

Minimal Mean Weight Cuts and Cycles in Directed Graphs

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1. Introduction

In discrete optimizations, a sort of problem arises in which the goal is to find a set with the minimum (or maximum) mean weight of an element among the sets of a given type. More formally, the problem is posed as follows. Let E be a finite set whose elements $e \in E$ have real weights $c(e) \in \mathbb{R}$, and let $\mathcal{F} \subset 2^E$ be a collection of nonempty subsets of E ; in applications, the sets in $\mathcal{F}$ usually have a certain combinatorial sense and can be given implicitly. By the *weight* of a subset $E' \subseteq E$ we mean the sum $c(E') := \sum_{e \in E'} c(e)$, and by its *mean weight* the ratio $m_c(E') := c(E')/|E'|$. It is required to find a set $F \in \mathcal{F}$ having the minimum (or the maximum) mean weight. Since we do not impose any restrictions on the sign of c , it is sufficient to consider only the minimization problem. We call it the *minimum mean problem* and denote it by $\mathcal{M}(\mathcal{F}, c)$. The value of the minimum is denoted by $m^* = m^*(\mathcal{F}, c)$.

In this paper we are interested in the minimum mean problem for some natural families of cuts and circuits of directed graphs. For some of them, the problem turns out to be intractable (NP-hard), while for some other it is solvable in polynomial time. Besides, we consider an analog for the minimum mean problem for linear cones and develop a general method to solve it; implementations of this method give rise to polynomial algorithms that solve $\mathcal{M}(\mathcal{F}, c)$ for a certain class of sets $\mathcal{F}$, including the set of all cuts and the set of all circuits of a digraph.

Let us start by specifying our terminology and notation. Consider a digraph $G = (V, E)$ with possible multiple edges. Given an edge $e \in E$, denote by $h(e)$ ($t(e)$) the vertex $v \in V$ such that e enters (respectively,

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