \cap B$.

(See Figure 3.) It follows from (ii)–(v) and the constructions of the graph $\mathscr{G}$ and the metric h that $h(S_j S_{j+1}) = \frac{1}{4}$ for $j = 0, \ldots, 3$. Let J be the set of pairs (j, j') such that $0 \leqslant j < j' \leqslant 4$ and $i(j-1) < i(j) = i(j+1) = \cdots = i(j'-1)$

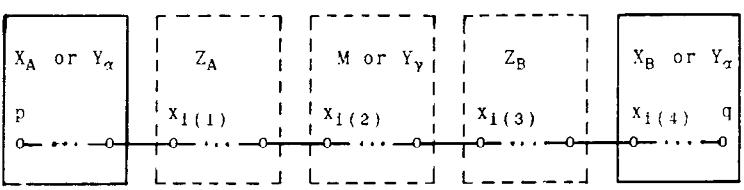

FIG. 3.

$< i(j')$ [letting $i(-1) := -1$]. Using (i) and the fact that m and h are metrics, we have

$$m(EL) = \sum\big(m(x_{i(j)}x_{i(j+1)}) : j = 0,\dots,3\big) = \sum\big(m(x_{i(j)}x_{i(j')}) : (j, j') \in J\big)$$

$$= \sum\big(h(s_j s_{j'}) : (j, j') \in J\big) \leqslant \sum\big(h(s_j s_{j+1}) : j = 0,\dots,3\big)$$

$$= 4 \times \tfrac{1}{4} = 1.$$

Now since $m(EL) \geqslant m(pq) = 1$, we obtain $m(EL) = 1$, as required.

This completes the proof of Theorem 1.

In conclusion we make several observations from the proof of Theorem 1; they will be used in Section 5. As before, we assume that H has property (P) and G is complete. It was established in the proof of Theorem 1 that the optimum in $M^*(G, c, H)$ is achieved on a metric m induced by a framework; such a framework is called *optimal* for G, c, and H. Thus, we obtain

STATEMENT 3.12. *Each vertex of the polyhedron $P(G, H)$ is the metric $m[\mathscr{R}]$ induced by a framework $\mathscr{R}$ for VG and H.*

The following is an obvious corollary of the construction of the metric $m[\mathscr{R}]$.

STATEMENT 3.13. *If $\mathscr{R} = (X_A; Y_\alpha; Z_A)$ is a framework for VG and H in which Z_A is empty for each $A \in \mathscr{A}$, then the metric $m[\mathscr{R}]$ is half-integral.*

For $A \in \mathscr{A}$, let $\mathscr{D}(A)$ denote the set of $\alpha \in \mathscr{D}$ such that $\alpha \subset A$. For $\alpha \in \mathscr{D}$, let d_α denote the number of $A \in \mathscr{A}$ such that $\alpha \subset A$. One can check that the metric $m[\mathscr{R}]$ is expressed as

$$m[\mathscr{R}] = \frac{1}{4}\Big(\sum(\rho X_A + \rho W_A : A \in \mathscr{A}) - \sum((d_\alpha - 2)\rho Y_\alpha : \alpha \in \mathscr{D})\Big), \quad (16)$$

where $W_A := X_A \cup Z_A \cup \bigcup(Y_\alpha : \alpha \in \mathscr{D}(A))$ ($\rho X'$ is the characteristic function of the "cut" $\delta X'$, $X' \subseteq V$). This yields a special minimax relation in which the maximum total value $v(G, c, H)$ is strictly bounded by certain linear combinations of capacities of cuts of G.

STATEMENT 3.14. *The following is true*:

$$v(G, c, H) = \frac{1}{4}\min\Big\{\sum\big(c(\delta X_A) + c(\delta W_A) : A \in \mathscr{A}(H)\big) - \sum\big((d_\alpha - 2)c(\delta Y_\alpha) : \alpha \in \mathscr{D}(H)\big)\Big\},$$

where the minimum is taken over all frameworks $\mathscr{R} = (X_A; Y_\alpha; Z_A)$ *for* VG *and* H.

STATEMENT 3.15. *Let* $\mathscr{R} = (X_A; Y_\alpha; Z_A)$ *be an optimal framework for* G, c, *and* H, *and suppose that, for some* $A \in \mathscr{A}$, *no more than one set among* X_A *and* Y_α, $\alpha \in \mathscr{D}(A)$, *is nonempty. Then the framework* $\mathscr{R}'(X'_A; Y'_\alpha; Z'_A)$ *obtained from* $\mathscr{R}$ *by setting* $Z'_A := \varnothing$ (*and keeping the other* $2|\mathscr{A}| + |\mathscr{D}| - 1$ *sets*) *is also optimal.*

Proof. Put $l := m[\mathscr{R}]$ and $l' := m[\mathscr{R}']$. It follows from (16) that if each of X_A and Y_α, $\alpha \in \mathscr{D}(A)$, is empty, then $l' \leqslant l$, whence $cl' \leqslant cl$ and $\mathscr{R}'$ is optimal. Now assume that there is a unique nonempty set among X_A and Y_α, $\alpha \in \mathscr{D}(A)$. Let $\mathscr{R}'' = (X''_A; Y''_\alpha; Z''_A)$ be the framework obtained from $\mathscr{R}$ by setting $X''_A := X_A \cup Z_A$ and $Z''_A := \varnothing$ if $X_A \neq \varnothing$ and by setting $Y''_\alpha := Y_\alpha \cup Z_A$ and $Z''_A := \varnothing$ if $Y_\alpha \neq \varnothing$. Put $l'' := m[\mathscr{R}'']$. One can check that $l(e) = \frac{1}{2}[l'(e) + l''(e)]$ holds for any $e \in EG$. This implies the optimality of both $\mathscr{R}'$ and $\mathscr{R}''$. ∎

4. PROOF OF THEOREM 2

We start with two statements. K' is an *induced subgraph* of a graph K if $VK' = X$ and $EK' = \{xy \in EK : x, y \in X\}$ for some $X \subseteq VK$; K' may be denoted as $K\langle X \rangle$. If K'' is an arbitrary subgraph of K, we write $K'' \subseteq K$.

STATEMENT 4.1. *If a scheme* H' *is an induced subgraph of a scheme* H, *then* $\nu(H) = k\nu(H')$ *for some positive integer* k.

Proof. Consider a problem $M^*(G', c', H')$. Assuming $(VH - VH') \cap VG' = \varnothing$, take the graph G with $VG = VG' \cup (VH - VH')$ and $EG = EG'$. Obviously, $M^*(G, c, H)$ and $M^*(G', c, H')$ have the same set of feasible solutions, whence the result follows. ∎

STATEMENT 4.2. *If a scheme* H *does not have property* (P), *then there exists a subset* $T \subseteq VH$ *of cardinality* 6 *such that* $H' := H\langle T \rangle$ *satisfies*

$$H^0 \subseteq H' \subseteq H^1, \tag{17}$$

where H^0 *and* H^1 *are graphs as drawn in Figure* 4.

(One can show that the converse is also true.)

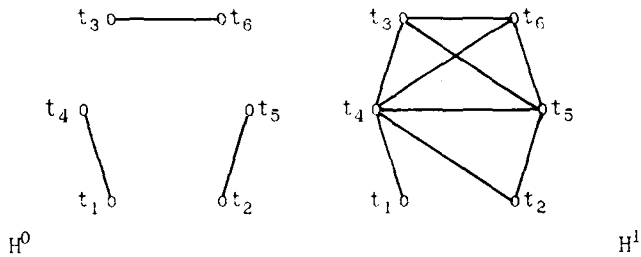

FIG. 4.

Proof. Let A_1, A_2, and A_3 be distinct pairwise intersecting anticliques in H such that $A_1 \cap A_2 \neq A_1 \cap A_3$. Two cases are possible.

(1) For $i = 1,2,3$, each $A'_i := (A_j \cap A_k) - A_i$ is nonempty, where $\{i, j, k\} = \{1,2,3\}$. Choose six terminals, say $t_1, \ldots, t_6$, so that $t_i \in A'_i$, $t_{i+3} \in A_i$, and $t_i t_{i+3} \in EH$ (t_{i+3} exists because A_i is a maximal independent set).

(2) There are A_i and A_j such that $A_i \cap A_j \subset A_k$ and $A' := (A_i \cap A_k) - A_j \neq \varnothing$, where $\{i, j, k\} = \{1,2,3\}$. Choose six terminals, say $t_1, \ldots, t_6$, so that $t_1 \in A_1 \cap A_2 \cap A_3$, $t_2 \in A'$, $t_3 \in A_i - A_k$, $t_5 \in A_j$, $t_6 \in A_k$, and $t_r t_{r+3} \in EH$, $r = 1,2,3$ (t_4 exists because H contains no isolated vertex).

A straightforward check shows that in both cases the terminals $t_1, \ldots, t_6$ are distinct and they induce the subgraph H' in H satisfying (17). ■

We shall assume that the vertices of the graphs H^0 and H^1 are labeled by $t_1, \ldots, t_6$ as shown in Figure 4. Let $\mathscr{H}$ be the set of schemes H' whose vertices are labeled by $t_1, \ldots, t_6$ and such that H' satisfies the inclusions in (17) with preserving the labels. We say that (labeled or unlabeled) graphs K and K' are *isomorphic* (denoted as $K \cong K'$) if their underlying unlabeled graphs are isomorphic.

Now we begin to prove Theorem 2. Let k be a positive integer $\geqslant 3$. According to Statements 4.1 and 4.2 it suffices to consider only the different (up to isomorphism) schemes H in $\mathscr{H}$ and for each of these H's to yield G and c with $\varphi(M^*(G, c, H)) > k$. Unfortunately, the list of such H's is still too long to consider them separately. We give here a common construction for G and c whose features enable us to reduce this list to six cases of H's. The approach developed here extends slightly that outlined in Section 2.

A basic fragment of the graph G is the graph K from Example 3 in Section 2 (for given k); let the vertices of K be labeled as in Figure 2. The

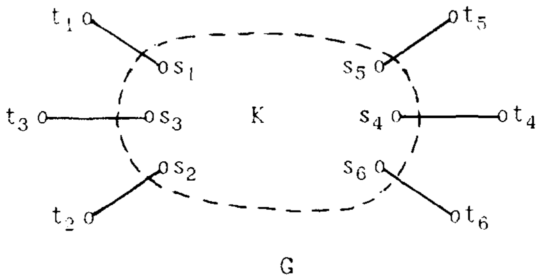

FIG. 5.

graph G is obtained from K by adding terminals $t_1, \dots, t_6$ and the edges $t_i s_i$, $i = 1, \dots, 6$; see Figure 5. It is easy to check that the following are true:

$$m_K(s_i s_j) = \begin{cases} k+1 & \text{for} \quad (i,j) = (2,5), \\ k & \text{for} \quad (i,j) = (1,4),(3,6),(2,4),(3,5), \\ k-1 & \text{for} \quad (i,j) = (3,4), \\ 2 & \text{for} \quad (i,j) = (5,6), \\ 1 & \text{for} \quad (i,j) = (4,5),(4,6) \end{cases} \tag{18}$$

[we point out that $m_K(s_i s_j)$ only for $1 \leqslant i < j \leqslant 6$ such that $t_i t_j \in EH^1$];

every $t_i t_j$-chain L in G, $i \neq j$, has view $t_i s_i \dots s_j t_j$, and for any function l on EG, L is a geodesic of l if and only if the part of L from s_i to s_j is a geodesic of $l|_{EK}$ (in K). (19)

Put $B := \{t_i s_i : i = 1, \dots, 6\}$. Suppose that some $H \in \mathscr{H}$ and $J \subseteq B$ are fixed. Define the capacities $c = c_{H,J}$ of edges of G by

$$c(e) := \begin{cases} c'(e) & \text{for} \quad e \in EG - J, \\ c'(e) + 1 & \text{for} \quad e \in J, \end{cases} \tag{20}$$

where c' is the capacity function for G generated by $\Gamma(G, EH)$ (see Section 2). We say that a function l on EG is *good* for H and J if

$$m_l(t_i t_j) = 1 \qquad \text{for all} \quad t_i t_j \in EH, \tag{21}$$

$$l \text{ is constant on } EK, \tag{22}$$

$$l(e) = 0 \qquad \text{for all} \quad e \in J. \tag{23}$$

Our proof is based on Claims 4.3–4.5 below (Claim 4.3 enables us, in particular, to eliminate many schemes in $\mathscr{H}$).

Claim 4.3. *Suppose that there exists a function l good for H and J. Let $U(l) := \{t_it_j : 1 \leqslant i < j \leqslant 6,\ m_l(t_it_j) \geqslant 1\}$, and let $H' \in \mathscr{H}$ be such that $EH \subseteq EH' \subseteq U(l)$. Then every optimal solution of $M^*(G, c_{H,J}, H')$ is a good function for H and J.*

Proof. Let $c := c_{H,J}$, and let f be the multiflow for G and H' generated by $\Gamma(G, EH)$. We observe that f and l are optimal solutions of $M(G, c, H')$ and its dual, respectively. Indeed, firstly, l is feasible for H' [by (21) and the definition of $U(l)$]. Secondly, (6) holds for f and l because of (23) and the fact that f saturates each edge in $EG - J$ [by (20)]. Thirdly, each chain $L \in \mathscr{L}(G, H')$ with $f(L) > 0$ is an EH-geodesic of G, and now, using (19) (twice) and (22), we obtain that L is an EH-geodesic of l, whence (7) holds for f and l. Now consider an arbitrary optimal solution l' of $M(G, c, H')$. The relations (6) and (7) for H', f, and l' show that (21) and (23) hold for l'. Finally, (22) for l' follows from (7) (for f and l'), (19), and the facts that $EH^0 \subseteq EH$ and K is U^0-primitive, where $U^0 := \{s_1s_4, s_2s_5, s_3s_6\}$. ■

For a good function l, let $a(l)$ denote $l(e)$ for $e \in EK$.

Claim 4.4. *If l is good for H and J and $a(l) > 0$, then l is not $1/k$-integral.*

Proof. Let L be a t_2t_5-geodesic of l, and let L' be the part of L from s_2 to s_5. Then (19) and (22) imply that L' is a geodesic of K. Therefore L' has $k+1$ edges [by (18)]. Since $t_2t_5 \in EH^0 \subseteq EH$, we have $l(EL) = 1$ [by (21)], and hence $1 = l(EL) \geqslant l(EL') = (k+1)a(l)$. So $0 < ka(l) < 1$ and the result follows. ■

Claim 4.5. *The following are equivalent:*

(i) *$a(l) > 0$ for each good function l for H and J;*
(ii) *the system $S(H, J)$ given by*

$$\tau_J(t_is_i)l(t_is_i) + \tau_J(t_js_j)l(t_js_j) = 1, \qquad t_it_j \in EH,$$

where $\tau_J(e)$ is 0 if $e \in J$ and 1 if $e \in B - J$, has no nonnegative solution.

The proof is obvious.

Now we consider six concrete schemes $H_0, \dots, H_5$ in $\mathscr{H}$, where $H_0 := H^0$ and $H_1, \dots, H_5$ are schemes drawn in Figure 6. For each $H = H_i$ we shall fix a subset $J = J_i \subset B$ and point out a function $l = l_i$ good for H and J. Next, we

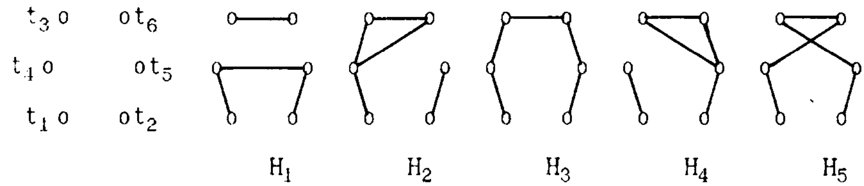

FIG. 6.

shall establish that the system $S(H, J)$ has no solution for each of these H's and J's. Thus, by Claims 4.3, 4.4, and 4.5, $M^*(G, c_{H,J}, H')$ has no $1/k$-integral optimal solution for each $H' \in \mathscr{H}(H, l)$, where $\mathscr{H}(H, l)$ is the set of schemes H' such that $EH \subseteq EH' \subseteq U(l)$ [$U(l)$ is defined as in Claim 4.3]. Finally, we shall show that each scheme in $\mathscr{H}$ is isomorphic to some scheme in

$$\mathscr{F} := \{H_0\} \cup \mathscr{H}(H_1, l_1) \cup \cdots \cup \mathscr{H}(H_5, l_5),$$

whence Theorem 2 will follow by the above arguments.

Case $H = H_0$ $(= H^0)$. Put $J := \{t_1s_1, t_2s_2, t_3s_3, t_5s_5\}$ and $l(e) := 1/(k+1)$ for all $e \in EG - J$ [$l(e) := 0$ for $e \in J$]. One can see from (18) that (21) holds for given H and l. Thus, l is good for H and J. The system $S(H, J)$ contains the equality $0 \cdot l(t_2s_2) + 0 \cdot 1(t_5s_5) = 1$; therefore, $S(H, J)$ has no solution.

Case $H = H_i$, $i = 1, \ldots, 5$. For each of these H's we put $J := \{t_1s_1, t_2s_2\}$ and

$$l(e) := \begin{cases} \dfrac{1}{2k} & \text{for} \quad e \in EK \cup \{t_3s_3\}, \\ \dfrac{k-1}{2k} & \text{for} \quad e = t_5s_5, t_6s_6, \\ \dfrac{1}{2} & \text{for} \quad e = t_4s_4. \end{cases}$$

Using (18), one can check that $U(l) = EH^1$ and $m_l(t_it_j) = 1$ for all $t_it_j \in U(l)$ (and so l is good for each of the H's and J's in question). For $H = H_1$ the

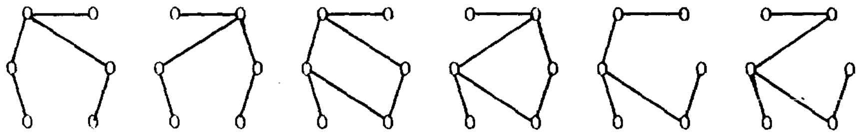

FIG. 7.

system $S(H, J)$ contains the equalities

$$0 \cdot l(t_1 s_1) + l(t_4 s_4) = 1,$$

$$0 \cdot l(t_2 s_2) + l(t_5 s_5) = 1,$$

$$l(t_4 s_4) + l(t_5 s_5) = 1;$$

therefore $S(H, J)$ has no solution. We leave it to the reader to verify the unsolvability of $S(H_i, J)$ for $i = 2, \ldots, 5$.

Now consider an arbitrary scheme $H \in \mathscr{H}$ not belonging to $\mathscr{F}$. Let $\Delta := EH - EH^0$. We have $H \neq H^0$ and $EH_i \not\subseteq EH$ for $i = 1, \ldots, 5$ [otherwise $H \in \mathscr{F}$ because $EH \subseteq EH^1 = U(l_i)$]. This implies that

$$1 \leqslant |\Delta| \leqslant 3 \quad \text{and} \quad \text{either } \Delta \subseteq \{t_2 t_4, t_3 t_4, t_3 t_5\} \text{ or } \Delta \subseteq \{t_2 t_4, t_4 t_6, t_5 t_6\}. \tag{24}$$

Furthermore, $|\Delta| > 1$ (otherwise $H \cong H_1$), and H is not a chain with 5 edges (otherwise $H \cong H_3$). This and (24) imply that H is one of the graphs shown in Figure 7. It is easy to check that each of these graphs is isomorphic to some graph in $\mathscr{F}$.

This completes the proof of Theorem 2.

5. INTEGRALITY AND HALF-INTEGRALITY

Here we describe the classes of schemes H for which $\nu(H) = 1$ and $\nu(H) = 2$. By Theorem 2 such schemes must have property (P). Three cases of schemes H with $\nu(H) = 1$ or 2 are known:

(E1) H is a complete bipartite graph;
(E2) EH consists of two nonadjacent edges;

(E3) there exists a partition $\{\mathscr{A}_1, \mathscr{A}_2\}$ of the set $\mathscr{A}(H)$ of anticliques of H such that each $\mathscr{A}_i$ consists of pairwise disjoint anticliques; in other words, H is the complement of the line graph of a bipartite graph.

(E1) and (E2) are special cases of (E3). In case (E1), $\varphi(M(H)) = \nu(H) = 1$ by a "multiterminal" version of the max-flow min-cut theorem of Ford and Fulkerson [5]. In case (E2), $\varphi(M(H)) = 2$ and $\nu(H) = 1$ by the max-two-commodity-flow min-cut theorem of Hu [7]. In case (E3), $\varphi(M(H)) = \nu(H) = 2$ by [13] (a detailed proof was given in [8, 16]; in [10] a strongly polynomial-time algorithm to solve the problem $M(G, c, H)$ and its dual with such schemes H was developed). Note also that an arbitrary complete graph H is a special case of (E3); the fact that $\varphi(M(H)) = \nu(H) \leqslant 2$ for such an H was independently established by Lovász [17], Mader [18], and Cherkassky [3] (Cherkassky's proof and algorithm were described also in [1]).

Remark. The existence of a half-integral optimal solution of $M^*(G, c, H)$ with H as in (E3) can be established directly from Statement 3.14 as follows. One may assume that the graph G is complete. First of all we observe that each terminal in H is in at most two anticliques, whence $d_\alpha = 2$ for all $\alpha \in \mathscr{D}$. Let $\mathscr{R} = (X_A; Y_\alpha; Z_A)$ be an optimal framework for G, c, and H. Form the families $\mathscr{R}^i = (X_A^i; Y_\alpha^i; Z_A^i)$, $i = 1,2$, by setting $Y_\alpha^i := Z_A^i := \varnothing$ for all $\alpha \in \mathscr{D}$ and $A \in \mathscr{A}$ and

$$X_A^i := \begin{cases} X_A & \text{if} \quad A \in \mathscr{A}_i, \\ W_A \left[= X_A \cup Z_A \cup \bigcup (Y_\alpha : \alpha \in \mathscr{D}(A)) \right] & \text{if} \quad A \in \mathscr{A}_{3-i}. \end{cases} \tag{25}$$

One can check that each $\mathscr{R}^i$ is a framework for G, c, and H. Considering (16) for $\mathscr{R}$, $\mathscr{R}^1$, and $\mathscr{R}^2$ and taking into account that $d_\alpha = 2$ for all $\alpha \in \mathscr{D}$, we obtain $m[\mathscr{R}] = \frac{1}{2}(m[\mathscr{R}^1] + m[\mathscr{R}^2])$. This implies that each $m[\mathscr{R}^i]$ is optimal. Furthermore, $m[\mathscr{R}^i]$ is half-integral, by Statement 3.13.

Let $K + K'$ denote the graph consisting of disjoint graphs K and K'. Let K_n be the complete graph with n vertices. It turns out there is one more scheme H with $\nu(H) = 2$.

Statement 5.1. *If $H = K_2 + K_3$ then $\nu(H) = 2$.*

[$\varphi(M(H)) = 2$ is also true, but a proof of this fact exceeds the limits of our paper.]

Proof. Let, for definiteness, $VH = \{s_1, \ldots, s_5\}$ and $EH = \{s_1s_2, s_3s_4, s_4s_5, s_5s_3\}$. Then $\mathscr{A}(H) = \{\{s_i, s_j\} : 1 \leqslant i \leqslant 2,\ 3 \leqslant j \leqslant 5\}$, $\mathscr{D}(H) = \{\{s_i\} : i = 1,\ldots,5\}$, and H has property (P). Consider a problem

$M^*(G, c, H)$ with a complete graph G. Let $\mathscr{R} = (X_A; Y_\alpha; Z_A)$ be an optimal framework for G, c, and H in which the number $\omega(\mathscr{R})$ of nonempty sets among Z_A, $A \in \mathscr{A}$, is minimum. Put $l := m[\mathscr{R}]$. We shall show that $\omega(\mathscr{R}) = 0$, whence l is half-integral, by Statement 3.13. Suppose, for a contradiction, that $\omega(\mathscr{R}) > 0$.

Consider an anticlique $A = \{s_i, s_j\}$. Let $S = S(A)$ be the collection of nonempty sets among X_A, $Y_{\{s_i\}}$, and $Y_{\{s_j\}}$. It follows from (14) that if X_A (Y_α) is nonempty, then $X_A \cap VH = A$ (respectively, $Y_\alpha \cap VH = \alpha$). Therefore, $|S| \leqslant 2$, and if $|S| = 2$ then $S = \{Y_{\{s_i\}}, Y_{\{s_j\}}\}$. Furthermore, by Statement 3.15, $Z_A = \varnothing$ if $|S| \leqslant 1$.

Let η be the number of $A \in \mathscr{A}$ such that $X_A \neq \varnothing$. Since $|VH| = 5$, $\eta \leqslant 2$. If $\eta = 2$ then $Y_{\{s\}}$ is nonempty for exactly one $s \in VH$, whence by the above arguments $|S(A)| \leqslant 1$ for each $A \in \mathscr{A}$, and all Z_A's are empty. It remains to consider two cases.

Case 1. $\eta = 0$. For $k = 1, 2$, form $\mathscr{R}^k = (X_A^k; Y_\alpha^k; Z_A^k)$ by setting $X_A^k := Z_A^k := \varnothing$ for all $A \in \mathscr{A}$ and

$$Y^k_{\{s_i\}} := \begin{cases} Y_{\{s_i\}} & \text{for } i = k, \\ Y_{\{s_i\}} \cup Z_{\{s_i, s_k\}} & \text{for } i = 3,4,5, \\ Y_{\{s_i\}} \cup M \cup \bigcup\left(Z_{\{s_i, s_j\}} : j = 3,4,5\right) & \text{for } i = 3 - k. \end{cases}$$

Case 2. $\eta = 1$. Let, for definiteness, $X_A = \{s_1, s_3\}$. Then $Y_\alpha = \varnothing$ for $\alpha = \{s_1\}, \{s_3\}$, and $Z_A = \varnothing$ for $A = \{s_1, s_3\}, \{s_1, s_4\}, \{s_1, s_5\}, \{s_2, s_3\}$. For $k = 1, 2$, form $\mathscr{R}^k = (X_A^k; Y_\alpha^k; Z_A^k)$ by setting

$$X_A^k := \begin{cases} X_A & \text{for } A = \{s_1, s_3\}, \\ Y_{\{s_2\}} \cup Y_{\{s_{3+k}\}} \cup Z_{\{s_2, s_{3+k}\}} & \text{for } A = \{s_2, s_{3+k}\}, \end{cases}$$

$$Y_\alpha^k := Y_\alpha \qquad \text{for} \quad \alpha = \{s_{6-k}\},$$

and setting $Q := \varnothing$ for the remaining Q's in $\mathscr{R}^k$.

One can verify that in both cases each $\mathscr{R}$ is a framework for VG and H. Let $l^k := m[\mathscr{R}^k]$. A straightforward though tiring check shows that in each case $l = \frac{1}{2}(l^1 + l^2)$, which implies that each $\mathscr{R}^k$ is optimal. But $\omega(\mathscr{R}^k) = 0$, a contradiction with the choice of $\mathscr{R}$. ■

Now we prove that there is no scheme H with $\nu(H) = 1$ or 2 different from those listed above.

THEOREM 4.

(i) $\nu(H)=1$ *if and only if* H *is as in* (E1) *or* (E2).

(ii) $\nu(H)=2$ *if and only if* H *is as in* (E3) *or Statement* 5.1 *and* H *is different from the scheme in* (E1) *and* (E2).

Proof. It suffices to show that

(a) $\nu(H)>1$ if H is not as in (E1) or (E2), and

(b) $\nu(H)>2$ if H is not as in (E3) or Statement 5.1.

Suppose that H is neither as in (E1) nor as in (E2). It is not difficult to show that there exist a subset $T \subseteq VH$ and a partition $\{T_1, T_2, T_3\}$ of T such that (i) the induced subgraph $H' := H\langle T\rangle$ contains exactly three edges and has no isolated vertices, and (ii) for $1 \leqslant i < j \leqslant 3$, T_i and T_j are joined by an edge. By Statement 4.1, it suffices to show that $\nu(H') \geqslant 2$. Let G be the graph with the vertex set $T \cup \{x_1, x_2, x_3, y\}$ and the edge set $E_0 \cup E_1 \cup E_2 \cup E_3$, where $E_0 := \{x_i y : i=1,2,3\}$ and $E_i := \{x_i p : p \in T_i\}$, $i=1,2,3$. Define $c(e)$ to be 1 for $e \in E_0$ and to be a large enough positive integer for $e \in EG - E_0$. It is easy to see that $M^*(G,c,H')$ has the unique optimal solution l, where $l(e) := \frac{1}{2}$ for $e \in E_0$ and 0 for $e \in EG - E_0$.

In order to prove (b) we need two auxiliary statements. We say that a graph K' is a *vertex minor* of a graph K if there is a sequence $K = K^1, K^2, \ldots, K^n = K'$ of graphs such that, for $i=1,\ldots,n-1$, either $K^{i+1} = K^i\langle X\rangle$ for some $X \subseteq VK^i$ or K^{i+1} is the graph $K^i/\{x,y\}$ obtained from K^i by identifying two nonadjacent vertices x and y in it (and then identifying multiple edges).

STATEMENT 5.2. *If* H' *is a vertex minor of* H, *then* $\nu(H) = k\nu(H')$ *for some integer* $k \geqslant 1$.

[It is also true that $\varphi(M(H)) = k'\varphi(M(H'))$, but it is not important for us.]

Proof. In view of Statement 4.1, it suffices to prove this for the case $H' = H/\{s,t\}$, where s and t are nonadjacent vertices in H. Let p be the vertex in H' arising by identifying s and t. Consider a vertex l' of a polyhedron $P(G',H')$, and choose a problem $M^*(G',c',H')$ having the unique optimal solution l'. Let G be the graph obtained from G' by adding "new" vertices s and t and the edges sp and tp; one can suppose that $VH = (VH' - \{p\}) \cup \{s,t\}$. Put $c(e) := c'(e)$ for $e \in EG'$ and $c(sp) := c(tp) := a$, where $a := c'(EG')+1$. Let l be an optimal solution of $M^*(G,c,H)$, and let b be the function on EG defined by $b(e) := l'(e)$ for $e \in EG'$ and

$b(sp) := b(tp) := 0$. We show that $l = b$, whence the result obviously follows.

Let g be the function on EG' defined by $g(e) := l(e)$ if e is not incident to p and $g(e) := l(e) + l(sp) + l(tp)$ otherwise. It is easy to check that b and g are feasible solutions of $M^*(G, c, H)$ and $M^*(G', c', H')$, respectively (using the facts that $st \notin EH$ and that, for $q \in VH' - \{p\}$, $pq \in EH'$ if and only if at least one of sq and tq is in EH). Obviously, $c'l' = cb$. Also we have

$$cl - c'g = a(l(sp) + l(tp))$$

$$- \sum (c'(e)[l(sp) + l(tp)] : e \in EG',\ e \text{ incident to } p)$$

$$\geqslant [a - c'(EG)][l(sp) + l(tp)] = l(sp) + l(tp),$$

by definition of a. Thus,

$$cl - l(sp) - l(tp) \geqslant c'g \geqslant c'l = cb \geqslant cl, \tag{26}$$

whence $l(sp) = l(tp) = 0$ and $c'g = c'l'$. This implies $g = l'$ (by the uniqueness of optimal l') and, finally, $l = b$. ■

Now we introduce three special cases of schemes H':

(H1) where k is an odd number $\geqslant 5$, $VH' = \{t_1, \dots, t_k\}$, and $t_i t_j \in EH'$ if and only if $2 \leqslant |i - j| \leqslant k - 2$;

(H2) $VH' = \{t_1, \dots, t_5\}$, $t_1t_2, t_2t_3, t_1t_3, t_4t_5 \in EH'$, $t_it_4 \notin EH'$ for $i = 1,2,3$, and $t_it_5 \in EH'$ for at least one $i \in \{1,2,3\}$;

(H3) $H' = K_3 + K_3$.

STATEMENT 5.3. *If H has property* (P) *and H is different from schemes of the form* (E3) *and Statement* 5.1, *then H has a vertex minor isomorphic to one of graphs H' as in* (H1)–(H3).

Proof. Two cases are possible.

Case 1. *Each vertex in H belongs to no more than two anticliques.* Let $A_1, \dots, A_k = A_0$ be a sequence of distinct anticliques of H such that (a) A_{i-1} meets A_i, $i = 1, \dots, k$, (b) k is odd $\geqslant 3$, (c) k is minimum subject to (a) and (b). Such a sequence exists; otherwise H would be as in (E3). Moreover, $k \geqslant 5$ (it is easy to see that if a graph contains three pairwise intersecting anticliques, then it also contains three anticliques having a common vertex). Choose a vertex, say t_i, in $A_{i-1} \cap A_i$, $i = 1, \dots, k$. Then the t_i's are distinct and the graph $H\langle\{t_1, \dots, t_k\}\rangle$ is as in (H1).

Case 2. There are three distinct anticliques, say A_1, A_2, and A_3, in H, having a common vertex s. Let $\alpha := A_1 \cap A_2$ ($= A_i \cap A_j$, $i \neq j$). Choose a vertex, say t_i, in $A_i - \alpha$, $i = 1,2,3$. Then $st_i \notin EH$, $i = 1,2,3$, and $t_i t_j \in EH$, $1 \leqslant i < j \leqslant 3$ [the latter follows from property (P)]. Let $W_1 := \{p \in VH : pt_i \notin EH,\ i = 1,2,3\}$ and $W_2 := VH - W_1 - \{t_1, t_2, t_3\}$. We observe that $W_1 \neq \varnothing$ (as $s \in W_1$) and each vertex in W_1 is joined by an edge with some vertex in $W_1 \cup W_2$ (as H has no isolated vertex). Suppose that $W_2 \neq \varnothing$.

(1) If there are $p \in W_1$ and $q \in W_2$ such that $pq \in EH$, then $H\langle\{t_1, \dots, t_5\}\rangle$ is as in (H2), where $t_4 := p$ and $t_5 := q$. Thus, we may assume that $pq \notin EH$ for any $p \in W_1$ and $q \in W_2$.
(2) If $p, q \in W_1$, $pq \in EH$, and $v \in W_2$, then $H\langle\{t_1, t_2, t_3, p, q, v\}\rangle / \{q, v\}$ is isomorphic to a graph as in (H2). Thus, one may assume that $W_2 = \varnothing$. Since H is not as in Statement 5.1, $|W_1| \geqslant 3$. Note that the graph $H\langle W_1\rangle$ is connected [otherwise H would have at least three components, contrary to the fact that H has property (P)]. So there are three vertices p, q, and v in W_1 such that $pq, pv \in EH$. Now if $qv \in EH$, then $H\langle\{t_1, t_2, t_3, p, q, v\}\rangle$ is as in (H3), and if $qv \notin EH$, then $H\langle\{t_1, t_2, t_3, p, q, v\}\rangle / \{t_1, v\}$ is isomorphic to a graph as in (H2). ■

According to Statements 5.2 and 5.3, it suffices to show that $\nu(H) > 2$ for the schemes $H = H'$ as in (H1)–(H3). We apply the approach set forth in Section 2. More precisely, in each of these cases of H's we construct a graph G, a subset $U \subseteq EH$, and a function l^0 on EG satisfying (9) and not being $\frac{1}{2}$-integral. We show that (8) with $\Gamma := \Gamma(l^0, U)$ has a unique solution.

(a) Let H be as in (H1). Let G be the graph drawn in Figure 8(a), $U := \{t_i t_{i+2} : i = 1, \dots, k\} \cup \{t_1 t_4\}$, and

$$l^0(e) := \begin{cases} \frac{1}{2} & \text{if} \quad e = t_i t_{i+1}, \quad i = 1,2,4,5,\dots,k, \\ \frac{1}{4} & \text{if} \quad e = t_3 v, t_4 v, \\ \frac{3}{4} & \text{if} \quad e = t_1 v \end{cases}$$

(setting $t_{k+j} := t_j$). Obviously, $U \subseteq EH$, (9) holds, and $\Gamma(l^0, U)$ consists of

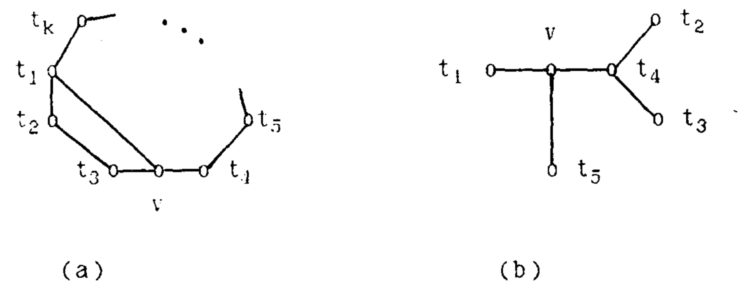

FIG. 8.

the chains $t_i t_{i+1} t_{i+2}$, $i = 1,4,5,\dots,k$, $t_2 t_3 v t_4$, $t_3 v t_4 t_5$, $t_1 v t_3$, and $t_1 v t_4$. Taking into account that k is odd, one can see that (8) has a unique solution.

(b) Let H be as in (H2), and let, for definiteness, $t_1 t_5 \in EH$. Let G be the graph drawn in Figure 8(b), $U := \{t_1t_2, t_2t_3, t_3t_1, t_4t_5, t_1t_5\}$, and

$$l^0(e) := \begin{cases} \frac{1}{2} & \text{if} \quad e = t_2t_4, t_3t_4, \\ \frac{1}{4} & \text{if} \quad e = t_1v, t_4v, \\ \frac{3}{4} & \text{if} \quad e = t_5v. \end{cases}$$

One can see that l^0 satisfies (9) (using the fact that $EH - U$ can contain only t_2t_5 or t_3t_5), $\Gamma(l^0, U)$ consists of the chains $t_1vt_4t_2$, $t_2t_4t_3$, $t_3t_4vt_1$, t_1vt_5, and t_4vt_5, and (8) has a unique solution.

(c) Let H be as in (H3), and let, for definiteness, H have the components with vertex sets $T_1 := \{t_1, t_2, t_3\}$ and $T_2 := \{t_4, t_5, t_6\}$. Take G to be the graph from Example 2 in Section 2, and let $U := EH$ and $l^0(e) := \frac{1}{4}$ for all $e \in EG$. Then (8) has a unique solution because of the U-primitivity of G.

This completes the proof of Theorem 4.

6. DUAL FEASIBILITY MULTIFLOW PROBLEMS

Statements 6.2–6.5 below give the fractionalities $\varphi(F^*(H))$ for all schemes H and, as a consequence, prove Theorem 3 from the Introduction. We start with some observations. It follows from claims in Section 2 that $\varphi(F^*(H))$ is the least common multiple of the numbers $\varphi(l, m[l, U])$, where

$$\varnothing \neq U \subseteq EH,\ l \text{ is a } U\text{-primitive function on the edge set of a connected graph } G \text{ with } [VG] \supseteq U, \text{ and } l \text{ is } U\text{-normalized.} \tag{27}$$

[l is called *U-normalized* if $\|(l, m[l, U])\|_\infty = 1$]. In particular, if H' is a subgraph of H, then $\varphi(F^*(H'))$ is a divisor of $\varphi(F^*(H))$. Also we have the following.

STATEMENT 6.1. *If G is a connected graph and G' is the complete graph with $VG' = VG$, then $\varphi(F^*(G', H)) = k\varphi(F^*(G, H))$ for some integer $k \geqslant 1$.*

Proof. Consider arbitrary l and U satisfying (27) for H and G, and let $q := m[l, U]$ and $l' := m_l$. Obviously, $q = m[l', U]$. By Statement 2.1(iv), l' is U-primitive, therefore (l', q) is an extremal vector of the cone $C(G', H)$. Next, Statement 2.1(i) implies that $\|l\|_\infty \leqslant \|q\|_\infty$ and $\|l'\|_\infty \leqslant \|q\|_\infty$, whence l' is U-normalized, and the result follows. ■

Thus, in order to determine $\varphi(F^*(H))$ we may consider only the set of U-primitive U-normalized metrics. It is easy fact that if $\varphi(l, m[l, U]) = 1$ for some l and U satisfying (27), then l is the characteristic function ρX of a simple cut δX of G. For $l := \rho X$ and $q := m[\rho X, U]$, the inequality in Statement 1.4 turns into the Ford-Fulkerson inequality

$$c(\delta X) \geqslant \sum \big(d(st): st \in EH, \quad |\{s, t\} \cap X| = 1\big). \tag{28}$$

Papernov generalized well-known results of Ford and Fulkerson [5] and Hu [7] by finding all the schemes H such that solvability of any problem $F(G, c, H, d)$ with given H depends only on the truth of (28) for each $X \subset VG$.

STATEMENT 6.2 [19]. *$\varphi(F^*(H)) = 1$ if and only if H is K_4 or C_5 (the circuit with five vertices) or a union of two stars.*

[See Figure 9(a),(b),(c).] (A *star* is a graph without isolated vertices whose edges meet a common vertex; a graph K is a *union* of graphs K^1 and K^2 if there are subgraphs G^1 and G^2 in K such that $G^1 \cup G^2 = K$ and G^i is isomorphic to K^i, $i = 1,2$.) In [11] the following theorem was proved.

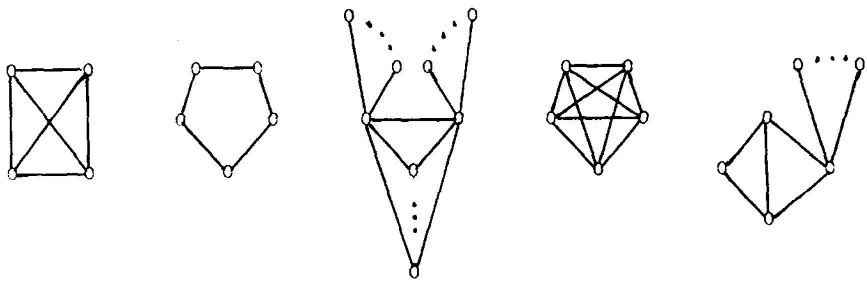

FIG. 9. (a) K_4; (b) C_5; (c) a union of two stars; (d) K_5; (e) a union of K_3 and a star.

THEOREM 6.3. *$\varphi(F^*(H)) = 2$ if H is different from the schemes in Statement 6.2, and it is a subgraph of K_5 (including K_5 itself) or a union of K_3 and a star.*

[See Figure 9(d), (e).]

STATEMENT 6.4. *If H contains a subgraph isomorphic to $K_2 + K_2 + K_2$ (i.e., H contains a matching of three edges), then $\phi(F^*(H)) = \infty$.*

Proof. By the arguments above, it suffices to consider only the scheme $H = K_2 + K_2 + K_2$. Fix an integer $k' \geqslant 3$. Let K be a graph from Example 4 in Section 2 with $p + q + r = k' + 3$, and let $EH = \{s_i s_i' : i = 1,2,3\}$. Consider the function l on EK taking identically the value $1/k'$. Then l is EH-primitive and EH-normalized. Hence $\varphi(F^*(H)) \geqslant \varphi(l, m[l, EH]) = k'$. ∎

One can check that there is a unique scheme different from that described in Statement 6.2, Theorem 6.3, and Statement 6.4, namely, $K_3 + K_3$. Our final statement is the following.

STATEMENT 6.5. *$\varphi(F^*(H)) = 12$ for $H = K_3 + K_3$.*

Proof. Let, for definiteness, $VH = T_1 \cup T_2$, $T_1 = \{s_1, s_2, s_3\}$, $T_2 = \{s_4, s_5, s_6\}$, and $EH = \{s_i s_j : 1 \leqslant i < j \leqslant 3 \text{ or } 4 \leqslant i < j \leqslant 6\}$. First of all we produce EH-primitive EH-normalized functions l_1 and l_2 such that $\varphi(l_1, m[l_1, EH]) = 3$ and $\varphi(l_2, m[l_2, EH]) = 4$.

(1) Let K be the graph with the distinguished vertices $s_1, \ldots, s_6$ from Example 1 in Section 2, and let l be the function on EK taking identically the value $\frac{1}{3}$. Then l is EH-primitive and EH-normalized [since $m_l(s_1 s_2) = 1$].

(2) Let K be the graph from Example 2 in Section 2 for given T_1 and T_2, and let l be the function on EK taking identically the value $\frac{1}{4}$. Then l is EH-primitive and EH-normalized.

Thus, $\varphi(F^*(H)) \geq 3 \times 4 = 12$. Now consider an arbitrary problem $F^*(G, c, H, d)$ with a complete graph G and the given scheme H. We show that its fractionality is less than or equal to 4, whence the result will follow.

For $i = 1, \dots, 6$, let J_i denote the pair $\{j, r\}$ of indices such that $s_i s_j, s_i s_r \in EH$. For $s_i s_j \in EH$, put $d_{ij} := d(s_i s_j)$. We reduce the problem $F(G, c, H, d)$ to $M(G', c', H')$, where:

(i) G' is the complete graph obtained by adding new vertices $t_1, \dots, t_6$ to VG;

(ii) $c'(e) := c(e)$ for $e \in EG$; $c'(t_i s_i) := d_{ij} + d_{ir}$; $\{j, r\} = J_i$, $i = 1, \dots, 6$; $c'(e) := 0$ for the remaining edges e in EG';

(iii) $VH' = \{t_1, \dots, t_6\}$ and $EH' = \{t_i t_j : 1 \leq i < j \leq 3$ or $4 \leq i < j \leq 6\}$ (thus, $H' \cong H$).

Let f' be an optimal solution of $M(G', c', H')$, and let f be the multiflow for G and H induced by f', i.e., $f(s_i \dots s_j) := f'(t_i s_i \dots s_j t_j)$ for each chain $L = s_i \dots s_j$ in $\mathscr{L}(G, H)$. Clearly $v(f', t_i t_j) = v(f, s_i s_j)$ for $s_i s_j \in EH$ [$v(f, st)$ is defined as in (2)]; denote this value by v_{ij}. We have

$$v_{ij} + v_{ir} = \zeta^{f}(t_i s_i) \leq c'(t_i s_i), \quad \{j, r\} = J_i, \qquad i = 1, \dots, 6,$$

or

$$v_{ij} + v_{ir} \leq d_{ij} + d_{ir}, \quad \{j, r\} = J_i, \qquad i = 1, \dots, 6. \tag{29}$$

Considering the inequalities in (29) for $i = 1,2,3$ and for $i = 4,5,6$, we obtain that $F(G, c, H, d)$ is solvable and f is its solution if and only if each inequality in (29) holds with equality (this is equivalent to that $v_{ij} = d_{ij}$ for all $s_i s_j \in EH$).

Now suppose $F^*(G, c, H, d)$ is solvable. Then $F(G, c, H, d)$ is not, and hence at least one inequality in (29) is strong. This implies

$$\sum \left(v_{ij} : s_i s_j \in EH \right) < \sum \left(d_{ij} : s_i s_j \in EH \right). \tag{30}$$

Next, as H' ($\cong H$) has property (P), $M^*(G', c', H')$ has a $\frac{1}{4}$-integral optimal solution l', by Theorem 1. Moreover, by arguments in Section 2, one may assume that l' is a metric on VG' and $\|l'\|_\infty = 1$. Put $l := l'|_{EG}$, $l'_i := l'(t_i s_i)$,

$c_i' := c'(t_i s_i)$, $i = 1,\ldots,6$, and $l_{ij} := l(s_i s_j)$ [$= m_l(s_i s_j)$, as l is a metric]. We assert that

$$cl < \sum \left(d_{ij} l_{ij} : s_i s_j \in EH\right) \tag{31}$$

[cf. (4)], whence $(\lambda l, \lambda q)$ is a solution of $F^*(G, c, H, d)$ with $\varphi(\lambda l, \lambda q) \leqslant 4$, where $q := m[l, EH]$ and $\lambda := 1/\|(l, q)\|_\infty$.

Firstly, since f' and l' are optimal, we have

$$\sum \left(v_{ij} : s_i s_j \in EH\right) = v(f) = c'l' = cl + \sum \left(c_i' l_i' : i = 1,\ldots,6\right). \tag{32}$$

Secondly, $l_i' + l_{ij} + l_j' \geqslant m_{l'}(t_i t_j) = 1$ for any $s_i s_j \in EH$; therefore

$$\begin{aligned} \sum d_{ij} - \sum d_{ij} l_{ij} &\leqslant \sum d_{ij}(l_i' + l_j') \\ &= \sum \left((d_{ij} + d_{ir}) l_i' : \{j, r\} = J_i,\ i = 1,\ldots,6\right) \\ &= \sum \left(c_i' l_i' : i = 1,\ldots,6\right) \end{aligned} \tag{33}$$

(where in the corresponding sums ij runs over the set $\{ij : s_i s_j \in EH\}$). Now, comparing (32) and (33) with (30), we obtain (31), as required. ∎

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