CYCLICAL GAMES WITH PROHIBITIONS

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Abstract. We consider a certain combinatorial game on a digraph for two cases of the price function. For one case the game in question extends the cyclical game studied in [1,2] which, in its turn, is a generalization of the well-known problem of finding a minimum mean cycle in an edge-weighted digraph. We prove the existence of optimal uniform stationary strategies for both cases and give algorithms to find such strategies.

Key words. Digraph, Cyclical Game, Stationary Strategy, Mean Cycle.

Abbreviated title. Cyclical Games.

1. INTRODUCTION

Let $G = (V, E)$ be a digraph whose vertex-set V is partitioned into two subsets A and B, that is, $A \cap B = \emptyset$ and $A \cup B = V$. We admit loops but not multiple edges in G; the edge leaving a vertex v and entering a vertex u is denoted by vu . Throughout the paper we assume that G has no *terminal* vertices, that is, for each $v \in V$ the set $E(v)$ of edges in G leaving v is nonempty. Let $V(v)$ denote the set of end vertices u of edges $vu \in E(v)$.

Let us interpret A (B) as the set of *positions* of a first (second) player, *White* (*Black*) say, and imagine that White and Black play the following game with opposite interests. There is a *chip* which is placed at some initial position $s \in V$. The players move the chip step by step along edges of G. If, at a current moment, the chip occurs at a position $v \in A$ $(v \in B)$ then White (Black) chooses some leaving edge $vu \in E(v)$ to move the chip along vu to the next position u.

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Suppose that the game continues infinitely long; thus the chip passes an infinite path in G. Denote by $\mathcal{P} := \mathcal{P}_s$ the set of all infinite (directed) paths in G starting at s. Let $f : \mathcal{P} \to \mathbb{R}$ be a function; $f(P)$ is called the *price* of a path $P \in \mathcal{P}$. The aim of White is to maximize the price $f(P)$ of the path P made by the chip, while that of Black is to minimize this value.

In what follows we consider certain price functions f which can be expressed in a compact form. Moreover, f in question will provide the existence of optimal behaviors for both players such that, after a finite numbers of moves, the chip will go along one and the same cycle in G. A *game* (an instance of the above game specified by f) having such a property is called *cyclical*.

One special case of cyclical games was studied in [1,2]. Let each edge $e \in E$ has an integer-valued *cost* $c(e) \in \mathbb{Z}$. For $P = v_0 v_1 \dots v_i \dots \in \mathcal{P}_s$ ($v_0 = s$) and an integer $t \geq 1$ denote by P_t the beginning part $v_0 v_1 \dots v_t$ of P. Define the *cost* $c(P_t)$ of P_t to be $\sum(c(v_i v_{i+1}) : i = 0, \dots, t-1)$, and the *mean cost* $\overline{c}(P_t)$ of it to be $c(P_t)/t$. Then the price $f(P)$ of P is defined as

$$f(P) := \overline{\lim}_{t \to \infty} \overline{c}(P_t). \tag{1}$$

The game with the price function (1) is called the *mean cost* game. This generalizes finite positional games and so-called ergodic extensions of matrix games [5]. It turns out that for each position $v \in V$ there exists an optimal choice of a move which depends neither on previous moves nor on an initial position s.

More precisely, we say that White applies a *stationary strategy* $\sigma_A : A \to V$, where $\sigma_A(v) \in V(v)$ for $v \in A$, if whenever the chip occurs at the position $v \in A$, White moves it every time to the same position $\sigma_A(v)$. A stationary strategy for Black is defined analogously. It is clear that if White applies a stationary strategy σ_A and Black does a stationary strategy σ_B then the resulting path P consists of a beginning part L of at most $|V|$ edges, followed by an infinitely repeated cycle C, that is, $P = LCC \dots C \dots$. Thus the price $\overline{c}(P)$ of P is the mean cost $\overline{c}(C)$ of C. We say that a stationary strategy is *uniform* if the player applies it independently of an initial position s.

In fact, the existence of uniform stationary optimal strategies for the mean cost game can be deduced from a general theorem on stationary optimal strategies in stochastic games (cf. [6]) whose proof relies on fixed point techniques. Another, more direct, proof occurred in [1]. In [2] a structural theorem was stated which enables us to observe such strategies explicitly. To formulate this, we need some terminology and notation.

Let $\varepsilon : V \to \mathbb{R}$ be a mapping, called a *potential*. Define the *ε-reduced cost* function c' by

$$c'(vu) := c(vu) + \varepsilon(v) - \varepsilon(u) \qquad \text{for } vu \in E;$$

we say that c' is obtained from c by the *ε-reduction*, denoting this as $c \xrightarrow{\varepsilon} c'$. Observe that an ε-reduction does not change, in essence, our game (since for $P_t = v_0 v_1 \dots v_t$ we have $\overline{c}'(P_t) = \overline{c}(P_t) + (\varepsilon(v_0) - \varepsilon(v_t))/t$, and the second term in the sum becomes infinitely small when t increases). We say that c' is *equivalent* to c if c' can be obtained from c by some ε-reduction. For $a : E \to \mathbb{Q}$ and $v \in V$ define the value

$$\begin{aligned} M(a,v) &:= \max\{a(vu) : u \in V(v)\} \qquad \text{if } v \in A, \\ &:= \min\{a(vu) : u \in V(v)\} \qquad \text{if } v \in B; \end{aligned}$$

and three subsets of $V(v)$:

$$\begin{aligned} V^0(a,v) &:= \{u \in V(v) : a(vu) = M(a,v)\}; \\ V^+(a,v) &:= \{u \in V(v) : a(vu) > M(a,v)\}; \\ V^-(a,v) &:= \{u \in V(v) : a(vu) < M(a,v)\}. \end{aligned}$$

Theorem 1 [2]. *Let the price function be as in (1). Then there exists a rational-valued function c' on E equivalent to c so that for any $v \in V$ the following hold:*

(i) $M(c',u) = M(c',v)$ *if* $u \in V^0(c',v)$,

(ii) $M(c',u) \geq M(c',v)$ *if* $u \in V^+(c',v)$,

(iii) $M(c',u) \leq M(c',v)$ *if* $u \in V^-(c',v)$.

The numbers $M(c',v)$, $v \in V$, are determined uniquely for G and c.

Moreover, it was shown in [2] that c' in Theorem 1 can be chosen not huge, namely, $\|c'\| \leq 2n\|c\|$ (where $\|a\|$ denotes the norm $\max\{|a_i| : i = 1, \dots, r\}$ of a vector $a = (a_1, \dots, a_r)$), and that the denominator of each component of c' does not exceed n^n.

We say that an ordered partition $(V_1, V_2, \dots, V_m)$ of V is *ergodic* if:

(2) for $i = 1, \dots, m$ the subgraph of G induced by V_i has no terminal vertices;

(3) for $1 \leq i < j \leq m$ there are no edges vu in G such that either $v \in A \cap V_i$ and $u \in V_j$, or $v \in B \cap V_j$ and $u \in V_i$.

Theorem 1 has the following obvious corollary.

Proposition 1.1 [2]. *Let c' be as in Theorem 1. Let $p_1 < p_2 < \dots < p_m$ be all different numbers among $M(c',v)$, $v \in V$, and let V_i be the subset of vertices v of G such that $M(c',v) = p_i$. Then $(V_1, \dots, V_m)$ is an ergodic partition of V.*

Theorem 1 (or Proposition 1.1) shows that if White follows a stationary "max"-strategy σ_A by taking for each $v \in A$ an edge vu with $c'(vu)$ maximum (that is, $u = \sigma_A(v) \in V^0(c', v)$) then he ensures his gain at least $p(s)$, where $p(s)$ is p_i for i such that $s \in V_i$. Similarly a stationary "min"-strategy σ_B for Black ensures his losses at most p_i. Thus

$$\inf_{\widetilde{\sigma}_B} \overline{c}(\sigma_A, \widetilde{\sigma}_B, s) = \sup_{\widetilde{\sigma}_A} \overline{c}(\widetilde{\sigma}_A, \sigma_B, s),$$

where the infinum (supremum) is taken over all (not necessarily stationary) strategies $\widetilde{\sigma}_B$ ($\widetilde{\sigma}_A$) of Black (White), and $\overline{c}(\sigma'_A, \sigma'_B, s)$ denotes the price of the path starting at s that arises when strategies σ'_A and σ'_B are applied. This implies the following discrete minimax relation:

$$\max_{\sigma_A} \min_{\sigma_B} \overline{c}(\sigma_A, \sigma_B, s) = \min_{\sigma_B} \max_{\sigma_A} \overline{c}(\sigma_A, \sigma_B, s) = p(s),$$

where σ_A (σ_B) runs over the set of all stationary strategies of White (Black).

The value $p(v)$ is called the *price* of a position $v \in V$ (when v is considered as the initial position in the game). Since $p(v)$ is the mean cost of a (simple) cycle, it is a rational with the denominator at most $|V|$. From Theorem 1 it follows that the problem of determining whether or not the price of a position v is more (less) than or equal to a given number $p \in \mathbb{Q}$ belongs to $NP \cap$co-NP. However, at present no polynomial algorithm is known for this problem. It should be noted that the proof of Theorem 1 given in [2] provides an algorithm for finding an ε-reduction $c \xrightarrow{\varepsilon} c'$ with c' as in this theorem. Lebedev [4] showed that this algorithm is not polynomial in worse case (though it turned out to be fast enough in computer experiments).

Note that if $A = \emptyset$ then the game turns, in fact, into the well-known problem of determining a minimum-mean cycle in G, which can be solved in strongly polynomial time (e.g., an algorithm of Karp [3] has complexity $O(|V||E|)$).

Proposition 1.1 has one combinatorial consequence mentioned in [2]. A bicolored digraph $G = (A, B; E)$ without terminal vertices is called *ergodicly simple* if for any cost function c on E all vertices of G have the same prices. From Proposition 1.1 it follows that G is ergodicly simple if and only if it has no ergodic partition $(V_1, \ldots, V_m)$ with $m > 1$. In particular, consider G such that for each $v \in A$ and $u \in B$ at least one of vu and uv is an edge of G. Then G has no ergodic partition with $m > 1$. This implies that for any $c : E \to \mathbb{Z}$ there exists an ε-reduction $c \xrightarrow{\varepsilon} c'$ such that the values $M(c', v)$ are the same for all $v \in V$ (this generalizes the result of Moulin [5] that, in our terms, any complete symmetric bipartite graph is ergodicly simple). An unexpected fact is that the problem of determining whether a bicolored digraph is ergodicly simple is NP-complete [4].

In the present paper we study a generalization of the mean-cost game as well as one more kind of cyclical games, the so-called maximum cost game.

Let $k : V \to \mathbb{Z}^+$ be a mapping such that $k(v) < |E(v)|$ for any $v \in V$. We call $k(v)$ the number of *prohibitions* in a vertex v. This means that if, at a current moment, the chip occurs at a position v of White (Black) then the other player, Black (White), forbids using on this step the edges of some subset $E' \subset E(v)$ of cardinality $|E'| \le k(v)$ (we also may interpret $k(v)$ as the maximum number of elements of $E(v)$ which can be "spoilt" at the same time).

If the price function f is defined as in (1), we get the *mean cost* game *with prohibitions*.

For $a : E \to \mathbb{R}$ and $v \in A$, let $M_k(a, v)$ denote the $(k(v)+1)$th maximum among the numbers $a(e)$, $e \in E(v)$, that is, $M_k(a, v) := a(e)$ for $e \in E(v)$ such that

$$|\{e' \in E(v) : a(e') > a(e)\}| \le k(v) \text{ and } |\{e' \in E(v) : a(e') < a(e)\}| \le |E(v)| - k(v) - 1.$$

Similarly for $v \in B$ define $M_k(a, v)$ to be the $(k(v)+1)$th minimum among $a(e)$, $e \in E(v)$. The set $V(v)$ is partitioned into three subsets:

$$V_k^0(a, v) := \{u \in V(v) : a(vu) = M_k(a, v)\};$$
$$V_k^+(a, v) := \{u \in V(v) : a(vu) > M_k(a, v)\};$$
$$V_k^-(a, v) := \{u \in V(v) : a(vu) < M_k(a, v)\}.$$

At the present paper we prove the following theorem.

Theorem 2. *Let the price function be as in (1), and let $k \to \mathbb{Z}^+$ be a mapping such that $k(v) < |E(v)|$ for any $v \in V$. Then there exists a rational-valued function c' on E equivalent to c so that for any $v \in V$ the following hold:*

(i) $M_k(c', u) = M_k(c', v)$ *if* $u \in V_k^0(c', v)$,

(ii) $M_k(c', u) \ge M_k(c', v)$ *if* $u \in V_k^+(c', v)$,

(iii) $M_k(c', u) \le M_k(c', v)$ *if* $u \in V_k^-(c', v)$.

The number $M_k(c', v)$, $v \in V$, are determined uniquely for G, c and k.

Theorem 2 easily implies the following result.

Proposition 1.2. *Let c' be as in Theorem 2. Let $p_1 < p_2 < \ldots < p_m$ be all different numbers among $M_k(c', v)$, $v \in V$, and let V_i be the subset of vertices v of G such that $M_k(c', v) = p_i$. Then for $i = 1, \ldots, m$ and $v \in V_i$ the following are true:*

(i) if $v \in A$ *then* $d_{i+1}(v) + \ldots + d_m(v) \le k(v) < d_i(v) + \ldots + d_m(v)$;

(ii) if $v \in B$ *then* $d_1(v) + \ldots + d_{i-1}(v) \le k(v) < d_1(v) + \ldots + d_i(v)$;

where $d_j(v)$ *is the number of edges* $vu \in E(v)$ *with* $u \in V_j$.

The value $p^k(v) := p_i$ is called the *k-price* of a vertex $v \in V_i$. When $k \equiv 0$ the mean cost game with prohibitions turns into the above "pure" mean cost game, and Theorem 2 (Proposition 1.2) turns into Theorem 1 (Proposition 1.1). Theorem 2 and Proposition 1.2 explicitly exhibit optimal uniform stationary strategies for both players. Here a stationary strategy for White means a choice of a subset $E'(v) \subseteq E(v)$ of $k(v) + 1$ edges for each $v \in A$ and a choice of a subset $E''(u) \subset E(u)$ of $k(u)$ edges for each $u \in B$ so that whenever the chip occurs at a position $v \in A$ $(u \in B)$ White is admitted to pick out for moving any edge among those in $E'(v)$ which are not forbidden by Black (respectively, forbids Black to use the edges in $E''(u)$). More formally, by a stationary strategy for White we mean a mapping $\sigma_A : V \to 2^V$ such that $\sigma_A(v) \subseteq V(v)$, $|\sigma_A(v)| = k(v) + 1$ if $v \in A$, and $|\sigma_A(v)| = k(v)$ if $v \in B$. A stationary strategy for Black is defined in a similar way.

Theorem 2 will be proved in Section 3 by use of an auxiliary theorem stated in Section 2. The proof uses ideas of that of Theorem 1 given in [2], and it provides an algorithm to find c' as in Theorem 2 (as well as optimal stationary strategies and the k-prices of the vertices).

In Section 4 we consider the game with another price function f, namely,

$$f(P) := \overline{\lim}_{t\to\infty}\, c(v_t v_{t+1}), \tag{4}$$

where $c : E \to \mathbb{Z}$ and $P = v_0 v_1 \ldots v_i \ldots \in \mathcal{P}_s$. In other words, the price of P is the maximum cost $c(e)$ over the edges e which occur infinitely many times in P. The game with the price function (4) is called the *maximum cost* game. We shall consider at once the corresponding game with prohibitions. We prove that this game has the property similar to that for the mean cost one: the game can be solved in uniform stationary strategies (and therefore, it is cyclical). Moreover, the proof will provide a *strongly polynomial* algorithm to find such strategies.

We shall use one observation concerning cyclical games with prohibitions. Given a k as above, define the function k' on V by

$$\begin{aligned} k'(v) &:= k(v) \quad \text{if } v \in A \\ &:= |E(v)| - k(v) - 1 \quad \text{if } v \in B. \end{aligned} \tag{5}$$

Proposition 1.3. *Let $G' = (A', B'; E)$ be the graph with $A' := A \cup B$ and $B' := \emptyset$, let f be a price function, and let k' be as in (5). Suppose that the game with prohibitions for G', s, k' and f has an optimal solution in stationary strategies $\sigma_{A'}$ and $\sigma_{B'}$ of White and Black respectively. Then σ_A and σ_B are optimal strategies for the game with $G = (A, B; E)$, s, k, f, where*

$$\sigma_A(v) := \sigma_{A'}(v) \text{ and } \sigma_B(v) := \sigma_{B'}(v) \text{ for } v \in A;$$
$$\sigma_B(v) := V(v) - \sigma_{B'}(v) \text{ and } \sigma_A(v) := V(v) - \sigma_{A'}(v) \text{ for } v \in B.$$

Thus for the price functions (1) and (4) it suffices to consider only the assymmetric version of the game with prohibitions in which for each position the choice of edges to move belongs to White, while the choice of edges to forbid belongs to Black. The proof of Proposition 1.3 appeals to obvious reasonings and we leave it to the reader.

2. AUXILIARY THEOREM AND ALGORITHM

Consider the assymetric version of the mean cost game with prohibitions with $A = V$ and $B = \emptyset$. One may assume that the graph G is connected. The proof of Theorem 2 is based on an auxiliary theorem that we state in this section. For brevity the symbol k in the terms $M_k(\cdot, v)$, $V_k^0(\cdot, v)$, $V_k^+(\cdot, v)$, $V_k^-(\cdot, v)$ will be omitted.

Theorem 2.1. *Let c be a rational-valued function on E. Define*

$$\mu := \mu(c) := \max_{v \in V} M(c, v); \qquad \nu := \nu(c) := \min_{v \in V} M(c, v).$$

Let $p \in \mathbb{Q}$ be a number such that $\nu \le p \le \mu$. Then there exist a function c' on E equivalent to c and a partition (V', V'') of V (possibly $V' = \emptyset$ or $V'' = \emptyset$) so that:

(6) $$\nu \le M(c', v) \le p \quad \text{for any } v \in V';$$

(7) $$p \le M(c', v) \le \mu \quad \text{for any } v \in V'';$$

(8) $$\text{if } v \in V' \text{ then } (V^0(c', v) \cup V^-(c', v)) \subseteq V';$$

(9) $$\text{if } v \in V'' \text{ then } (V^0(c', v) \cup V^+(c', v)) \subseteq V''.$$

Proof. We construct a finite sequence $h_0, h_1, \ldots, h_t$ of functions on E such that $h_0 := c$, each h_i is obtained from h_{i-1} by an ε-reduction, and $c' := h_t$ together with some partition (V', V'') satisfies (6)-(9). In fact, an algorithm will be clearly seen behind the proof; the ith iteration of the algorithm consists in forming h_i from h_{i-1}.

Suppose that a function h_{i-1} $(i \geq 1)$ has been already determined. For brevity we denote h_{i-1} by h, and denote $M(h,v)$, $V^0(h,v)$, $V^+(h,v)$, $V^-(h,v)$ by $M(v)$, $V^0(v)$, $V^+(v)$, $V^-(v)$ respectively. The new function h_i that we construct is denoted by h'.

Introduce the following sets:

$$D^0 := \{v \in V : M(v) = p\};$$
$$D^+ := \{v \in V : M(v) > p\}.$$

If $D^+ = \emptyset$ then $c' := h$ satisfies (6)-(9) with $V' = V$ and $V'' := \emptyset$. Thus one may assume that $D^+ \neq \emptyset$ (note that if the case $D_+ \cup D_0 = V$ takes place then $c' := h$ satisfies (6)-(9) with $V' := \emptyset$ and $V'' := V$, but we do not distinguish this case to get a stronger version of the theorem (Proposition 2.3)). An edge $e \in E$ is called *tight* (*exceeding, deficit*) if $h(e)$ is equal to (respectively, more, less than) p.

The function h' arises from h by an ε-reduction of a special kind. We say that h' is the L, δ-*shift* of h, where $L \subseteq V$ and $\delta \in \mathbb{Q}$ if

$$\begin{aligned} h'(vu) &:= h(vu) - \delta \quad \text{if } v \in L \text{ and } u \in \overline{L} \\ &:= h(vu) + \delta \quad \text{if } v \in \overline{L} \text{ and } u \in L \\ &:= h(vu) \qquad \text{otherwise,} \end{aligned} \tag{10}$$

where $\overline{L} := V - L$. Clearly h' is obtained from h by the ε-reduction with the potential ε defined as $\varepsilon(v) := 0$ if $v \in L$ and $\varepsilon(v) := \delta$ if $v \in \overline{L}$.

To determine L, we apply the following labelling process. At first we label the set $X_0 := D^+$. Suppose that we have already labelled sets $X_0, X_1, \ldots, X_j$, where $X_1, \ldots, X_j$ are pairwise disjoint nonempty subsets of D^0. Put $L_j := X_0 \cup X_1 \cup \ldots X_j$. Define X_{j+1} to be the set of all still unlabelled vertices v in D^0 such that there are at least $k(v) + 1 - |V^+(v)|$ tight edges from v to L_j, that is,

$$X_{j+1} := \{v \in D^0 - L_j \ : \ |V^+(v)| + |V^0(v) \cap L_j| \geq k(v) + 1\}. \tag{11}$$

If $X_{j+1} = \emptyset$, the labelling process is completed, and L_j is the resulting set L.

From the labelling process it is easy to see that $D^+ \subseteq L \subseteq D^+ \cup D^0$, and

(12) $v \in V$ belongs to L if and only if there are at least $k(v) + 1 - |V^+(v)|$ tight edges from v to L, that is, $|V^+(v)| + |V^0(v) \cap L| \geq k(v) + 1$.

Now we explain how to determine δ.

(i) For $v \in L$ let $\rho(v)$ be the number of non-deficit edges from v to L, that is, $\rho(v) := |(V^0(v) \cup V^+(v)) \cap L|$. If $\rho(v) \le k(v)$, define $\pi(v)$ to be the $(k(v)+1-\rho(v))$th maximum among the numbers $h(vu)$ for $u \in \overline{L}$ (by (12), there are at least $k(v)+1-\rho(v)$ exceeding edges from v to $\overline{L}$). If $\rho(v) \ge k(v)+1$, put $\pi(v) := +\infty$.

(ii) For $v \in \overline{L}$ let $\omega(v)$ be the number of exceeding edges from v to $\overline{L}$, that is, $\omega(v) := |V^+(v) \cap \overline{L}|$, and let $\lambda(v)$ be the number of edges from v to L. If $\omega(v)+\lambda(v) \ge k(v)+1$, define $\pi(v)$ to be the $(k(v)+1-\omega(v))$th maximum among the numbers $h(vu)$ for $u \in L$ (by (12), there are at most $k(v)-\omega(v)$ non-deficit edges from v to L). Otherwise put $\pi(v) := -\infty$.

From (12) and the definition of π one can see that

$$\pi(v) > p \ \text{ if } \ v \in L, \quad \text{and} \ \ \pi(v) < p \ \text{ if } \ v \in \overline{L}. \tag{13}$$

Consider three possible cases.

Case 1. $\pi(v)$ is finite for at least one vertex $v \in V$. Put

$$\delta := \min\{\min\{\pi(v) - p : v \in L\}, \min\{p - \pi(v) : v \in \overline{L}\}\}. \tag{14}$$

Case 2. $L \ne V$ and $\pi(v)$ is $+\infty$ or $-\infty$ for all $v \in V$. Put

$$\delta := 1 + 2\max\{|h(e)| : e \in E\}. \tag{15}$$

Case 3. $L = V$ (this case is possible only if $D^+ \cup D^0 = V$). Put $\delta := 1$.

Let h' be the L,δ-shift of h. In cases 2 and 3, the process of constructing $h_0, h_1, \ldots$ is completed, and we finish with $c' := h'$. In case 1, we continue the process by forming h_{i+1} from $h_i := h'$, and so on.

(To illustrate the above process, consider the graph G drawn in Fig. 1. In fragment (A0), the numbers on edges e indicate their original costs $c(e) =: h_0$. Let $k(x) = k(y) = 1$ and $k(z) = k(w) = 0$; then $M(c,x) = M(c,y) = M(c,z) = -1$ and $M(c,w) = 1$. Let $p = 0$. Fragment (Ai), $i = 1,2,3$, shows the numbers $h_i(e)$, $e \in E$, obtained on ith iteration and the labelled set $L = L(i+1)$ on the next, $(i+1)$th, one. After four iterations we get the resulting function $c' := h_4$ and the partition (V', V'') with $V' = \{x,y\}$ and $V'' = \{z,w\}$, as indicated in fragment (A4). One can see that $M(c',x) = -1$, $M(c',y) = M(c',z) = 0$, $M(c',w) = 1$, and (6)-(9) hold.)

We have to prove that: (a) the sequence $Q := (h_0, h_1, \ldots, h_i, \ldots)$ is finite, and (b) the final function $h_t =: c'$ in Q satisfies (6)-(9) with some partition (V', V'').

First we prove (b) (assuming validity of (a)). Consider $h := h_{i-1}$ for some $i \geq 1$. By induction on i one may assume that $\nu \leq M(h,v) \leq \mu$ for all $v \in V$. If $D^+ = \emptyset$ then $c' := h$ satisfies (6)-(9) with $V' := V$ and $V'' := \emptyset$. Let $D^+ \neq \emptyset$, and let L, δ and h' arise from h according to the above labelling process. If $L = V$ then (6)-(9) hold for $V' := \emptyset$ and $V'' := V$. Let $L \neq V$. Consider $v \in V$. Since $\delta > 0$ (by (13)-(15)), the definition (10) shows that for $e \in E(v)$, $h'(e) \leq h(e)$ if $v \in L$, and $h'(e) \geq h(e)$ if $v \in \overline{L}$. Hence,

(16) $\quad M(h',v) \leq M(h,v) \leq \mu \ \text{ for } \ v \in L, \quad \text{and } \ M(h',v) \geq M(h,v) \geq \nu \ \text{ for } \ v \in \overline{L}.$

Fig. 1

To prove that

(17) $\qquad\qquad M(h',v) \geq p \ \text{ for } \ v \in L, \quad \text{and } \ M(h',v) \leq p \ \text{ for } \ v \in \overline{L},$

denote by E' (E'') the set of edges from v to L (respectively, to $\overline{L}$).

(i) Let $v \in L$. Then $h'(e) = h(e)$ for $e \in E'$, and $h'(e) = h(e) - \delta < h(e)$ for $e \in E''$. If $\rho(v) \geq k(v) + 1$ then $M(h', v) \geq p$ is obvious. If $\rho(v) \leq k(v)$ then $M(h', v) \geq \min\{m_1, m_2\} \geq \min\{p, m_2\}$, where m_1 is the $\rho(v)$th maximum among $h'(e)$, $e \in E'$, and m_2 is the $(k(v) + 1 - \rho(v))$th maximum among $h'(e)$, $e \in E''$. From the definitions of $\pi(v)$ and δ we observe that $m_2 = \pi(v) - \delta \geq p$, whence $M(h', v) \geq p$.

(ii) Let $v \in \overline{L}$. Then $h'(e) = h(e)$ for $e \in E''$, and $h'(e) = h(e) + \delta > h(e)$ for $e \in E'$. If $\omega(v) + \lambda(v) \leq k(v)$ then there are at most $k(v)$ edges $e \in E(v)$ for which $h'(e) > p$, whence $M(h', v) \leq p$. Let $\omega(v) + \lambda(v) \geq k(v) + 1$. By (13), (14) and (10), the $(k(v) + 1 - \omega(v))$th maximum among $h'(e)$, $e \in E'$, is at most p. Since there are exactly $\omega(v)$ edges e in E'' for which $h'(e) > p$, we conclude that $M(h', v) \leq p$.

Thus (17) is true. Suppose that $\pi(v)$ is $+\infty$ or $-\infty$ for all $v \in V$. Put $V' := \overline{L}$, $V'' := L$ and $c' := h'$. Then (6)-(7) follow from (16)-(17). Next, observe from (10) and (15) that $c'(e) < p$ if $v \in L$ and $e \in E''$, and that $c'(e) > p$ if $v \in \overline{L}$ and $e \in E'$. Hence, (8)-(9) hold.

Now we prove that the sequence Q is finite. Consider two consequtive functions $h := h_{i-1}$ and $h' := h_i$ for $1 \leq i < |Q|$. Let h' be an L, δ-shift of h, and let L be the union of sets $X_0, X_1, \ldots, X_r$ arising in the labelling process for h according to (11). For $j = 0, \ldots, r$ define U_j to be the set of non-deficit edges from $V - L_j$ to X_j, that is,

$$U_j := \{vu \in E : v \in V - L_j,\ u \in X_j,\ h(vu) \geq p\},$$

where $L_j := X_0 \cup X_1 \cup \ldots \cup X_j$. We associate with X_j's and U_j's the sequence

$$S = \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_r, \beta_r$$

of numbers such that $\alpha_j = |X_j|$ and $\beta_j = -|U_j|$. Similarly, let h_{i+1} be the L', δ'-shift of h', and let $(X'_0, X'_1, \ldots, X'_{r'})$, $(U'_0, U'_1, \ldots, U'_{r'})$ and $S' = \alpha'_0, \beta'_0, \alpha'_1, \beta'_1, \ldots, \alpha'_{r'}, \beta'_{r'}$ be the corresponding sequences for h'.

The finiteness of Q is implied by the following lemma.

Lemma 2.2. *S' is lexicographically less than S.*

Proof. Since X_0 (X'_0) is the set of vertices $v \in V$ with $M(h, v) > p$ (respectively, $M(h', v) > p$), and $M(h', v) \leq M(h, v)$ holds for any $v \in V$ with $M(h, v) \geq p$, we have $X'_0 \subseteq X_0$. Thus one may assume that $X'_0 = X_0$. Let q be the maximum index such that $X'_j = X_j$ for $j = 0, \ldots, q$. We know that $h'(e) = h(e)$ for any edge e with both ends in L and that $h'(e) > h(e)$ for any edge e from $\overline{L}$ to L (since $\delta > 0$), therefore, $U_j \subseteq U'_j$ for $j = 0, \ldots, q$. Thus one may assume that $U'_j = U_j$ for $j = 0, \ldots, q$. Let z be a vertex achieving the maximum in (14). From the definition of π it follows that:

(18) if $z \in L$ then $|V^+(h', z) + V^0(h', z) \cap L| \leq k(z)$;

(19) if $z \in \overline{L}$ then $|V^+(h', z) + V^0(h', z) \cap L| \geq k(z) + 1$.

In particular, (18)-(19) imply that the case $q = r = r'$ is impossible. Indeed, if $q = r = r'$ then $L = L'$. But, according to the labelling process for h', z belongs to L' if and only if $|V^+(h', z) + V^0(h', z) \cap L'| \geq k(z) + 1$ (cf. (12)), which is impossible because of (18) and (19). Thus at least one of r and r' is greater than q.

Suppose that S' is lexicographically more than or equal to S. If $r = q$, we may assume by definition that $X_{q+1} := \emptyset$ and $\alpha_{q+1} := 0$. Since $\alpha'_j = \alpha_j$ and $\beta'_j = \beta_j$ for $j = 0, \ldots, q$, $\alpha'_{q+1} > 0$ and $\alpha'_{q+1} \geq \alpha_{q+1}$. Moreover, from the maximal choice of q it follows that there is a vertex $z \in X'_{q+1}$ which is not in X_{q+1}. We assert that there is an edge zu such that $u \in X_q$, $h(zu) < p$ and $h'(zu) = p$, which contradicts with $U'_q = U_q$.

For $g : E \to \mathbb{Q}$ and $j = 0, \ldots, q$ let $\gamma_j(g)$ denote $|V^+(g, z)| + |V^0(g, z) \cap L_j|$. From the labelling processes for h and h' one has:

(20) $$\gamma_q(h) \leq k(z), \quad \gamma_q(h') \geq k(z) + 1 \quad \text{and} \quad \gamma_{q-1}(h') \leq k(z)$$

(putting $\gamma_{-1}(h') := 0$). Consider two cases.

(i) $z \in L$. Then $h'(e) \leq h(e)$ for any $e \in E(z)$. This implies $\gamma_q(h') \leq \gamma_q(h)$, contrary to (20).

(ii) $z \in \overline{L}$. Let $W := V^0(h', z) \cap X_q$. We know that $h'(e) > h(e)$ for any edge e from z to L. This implies that if W is nonempty then $U'_q \neq U_q$ (as $p = h'(zw) > h(zw)$ for $w \in W$). Suppose that $W = \emptyset$. If $q \geq 1$ then $W = \emptyset$ implies $\gamma_q(h') = \gamma_{q-1}(h')$, contrary to (20). Thus $q = 0$. Then $W = \emptyset$ and $\gamma_0(h') \geq k(z) + 1$ imply that $|V^+(h', z)| \geq k(z) + 1$, and therefore, $M(h', z) > p$, which is impossible by (17).

Thus $U'_q \neq U_q$. This contradiction proves the lemma. •

The proof of Theorem 2.1 is completed. ••

The above proof obviously provides an algorithm for finding c' as in Theorem 2.1. However, Lemma 2.2 gives an upper bound for the number of iteration of the algorithm which is only exponential in $|V|$.

In fact, the above proof enables us to get a slightly stronger version of Theorem 2.1; it will be used in the proof of Theorem 2 in the next section.

Proposition 2.3. *Let c, μ, ν and p be as in the hypotheses of Theorem 2.1. Then there exist a function c' on E equivalent to c and a partition (V', V'') of V which satisfy (6)-(9) and the following property:*

(21) if the set $D^+ := \{v \in V : M(c', v) > p\}$ is nonempty then there exists a (simple) cycle C in G such that:

(i) C contains at least one vertex in D^+;

(ii) for each edge vu in C, $c'(vu) = M(c', v) \geq p$.

Proof. Let $c = h_0, h_1, \ldots, h_t = c'$ be the sequence determined in the proof of Theorem 2.1. We show that c' satisfies (21). If $t = 0$ then $M(c, v) \leq p$ for all $v \in V$, and we are done. Let $t \geq 1$. Then h_t is an L, δ-shift of $h := h_{t-1}$; let L be the union of sets $X_0, X_1, \ldots X_r$ arising in the labelling process for h. Since $\pi(v) = \infty$ for all $v \in L = V''$ (as h_t is the last member of the above sequence), we have:

(i) $X_0 = \{v \in V : M(c', v) > p\}$;

(ii) for any $v \in V''$, $M(c', v) = M(h, v) \geq p$ and if vu is an edge such that $v \in V''$ and $c'(vu) = M(c', v)$ then $u \in V''$ and $c'(vu) = h(vu)$.

Moreover, from the labelling process it follows that for any $v \in X_j$, $j = 1, \ldots, r$, there is an edge vu with $u \in X_{j-1}$ and $h(vu) = p$. Now, for each $v \in V''$ choose an edge $e_v = vu \in E(v)$ such that: (a) $c'(e_v) = M(h, v)$, and (b) if $v \in X_j$ for $j \geq 1$ then $u \in X_{j-1}$. Let $U := \{e_v : v \in V''\}$. We know that both ends of each e_v are in V''. Let C be a simple cycle in the subgraph induced by U. Then the choice of edges e_v for $v \in V'' - X_0$ implies that C contains a vertex in X_0, whence (21) follows. •

3. PROOF OF THEOREM 2.

First of all we prove two lemmas. As before, we assume that $A = V$ and $B = \emptyset$. We omit the symbol k in the term $M_k(\cdot, v)$.

Lemma 3.1. *Let g be a function on E equivalent to c. Let μ (ν) be the maximum (respectively, minimum) of $M(g, v)$ for all $v \in V$. Then there exists a rational number p such that $\nu \leq p \leq \mu$ and the denominator of p is at most $n := |V|$.*

Proof. For each $v \in V$ choose an edge $e_v \in E(v)$ such that $g(e_v) = M(g, v)$. Let $U := \{e_v : v \in V\}$, and let C be a cycle (considered as an edge-set) in U. Then $\nu \leq \overline{g}(C) \leq \mu$. Since g is equivalent to c, $\overline{g}(C) = \overline{c}(C)$. Now the lemma follows from the integrality of c and the fact that $|C| \leq n$. •

Lemma 3.2. *Let g be a function on E equivalent to c and such that $\mu - \nu \leq 1/n^2$, where μ, ν and n are defined as in Lemma 3.1. Then there exist a rational p and a function h on E equivalent to c such that $\nu \leq p \leq \mu$ and $M(h, v) = p$ for all $v \in V$.*

Proof. Let p be a rational with the denominator at most n such that $\nu \leq p \leq \mu$ (existing by Lemma 3.1). By Proposition 2.3 applied to g, there exist a function g' equivalent to g and a partition (V', V'') of V satisfying (6)-(9) and (21) (with g' instead of c').

Suppose that the set $\{v \in V : M(g', v) > p\}$ is nonempty. Consider a cycle C as in (21). Since $g'(e) \geq p$ for any edge e of C and this inequality is strong for at least one e, $\overline{c}(C) = \overline{g}'(C) > p$. The denominator of $\overline{c}(C)$ is at most n (as c is integral), hence $\overline{c}(C) - p > 1/n^2$. On the other hand, $\nu \leq \overline{g}'(C) \leq \mu$ (as $\nu \leq M(g', v) \leq \mu$ for all $v \in V$, by (6)-(7)), therefore $\mu - \nu \geq \overline{c}(C) - p$. Thus $\mu - \nu > 1/n^2$; a contradiction.

Thus $\nu \leq M(g', v) \leq p$ for all $v \in V$. Define $f(e) := -g'(e)$ for $e \in E$ and $k'(v) := |E(v)| - k(v) - 1$. Applying to f and k' similar arguments as to g and k we deduce that there exists a function f' such that $M_{k'}(f', v) = p$ for all $v \in V$. Then $h := -f'$ is as required. •

To prove Theorem 2, we now describe a process of splitting the graph G. For a subset $X \subseteq V$ let $\langle X \rangle$ denote the subgraph of G induced by X. Suppose that there have been obtained a partition $\Pi = (V_1, V_2, \ldots, V_q)$ of V and functions g_i on E_i, $i = 1, \ldots, q$, where E_i is the edge-set of $G_i := \langle V_i \rangle$, such that the following hold:

(22) for $i = 1, \ldots, q$, g_i is equivalent (in G_i) to the restriction of c on E_i;

(23) for $i = 1, \ldots, q$ and $v \in V_i$, the number $a(v)$ of edges from v to $V_{i+1} \cup \ldots \cup V_q$ is at most $k(v)$, while the number of edges from v to $V_i \cup \ldots \cup V_q$ is at least $k(v) + 1$;

(24) $\nu_1 \leq \mu_1 \leq \nu_2 \leq \mu_2 \leq \ldots \leq \nu_q \leq \mu_q$, where μ_i (ν_i) is the maximum (respectively, minimum) of values $M_{k'}(g_i, v)$ (concerning G_i) for all $v \in V_i$, and $k'(v) := k(v) - a(v)$.

(The process starts with $q := 1$ and $g_1 := c$.) Consider a graph G_i such that $\nu_i \neq \mu_i$. Two cases are possible.

Case 1. $\mu_i - \nu_i \leq 1/n_i^2$, where $n_i := |V_i|$. Then, by Lemma 3.2 (applied to G_i, g_i and k'), there exist a rational p and a function h on E_i equivalent to g_i such that $\nu_i \leq p \leq \mu_i$ and $M_{k'}(h, v) = p$ for all $v \in V_i$.

Case 2. $\mu_i - \nu_i > 1/n^2$. Put $p := (\mu_i + \nu_i)/2$. By Theorem 2.1 (applied to G_i, g_i and k'), there exist a function h on E_i equivalent to g_i and a partition (V', V'') of V_i satisfying (6)-(9) (with h and k' instead of c' and k).

In Case 1 we replace g_i by the new function h. In Case 2 we replace Π by the new partition $\Pi' := (V_1, \ldots, V_{i-1}, V', V'', V_{i+1}, \ldots, V_q)$; define the function g' on the edge-set E' of $\langle V' \rangle$ to be the restriction of h on E', and similarly define g'' on the edge-set E'' of $\langle V'' \rangle$. Then Π' and the functions $g_1, \ldots, g_{i-1}, g', g'', g_{i+1}, \ldots, g_q$ (after a corresponding renumeration) maintain (22)-(24).

Eventually, after a polynomial (in $|V|$ and $\log\|c\|$) number of steps, we get a partition $\Pi = (V_1, \ldots, V_q)$ and functions $g_1, \ldots, g_q$ satisfying (22)-(24) and

$$\mu_i = \nu_i \quad \text{for} \;\; i = 1, \ldots, q. \tag{25}$$

Put $p_i := \mu_i$. By (22), g_i can be obtained by an ε_i- reduction from the restriction of c on E_i. Define the potential ε' on V by $\varepsilon'(v) := \varepsilon_i(v)$ for $i = 1, \ldots, q$ and $v \in V_i$, and let g be the function on E obtained from c by the ε'-reduction. In order to get a function c' as in Theorem 2 we correct g by "shifting" it as follows. For $1 \leq i < j \leq q$ and an edge e from V_i to V_j (from V_j to V_i), define the "discrepance" $\Delta(e)$ to be $\max\{0, p_i - g(e)\}$ (respectively, $\Delta(e) := \max\{0, g(e) - p_j\}$. Let Δ_{ij} (respectively, Δ_{ji}) be the maximum of $\Delta(e)$ over the edges e from V_i to V_j (from V_j to V_i). Choose numbers $\gamma_1, \gamma_2, \ldots, \gamma_{q-1}, \gamma_q = 0$ so that they satisfy

$$\gamma_i + \gamma_{i+1} + \ldots + \gamma_{j-1} > \max\{\Delta_{ij}, \Delta_{ji}\}$$

for all $1 \leq i < j \leq q$. Now execute the ε''- reduction $g \xrightarrow{\varepsilon''} c'$ with ε'' defined by

$$\varepsilon''(v) = \gamma_i + \gamma_{i+1} + \ldots \gamma_q \quad \text{for} \;\; v \in V_i, \;\; i = 1, \ldots, q.$$

A straightforward check-up shows that, for any edge $e \in E$ from V_i to V_j, $i \neq j$, one has $c'(e) > p_i$ if $i < j$, and $c'(e) < p_i$ if $i > j$. This and (22)-(25) imply that c' satisfies (i)-(iii) in Theorem 2.

This completes the proof of Theorem 2. ••

Remark. One can show that there exists a function $\widetilde{c}'$ satisfying (i)-(iii) in Theorem 2 such that $\|\widetilde{c}'\| < 2n\|c\| + n$, and the denominator of each component of $\widetilde{c}'$ does not exceed n^n. Such a $\widetilde{c}'$ can be obtained from c' as above in the following way. For each $v \in V$ fix an edge $e_v \in E(v)$ such that $c'(e_v) = M_k(c', v)$; let $U := \{e_v : v \in V\}$. Consider the weak components $G_i = (V_i, U_i)$, $i = 1, \ldots, r$, of the graph induced by U. By Theorem 2 the values $c'(e_v)$ are equal to the same number p_i for all $v \in V_i$; let for definiteness $p_1 \leq p_2 \leq \ldots \leq p_r$. One can show that there exists an ε'-reduction (which is the composition of certain L, δ-shifts) of c such that the resulting function $\widetilde{c}'$ satisfies (i)-(iii) in Theorem 2 and has the following additional properties:

(i) for $i = 1, \ldots, r$, each edge $vu \in E$ with $v, u \in V_i$ satisfies $\widetilde{c}'(vu) = c'(vu)$;

(ii) there is a tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, \ldots, r\}$ such that for each $\{i, j\} \in \mathcal{E},\ i < j$, the value

$$\min\{\, \min\{\widetilde{c}'(vu) - p_i : uv \in E, v \in V_i, u \in V_j\},$$
$$\min\{p_j - \widetilde{c}'(vu) : uv \in E, v \in V_j, u \in V_i\}\}$$

is equal to 1 if $p_i < p_j$, and 0 if $p_i = p_j$.

For each $\{i, j\} \in \mathcal{E}$ let e_{ij} be an edge in G connecting V_i and V_j which achieves the minimum in the above expression; put $W := \{e_{ij} : \{i, j\} \in \mathcal{E}\}$. Observe that for any two vertices $v, u \in V$ there is a sequence $v = v_0, v_1, \ldots, v_q = u$ of distinct vertices of G such that v_i and v_{i+1} are connected by an edge, say b_i, either in U or in W. Now the required property for $\widetilde{c}'$ follows from the facts that $|\varepsilon'(v_{i+1}) - \varepsilon'(v_i)| = |c(b_i) - \widetilde{c}'(b_i)|$, and that $|\widetilde{c}'(b_i)|$ is a rational not exceeding $\|c\| + 1$ whose denominator is at most n.

4. MAXIMUM COST GAME

In this section we prove the following theorem on the maximum cost game with prohibitions.

Theorem 4.1. *Let the price function f be as in (4), and let $k : V \to \mathbb{Z}^+$ be a mapping such that $k(v) < |E(v)|$ for any $v \in V$. Then the game for $G = (A, B; E)$, c and k has an optimal solution in uniform stationary strategies.*

Proof. According to Proposition 1.3, it suffices to consider the case $A = V$ and $B = \emptyset$.

Let $p_1 < p_2 < \ldots < p_m$ be all different values among $c(e)$, $e \in E$. Fix some $i \in \{1, \ldots, m\}$, and let $p := p_i$. The core of our proof is to show the existence of a partition $\Pi_i = (V_i^-, V_i^+)$ of V (possibly $V_i^- = \emptyset$ or $V_i^+ = \emptyset$) such that for each vertex $v \in V_i^+$ ($v \in V_i^-$), considered as the initial position in the game, White ensures his gain at least p (respectively, Black ensures his losses less than p) using a uniform stationary strategy. More precisely, we prove the existence of a partition Π_i, a stationary strategy $\sigma_A : V \to 2^V$ of White (that is, $\sigma_A \subseteq V(v)$ and $|\sigma_A(v)| = k(v) + 1$) and a stationary strategy $\sigma_B : V \to 2^V$ of Black (that is, $\sigma_B \subseteq V(v)$ and $|\sigma_A(v)| = k(v)$) so that:

(26) $\sigma_A(v) \subseteq V_i^+$ for $v \in V_i^+$; and $(V(v) \cap V_i^+) \subseteq \sigma_B(v)$ for $v \in V_i^-$;

(27) if $v \in V_i^+$ and if $P = v_0 v_1 \ldots v_t \ldots \in \mathcal{P}_v$ is an arbitrary path *satisfying* σ_A (that is, if $v_{t+1} \in \sigma_A(v_t)$ for any t) then $f(P) \geq p$;

(28) if $v \in V_i^-$ and if $P = v_0 v_1 \ldots v_t \ldots \in \mathcal{P}_v$ is an arbitrary path *satisfying* σ_B (that is, if $v_{t+1} \notin \sigma_B(v_t)$ for any t) then $f(P) < p$.

For a vertex v and a stationary strategy σ of White (Black) define $l(\sigma, v)$ to be the minimum (respectively, define $u(\sigma, v)$ to be the maximum) of $f(P)$ over all paths $P \in \mathcal{P}_v$ satisfying σ. Then (27) and (28) can be rewritten as:

$$l(\sigma_A, v) \geq p \quad \text{for all } v \in V_i^+; \tag{29}$$

$$u(\sigma_B, v) < p \quad \text{for all } v \in V_i^-. \tag{30}$$

We proceed by induction on $|V|$. For $v \in V$ denote by $T(v)$ the set of vertices $u \in V(v)$ such that $c(vu) \geq p$.

The process of forming Π_i falls into several stages. First of all define the set

$$X := \{v \in V : |T(v)| \geq k(v) + 1\} \tag{31}$$

Let $y_1, y_2, \ldots, y_r$ be a maximal sequence of distinct vertices in $\overline{X}$ $(:= V - X)$ such that for each y_i and $Y_i := X \cup \{y_1, \ldots, y_i\}$, $i = 1, \ldots, r$:

(32) the number of edges from y_i to Y_{i-1} and "non-deficit" edges from y_i to $V - Y_{i-1}$ is at least $k(v) + 1$, that is, $|V(y_i) \cap Y_{i-1}| + |T(y_i) \cap \overline{Y}_{i-1}| \geq k(y_i) + 1$

(taking $Y_0 := X$). Put $Y := Y_r$. For an infinite path P in G denote by $I(P)$ the set of edges of G which occur infinitely many times in P.

Claim 1. *Suppose that $Y = V$. Let σ_A be a stationary strategy for White such that $\sigma_A(v) \subseteq T(v)$ for any $v \in X$ and $\sigma_A(y_i) \subseteq T(y_i) \cup Y_{i-1}$ for $i = 1, \ldots, q$. Then $l(\sigma_A, v) \geq p$ for all $v \in V$.*

(The existence of σ_A obviously follows from (31) and (32).)

Proof. Consider $v \in V$ and a path $P = v_0 v_1 \ldots v_t \ldots \in \mathcal{P}_v$ $(v_0 = v)$ satisfying σ_A. Suppose that $c(e) < p$ for all $e \in I(P)$. Let t be an index such that $v_j v_{j+1} \in I(P)$ for any $j \geq t$. Clearly $v_j \notin X$ for $j \geq t$, therefore, $v_j \in Y - X$. Take the minimum number i such that $y_i = v_j$ for some $j \geq t$. Since $c(v_j v_{j+1}) < p$, $v_{j+1} \notin T(y_i)$, whence $v_{j+1} \in Y_{i-1}$. Thus $v_{j+1} = y_{i'}$ for some $i' < i$; a contradiction. •

Thus if $Y = V$ then (26),(29) and (30) hold for $V_i^+ := V$, $V_i^- := \emptyset$ and a strategy σ_A as in Claim 1.

Now suppose that $Y \neq V$. From the maximality of $(y_1, \ldots, y_r)$ it follows that

$$|V(v) \cap Y| + |T(v) \cap \overline{Y}| \leq k(v) \quad \text{for any } v \in \overline{Y}. \tag{33}$$

Let $z_1, z_2, \ldots, z_q$ be a maximal sequence of distinct vertices in Y such that for each z_i and $Z_i := \overline{Y} \cup \{z_1, \ldots, z_i\}$, $i = 1, \ldots, q$:

$$|V(z_i) \cap \overline{Z}_{i-1}| \leq k(z_i) \tag{34}$$

(taking $Z_0 := \overline{Y}$). Put $Z := Z_q$.

Claim 2. *Let $\sigma_B(v) \supseteq T(v) \cup (V(v) \cap Y)$ for $v \in \overline{Y}$ and $\sigma_B(z_i) \supseteq (V(z_i) \cap \overline{Z}_{i-1})$ for $i = 1, \ldots, q$. Then $u(\sigma_B, v) < p$ for all $v \in Z$.*

(The existence of σ_B obviously follows from (33) and (34).)

Proof. Consider $v \in Z$ and an arbitrary path $P = v_0 v_1 \ldots v_t \ldots \in \mathcal{P}_v$ ($v_0 = v$) satisfying σ_B. Suppose that there is an edge $wu \in I(P)$ with $c(wu) \geq p$. Since $wu \notin \sigma_B(w)$ and $T(v') \subseteq \sigma_B(v')$ for all $v' \in \overline{Y}$, we have $w \in Z - \overline{Y}$. We may assume that wu is chosen so that $w = z_j$ with j maximum. Consider an edge $v'w$ belonging to $I(P)$ (such an edge obviously exists). It follows from the definition of σ_B that neither $v' \in V - Z$ nor $v' \in Z_{j'}$ with $j' \leq j$ are possible. Therefore, $v' = z_{j'}$ for some $j' > j$; a contradiction. •

From the maximality of $(z_1, \ldots, z_q)$ it follows that

$$|V(v) \cap \overline{Z}| \geq k(v) + 1 \quad \text{for any } \ v \in \overline{Z}. \tag{35}$$

If $Z = V$ then (26),(29) and (30) hold for $V_i^+ := \emptyset$ and $V_i^- := V$. Otherwise put $V' := \overline{Z}$. Define c' to be the restriction of c on the edge-set of the subgraph G' of G induced by V', and define k' to be the restriction of k on V'. By (35), $|V'(v)| \geq k(v) + 1 = k'(v) + 1$ for all $v \in V'$ where $V'(v) := V(v) \cap V'$. Since $|V'| < |V|$, by induction, for G', c' and k' there are a partition $(V_i'^+, V_i'^-)$ of V' and stationary strategies σ'_A and σ'_B satisfying (26),(29) and (30).

Finally, put $V_i^- := Z \cup V_i'^-$ and $V_i^+ := V_i'^+$. Choose stationary strategies σ_A and σ_B in G so that $\sigma_A(v) = \sigma'_A(v)$ for $v \in V_i^+$; $\sigma_B(v) = \sigma'_B(v)$ for $v \in V_i'^-$; and $\sigma_B(v)$ is as in Claim 2 for $v \in Z$ (σ_A and σ_B are chosen arbitrarily for the other vertices in G). It is easy to see that the resulting partition and strategies satisfy (26),(29) and (30).

Now we finish to prove the theorem. Clearly $\emptyset = V_1^- \subseteq V_2^- \subseteq \ldots \subseteq V_m^-$. Let J be the set of indices $i \in \{1, \ldots, m\}$ such that the set $V_i := V_i^+ \cap V_{i+1}^-$ is nonempty (taking $V_{m+1}^- := V$). Let σ_A^i (σ_B^i) stand for a strategy σ_A (σ_B) as in (26),(29) and (30) for a corresponding i. Define σ_A^* and σ_B^* as

$$\text{for } i \in J \text{ and } v \in V_i, \quad \sigma_A^*(v) := \sigma_A^i(v) \ \text{ and } \ \sigma_B^*(v) := \sigma_B^{i+1}(v). \tag{36}$$

We assert that for each $i \in J$ and $v \in V_i$ the equality $l(\sigma_A^*, v) = u(\sigma_B^*, v) = p_i$ holds, and therefore, σ_A^* and σ_B^* are optimal stationary strategies for White and Black respectively. First of all we observe from (26) that

$$(37) \quad \sigma_A^*(v') \subseteq (V_j \cup \ldots \cup V_m) \quad \text{and} \quad V(v') \cap (V_{j+1} \cup \ldots \cup V_m) \subseteq \sigma_B^*(v') \quad \text{for any } j \in J \text{ and } v' \in V_j.$$

Consider arbitrary $i \in J$, $v \in V_i$ and a path $P = v_0 v_1 \ldots v_t \ldots \in \mathcal{P}_v$ $(v_0 = v)$. Suppose that P satisfies σ_A^*. Then, by (37), each vertex of P is contained in some V_j with $j \geq i$. Therefore, there are numbers $j \geq i$ and t such that $v_{t'} \in V_j$ for all $t' \geq t$. In other words, after t steps the chip is moved only within one set V_j according to the strategy σ_A^j (by (36)). In view of (29), this implies that $f(P) \geq p_j \geq p_i$. Similarly, if P satisfies σ_B^* then, by (37), after a finite number of steps the chip is moved only within one set V_j, $j \leq i$, according to the strategy σ_B^{j+1}. Then, by (30), we have $f(P) < p_{j+1} \leq p_{i+1}$. Now, since there is no edge $e \in E$ with $p_i < c(e) < p_{i+1}$, we conclude that if P satisfies both σ_A^* and σ_B^* then $f(P) = p_i$.

This completes the proof of the theorem. •

The above proof can be easily turned into a strongly polynomial algorithm for finding optimal stationary strategies in this game.

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