A GENERALIZED MFMC-PROPERTY AND MULTICOMMODITY CUT PROBLEMS

A.V. KARZANOV

ABSTRACT

Let K_V be the complete undirected graph with the vertex-set V whose edges have nonnegative weights (lengths) l, T be a distinguished subset of its vertices and $\mathscr{C}$ be a family of cuts in K_V. In the work mainly the maximum fractional packing problem for $\mathscr{C}$ is considered and the question is studied: if T and $\mathscr{C}$ have the property that, for any l, the maximum total value of packings is completely determined by distances (with respect to l) between elements of T. It is known that the answer is affirmative when

(i) $T = \{s, s', t, t'\}$ and $\mathscr{C}$ consists of all cuts containing both edges ss' and tt' (such a $\mathscr{C}$ is said to be a two-commodity cut family)

(ii) $|T|$ is even and $\mathscr{C}$ is the set of all cuts induced by subsets $X \subset V$ such that $X \cap T$ is odd (so-called T-cut family). As main result, a large class (complete under some additional assumptions) of T and $\mathscr{C}$ having this property is described. To that end we introduce a concept of a generalized MFMC-property for abstract families of subsets of a set and

establish some interconnection between multicommodity cut problems and multicommodity flow ones. Also other results about cut packing problems are presented.

1. INTRODUCTION

For a finite set Y, let K_Y denote the complete undirected graph with the vertex-set Y and $E(Y)$ denote the set of edges of K_Y; xy (or yx) denotes the edge with ends x and y. For $X \subseteq Y$, $\partial X = \partial^Y X$ is the set of edges in K_Y with one end in X and the other in $Y - X$. $E \subseteq E(Y)$ is called a *cut* if $E = \partial X$ for some *proper* subset X of Y (i.e. $\phi \neq X \neq Y$). Obviously, $X \neq X'$ and $\partial X = \partial X'$ imply $X' = Y - X$.

We consider a quadruple (V, l, T, S) consisting of a (basic) finite set V, a function $l\colon E(V) \to \mathbf{R}_+$ ($\mathbf{R}_+$ is the set of nonnegative reals), a distinguished subset $T \subseteq V$ and a nonempty collection $S \subset 2^T$ of proper subsets of T; we refer to l, T and S as an *edge length* function, a set of *terminals* and a *scheme*, respectively. Let $\mathscr{X}(V, S)$ denote the family of sets $X \subset V$ such that $X \cap T \in S$, and $\mathscr{C}(V, S)$ the set $\{\partial^V X\colon X \in \mathscr{X}(V, S)\}$ of cuts (note that $\mathscr{C}(V, S)$ may contain repeated members, clearly this is so if and only if there are two complementary members A and $T - A$ in S). A function $\alpha\colon \mathscr{X}(V, S) \to \mathbf{R}_+$ is called *l-admissible* if, for any $e \in E(V)$, the value

$$\lambda^\alpha(e) := \sum(\alpha(X)\colon X \in \mathscr{X}(V, S),\ e \in \partial X)$$

does not exceed $l(e)$ (in other words, the function f on $\mathscr{C}(V, S)$ defined by $f(\partial X) = \alpha(X)$ ($X \in \mathscr{X}(V, S)$) is a packing of $\mathscr{C}(V, S)$ into $E(V)$ weighted by l). Also l-admissible α will be referred to as a *multicommodity cut,* or a *multicut* for the sake of brevity.

In this work we study mainly the maximum packing problem for $\mathscr{C}(V, S)$ and l: find l-admissible $\alpha\colon \mathscr{X}(V, S) \to \mathbf{R}_+$ whose total value $1 \cdot \alpha$ is maximum; this maximum value is denoted by $p(V, S, l)$. Such a problem will be shortly called a *multicut* max-$\sum$ *problem* and denoted by $\sum(V, S, l)$.

Applying the linear programming (l.p.) duality theorem to $\sum(V, S, l)$ we have

(1.1) $$p(V, S, l) = \min\{l \cdot w\colon w \in \mathbf{R}_+^{E(V)},\ w(\partial X) \geq 1\ \ \forall X \in \mathscr{X}(V, S)\}.$$

(For functions $f, g \in \mathbf{R}^Y$, $f \cdot g$ denotes $\sum(f(y)g(y)\colon y \in Y)$ and if $Y' \subseteq Y$ then $f(Y')$ denotes $\sum(f(y)\colon y \in Y')$.)

A number of families of cuts is known for which there is a minimax relation of a more special kind than (1.1). We need some known notions. Let $\mathscr{F} \neq \phi$ be a collection of nonempty subsets of a set E (we admit that $F \subset F'$ for some $F, F' \in \mathscr{F}$ and that $\mathscr{F}$ contains repeated members). The *blocker* $b(\mathscr{F})$ of $\mathscr{F}$ is the set of minimal subsets of E meeting each member of $\mathscr{F}$. $\mathscr{F}$ is said to have the (weak) *MFMC-property* (cf. [1]) if, for any $c \in \mathbf{R}_+^E$,

$$\max(1 \cdot f) = \min\{c(B)\colon B \in b(\mathscr{F})\},$$

where the maximum is taken over all $f\colon \mathscr{F} \to \mathbf{R}_+$ satisfying the packing condition $\sum(f(F)\colon e \in F \in \mathscr{F}) \leq c(e)$ for all $e \in E$.

According to the definition a cut family $\mathscr{C}(V, S)$ has the MFMC-property if the equality

(1.2) $$p(V, S, l) = \min\{l(B)\colon B \in b(\mathscr{C}(V, S))\}$$

holds for any $l \in \mathbf{R}_+^{E(V)}$. It can be shown (see Theorem 7.1 and Corollary 7.2) that if $\mathscr{C}(V, S)$ has the MFMC-property then any member B of $b(\mathscr{C}(V, S))$ is the union of some disjoint T-terminus chains in K_V and, moreover, for any $l \in \mathbf{R}_+^{E(V)}$, (1.2) may be rewritten in the following more nice form:

(1.3) $$p(V, S, l) = \min\{\sum(\mu_l(u)\colon u \in U)\},$$

where the minimum is taken over all $U \subseteq E(T)$ such that $U \cap \partial^T A \neq \phi$ for each $A \in S$.

[Some terminology and notation throughout the paper: a *chain* in K_Y is a set $L \subseteq E(Y)$ of *edges* such that a subgraph in K_Y induced by L is connected and it has the vertices of valency 2 except two vertices x and y which are of valency 1; these x and y are called the ends of L and the edge xy is denoted by eL; a *circuit* is a similar nonempty subset such that each vertex in the induced (connected) subgraph is of valency 2;

a chain L in K_V is called *T-terminus* (for a distinguished T) if $eL \in E(T)$; for $l \in \mathbf{R}_+^{E(V)}$, $\mu_l \in \mathbf{R}_+^{E(V)}$ denotes the distance function induced by l, i.e. $\mu_l(xy)$ is the minimum of $l(L)$ over all chains L in K_V with $eL = xy$.]

Here are some known examples of cut families having the MFMC-property (Examples 2, 3 will occur in Section 5, 6).

Example 1. $T = \{s, t\}$ and $S = \{\{s\}\}$, i.e. $\mathscr{C}(V, S)$ consists of the cuts in K_V "separating" s and t. The maximum packing problem for $\mathscr{C}(V, S)$ with such S is usually called one-commodity cut problem. Robacker [2] proved that $p(V, S, l) = \mu_l(st)$.

Example 2. $T = \{s, s', t, t'\}$ and $S = \{\{s, t\}, \{s, t'\}\}$, i.e. $\mathscr{C}(V, S)$ consists of the cuts in K_V each "separating" s and s' as well as t and t'. The corresponding maximum packing problem is known as two-commodity cut problem. It follows from results of Lehman [3] and Hu [4] that $p(V, S, l) = \min\{\mu_l(ss'), \mu_l(tt')\}$ (see also [5], [6]).

Example 3. $|T|$ is a positive even integer and S consists of the odd-size subsets of T ($\mathscr{C}(V, S)$ for such T and S is usually called the family of *T-cuts*, or *odd-terminus cuts*). Edmonds and Johnson [7] (see also [8]) proved that $p(V, S, l)$ is equal to the minimum of $l(J)$ over all T-joins J in K_V (a *T-join* is a minimal subset $J \subseteq E(V)$ such that the odd valency vertex-set of the subgraph in K_V induced by J is exactly T). It is easy to show that Edmonds–Johnson's relation can be expressed also in the form (1.3), where U ranges over all perfect matchings in K_T.

Unfortunately, the list of cut families $\mathscr{C}(V, S)$ having the MFMC-property is rather short (the complete collection of such families for $|V| \geqslant |T| + 2$ will be given (without proof) in Section 7). In this paper, we are interested in the cut families for which a special minimax relation, weaker than (1.3), hold. Namely, we study such triples V, T, S for which, for any $l \in \mathbf{R}_+^{E(V)}$, the value $p(V, S, l)$ is determined by certain non-negative linear combinations of distances between terminals. More precisely, let β: $E(T) \to \mathbf{R}_+$ be a function satisfying

$$\beta(\partial^T A) (= \Sigma(\beta(u): u \in \partial^T A)) \geqslant 1 \quad \text{for all} \quad A \in S,$$

and let L^u $(u \in E(T))$ be a shortest chain in K_V (i.e. $l(L) = \mu_l(eL)$) with $eL = u$. Then for arbitrary l-admissible α: $\mathscr{X}(V, S) \to \mathbf{R}_+$ we have

$$\begin{aligned} 1 \cdot \alpha = {} & \Sigma(\alpha(X): X \in \mathscr{X}(V, S)) \leqslant \\ & \leqslant \Sigma(\alpha(X)\beta(\partial^T(X \cap T)): X \in \mathscr{X}(V, S)) = \\ & = \sum_{u \in E(T)} \beta(u) \Sigma(\alpha(X): X \in \mathscr{X}(V, S),\ u \in \partial^V X) \leqslant \\ & \leqslant \sum_{u \in E(T)} \beta(u) \Sigma(\lambda^\alpha(e): e \in L^u) \leqslant \sum_{u \in E(T)} \beta(u) l(L^u), \end{aligned}$$

whence

$$p(V, S, l) \leqslant \Sigma(\beta(u)\mu_l(u): u \in E(T)).$$

Definition. We say a scheme $S \subset 2^T$ belongs to the DSC-class with respect to the multicut max-Σ problems (shortly, to the *Σ-DSC-class*) if, for any $V \supseteq T$ and $l \in \mathbf{R}_+^{E(V)}$, the following equality is true:

$$(1.4) \qquad \begin{aligned} & p(V, S, l) = \\ & = \min\Big\{\sum_{u \in E(T)} \beta(u)\mu_l(u): \beta \in \mathbf{R}_+^{E(T)},\ \beta(\partial^T A) \geqslant 1\ \forall A \in S\Big\}. \end{aligned}$$

(The term "DSC-class" abbreviates that of "the class of schemes for which the output of any corresponding packing problem is *determined* by the lengths of *shortest chains* connecting terminals".)

Clearly, a scheme $S \subset 2^T$ belongs to the Σ-DSC-class if $\mathscr{C}(V, S)$ has the MFMC-property for each $V \supseteq T$ since the relation (1.3) (subject to $U \cap \partial^T A \neq \phi$ for all $A \in S$) is a special case of (1.4) with 0, 1-functions β.

Example 4. S is the set of 1-element subsets of T. For any $V \supseteq T$ and l, the equality (1.4) is known to hold for some β taking values 0, 1 and $\frac{1}{2}$; thus, such S belongs to the Σ-DSC-class. At the same time $\mathscr{C}(V, S)$ does not have the MFMC-property if $3 \leqslant |T| \neq 4$.

Our aim is to characterize the Σ-DSC-class. We need some definitions. Two subsets $X, Y \subseteq W$ are said to be *crossing* if none of $X \cap Y, X - Y, Y - X$ and $W - (X \cup Y)$ is empty and to be *laminar* otherwise. A collection $\mathscr{X} \subset 2^W$ is called *i-crossing* (for an integer $i \geq 2$) if there are at least i members of $\mathscr{X}$ each two of which are crossing. For example, the collection $\{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{3, 4, 5\}, \{5\}\}$ of subsets of the five elements set is 2- and 3-crossing but not 4-crossing. A collection is called *laminar* if it is not 2-crossing.

Definition. A scheme $S \subset 2^T$ is called *3-complete* if, for any three pairwise crossing members $A_1, A_2, A_3 \in S$ (if any), there exist non-negative reals $\gamma(A_i)$ $(i = 1, 2, 3)$, and a function ϵ: $S \to \mathbf{R}_+$ such that

$$(1.5)\quad \begin{aligned} &\gamma(A_1) + \gamma(A_2) + \gamma(A_3) \leq \Sigma(\epsilon(A)\colon A \in S),\\ &\Sigma(\gamma(A_i)\rho_{A_i}\colon i = 1, 2, 3) > \Sigma(\epsilon(A)\rho_A\colon A \in S),\end{aligned}$$

where, for $B \subseteq T$, ρ_B denotes the characteristic vector of the subset $\partial^T B$ in $R^{E(T)}$, and the vector inequality $>$ in (1.5) means that the left hand side value is no less than the right hand side value for any component $u \in E(T)$ and it is strictly more for at least one component u.

For example, where $T = \{0, \dots, 5\}$, the scheme S consisting of six subsets $A_i = \{i, i+1, i+2\}$ $(i = 1, 2, 3)$, $A_4 = \{1\}$, $A_5 = \{3\}$, $A_6 = \{5\}$ is 3-complete because for the triple of pairwise crossing members $\{A_1, A_2, A_3\}$ (it is the unique triple in S with such a property) (1.5) holds with $\gamma(A_i) = 1$, $\epsilon(A_i) = 0$ $(i = 1, 2, 3)$ and $\epsilon(A_j) = 1$ $(j = 4, 5, 6)$ (the corresponding component inequality is strict for the edge $u = 02$). Also the schemes in Examples 1, 2 are obviously 3-complete. One can verify that the schemes in Example 3 are also 3-complete (see Section 5).

Theorem A.

(i) *A scheme belongs to the Σ-DSC-class if it is 3-complete.*

(ii) *Let a scheme $S \subset 2^T$ be not 3-complete. Then S does not belong to the Σ-DSC-class. Moreover, for any $V \supset T$ with $|V| \geq |T| + 2$ there exists $l \in \mathbf{R}_+^{E(V)}$ such that (1.4) is not true.*

This theorem is central in the paper. It will be proved in Section 4. Note that there is a good characterization of the Σ-DSC-class since a polynomial (in $|T|$, $|S|$) algorithm can be produce which decides if a scheme S is 3-complete. Indeed, we can generate all triples in S and select the triples of pairwise crossing members. Further each triple $\{A_1, A_2, A_3\}$ of pairwise crossing members can be examined in a constant time whether γ and ϵ exist for it satisfying (1.5). For one can identify (in a new vertex) each maximal subset of terminals having the property that $st \notin \partial^T A_i$ for $i = 1, 2, 3$ and any two elements s and t of the subset, and next solve a corresponding linear program for a graph with at most $2^3 = 8$ vertices.

Now we introduce one more type of cut packing problems. Such problems will arise in the proof of Theorem A, but they are interesting also by themselves. For arbitrary α: $\mathscr{X}(V, S) \to \mathbf{R}_+$ and $A \in S$ define the *partial value* of α to be

$$\zeta^\alpha(A) \triangleq \Sigma(\alpha(X)\colon X \subset V,\ X \cap T = A).$$

Multicut existence problem EX (V, S, l, d): given V, T, S, l (defined as above) and a function d: $S \to \mathbf{R}_+$ (as demands on partial values), find l-admissible α: $\mathscr{X}(V, S) \to \mathbf{R}_+$ satisfying $\zeta^\alpha(A) \geq d(A)$ for all $A \in S$ (or establish that such α does not exist).

If α is l-admissible and L is a shortest chain joining terminals s and t, then

$$\begin{aligned}\mu_l(st) = l(L) &\geq \Sigma(\lambda^\alpha(e)\colon e \in L) =\\ &= \Sigma(\alpha(X)|\partial X \cap L|\colon X \in \mathscr{X}(V, S)) \geq\\ &\geq \Sigma(\alpha(X)\colon X \in \mathscr{X}(V, S),\ st \in \partial^V X) =\\ &= \Sigma(\zeta^\alpha(A)\colon A \in S,\ st \in \partial^T A).\end{aligned}$$

And so, in order to have a solution of a problem EX (V, S, l, d) it is necessary (but, in general, not sufficient) that the inequality

$$(1.6)\quad \mu_l(u) \geq \Sigma(d(A)\colon A \in S,\ u \in \partial^T A)$$

should hold for all $u \in E(T)$.

Definition. We say a scheme $S \subset 2^T$ belongs to the DSC-class with respect to the multicut existence problems (shortly, to the EX-DSC-class) if, for any $V \supseteq T$, $l \in \mathbf{R}_+^{E(V)}$ and $d \in \mathbf{R}_+^S$, EX (V, S, l, d) has a solution whenever the inequality (1.6) holds for each $u \in E(T)$.

In Section 3 we prove the following theorem which gives a complete description of the EX-DSC-class.

Theorem B.

(i) *A scheme belongs to the* EX-DSC-*class if it is not* 3-*crossing.*

(ii) *Let a scheme* $S \subset 2^T$ *be* 3-*crossing,* $V \supset T$ *and* $|V| \geqslant |T| + 2$. *Then there exist* $l \in \mathbf{R}_+^{E(V)}$ *and* $d \in \mathbf{R}_+^S$ *such that* (1.6) *holds for each* $u \in E(T)$ *but* EX (V, S, l, d) *has no solution.*

(Note that there is a stronger, half-integer, version of this theorem, it will be mentioned in Section 5.) Theorem B is one of statements on which the proof of Theorem A is based. The proof of Theorem B uses in turn some relationship between multicut existence problems and one class of multicommodity flow problems, it allow us to apply one known result about multicommodity flows. Such a relationship can be shown in abstract terms. To that end, in Section 2, we consider an arbitrary family of subsets of a set with a fixed partition into subfamilies (such a family is named us to be a *compound* one), introduce concepts of the generalized Σ-MFMC- and EX-MFMC-properties for compound families and establish some facts about families having such properties. It will follow from the definitions for our special case that $S \subset 2^T$ belongs to the Σ-DSC- (resp., EX-DSC-) class if and only if the cut family $\mathscr{C}(V, S)$ (divided into subfamilies $\mathscr{C}^V(A) = \{\partial^V X\colon X \subset V,\ X \cap T = A\}$ $(A \in S)$) has the generalized Σ-MFMC- (resp., EX-MFMC-) property for every $V \supseteq T$.

In Section 5 we study a special subclass of 3-complete schemes, so-called 2-complete ones, and show that multicut max-Σ problems with such schemes have optimum solutions with some nice features. As an application of this, we give a simple proof of one of Seymour's theorems on multicommodity cuts announced in [9] and proved in [10]. Also here we show that unbounded least "fractionality" of optimum solutions in

multicut max-Σ problems with integer-valued lengths and laminar schemes is possible. Finally, in Section 6 the concepts of the generalized MFMC-properties are illustrated with multicommodity flows and some known results are surveyed.

2. COMPOUND FAMILIES AND THE GENERALIZED MFMC-PROPERTIES

Let $\mathscr{F}$ be a family of nonempty subsets of a finite set E. $\mathscr{F}$ is called *compound* and denoted by $(\mathscr{F}_i\colon i \in I)$ if some partition of $\mathscr{F}$ into nonempty subfamilies $\mathscr{F}_i$ $(i \in I)$ is fixed. For $c \in \mathbf{R}_+^E$, a function $f\colon \mathscr{F} \to \mathbf{R}_+$ is c-admissible if

(2.1) $\Sigma(f(F)\colon e \in F \in \mathscr{F}) \leqslant c(e)$ for all $e \in E$.

We consider the following two packing problems.

A. *Max-Σ problems* $\Sigma(\mathscr{F}, c)$: given $c \in \mathbf{R}_+^E$, find c-admissible f on $\mathscr{F}$ with $1 \cdot f$ maximum (this maximum is denoted by $p(\mathscr{F}, c)$).

B. *Existence problem* EX $(\mathscr{F}, c, d)$: given $c \in \mathbf{R}_+^E$ and $d \in \mathbf{R}_+^I$, find c-admissible f on F such that

(2.2) $\Sigma(f(F)\colon F \in \mathscr{F}_i) \geqslant d(i)$ for all $i \in I$

(or establish that such f does not exist).

Applying the l.p. duality theorem to $\Sigma(\mathscr{F}, c)$,

(2.3) $p(\mathscr{F}, c) = \min\{c \cdot w\colon w \in \mathbf{R}_+^E,\ w(F) \geqslant 1$ for all $F \in \mathscr{F}\}$.

For a function w on E and $i \in I$, let w^i denote the value $\min\{w(F)\colon F \in \mathscr{F}_i\}$. The following gives a criterion of solvability of existence problems.

Proposition 2.1. EX $(\mathscr{F}, c, d)$ *has a solution if and only if the inequality*

(2.4) $c \cdot w \geqslant \Sigma(d(i) w^i\colon i \in I)$

holds for any $w \in \mathbf{R}_+^E$.

Proof. Solvability of $\mathrm{EX}(\mathscr{F}, c, d)$ means that the system of the linear inequalities (2.1)–(2.2) has a feasible solution. Farkas' lemma applying to this system implies that $\mathrm{EX}(\mathscr{F}, c, d)$ has a solution if and only if the inequality

$$c \cdot w - \Sigma(d(i)\nu(i)\colon\ i \in I) \geq 0$$

holds for any $w \in \mathbf{R}_+^E$ and $\nu \in \mathbf{R}_+^I$ satisfying

$$w(F) - \nu(i) \geq 0 \quad \text{for all } i \in I \text{ and } F \in \mathscr{F}_i,$$

whence the result easily follows. ∎

Now we define the collection

$$\hat{b}(\mathscr{F}) = \cup\,(b(\mathscr{F}_i)\colon\ i \in I)$$

(where $b(\mathscr{F}_i)$ is the blocker of $\mathscr{F}_i$) which is called the *imperfect blocker* of the compound family $(\mathscr{F}_i\colon\ i \in I)$. For $B \subseteq E$, we say that B meets $\mathscr{F}_i$ if B meets each member of $\mathscr{F}_i$.

Let $\theta(E') = \theta^E(E')$ denote the characteristic vector in $\mathbf{R}^E$ of a subset $E' \subseteq E$. Consider a function $\delta\colon \hat{b}(\mathscr{F}) \to \mathbf{R}_+$ satisfying

(2.5) $\quad \Sigma(\delta(B)\colon\ B \in \hat{b}(\mathscr{F}),\ B \text{ meets } \mathscr{F}_i) \geq 1 \quad \text{for all } i \in I.$

For $w = \Sigma(\delta(B)\theta(B)\colon\ B \in \hat{b}(\mathscr{F}))$ and any $F \in \mathscr{F}_i$, $i \in I$, we have

$$w(F) = \Sigma(\delta(B)\theta(B)\theta(F)\colon\ B \in \hat{b}(\mathscr{F})) \geq$$

$$\geq \Sigma(\delta(B)\colon\ B \in \hat{b}(\mathscr{F}),\ B \text{ meets } \mathscr{F}_i) \geq 1$$

and hence (2.3) implies

$$p(\mathscr{F}, c) \leq \Sigma(\delta(B)c(B)\colon\ B \in \hat{b}(\mathscr{F})).$$

Definition. We say a compound family $\mathscr{F}$ has the generalized MFMC-property with respect to the max-Σ problems (shortly, the *gen. Σ-MFMC-property*) if the equality

(2.6) $\quad p(\mathscr{F}, c) = \min\{\Sigma(\delta(B)c(B)\colon\ B \in \hat{b}(\mathscr{F}))\}$

holds for any $c \in \mathbf{R}_+^E$, where the minimum is taken over all $\delta\colon \hat{b}(\mathscr{F}) \to \mathbf{R}_+$ satisfying (2.5). (Obviously, the notion of the gen. Σ-MFMC-property is equivalent to that of the MFMC-property in the case $|I| = 1$, i.e. when an ordinary (noncompound) family is considered.)

Now consider an existence problem $\mathrm{EX}(\mathscr{F}, c, d)$. It follows from Proposition 2.1 that it is necessary (but generally not sufficient) for solvability of this problem that the inequality

(2.7) $\quad c(B) \geq \Sigma(d(i)\colon\ i \in I,\ B \text{ meets } \mathscr{F}_i)$

holds for all $B \in \hat{b}(\mathscr{F})$, because, for $B \in \hat{b}(\mathscr{F})$ and $w = \theta(B)$, we have $c \cdot w = c(B)$ and $w^i = \min\{|B \cap F|\colon\ F \in \mathscr{F}_i\} \geq 1$ for all $i \in I$ such that B meets $\mathscr{F}_i$.

Definition. We say a compound family $\mathscr{F}$ has the generalized MFMC-property with respect to the existence problems (shortly, the *gen. EX-MFMC-property*) if, for any $c \in \mathbf{R}_+^E$ and $d \in \mathbf{R}_+^I$, $\mathrm{EX}(\mathscr{F}, c, d)$ has a solution whenever (2.7) holds for all $B \in \hat{b}(\mathscr{F})$.

(It is easy to show that in the case $|I| = 1$ the notions of the gen. EX-MFMC- and Σ-MFMC-properties are equivalent and, therefore, both of them are equivalent to the MFMC-property.)

Now we introduce some packing problem for the imperfect blockers. Given $w \in \mathbf{R}_+^E$, let $\kappa\colon \hat{b}(\mathscr{F}) \to \mathbf{R}_+$ be a w-admissible function, i.e.

(2.8) $\quad \Sigma(\kappa(B)\theta(B)\colon\ B \in \hat{b}(\mathscr{F})) \leq w.$

Then, for any $F \in \mathscr{F}$, we get

$$w(F) \geq \theta(F)\,\Sigma(\kappa(B)\theta(B)\colon\ B \in \hat{b}(\mathscr{F})) \geq$$

$$\geq \Sigma(\kappa(B)\colon\ B \in \hat{b}(\mathscr{F}),\ B \cap F \neq \phi),$$

therefore, for any $i \in I$,

(2.9) $\quad w^i\ (= \min\{w(F)\colon\ F \in \mathscr{F}_i) \geq \Sigma(\kappa(B)\colon\ B \in \hat{b}(\mathscr{F}),\ B \text{ meets } \mathscr{F}_i).$

We say that a w-admissible κ *locks* $\mathscr{F}_i$ if

(2.10) $\quad w^i = \Sigma(\kappa(B)\colon\ B \in \hat{b}(\mathscr{F}),\ B \text{ meets } \mathscr{F}_i).$

Locking problem $\mathrm{LOCK}(\mathscr{F}, w)$: given $w \in \mathbf{R}_+^E$, find a w-admissible $\kappa\colon \hat{b}(\mathscr{F}) \to \mathbf{R}_+$ which locks $\mathscr{F}_i$ for all $i \in I$ (or establish that such a

κ does not exist). $\mathscr{F}$ is called *lockable* if LOCK $(\mathscr{F}, w)$ has a solution (i.e. such a κ exists) for any $w \in \mathbf{R}_+^E$.

Lemma 2.2. *The following statements are equivalent:*

(i) *$\mathscr{F}$ has the gen. EX-MFMC-property;*

(ii) *$\mathscr{F}$ is lockable.*

Proof. (i) → (ii). In view of Proposition 2.1, the fact that $\mathscr{F}$ has the gen. EX-MFMC-property means that, for any fixed $w \in \mathbf{R}_+^E$, (2.4) is true for every $c \in \mathbf{R}_+^E$ and $d \in \mathbf{R}_+^I$ satisfying (2.7) for all $B \in \hat{b}(\mathscr{F})$. Applying Farkas' lemma to the implication $\forall c \geq 0$, $d \geq 0$ $(\forall B (2.7) \to (2.4))$ we get that there is a $\kappa\colon \hat{b}(\mathscr{F}) \to \mathbf{R}_+$ satisfying (2.8) and

(2.11) $\quad -\sum(\kappa(B)\colon B \in \hat{b}(\mathscr{F}),\ B \text{ meets } \mathscr{F}_i) \leq -w^i$ for all $i \in I$.

But the w-admissibility of κ implies (2.9) for any $i \in I$, and (2.10) holds because of (2.9) and (2.11). Considering all $w \in \mathbf{R}_+^E$ we conclude that $\mathscr{F}$ is lockable. (ii) → (i) is proved by conversion of the above arguments. ∎

Lemma 2.2 will be used in the proof of Theorem B. The following lemma will be needed for proving Theorem A.

Lemma 2.3. *Let $\mathscr{F}$ be a compound family and $c \in \mathbf{R}_+^E$. The following statements are equivalent:*

(i) *the equality (2.6) is true;*

(ii) *the inequality*

(2.12) $\quad c \cdot w \geq \sum(d(i)\colon i \in I)$

is valid for any $d \in \mathbf{R}_+^I$ and $w \in \mathbf{R}_+^E$ such that d satisfies (2.7) for all $B \in \hat{b}(\mathscr{F})$ and w does

(2.13) $\quad w(F) \geq 1$ *for all* $F \in \mathscr{F}$.

Proof. First of all, we observe that validity of (2.6) is equivalent to feasibility (by $f\colon \mathscr{F} \to \mathbf{R}_+$ and $\delta\colon \hat{b}(\mathscr{F}) \to \mathbf{R}_+$) of the system of linear inequalities {(2.1), (2.5)} together with the inequality

(2.14) $\quad -\sum(f(F)\colon F \in \mathscr{F}) + \sum(c(B)\delta(B)\colon B \in \hat{b}(\mathscr{F})) \leq 0.$

By Farkas' lemma, the system {(2.1), (2.5), (2.14)} has a feasible solution if and only if the inequality

(2.15) $\quad c \cdot \tilde{w} - \sum(\tilde{d}(i)\colon i \in I) \geq 0$

holds for any $\tilde{w} \in \mathbf{R}_+^E$, $\tilde{d} \in \mathbf{R}_+^I$ and $\psi \in \mathbf{R}_+$ satisfying

(2.16) $\quad \tilde{w}(F) - \psi \geq 0$ for all $F \in \mathscr{F}$,

(2.17) $\quad -\sum(\tilde{d}(i)\colon i \in I,\ B \text{ meets } \mathscr{F}_i) + c(B)\psi \geq 0$ for all $B \in \hat{b}(\mathscr{F})$

($\tilde{w}, \tilde{d}$ and ψ are variables dual to (2.1), (2.5) and (2.14), respectively). But the implication {(2.16), (2.17)} → (2.15) is equivalent to {(2.13), (2.7)} → (2.12). For, partly, if $\psi = 0$ then (2.17) implies $\tilde{d} = 0$ (as $\tilde{d} \geq 0$, each $F \in \mathscr{F}$ is nonempty and each $\mathscr{F}_i$ is nonempty), whence (2.15) trivially follows, partly, for arbitrary $\psi > 0$, the mapping $(\tilde{w}, \tilde{d}, \psi) \to (w, d)$ defined by $w = \tilde{w}/\psi$ and $d = \tilde{d}/\psi$ turns the former implication into the latter one. ∎

Remark 2.4. It follows from Lemmas 2.1 and 2.3 that if $\mathscr{F}$ has the gen. EX-MFMC-property then it has the gen. Σ-MFMC-property as well (because (2.4) and (2.13) together imply (2.12)).

Now we return to cut packing problem. A cut family $\mathscr{C}(V, S)$ has a natural representation as a compound one, namely $(\mathscr{C}^V(A)\colon A \in S)$, where $\mathscr{C}^V(A)$ denotes $\{\partial^V X\colon X \subset V,\ X \cap T = A\}$ for $A \in S$. Obviously, the blocker of $\mathscr{C}^V(A)$ consists of all the minimal chains in K_V with one end in A and the other one in $T - A$. Thus $\hat{b}(\mathscr{C}(V, S)) = \mathscr{L}^S$, where $\mathscr{L}^S$ is the set of minimal T-terminus chains L such that $eL \in \partial^T A$ for some $A \in S$ (such a chain L is called an *S-chain*). According to the definition, $\mathscr{C}(V, S)$ has the gen. Σ-MFMC-property if, for any $l \in \mathbf{R}_+^{E(V)}$,

(2.18) $\quad p(V, S, l) = \min\{\sum(\delta(L)l(L)\colon L \in \mathscr{L})\}$,

where $\mathscr{L} = \mathscr{L}^S$ and the minimum is taken over all $\delta\colon \mathscr{L} \to \mathbf{R}_+$ satisfying

(2.19) $\quad \sum(\delta(L)\colon L \in \mathscr{L},\ eL \in \partial^T A) \geq 1$ for all $A \in S$.

Let $\mathscr{L}(V,T)$ denote the set of T-terminus chains in K_V. Note that if $eL \in \partial^T A$ for some T-terminus L and $A \in S$, then there is an S-chain $L' \subseteq L$ for which also $eL' \in \partial^T A$. It easily follows from this that if (2.18) (subject to (2.19)) holds for $\mathscr{L} = \mathscr{L}(V,T)$, then the same is true for $\mathscr{L} = \mathscr{L}^S$, and vice versa. Next, let $\mathscr{L} = \mathscr{L}(V,T)$, and assume that (2.18) (subject to (2.19)) holds and δ^* attains the minimum in (2.18). Then, for $L \in \mathscr{L}(V,T)$, $\delta^*(L) > 0$ implies that L is a shortest chain (since if $l(L') < l(L)$ and $eL' = eL$ for some chain L', the the function δ' defined by $\delta'(L) = 0$, $\delta'(L') = \delta^*(L) + \delta^*(L')$ and $\delta'(L'') = \delta^*(L'')$ for the remaining L'''s in $\mathscr{L}(V,T)$ would satisfy (2.19), which would lead to a contradiction to the minimality of δ^*). Put $\beta^*(u) = \Sigma(\delta^*(L)$: $L \in \mathscr{L}(V,T)$, $eL = u)$ for each $u \in E(T)$. Then β^* attains the equality in (1.4). Conversely, validity of (1.4) easily implies that of (2.18) (subject to (2.19)) for $\mathscr{L} = \mathscr{L}(V,S)$. Thus, a scheme $S \subset 2^T$ belongs to the Σ-DSC-class if and only if $\mathscr{L}(V,S)$ has the gen. Σ-MFMC-property for all $V \supseteq T$.

Next, for $\mathscr{F} = \mathscr{C}(V,S)$, $c = l \in \mathbf{R}_+^{E(V)}$ and $d \in \mathbf{R}_+^S$, the inequality (2.7) is specified as

$$(2.20) \qquad l(L) \geqslant \Sigma(d(A):\ A \in S,\ eL \in \partial^T A),$$

where L is an S-chain in K_V. Easy arguments show that if, given l and d, the inequality (2.20) holds for any S-chain L, then it does for any T-terminus chain L, whence validity of (2.20) for all $L \in \mathscr{L}^S$ is equivalent to that of (1.6) for all $u \in E(T)$. Thus, a scheme $S \subset 2^T$ belongs to the EX-DSC-class if and only if $\mathscr{C}(V,S)$ has the gen. EX-MFMC-property for all $V \supseteq T$.

For $w \in \mathbf{R}_+^{E(V)}$ and $A \in S$, let w^A denote the value $\min\{w(E')$: $E' \in \mathscr{C}^V(A)\}$. According to the above definition, a w-admissible function φ: $\mathscr{L}^S \to \mathbf{R}_+$ locks $\mathscr{C}^V(A)$ if

$$(2.21) \qquad w^A = \Sigma(\varphi(L):\ L \in \mathscr{L},\ eL \in \partial^T A),$$

where $\mathscr{L} = \mathscr{L}^S$. As above, we extend $\mathscr{L}$ to $\mathscr{L}(V,T)$, and let φ: $\mathscr{L} \to \mathbf{R}_+$ (where $\mathscr{L} = \mathscr{L}(V,T)$) be a w-admissible functions satisfying (2.21). Then the function φ' on $\mathscr{L}^S$ defined by

$$\varphi'(L') = \Sigma(\varphi(L):\ L \in \mathscr{L}(V,T),\ L' \subseteq L), \qquad L' \in \mathscr{L}^S,$$

is also w-admissible and

$$\Sigma(\varphi'(L'):\ L \in \mathscr{L}^S,\ eL' \in \partial^T A) \geqslant$$
$$\geqslant \Sigma(\varphi(L):\ L \in \mathscr{L}(V,T),\ eL \in \partial^T A),$$

whence it follows that φ' locks $\mathscr{C}^V(A)$. Therefore, we may consider the following slightly different (but equivalent, in essence) form of the locking problem for $\mathscr{C}(V,S)$: given $w \in \mathbf{R}_+^{E(V)}$, find a w-admissible φ: $\mathscr{L}(V,T) \to \mathbf{R}_+$ satisfying (2.21) for all $A \in S$ (or establish that such φ does not exist). This form is denoted by $\mathrm{LOCK}^f(V,S,w)$ and called a *multiflow locking problem.*

The problem $\mathrm{LOCK}^f(V,S,w)$ will appear in the proof of Theorem B. Note that such a problem has also other applications, in particular, it is used for solving a number of multicommodity flow problems (see [11], [12]).

3. PROOF OF THEOREM B

A proof of Theorem B can be immediately obtained from Lemma 2.2 and the following theorem.

Theorem 3.1.

(i) *Let a scheme* $S \subset 2^T$ *be not* 3*-crossing. Then, for any* $V \supseteq T$, *the cut family* $\mathscr{C}(V,S)$ *is lockable.*

(ii) *Let a scheme* $S \subset 2^T$ *be* 3*-crossing,* $V \supset T$ *and* $|V| \geqslant |T| + 2$. *Then* $\mathscr{C}(V,S)$ *is not lockable.*

Theorem 3.1 is due to M.V. Lomonosov and the author (see [13], [12]). However, in order to make the paper self-contained I give another proof of Theorem B. Namely, a direct proof of the part (ii) of Theorem B and a new proof of the part (i) Theorem 3.1 are given here (the part (i) of Theorem B follows from the part (i) of Theorem 3.1 and Lemma 2.2).

Proof of the part (ii) of Theorem B. Let $S \subset 2^T$ be 3-crossing, $V \supset T$ and $|V| \geqslant |T| + 2$. Choose an arbitrary triple of pairwise crossing members in S, say $S^* = \{A_1, A_2, A_3\}$. Put $d(A_i) = 1$ $(i = 1, 2, 3)$ and $d(A) = 0$ for all $A \in S - S^*$. Now our purpose is to determine a function l on $E(V)$ such that the inequality (1.6) holds for any $u \in E(T)$ but there exists a function $w \in \mathbf{R}_+^{E(V)}$ such that

$$l \cdot w < \sum (d(A) w^A : A \in S)$$

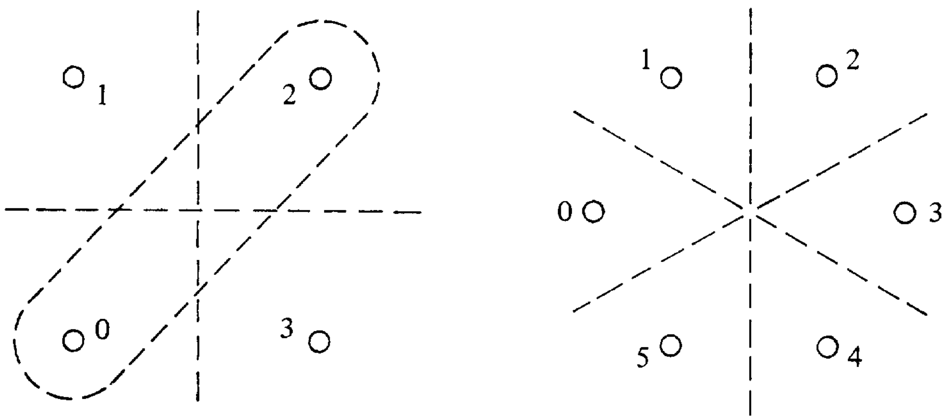
(then by Proposition 2.1, EX (V, S, l, d) has no solution).

First of all we introduce two sorts of triples of pairwise crossing sets:

1. $S_1 = \{\{0, i\}: i = 1, 2, 3\} \subset 2^{T_1}$, where $T_1 = \{0, 1, 2, 3\}$,
2. $S_2 = \{\{i, i+1, i+2\}: i = 1, 2, 3\} \subset 2^{T_2}$, where $T_2 = \{0, \ldots, 5\}$.

(see Fig. 3.1, where the corresponding families of cuts are shown).

Proposition 3.2. *Let T' be a minimal subset of T having the property that the subsets $A_i' = A_i \cap T'$ $(i = 1, 2, 3)$, are still pairwise crossing with respect to T'. Then the triple $S' = \{A_1', A_2', A_3'\} \subset 2^{T'}$ is equivalent to either S_1 or S_2, that is, there is a one-to-one mapping $\chi: T' \to T_1$ (resp., $\to T_2$) such that, $\chi(A_i')$ is either $\{0, i\}$ or $T_1 - \{0, i\}$ (resp., is either $\{i, i+1, i+2\}$ or $T_2 - \{i, i+1, i+2\}$) for $i = 1, 2, 3$.*



(a) (b)

Figure 3.1

This statement occurred in [12]. A sketch of that proof is the following. Obviously, $|A_1^* \cap A_2^* \cap A_3^*| \leq 1$ for arbitrary $A_i^* \in \{A_i', T' - A_i'\}$ $(i = 1, 2, 3)$. Without loss of generality we may assume $A_1' \cap A_2' \cap A_3' \neq \neq \phi$. The following two cases are possible:

1. for $j = 1, 2, 3$, A_j' does not lie in $A_p' \cup A_q'$, where $\{p, q\} = \{1, 2, 3\} - \{j\}$,
2. $A_j' \subseteq A_p' \cup A_q'$ for some $j \in \{1, 2, 3\}$ and $\{p, q\} = \{1, 2, 3\} - \{j\}$.

Using the minimality of T' one shows that S' is equivalent to S_1 in the first case and S' is equivalent to S_2 in the second case. ∎

Now, let T' and S' be defined for above T and S^* as in Proposition 3.2.

Case 1. $|T'| = 4$ (and S' is equivalent to S_1).

We may assume that T' is identified with T_1 by use of the map χ. Take an arbitrary element x in $V - T$, and let $G_1 = (V_1, E_1)$ be the subgraph of the graph K_V drawn in Fig. 3.2a. Put $l(e) = 1$ for all $e \in E_1$ and $l(e) \geq 2$ for all $e \in E(V) - E_1$.

Case 2. $|T'| = 6$ (and S' is equivalent to S_2).

Similarly we assume that T' is identified with T_2. Take arbitrary two elements x and y in $V - T$ (which exist because $|V| \geq |T| + 2$), and let $G_2 = (V_2, E_2)$ be the subgraph of K_V drawn in Fig. 3.2b. Put $l(e) = 1$ for all $e \in E_2$ and $l(e) \geq 3$ for all $e \in E(V) - E_2$.

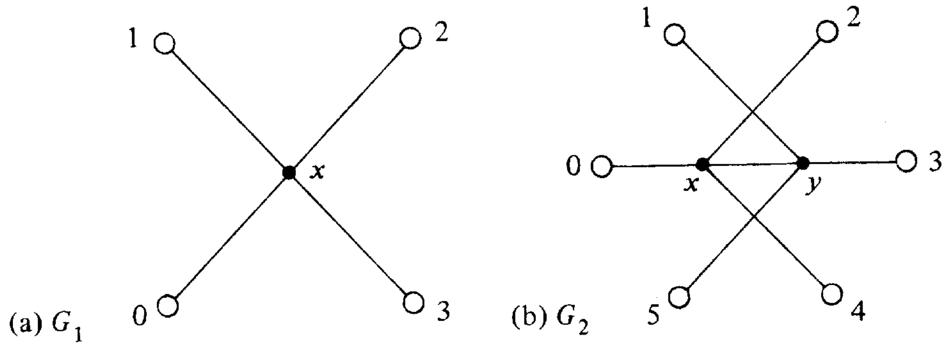


Figure 3.2

One can check that in both cases the inequality (1.6) holds for all $u \in E(T)$. However, in both cases the problem EX (V, S, l, d) has no solution. Indeed, consider the function w on $E(V)$ defined to be 1 on E_1 (respectively, on E_2) and 0 on the remaining edges in $E(V)$. It is easy to check that $w^{A_i} = 2$ in Case 1 and $w^{A_i} = 3$ in Case 2 $(i = 1, 2, 3)$. If Case 1 takes place we have $l \cdot w = 4 < 6 = \Sigma(d(A)w^A : A \in S)$ and in Case 2 does we have $l \cdot w = 7 < 9 = \Sigma(d(A)w^A : A \in S)$. ∎

Proof of the part (i) of Theorem 3.1. We say that a nonnegative integer-valued function w on $E(V)$ is *inner cut even* if the value $w(\partial^V\{x\})$ is even for all $x \in V - T$.

Lemma 3.3. *Let* $S \subset 2^T$ *be not 3-crossing.* $V \supseteq T$, *and let* w *be inner cut even function on* $E(V)$. *Then there exists a* w*-admissible* φ: $\mathscr{L}(V, T) \to \mathbf{Z}_+$ *locking all* $A \in S$ ($\mathbf{Z}_+$ *is the set of nonnegative integers).*

(Clearly this lemma is a strengthening of (i) in Theorem 3.1. It is due to L o m o n o s o v and the author [13], [12]; their proof is based on an algorithm whose running time is bounded by $w(E(V))$ times a polynomial in $|V|$ and $|S|$. In [12] also another algorithm is developed which works with arbitrary nonnegative real-valued w and uses $O(|T|^5 |V|^5)$ operations.) Here I present another and simpler proof of this lemma.

Proof. We proceed by induction on $\eta(w) = \Sigma(w(\partial\{x\}: x \in V - T)$. Assume that $\eta(w) = 0$, and put $\varphi(L) = w(u)$ for all $L = \{u\}$. $(u \in E(T))$ and $\varphi(L) = 0$ for the remaining L's in $\mathscr{L}(V, T)$. Then $w^A = \Sigma(w(u)$: $u \in \partial^T A)$ and (2.21) is obviously true for all $A \in S$. Now we assume that $\eta(w) > 0$, and let $x \in V - T$ be such a vertex that $w(\partial\{x\}) > 0$. Define the set $V(x)$ to be $\{y$: $w(xy) > 0\}$. For $X \subset V$, let X^* denote X if $x \notin X$ and it do $V - X$ if $x \in X$.

Assume that $|V(x)| = 1$, and let $V(x) = \{y\}$. Put $w'(xy) = 0$ and $w'(e) = w(e)$ $(e \in E(V) - \{xy\})$. Since $w(xy)$ $(= w(\partial\{x\}))$ is even, w' is inner cut even. We show that $w'^A = w^A$ for all $A \in S$ (then, by induction, there is a w'-admissible φ: $\mathscr{L}(V, T) \to \mathbf{Z}_+$ locking all $A \in S$ (concerning w'), and hence φ is w-admissible and it locks all $A \in S$ for w). Let $A \in S$, $X \subset V$, $X \cap T = A$ and $w'(\partial X) = w'^A$. If $xy \notin \partial X$, then $w'(\partial X) = w(\partial X) \geqslant w^A$. Suppose $xy \in \partial X$, and let $Y = X^* \cup \{x\}$. Then $Y \cap T = X^* \cap T$ and $w'(\partial X) = w'(\partial Y) = w(\partial Y) \geqslant w^A$, therefore $w'^A = w^A$.

Thus, we may assume that $|V(x)| \geqslant 2$. For two distinct $y, z \in V(x)$, we say that w' is obtained from w by *righting* on y, z if $w'(xy) = w(xy) - 1$, $w'(xz) = w(xz) - 1$, $w'(yz) = w(yz) + 1$ and $w'(e) = w(e)$ for the remaining e's in $E(V)$. Obviously, $\eta(w') < \eta(w)$ and w' is inner cut even when w does so. We shall show that there are $y, z \in V(x)$ such that the function w' obtained from w by righting on y, z satisfies $w'^A = w^A$ for all $A \in S$. Then, by induction, there exists a w'-admissible φ': $\mathscr{L}(V, T) \to \mathbf{Z}_+$ which locks all $A \in S$ (for w'). If $w'(yz) > \Sigma(\varphi'(L)$: $L \in \mathscr{L}(V, T)$, $yz \in L)$, then φ' is also w-admissible and hence φ' locks all $A \in S$ for w. Otherwise, we take an arbitrary $L \in \mathscr{L}(V, T)$ such that $yz \in L$ and yield the chain L' for which $eL' = eL$ and $L' \subseteq (L - \{yz\}) \cup \{xy, xz\}$. Then the function φ defined by $\varphi(L) = \varphi'(L) - 1$, $\varphi(L') = \varphi'(L') + 1$ and $\varphi(L'') = \varphi'(L'')$ $(L'' \in \mathscr{L}(V, T) - \{L, L'\})$ is obviously w-admissible and it locks all $A \in S$ for w.

We need three simple claims. We say that X *separates* Y and Z if either $Y \subseteq X$, $Z \cap X = \phi$ or $Z \subseteq X$, $Y \cap X = \phi$.

(1) Let w be inner cut even, and w' be obtained from w by righting on y, z $(y, z \in V(x))$. Next, let $w'^A < w^A$ for some $A \in S$, and $X \subset V$ be such that $X \cap T = A$ and $w'(\partial X) = w'^A$. Then X separates $\{x\}$ and $\{y, z\}$, and $w(\partial X) = w^A$.

Indeed, let $E_1 = \partial^V X$. $w'(E_1) = w'^A < w^A \leqslant w(E_1)$ easily implies that $xy, xz \in E_1$ and $yz \notin E_1$; and so X separates $\{x\}$ and $\{y, z\}$, and $w'(E_1) = w(E_1) - 2$, whence $w'^A \geqslant w^A - 2$. Let $Y \subset V$ be such that $Y \cap T = A$ and $w(E_2) = w^A$, where $E_2 = \partial Y$. Since $X \cap T = Y \cap T$ and w is inner cut even, then $w(E_1) - w(E_2)$ is even. Hence $w'^A = w^A - 2$ and $w(E_1) = w^A$.

X is called *$V(x)$-maximal* if $X \in \mathscr{X}(V, S)$, $w(\partial X) = w^{X \cap T}$ and the set $X^* \cap V(x)$ is maximal under these conditions (X^* is defined as above).

(2) Let X_1 and X_2 be two $V(x)$-maximal sets such that $X_1^* \cap \cap V(x) \neq X_2^* \cap V(x)$ and $X_1^* \cap X_2^* \cap V(x) \neq \phi$, and let $A_i = X_i \cap T$ $(i = 1, 2)$. Then A_1 and A_2 (considered as subsets of T) are crossing.

Suppose, for a contradiction, that A_1 and A_2 are laminar. Let $A_i' = = X_i^* \cap T$ $(i = 1, 2)$. Assume that $A_1' \cap A_2' = \phi$, and let $Y_1 = X_1^* - X_2^*$ and $Y_2 = X_2^* - X_1^*$. $A_1' \cap A_2' = \phi$ implies $Y_i \cap T = A_i'$, and we have $w(\partial X_i^*)) = w^{A_i} \leqslant w(\partial Y_i)$ $(i = 1, 2)$. Hence, for any $e \in E(V)$ with $w(e) > 0$, the obvious submodular inequality

$$|\{e\} \cap \partial X_1^*| + |\{e\} \cap \partial X_2^*| \geqslant |\{e\} \cap \partial Y_1| + |\{e\} \cap \partial Y_2|$$

holds as equality. Thus, $w(zv) = 0$ for any $z \in X_1^* \cap X_2^*$ and $v \in V - - (X_1^* \cup X_2^*)$ which contradicts to $X_1^* \cap X_2^* \cap V(x) \neq \phi$ and $x \in V - - (X_1^* \cup X_2^*)$. A similar contradiction is produced when $A_1' \cup A_2' = T$. It remains to consider the case $\{A_1' \cap A_2', A_1' \cup A_2'\} = \{A_1', A_2'\}$. Let, for definiteness, $A_1' \cup A_2' = A_1'$, and let $Y = X_1^* \cup X_2^*$. Then, by arguments as above, $w(\partial Y) = w^{A'_1}$. But the set $Y \cap V(x)$ strongly includes $X_i^* \cap V(x)$ $(i = 1, 2)$, contrary to the $V(x)$-maximality of X_i.

(3) Let $X \subset V$, $X \cap T = A$ and $w(\partial X) = w^A$. Then $w(E') \leqslant w(E'')$, where $E' = \{xy: y \in X^* \cap V(x)\}$ and $E'' = \{xy: y \in V(x) - X^*\}$.

Indeed, let $Y = X^* \cup \{x\}$. Then $Y \cap T = X^* \cap T$, whence $w(\partial X^*) = w^A \leqslant w(\partial Y)$, where $A = X \cap T$. We have $\partial X^* - \partial Y = E'$ and $\partial Y - \partial X^* = E''$, and the result follows.

Now we continue proving the lemma. Suppose, for a contradiction, that each two $y, z \in V(x)$ have the property that $w'^A < w^A$ for some $A \in S$ and the function w' obtained from w by righting on y, z. Then, by (1), for each two $y, z \in V(x)$, there are $A \in S$ and $X \subset V$ such that $X \cap T = A$, $w(\partial X) = w^A$ and X separates $\{x\}$ and $\{y, z\}$. Choose a collection Q of $V(x)$-maximal sets such that $X \cap V(x) \neq Y \cap V(x)$ for any distinct $X, Y \in Q$ and, for any $V(x)$-maximal Z, there is an $X \in Q$ for which $X \cap V(x) = Z \cap V(x)$. Assuming that $|Q| \leqslant 2$ we get that there is an $X \in Q$ such that $\Sigma(w(xy): y \in X^* \cap V(x)) > \frac{1}{2}\Sigma(w(xy): y \in V(x))$, contrary to (3). Thus, $|Q| \geqslant 3$. Let X_1, X_2 be elements of Q such that $X_1^* \cap X_2^* \neq \phi$, and let $y \in (X_1^* - X_2^*) \cap V(x)$ and $z \in (X_2^* - X_1^*) \cap V(x)$. Take an $X_3 \in Q$ such that X_3 separates $\{x\}$ and $\{y, z\}$. Finally, let $A_i = X_i \cap T$ $(i = 1, 2, 3)$. By (2), any two members of $\{A_1, A_2, A_3\}$ are crossing. This contradiction completes the proof of the lemma.

4. PROOF OF THEOREM A

Proof of the part (i). Consider an arbitrary 3-complete scheme $S \subset 2^T$, a set $V \supseteq T$ and an edge length function $l \in \mathbf{R}_+^{E(V)}$. We have to prove that

(4.1) $$p(V, S, l) = \Sigma(\beta^*(u)\mu_l(u): u \in E(T))$$

for some $\beta^*: E(T) \to \mathbf{R}_+$ satisfying

(4.2) $$\beta^*(\partial^T A) \geqslant 1 \text{ for all } A \in S.$$

Define $\mu \in \mathbf{R}_+^{E(T)}$ by $\mu(u) = \mu_l(u)$ $(u \in E(T))$, and consider the "reduced" multicut max-Σ problem $\Sigma(T, S, \mu)$. By linear programming duality there is a $\beta^*: E(T) \to \mathbf{R}_+$ which satisfies (4.2) and

(4.3) $$p(T, S, \mu) = \Sigma(\beta^*(u)\mu_l(u): u \in E(T)).$$

It suffices to prove that $p(V, S, l) \geqslant p(T, S, \mu)$ (then (4.3) implies (4.1) since $p(V, S, l) \leqslant \Sigma(\beta^*(u)\mu_l(u): u \in E(T))$ always holds subject to (4.2)). Let $\tilde{\alpha}: S \to \mathbf{R}_+$ be an optimum solution of $\Sigma(T, S, \mu)$, i.e.

(4.4) $$\lambda^{\tilde{\alpha}}(u) \, (= \Sigma(\tilde{\alpha}(A): A \in S, u \in \partial^T A)) \leqslant \mu(u) \quad \text{for all } u \in E(T)$$

and $1 \cdot \tilde{\alpha} = p(T, S, \mu)$. Assume, in addition, that $\tilde{\alpha}$ is chosen so that the value $\psi(\tilde{\alpha}) = \Sigma(\lambda^{\tilde{\alpha}}(u): u \in E(T))$ is minimum.

Claim 4.1. *Let $S^* = \{A_1, A_2, A_3\}$ be a triple of pairwise crossing members in S. Then $\tilde{\alpha}(A_i) = 0$ for at least one $i \in \{1, 2, 3\}$.*

Indeed, let $a = \min\{\tilde{\alpha}(A_i): i = 1, 2, 3\}$, and suppose, for a contradiction, that $a > 0$. Since S is 3-complete, there is $\gamma: S^* \to \mathbf{R}_+$ and $\epsilon: S \to \mathbf{R}_+$ satisfying (1.5). Put $\tilde{\alpha}'(A_i) = \tilde{\alpha}(A_i) - b\gamma(A_i) + b\epsilon(A_i)$

$(i = 1, 2, 3)$ and $\tilde{\alpha}'(A) = \tilde{\alpha}(A) + b\epsilon(A)$ $(A \in S - S^*)$, where $b = \dfrac{a}{\max\{\gamma(A_i)\colon i = 1, 2, 3\}}$ $(0 < b < \infty$ since $a > 0$ and $\gamma(A_i) > 0$ for at least one i, because of (1.5)). Obviously, $\tilde{\alpha}' \geqslant 0$. From (1.5) we get $1 \cdot \tilde{\alpha}' \geqslant 1 \cdot \tilde{\alpha}$, $\lambda^{\tilde{\alpha}'} \leqslant \lambda^{\tilde{\alpha}}$ and $\lambda^{\tilde{\alpha}'}(u) < \lambda^{\tilde{\alpha}}(u)$ for some $u \in E(T)$, contradicting the choice of $\tilde{\alpha}$. ∎

Now let $S^+ = \{A \in S\colon \tilde{\alpha}(A) > 0\}$. By Claim 4.1, S^+ is not 3-crossing. Define the demand function $d\colon S^+ \to \mathbf{R}_+$ by $d(A) = \tilde{\alpha}(A)$ $(A \in S^+)$, and consider the multicut existence problem $\mathrm{EX}(V, S^+, l, d)$. Since S^+ is not 3-crossing and the inequalities in (4.4) hold (compare with (1.6)), then, by Theorem B, this problem has a solution $\alpha'\colon \mathscr{X}(V, S^+) \to \mathbf{R}_+$. Let the function α be the extension of α' with zero on $\mathscr{X}(V, S - S^+)$. Then, obviously, $1 \cdot \alpha = 1 \cdot \tilde{\alpha}$, hence $p(V, S, l) \geqslant 1 \cdot \alpha = p(T, S, \mu)$, as required.

Proof of the part (ii). This proof is more complicated. We are based here on Lemma 2.3. Let $S \subset 2^T$ be not 3-complete, $V \supset T$ and $|V| \geqslant |T| + 2$. According to Lemma 2.3 we have to prove that there are $l \in \mathbf{R}_+^{E(V)}$, $d \in \mathbf{R}_+^S$ and $w \in \mathbf{R}_+^{E(V)}$ such that

(4.5) $\quad w(\partial^V X) \geqslant 1$ for all $X \in \mathscr{X}(V, S)$,

(4.6) $\quad \mu_l(u) \geqslant \sum(d(A)\colon A \in S,\ u \in \partial^T A)$ for all $u \in E(T)$

but

(4.7) $\quad l \cdot w < \ (d(A)\colon A \in S)$

((4.6) is equivalent to $\{(2.7)\ \forall B \in \mathscr{b}(\mathscr{F})\}$ by $\mathscr{F} = \mathscr{C}(V, S)$ and $l = c$, as it was explained in Section 2.)

Let $S^* = \{A_1, A_2, A_3\}$ be a triple of pairwise crossing members in S such that there are no $\gamma\colon S^* \to \mathbf{R}_+$ and $\epsilon\colon S \to \mathbf{R}_+$ satisfying (1.5). Let us choose a minimal subset $T' \subseteq T$ such that $A_i' = A_i \cap T'$ $(i = 1, 2, 3)$, are still pairwise crossing (with respect to T'). By Proposition 3.2, we may assume that either

(1) $T' = T_1 = \{0, 1, 2, 3\}$ and $S' = S_1 = \{\{0, i\}\colon i = 1, 2, 3\}$

or

(2) $T' = T_2 = \{0, \ldots, 5\}$ and $S' = S_2 = \{\{i, i+1, i+2\}\colon i = 1, 2, 3\}$, where $S' = \{A_1', A_2', A_3'\}$.

First of all, we define a suitable function l as follows.

Case 1. $S' = S_1$.

Choose an arbitrary vertex $x \in V - T$, and let $G_1 = (V_1, E_1)$ be the graph with the vertex-set $\{0, 1, 2, 3, x\}$ as in Fig. 3.2a. Put

$$l(e) = \begin{cases} 1, & e \in E_1 \\ \rho_{A_1}(e) + \rho_{A_2}(e) + \rho_{A_3}(e), & e \in E(T), \\ \text{a large positive number}, & e \in E(V) - (E_1 \cup E(T)), \end{cases}$$

where ρ_A denote the characteristic vector of the subset $\partial^T A$ in $\mathbf{R}^{E(T)}$.

Case 2. $S' = S_2$.

Choose arbitrary $x, y \in V - T$ (existing because of $|V| \geqslant |T| + 2$, and let $G_2 = (V_2, E_2)$ be the graph with vertex-set $\{0, \ldots, 5, x, y\}$ as in Fig. 3.2b. l is defined similarly to the previous case (with G_2 instead of G_1).

Next, d is defined by $d(A_i) = 1$ $(i = 1, 2, 3)$ and $d(A) = 0$ $(A \in S - S^*)$. It is not difficult to verify that in both cases, (4.6) holds with given l and d. Now our aim is to define a function $w \in \mathbf{R}_+^{E(V)}$ that (4.5) and (4.7) should be true (with given l and d). At the beginning, we define the function w' as follows:

$$w'(e) = \begin{cases} \frac{1}{2}, & e \in E_1 \\ 0, & e \in E(V) - E_1 \end{cases} \quad \text{in Case 1;}$$

$$w'(e) = \begin{cases} \frac{1}{3}, & e \in E_2 \\ 0, & e \in E(V) - E_2 \end{cases} \quad \text{in Case 2.}$$

One can check that

(4.8) $\quad w'(\partial X) \geqslant 1$ for all $X \in \mathscr{X}(V, S^*)$

in both cases. Furthermore, in Case 1, we have

$$l \cdot w' = 4 \cdot 1 \cdot \frac{1}{2} < 3 = \Sigma(d(A)\colon A \in S)$$

and in Case 2, we have

$$l \cdot w' = 7 \cdot 1 \cdot \frac{1}{3} < 3 = \Sigma(d(A)\colon A \in S).$$

The only reason why it is not fit, in general, to put $w = w'$ is that the inequality $w'(\partial X) < 1$ is possible for some $X \in \mathscr{X}(V, S - S^*)$. However, the following lemma is valid (its proof is the central point in our process and it will be given later).

Lemma 4.2. *There exists a* $\beta \in \mathbf{R}_+^{E(T)}$ *such that*

$$\beta(\partial^T A_i) = 1 \quad (i = 1, 2, 3),$$
$$\beta(\partial^T A) > 1 \quad \text{for all } A \in S - S^*. \tag{4.9}$$

Assuming that this lemma is valid we define suitable w as follows. Put $w''(e) = \beta(e)$ $(e \in E(T))$ and $w''(e) = 0$ $(e \in E(V) - E(T))$. Let $\kappa = \min\{\beta(\partial^T A) - 1\colon A \in S - S^*\}$ and $\xi = \frac{1}{2}\min\{1, \kappa\}$, then $0 < \xi \leqslant \frac{1}{2}$. Now put $w = \xi w' + (1 - \xi) w''$. We observe that $w''(\partial X) = \beta(\partial^T A)$ for any $X \subset V$ and $A = X \cap T$, and hence, by (4.8) and (4.9), $w(\partial X) \geqslant 1$ for all $X \in \mathscr{X}(V, S^*)$. If $X \in \mathscr{X}(V, S - S^*)$ and $A = X \cap T$, then

$$w(\partial X) \geqslant (1 - \xi)\beta(\partial^T A) \geqslant (1 - \xi)(1 + \kappa) \geqslant 1.$$

Thus, (4.5) holds for given w. Next, by (4.9) and the definition of l, we have

$$l \cdot w'' = \Sigma(\rho'_{A_i}\beta\colon i = 1, 2, 3) = 3,$$

and now, since $l \cdot w' < 3$ and $\Sigma(d(A)\colon A \in S) = 3$, we obtain

$$l \cdot w = \xi l \cdot w' + (1 - \xi) l \cdot w'' < 3 = \Sigma(d(A)\colon A \in S),$$

i.e. (4.7) is true.

Proof of Lemma 4.2. We prove that the linear program $\mathscr{P}$: max δ subject to

$$\beta(\partial^T A_i) = 1 \quad (i = 1, 2, 3), \qquad | \tilde{\gamma}$$
$$-\beta(\partial^T A) + \delta \leqslant -1 \quad (A \in S - S^*), \qquad | \epsilon$$
$$\beta \in \mathbf{R}_+^{E(T)}, \quad \delta \in \mathbf{R}_+$$

has an optimum solution (β, δ) with either $\delta > 0$ or $\delta = \infty$. The program $\mathscr{P}^*$ dual to $\mathscr{P}$ is:

$$q(\tilde{\gamma}, \epsilon) = \Sigma(\tilde{\gamma}(A_i)\colon i = 1, 2, 3) - \Sigma(\epsilon(A)\colon A \in S - S^*) \to \min$$
$$\Sigma(\tilde{\gamma}(A_i)\rho_{A_i}\colon i = 1, 2, 3) - \Sigma(\epsilon(A)\rho_A\colon A \in S - S^*) \geqslant 0$$
$$\Sigma(\epsilon(A)\colon A \in S - S^*) \geqslant 1$$
$$\gamma(A_i) \lessgtr 0 \quad (i = 1, 2, 3), \quad \epsilon(A) \geqslant 0 \quad (A \in S - S^*).$$

Suppose that $\mathscr{P}^*$ has no feasible solution. Then one can see that, for any $A \in S - S^*$, there is an edge $u \in \partial^T A$ such that $u \notin \partial^T A_i$ $(i = 1, 2, 3)$. Define β as follows. Let $T' \subseteq T$ and $S' = \{A'_1, A'_2, A'_3\}$ be defined as in Proposition 3.2. If $T' = \{0, 1, 2, 3\}$, put $\beta(0, i) = \frac{1}{2}$ $(i = 1, 2, 3)$, and if $T' = \{0, \ldots, 5\}$, put $\beta(0, 3) = 1$. Next, put $\beta(u) = 2$ for all $u \in E(T)$ such that $u \notin \partial^T A_i$ $(i = 1, 2, 3)$, and $\beta(u) = 0$ for the remaining u's in $E(T)$. Then, as it is easy to see, $\beta(\partial^T A_i) = 1$ $(i = 1, 2, 3)$ and $\beta(\partial^T A) \geqslant 2$ $(A \in S - S^*)$, hence $(\beta, 1)$ is a feasible solution of $\mathscr{P}$.

Next, suppose that $\mathscr{P}^*$ has an unbounded solution, and let $(\tilde{\gamma}, \epsilon)$ be a feasible solution of $\mathscr{P}^*$ such that $q(\tilde{\gamma}, \epsilon) < 0$. Put $\gamma(A_1) = \max\{\tilde{\gamma}(A_1), 0\} - q(\tilde{\gamma}, \epsilon)$, $\gamma(A_i) = \max\{\tilde{\gamma}(A_i), 0\}$ $(i = 2, 3)$ and $\epsilon(A_i) = \max\{-\tilde{\gamma}(A_i), 0\}$ $(i = 1, 2, 3)$. Then $\gamma \geqslant 0$, $\epsilon \geqslant 0$ and (1.5) holds for these γ and ϵ, contradicting the choice of S^*.

Now suppose that $\mathscr{P}^*$ has an optimum solution $(\tilde{\gamma}, \epsilon)$ with $q(\tilde{\gamma}, \epsilon) = 0$. Note that feasibility of $(\tilde{\gamma}, \epsilon)$ implies that $\epsilon(A) > 0$ for some $A \in S - S^*$, whence $\tilde{\gamma}(A_i) > 0$ for some $A_i \in S^*$. But (1.5) holds for no (γ, ϵ), and now a contradiction to the supposition is immediately obtained from the following lemma (thus, it remains to consider only the situation when $\mathscr{P}^*$ has an optimum solution $(\tilde{\gamma}, \epsilon)$ with $q(\tilde{\gamma}, \epsilon) > 0$; then, by l.p. duality, there is a feasible solution (β, δ) of $\mathscr{P}$ with $\delta > 0$, as required).

Lemma 4.3. *Let* $\phi \neq I \subseteq \{1, 2, 3\}$, $\tilde{S}^* = \{A_i: i \in I\}$ *and* $\gamma(A_i) > 0$ *for* $i \in I$. *Further, let* $S(T)$ *be a collection of proper subsets of* T *such that* $\tilde{S}^* \subseteq S(T)$ *and, for any proper subset* A *of* T, *exactly one of* A *and* $T - A$ *is in* $S(T)$. *Then there exists no function* $\epsilon: S(T) - \tilde{S}^* \to \mathbf{R}_+$ *such that both of the following equalities are true:*

$$(4.10) \qquad \Sigma(\gamma(A_i)\rho_{A_i}: i \in I) = \Sigma(\epsilon(A)\rho_A : A \in S(T) - \tilde{S}^*),$$

$$(4.11) \qquad \Sigma(\gamma(A_i): i \in I) = \Sigma(\epsilon(A): A \in S(T) - \tilde{S}^*).$$

Proof. We need some known easy assertions about metrics. A function $\mu: E(T') \to \mathbf{R}_+$ is said to be a *metric on* T' if $\mu(xy) + \mu(yz) \geqslant \mu(xz)$ for any distinct $x, y, z \in T'$. A chain L in $K_{T'}$ is called a *geodetic* of μ if $\mu(eL) = \mu(L)$ $(= \Sigma(\mu(e): e \in L))$. Let $\mathscr{T}(\mu)$ denote the set of geodetics of μ.

(1) If $\mu, \mu', \mu'' \in E(T')$ are metrics and $\mu = \mu' + \mu''$, then $\mathscr{T}(\mu) = \mathscr{T}(\mu') \cap \mathscr{T}(\mu'')$.

(2) Let $S' \subset 2^{T'}$ and $\nu \in \mathbf{R}_+^{S'}$. Then $\mu = \Sigma(\nu(A)\rho'_A : A \in S')$ is a metric, and a chain $L \subseteq E(T')$ is a geodetic of μ if and only if $|L \cap \partial^{T'} A| \leqslant 1$ for each $A \in S'$ such that $\nu(A) > 0$ (ρ'_A is the characteristic vector of $\partial^{T'} A$ in $\mathbf{R}^{T'}$).

(3) Let μ be a metric on T', and let $\mu(st) = 0$ for some $st \in E(T')$. Then $\mu(sp) = \mu(tp)$ for any $p \in T - \{s, t\}$.

(4) (a corollary of (2) and (3)). Let μ be a metric on T', and let $\{T_1, T_2, \ldots, T_m\}$ be the partition of T' into maximal subsets T_i such that $\mu(u) = 0$ for any $u \in E(T_i)$. Define $\mu^P(ij) = \mu(st)$ for $ij \in E(P)$, $s \in T_i$, $t \in T_j$, where $P = \{1, \ldots, m\}$. Then

(a) μ^P is a metric on P,

(b) if $\mu = \Sigma(\nu(A)\rho'_A : A \in S')$, where $S' \subset 2^{T'}$, $\nu \in \mathbf{R}^{S'}$ and $\nu(A) > 0$ for all $A \in S'$, then, for any $A \in S'$, $A = \bigcup (T_i: i \in A^P)$ for some $A^P \subset P$, and $\mu^P = \Sigma(\nu(A)\rho''_{A^P}: A \in S')$, where ρ''_{A^P} denotes the characteristic vector of $\partial^P A^P$ in $\mathbf{R}^P$.

Now we start proving the lemma. Suppose, for a contradiction, that there is ϵ satisfying (together with given γ) (4.10) and (4.11). Let $\mu = \Sigma(\gamma(A_i)\rho_{A_i}: i \in I)$. By (2), μ is a metric. According to (4), without loss of generality we may assume that $\mu(u) > 0$ for all $u \in E(T)$ (as otherwise, we could consider P and μ^P instead of T and μ, where P is the partition for μ as in (4)). Thus, we have $2 \leqslant |T| \leqslant 2^{|I|} \leqslant 8$.

Case 1. $|I| = 1$.

Then $|T| = 2$, $\tilde{S}^* = S(T)$, and (4.10) is impossible.

Case 2. $|I| = 2$.

Let $\tilde{S}^* = \{A_1, A_3\}$, say. Since A_1 and A_3 are crossing, we have $|T| = 4$. Let, for definiteness, $T = \{0, 1, 2, 3\}$, $A_1 = \{0, 1\}$, $A_3 = \{0, 3\}$ and $S(T) - \tilde{S}^* = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0, 2\}\}$. It is easy to see that:

(i) each chain $L_j = \{j(j + 1), (j + 1)(j + 2)\}$ (indices are taken by modulo 4) is a geodetic of μ (see Fig. 4.1),

(ii) for each $A \in S(T) - \tilde{S}^*$ there is some geodetic L_j such that $|L_j \cap \partial^T A| = 2$. Thus, by (1) and (2), $\epsilon = 0$, a contradiction.

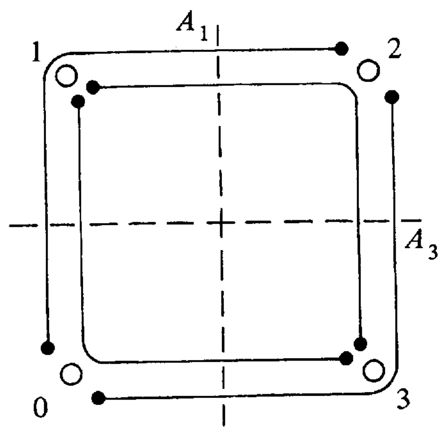


Figure 4.1

Case 3. $|I| = 3$.

Then $\widetilde{S}^* = S^*$ and $4 \leqslant |T| \leqslant 8$. Let $T' \subseteq T$ and $S' = \{A_i' = A_i \cap T' \colon i = 1, 2, 3\}$ be defined as in Proposition 3.2. Let ρ_B' be the characteristic vector of subset $\partial^{T'} B$ in $\mathbf{R}^{T'}$ for $B \subset T'$.

Claim 4.4. *The inequalities*

(4.12) $\quad \Sigma(\psi(A_i')\rho_{A_i'}' \colon i = 1, 2, 3) = \Sigma(\xi(A')\rho_{A'}' \colon A' \in S(T'))$,

(4.13) $\quad \Sigma(\psi(A_i') \colon i = 1, 2, 3) = \Sigma(\xi(A') \colon A' \in S(T'))$

imply

(4.14) $\quad \xi(A') = 0 \qquad$ *for all* $A' \in S(T') - S'$,

(4.15) $\quad \xi(A_i') = \psi(A_i')$ *for* $i = 1, 2, 3$.

Proof. Assume that $|T'| = 4$, and let $T' = \{0, 1, 2, 3\}$ and $A_i' = \{0, i\}$ $(i = 1, 2, 3)$. Put $A^j = \{j\}$ $(j = 0, 1, 2, 3)$. The vector equality (4.12) may be written as

$$\Sigma(\psi(A_i') - \xi(A_i') \colon i \in \{1, 2, 3\},\ u \in \partial^{T'} A_i') = \\ = \Sigma(\xi(A^j) \colon j \in \{0, 1, 2, 3\},\ u \in \partial^{T'} A^j) \quad \text{for all } u \in E(T'). \tag{4.16}$$

Summing up all the six equalities in (4.16) we get

$$4\Sigma(\psi(A_i') - \xi(A_i') \colon i = 1, 2, 3) = 3\Sigma(\xi(A^j) \colon j = 0, 1, 2, 3),$$

whence, taking into account (4.13), we obtain (4.14). Now (4.16) easily implies (4.15).

Now assume that $|T'| = 6$, and let $T' = \{0, \ldots, 5\}$, $A_i' = \{i, i+1, i+2\}$ $(i = 1, 2, 3)$. One can see that:

(a) each chain $L_j = \{j(j+1), (j+1)(j+2), (j+2)(j+3)\}$ $(j = 0, \ldots, 5)$ (indices are taken by modulo 6) is a geodetic of μ (see Fig. 4.2),

(b) for any $A' \in S(T') - S'$ there is a geodetic L_j such that $|L_j \cap \partial^{T'} A'| \geqslant 2$. Thus, by (b), (1) and (2), (4.14) is valid. Now,

considering the equality (4.12) for the edge $(i-1)i$ $(i = 1, 2, 3)$ we immediately obtain $\psi(A_i') = \xi(A_i')$, as required. ∎

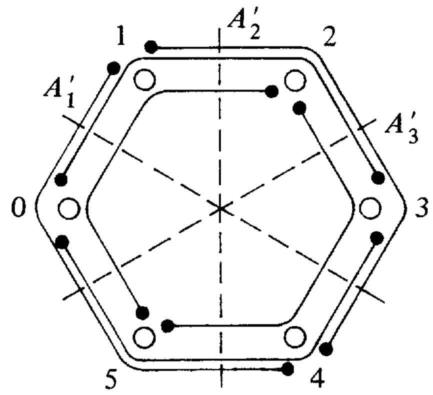


Figure 4.2

Thus, if $T' = T$, the result immediately follows from Claim 4.4. Now assume that $T' \subset T$. In view of Claim 4.4, in order to finish the proof it is enough to show that, if $A \in S(T)$ is such that $A \notin S^*$ and $A \cap T' = A_i'$ (or $A \cap T' = T' - A_i'$) for some i, then there is a geodetic $L \subset E(T)$ of μ for which $|L \cap \partial^T A| \geqslant 2$ (then, by (1) and (2), it must be $\epsilon(A) = 0$). Let, for definiteness, $A \cap T' = A_1'$ (and $A \neq A_1$), and choose a vertex $v \in T - T'$ such that either $v \in A$ and $v \notin A_1$ or $v \notin A$ and $v \in A_1$. Take $A^* \in \{A, T - A\}$ and $A_i^* \in \{A_i, T - A_i\}$ $(i = 1, 2, 3)$ so that $A^* \cap T' = A_1^* \cap T'$, $v \notin A^*$ and $v \in A_i^*$ $(i = 1, 2, 3)$. Since $\mu(xy) > 0$ for any $xy \in E(T)$, then $|A_1^* \cap A_2^* \cap A_3^*| \leqslant 1$, and so $A_1^* \cap A_2^* \cap A_3^* = \{v\}$. Hence $A_1^{*\prime} \cap A_2^{*\prime} \cap A_3^{*\prime} = \phi$, where $A_i^{*\prime} = A_i^* \cap T'$. Choose $p \in A_1^{*\prime} \cap A_2^{*\prime}$ and $q \in A_1^{*\prime} \cap A_3^{*\prime}$ (see Fig. 4.3) (every $A_i^{*\prime} \cap A_j^{*\prime}$ is nonempty because $A_i^{*\prime}$ and $A_j^{*\prime}$ are crossing). Now since $p \notin A_3^{*\prime}$ and $q \notin A_2^{*\prime}$, we have $\mu(pv) = \gamma(A_3)$, $\mu(vq) = \gamma(A_2)$ and $\mu(pq) = \gamma(A_2) + \gamma(A_3)$, therefore the chain $L = \{pv, vq\}$ is a geodetic of μ. But $p, q \in A^*$ and $v \notin A^*$, hence $|L \cap \partial^T A| = 2$.

This completes the proof of Lemmas 4.3, 4.2 and Theorem 1. ∎

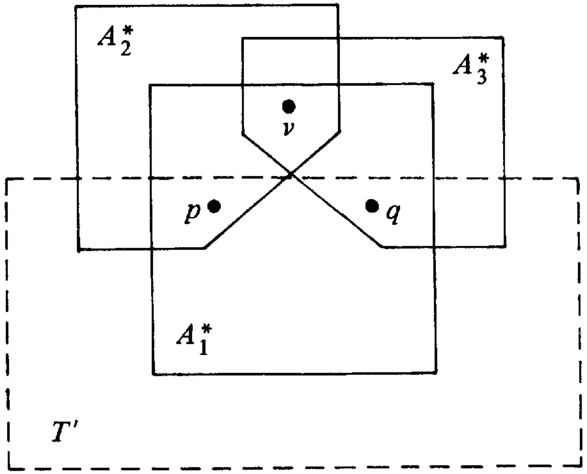


Figure 4.3

5. 2-COMPLETE SCHEMES

Consider a multicut max-Σ problem $\Sigma(V, S, l)$, where $S \subset 2^T$, and let $\mu = \mu_l|_{E(T)}$. It was explained in Section 4 that the minimax relation (1.4) holds for such a problem if and only if

(5.1) $\quad p(V, S, l) = p(T, S, \mu)$.

Thus, given $S \subset 2^T$, (5.1) is true for any $V \supseteq T$ and $l \in \mathbf{R}_+^{E(V)}$ if and only if S belongs to the Σ-DSC-class which, by Theorem A, is equivalent to that S is 3-complete. Moreover, the idea behind the proof of the part (i) of Theorem A prompts the following approach for solving $\Sigma(V, S, l)$ with 3-complete S. First we must solve the reduced multicut max-Σ problem $\Sigma(T, S, \mu)$ (which may be named a *pre-problem* of $\Sigma(V, S, l)$). Next, we reform the found optimum solution $\gamma \in \mathbf{R}_+^S$ of $\Sigma(T, S, \mu)$ into such an optimum solution γ' that the family $S^+(\gamma') = \{A \in S: \gamma'(A) > 0\}$ is not 3-crossing. Finally, we solve the multicut existence problem EX $(V, S^+(\gamma'), l, d)$, where $d(A) = \gamma'(A)$ $(A \in S^+(\gamma'))$, then the found solution α is an optimum solution of $\Sigma(V, S, l)$.

There is a greedy algorithm for solving any existence problem EX (V, S, l, d) with real-valued l and d and non-3-crossing S [18]. The complexity of the algorithm is $O(|T|^3|V|^3)$ operations. As a direct corollary of the algorithm, the following half-integerity theorem is stated. We say that a function l on $E(V)$ is *cyclically even* if it is nonnegative integer-valued and the value $l(C) = \Sigma(l(e): e \in C)$ is even for any circuit C in K_V.

Theorem 5.1. *Let $S \subset 2^T$ be not 3-crossing, $V \supseteq T$, $l \in \mathbf{Z}_+^{E(V)}$ and $d \in \mathbf{Z}_+^S$, and let* (1.6) *hold for all $u \in E(T)$. Further, let the function l', defined by $l'(e) = \Sigma(d(A): A \in S, e \in \partial^T A)$ $(e \in E(T))$ and $l'(e) = l(e)$ $(e \in E(V) - E(T))$, be cyclically even. Then* EX (V, S, l, d) *has an integer solution.*

Return to the max-Σ problem $\Sigma(V, S, l)$ with 3-complete S, and suppose that we succeed in finding such an optimum solution γ of the pre-problem $\Sigma(T, S, \mu)$ that $S^+(\gamma)$ is not 3-crossing and both ml and $m\gamma$ are integer-valued, for some integer m. Then, by Theorem 5.1, there exists an optimum solution α of $\Sigma(V, S, l)$ such that $2m\alpha$ is integer-valued.

Now we introduce one special class of 3-complete schemes. A scheme $S \subset 2^T$ is called *2-complete* if, for any two crossing $A_1, A_2 \in S$ (if any), there are $B_1, B_2 \in S$ such that $B_1^* = A_1^* - A_2^*$ and $B_2^* = A_2^* - A_1^*$ for some $A_i^* \in \{A_i, T - A_i\}$ and $B_i^* \in \{B_i, T - B_i\}$ $(i = 1, 2)$. It is easy to see that such a relation can be expressed as the vector inequality

(5.2) $\quad \rho_{A_1} + \rho_{A_2} > \rho_{B_1} + \rho_{B_2}$.

Obviously, any laminar scheme (i.e. consisting of pairwise laminar subsets) is 2-complete. It is easy to show that, for any T, the scheme of all the odd-size subsets of T (Ex. 3 in Introduction) is 2-complete. Here are other examples of 2-complete schemes S (their examination is left to the reader).

Example 5. S consist of all proper subsets of T.

Example 6. $S = \{A \subset T: 1 \leqslant |A| \leqslant k\}$ for arbitrary $k < |T|$.

Example 7. $S = \{A \subset T: |\partial^T A \cap U| = 1\}$ for arbitrary $U \subseteq E(T)$.

Example 8. $S = \{A \subset T: a(\partial^T A)$ odd$\}$ for arbitrary integer-valued function a on $E(T)$.

Proposition 5.2. *If a scheme is 2-complete, then it is also 3-complete.*

Indeed, let A_1, A_2, A_3 be pairwise crossing members of S. As S is 2-complete, there are $B_1, B_2 \in S$ for which (5.2) holds. Then (1.5) holds for $\gamma(A_1) = \gamma(A_2) = \epsilon(B_1) = \epsilon(B_2) = 1$, $\gamma(A_3) = 0$ and $\epsilon(A) = 0$ for $A \in S - \{B_1, B_2\}$. ∎

Thus, by Theorem A, each 2-complete schemes belong to the Σ-DSC-class. The 2-complete schemes have the following important property.

Theorem 5.3. *Let* $S \subset 2^T$ *be 2-complete,* $V \supseteq T$, $l \in \mathbf{R}_+^{E(V)}$, *and let* $\mu = \mu_l|_{E(T)}$. *Further, let* $\gamma\colon S \to \mathbf{R}_+$ *be* μ*-admissible. Then there exists an* l*-admissible* $\alpha\colon \mathscr{X}(V, S) \to \mathbf{R}_+$ *such that*

(i) $1 \cdot \alpha = 1 \cdot \gamma$,

(ii) α *is integer-valued if* l *and* γ *are integer-valued,*

(iii) *the family* $\mathscr{X}^+(\alpha) = \{X \in \mathscr{X}(V, S)\colon \alpha(X) > 0\}$ *is laminar,*

(iv) *if the family* $S^+(\gamma)$ *is laminar, then*

$$\zeta^\alpha(A)\ (= \Sigma(\alpha(X)\colon X \subset V,\ X \cap T = A)) = \gamma(A) \quad \textit{for all } A \in S.$$

Proof. Supposing that $S^+(\gamma)$ is not laminar, let γ' be μ-admissible and such that $1 \cdot \gamma' = 1 \cdot \gamma$, γ' is integral if γ is that, and the value $\omega(\gamma') = \Sigma(\lambda^{\gamma'}(u)\colon u \in E(T))$ is minimum. We assert that $S^+(\gamma')$ is laminar. For suppose that $A_1, A_2 \in S^+(\gamma')$ are crossing, and let $B_1, B_2 \in S$ be such that (5.2) holds. Define γ'' by $\gamma''(A_i) = \gamma'(A_i) - a$, $\gamma''(B_i) = \gamma'(B_i) + a$ $(i = 1, 2)$, where $a = \min\{\gamma'(A_1), \gamma'(A_2)\}$, and $\gamma''(A) = \gamma'(A)$ for the remaining A's in S. Then γ'' is μ-admissible and $1 \cdot \gamma'' = 1 \cdot \gamma'$. But $\omega(\gamma'') < \omega(\gamma')$, a contradiction to the choice of γ'.

Thus, we may assume that $S^+(\gamma)$ is laminar and now we prove (iv). We proceed by induction on $\gamma(l, \gamma) = |S^+(\gamma)| + |\{e \in E(V)\colon l(e) > 0\}|$ (by fixed V, T and S). The result is obvious when $S^+(\gamma) = \phi$. Let $S^+(\gamma) \neq \phi$. Choose a minimal set A^* among all $A \in S^+(\gamma)$ and their complements $T - A$. Without loss of generality, we may assume that $A^* \in S^+(\gamma)$. Put

$$X = A^* \cup \{x \in V - A^*\colon \min\{\mu_l(sx)\colon s \in A^*\} = 0\},$$

$$\Delta = \min\{\gamma(A^*),\ \min\{l(e)\colon e \in \partial^V X\}\}.$$

Since γ is μ-admissible, then $l(st) \geq \mu(st) \geq \lambda^\gamma(st) \geq \gamma(A^*) > 0$ for any $st \in \partial^T A^*$, hence $X \cap T = A^*$. Define l' and γ' by $l'(e) = l(e) - \Delta$ $(e \in \partial X)$, $l'(e) = l(e)$ $(e \in E(V) - \partial X)$, $\gamma'(A^*) = \gamma(A^*) - \Delta$ and $\gamma'(A) = \gamma(A)$ $(A \in S - \{A^*\})$, and let $\mu' = \mu_{l'}|_{E(T)}$. Obviously, $\Delta > 0$ and $\eta(l', \gamma') < \eta(l, \gamma)$. We have to prove that γ' is μ'-admissible, i.e.

$$(*) \qquad \lambda^{\gamma'}(pq)\ (= \Sigma(\gamma'(A)\colon A \in S,\ pq \in \partial^T A)) \leq \mu'(pq)$$

for any $pq \in \partial^T A$. Then, by induction, there exists an l'-admissible $\alpha'\colon \mathscr{X}(V, S) \to \mathbf{R}_+$ for which $\zeta^{\alpha'}(A) = \gamma'(A)$ for all $A \in S$, and required α is defined by $\alpha(X) = \alpha'(X) + \Delta$ and $\alpha(Y) = \alpha'(Y)$ $(Y \in \mathscr{X}(V, S) - \{X\})$.

(1) $p, q \in A^*$.

The choice of A^* implies $\lambda^\gamma(pq) = 0$, and $(*)$ is obvious.

(2) $p, q \in T - A^*$.

Let $L = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\} \subseteq E(V)$ be a shortest chain with $x_0 = p$ and $x_k = q$, i.e. $l'(L) = \mu'(pq)$. If $L \cap \partial X = \phi$, then $l'(L) = l(L) = \mu(pq)$, and the result follows. Assume that $L \cap \partial X \neq \phi$, and let $0 = i(0) < i(1) < \ldots < i(m) = k$ be such a sequence that $m = |L \cap \partial X|$ and $x_{i(j)}$ is in X if and only if j is odd. According to the definition of X, for any odd j, either $x_{i(j)} = s_j \in A^*$ or there is $s_j \in A^*$ such that $\mu_l(s_j, x_{i(j)}) = 0$. We observe that $\mu_l(e) \geq \Delta$ for any $e \in \partial X$. Further, the laminarity of $S^+(\gamma)$ and the minimal choice of A^* imply

$$\lambda^\gamma(ps_1) + \lambda^\gamma(s_{m-1}q) - \lambda^\gamma(pq) \geq 2\gamma(A^*) \geq 2\Delta.$$

Using these facts, we have

$$\mu'(pq) = l'(L) = l(L) - m\Delta \geq \sum_{j=0}^{m-1} \mu(x_{i(j)}x_{i(j+1)}) - m\Delta \geq$$

$$\geq \mu(ps_1) + \mu(s_{m-1}q) - 2\Delta \geq \lambda^\gamma(ps_1) + \lambda^\gamma(s_{m-1}q) - 2\Delta \geq$$

$$\geq \lambda^\gamma(pq) = \lambda^{\gamma'}(pq).$$

(3) $p \in T - A^*$, $q \in A^*$.

In this case the proof of (*) is carried out similarly to the previous one (one should take into account that $m = |L \cap \partial X|$ is odd and $\lambda^{\gamma'}(pq) = \lambda^{\gamma}(pq) - \Delta$).

It is easy to see that (ii) and (iii) are also true. (i) immediately follows from (iv). ∎

Corollary 5.4. *If S is 2-complete, l is integer-valued and $\Sigma(T, S, \mu)$ has an integer optimum solution, then $p(V, S, l) = p(T, S, \mu)$ and $\Sigma(V, S, l)$ has an integer-valued optimum solution α such that $\mathscr{X}^+(\alpha)$ is laminar.*

Remark. The above proof contains, in essence, an efficient algorithm solving $\Sigma(V, S, l)$ with 2-complete S and $l \in \mathbf{R}_+^{E(V)}$ in assumption that an optimum solution γ of the pre-problem $\Sigma(T, S, \mu)$ with laminar $S^+(\gamma)$ is given. Note that the following specification should be introduced in this algorithm for the efficiency: if A^* is chosen on some iteration, then subsequent iterations are accomplished for the same A^* while $\gamma(A^*) > 0$, where γ is considered as a current (decreased) function. Then complexity of the algorithm is shown to be $O(|T||V|^3)$ operations (note that this algorithm can be improved to be of complexity $O(|V|^2|T|\log|T|)$).

Now we demonstrate some applications. Let $U \subseteq E(V)$ and $\mathscr{X}_U = \{X \subset V: |\partial X \cap U| = 1\}$. Consider the following problem $P(V, U, l)$: given $l \in \mathbf{R}_+^{E(V)}$, find an l-admissible $\alpha: \mathscr{X}_U \to \mathbf{R}_+$ such that

(5.3) $\lambda^{\alpha}(u) = l(u)$ for all $u \in U$.

For a circuit $C \subseteq E(V)$ in K_V, let $\Delta(l, C)$ denote the value $l(C - U) - l(C \cap U)$. The problem $P(V, U, l)$ was studied by Seymour [9], [10] who proved that

(i) $P(V, U, l)$ has a solution α if and only if $\Delta(l, C)$ is nonnegative for each circuit C in K_V,

(ii) if l is integer-valued and $\Delta(l, C)$ is a nonnegative even integer for each circuit C in K_V then the problem has an integer solution.

We show that (ii) can be directly obtained from (i). Let T be the set of ends of edges of U, and let $S_U = \{A \subset T: |\partial^T A \cap U| = 1\}$ (cf. Exam-

ple 7). It is easy to see that $\mathscr{X}_U = \mathscr{X}(V, S_U)$. Assume that l is integer-valued and $\Delta(l, C)$ is a nonnegative even integer for every circuit C, and let $\mu = \mu_l|_{E(T)}$. First, we observe that μ is integer-valued, $\mu(u) = l(u)$ for all $u \in U$ and $\Delta(\mu, C)$ is a nonnegative even integer for every circuit C in K_T. Next, applying (i) to $P(T, U, \mu)$, there is a μ-admissible $\gamma: S_U \to \mathbf{R}_+$ such that

(*) $\lambda^{\delta}(u) = \mu(u) = l(u)$ for all $u \in U$,

where $\delta = \gamma$. Since S_U is 2-complete, we may assume that $S^+(\gamma) = \{A \in S_U: \gamma(A) > 0\}$ is laminar. It suffices to prove that there exists integer-valued μ-admissible $\tilde{\gamma}$ satisfying (*), where $\delta = \tilde{\gamma}$. Then, by Theorem 5.3, there exists l-admissible $\alpha: \mathscr{X}_U \to \mathbf{Z}_+$ such that $\zeta^{\alpha}(A) = \gamma(A)$ for all $A \in S_U$ (such an α is a solution of $P(V, U, l)$ obviously). We proceed by induction on $\psi(T, \mu) = |T| + \Sigma(\mu(e): e \in E(T))$.

If $\mu(st) = 0$ for some $st \in E(T)$ we reduce the problem to that with T', U' and μ', where T' is obtained from T by identifying s and t (i.e. by replacing s and t by a new vertex) and U' is the image of U under this identification; μ' is defined naturally. Then $\psi(T', \mu') < \psi(T, \mu)$, and the result follows by induction. Thus, we may assume that $\mu(e) > 0$ for all $e \in E(T)$. Choose a minimal set A^* among all $A \in S^+(\gamma)$ and their complements $T - A$. Without loss of generality, we may assume that $A^* \in S^+(\gamma)$. The laminarity of $S^+(\gamma)$, the minimal choice of A^* and the positivity of μ imply that there is no $pq \in U$ such that $p, q \in A^*$, whence easily follows $|A^*| = 1$. And so, let $A^* = \{s\}$ and $\partial^T A^* \cap U = \{st\}$, say. Put $\nu = \mu - \rho_{A^*}$, and define μ' to be the distance function on $E(T)$ induced by the edge length function ν. Clearly that $\nu \geq 0$, $\psi(T, \mu') < \psi(T, \mu)$ and $\Delta(\nu, C)$ is even for any circuit C in K_T. Let C be an arbitrary circuit in K_T. We show that $\Delta(\nu, C) \geq 0$. (Then $\Delta(\mu', C')$ is a nonnegative even integer for any circuit C', whence $\mu'(u) = \nu(u) = \mu(u)$ $(u \in U - \{st\})$ and $\mu'(st) = \nu(st) = \mu(st) - 1$. By induction, there is integer-valued μ'-admissible $\tilde{\gamma}'$ satisfying (*) (for $\delta = \tilde{\gamma}'$), therefore $\tilde{\gamma}$, defined by $\tilde{\gamma}(A^*) = \tilde{\gamma}'(A^*) + 1$ and $\tilde{\gamma}(A) = \tilde{\gamma}'(A)$ $(A \in S_U - \{A^*\})$, is a required μ-admissible function.) Suppose, for a contradiction, that $\Delta(\nu, C) < 0$. Then $C \cap \partial^T A^* \neq \phi$, hence C contains exactly two edges, say, sp and sq in $\partial^T A^*$. Since $\Delta(\nu, C)$ is even, we obtain $\Delta(\mu, C) = \Delta(\nu, C) + 2 = 0$, and so $sp, sq \in E(T) - U$. Now, because $\Delta(\mu, C) = 0$ and (*) holds for $\delta = \gamma$ we have $|(C - U) \cap \partial^T A| = |C \cap U \cap \partial^T A| \leqslant 1$ for any $A \in S^+(\gamma)$. But $(C - U) \cap \partial^T A^* = \{sp, sq\}$. This contradiction proves (i) → (ii). ∎

Now consider a max-Σ problem $\Sigma(V, S_U, l)$, and let α be its optimum solution. The definition of S_U implies $1 \cdot \alpha = \Sigma(\lambda^\alpha(u)\colon u \in U) \leqslant l(U)$. Thus, α is a solution of the problem $P(V, U, l)$ if $p(V, S_U, l) = l(U)$, and $P(V, U, l)$ has no solution if $p(V, S_U, l) < l(U)$. Therefore, $\Sigma(V, S_U, l)$ may be regarded as a generalization of $P(V, U, l)$. However, unlike the latter, $\Sigma(V, S_U, l)$ has the following rather unexpected property.

Proposition 5.5. *For any integer $k \geqslant 2$, there are T, $U \subset E(T)$ and $l\colon E(T) \to \mathbf{Z}_+$ such that $|T| = 2k$ and $p(T, S_U, l)$ has the denominator 2^{k-1}.*

Proof. Given k, let $G = (T, E)$ be the graph as in Fig. 5.1, and let U be the set of its thick edges. Put $l(e) = 1$ for $e \in E$ and $l(e) \geqslant 2k$ for $e \in E(T) - E$. Let $A_i = \{s_i\}$ $(i = 1, \dots, k)$, and $B_j = \{s_1, \dots, s_j, t_1, \dots, t_{j-1}\}$ $(j = 2, \dots, k)$. It is easy to see that U generates a spanning tree in K_T, whence S_U consists just of such A_i, B_j and their complements. Define γ by $\gamma(A_i) = \gamma(B_i) = m_i$, where

$$m_i = \frac{2^{k-i+1} - (-1)^{k-i+1}}{3 \cdot 2^{k-i+1}} \qquad (i \geqslant 2)$$

and

$$m_1 = \frac{2^k + (-1)^{k-1}}{3 \cdot 2^{k-1}}.$$

Then the denominator of $1 \cdot \gamma$ is 2^{k-1}. One can check that γ is l-admissible. Finally, γ is optimal since the "dual" function w on $E(T)$, defined by $w(s_i s_{i+1}) = w(s_i t_{i+1}) = m_{k-i+1}$ $(i = 1, \dots, k-1)$, $w(s_k t_k) = m_1$ and $w(e) = 0$ for the remaining e's in $E(T)$, satisfies $w(\partial^T A_i) = w(\partial^T B_j) = 1$ and $1 \cdot \gamma = 1 \cdot w$. ∎

In conclusion we consider a max-Σ problem $\Sigma(V, S, l)$ for S consisting of all the odd-size subsets of T, $|T|$ is even. Edmonds and

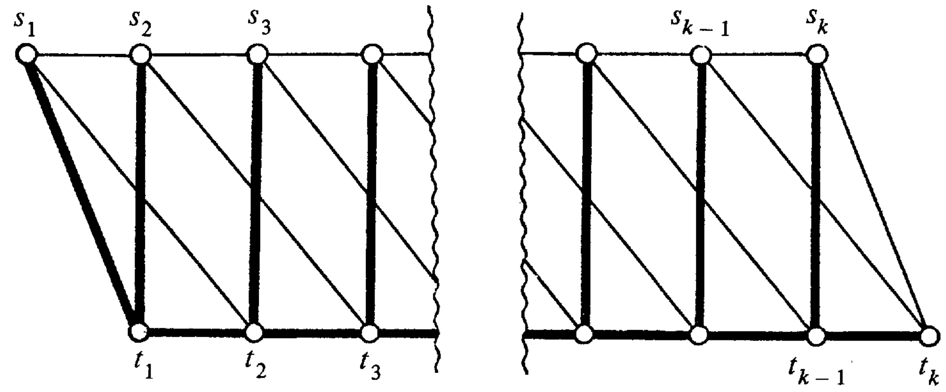


Figure 5.1

Johnson [7] constructed an algorithm for solving such a problem and proved that it has a half-integer optimum solution whenever l is integer-valued (cf. [8]). This result was strengthened by Seymour [10] who proved that if l is cyclically even then the problem has an integer optimum solution (unfortunately his proof is not constructive). Since S is 2-complete, as it is said above, it suffices to solve the pre-problem $\Sigma(T, S, \mu)$. Specific properties of the latter one (in particular, that μ is a metric) permit to construct an algorithm (using a matching technique) solving it which is much simpler than that in [7]. As a result, an algorithm of complexity $O((|T|^3 + |V|^2|T|) \log T)$ is developed for solving $\Sigma(V, S, l)$ with $l \in \mathbf{R}_+^{E(V)}$ and S as above which finds an integer optimum solution whenever l is cyclically even [19].

6. MULTICOMMODITY FLOWS

For $U \subseteq E(V)$, let $\mathscr{L}^V(U)$ denote the set of chains L in K_V such that $eL \in U$. $\mathscr{L}^V(U)$ may be considered as the compound family $(\mathscr{L}^V(u)\colon u \in U)$, where $\mathscr{L}^V(u)$ stands for $\mathscr{L}^V(\{u\})$. Clearly the imperfect blocker of $\mathscr{L}^V(U)$ consists of all subsets $E' = \partial X$, $X \in \mathscr{X}^V(U)$, where $\mathscr{X}^V(U) = \{X \subset V\colon \partial X \cap U \neq \phi\}$.

Now let a *capacity* function $c\colon E(V) \to \mathbf{R}_+$ and a *demand* function $a\colon U \to \mathbf{R}_+$ be given. A known multicommodity flow problem (here called a multiflow existence problem and denoted by $\mathrm{EX}^f(V, U, c, a)$) is: find c-admissible φ: $\mathscr{L}^V(U) \to \mathbf{R}_+$ such that

$$\Sigma(\varphi(L): L \in \mathscr{L}^V(u)) \geq a(u) \quad \text{for all } u \in U$$

(or establish that such a φ does not exist). According to the definition in Section 2, $\mathscr{L}^V(U)$ has the gen. EX-MFMC-property if, for any c and d, $\mathrm{EX}^f(V, U, c, a)$ is solvable provided that the inequality of Ford—Fulkerson's type

$$c(\partial X) \geq \Sigma(a(u): u \in U \cap \partial X)$$

holds for each $X \in \mathscr{X}^V(U)$. Let T^U denote the set of ends of edges of U, and H^U denote the graph (T^U, U). A complete description of families $\mathscr{L}^V(U)$ having the gen. EX-MFMC-property is stated in the following theorem due to Papernov.

Theorem 6.1 [14]. *$\mathscr{L}^V(U)$ has the gen. EX-MFMC-property if and only if H^U is K_4 or C_5 or a union of two stars.*

(Lomonosov (see [15]) proved algorithmically that if H^U is K_4 or C_5 or a union of two stars, c and a are integer-valued and the value $c(\partial X) - \Sigma(a(u): u \in U \cap \partial X)$ is a nonnegative even integer for any $X \subset V$, then the problem $\mathrm{EX}^f(V, U, c, a)$ has an integer solution. In [12], it is shown how to use some multiflow locking problems $\mathrm{LOCK}^f(V, S, w)$ with lockable $\mathscr{C}(V, S)$ in order to solve $\mathrm{EX}^f(V, U, c, a)$ for $H^U \in \{K_4, C_5\}$, and, as a result, an algorithm of $O(|V|^5)$-complexity is developed for solving such multiflow existence problems.)

Now consider a multiflow max-Σ problem (denoted by $\Sigma^f(V, U, c)$): given $V, U \subseteq E(V)$ and $c \in \mathbf{R}_+^{E(V)}$, find c-admissible φ: $\mathscr{L}^V(U) \to \mathbf{R}_+$ with $1 \cdot \varphi$ maximum. By the definition in Section 2, $\mathscr{L}^V(U)$ has the gen. Σ-MFMC-property if, for any c,

$$\max(1 \cdot \varphi) = \min\{\Sigma(\alpha(X)c(\partial X): X \in \mathscr{X}^V(U))\},$$

where the maximum is taken over all c-admissible φ and the minimum is taken over all α: $\mathscr{X}^V(U) \to \mathbf{R}_+$ satisfying

$$\Sigma(\alpha(X): X \in \mathscr{X}^V(U), u \in \partial X) \geq 1 \quad \text{for all } u \in U.$$

Let $\mathscr{A}(H)$ denote the collection of maximal independent subset of vertices in a graph H. We say that H is *even* if there is a partition $\{\mathscr{A}_1, \mathscr{A}_2\}$ of $\mathscr{A}(H)$ for which each $\mathscr{A}_i$ consists of pairwise disjoint members. The following theorem describes a class of families $\mathscr{L}^V(U)$ having the gen. Σ-MFMC-property [the part (i) was proved algorithmically by Lomonosov and the author (a detail construction of their algorithm is given in [12]) and the part (ii) was proved by Poevzner and the author].

Theorem 6.2 ([11], [16]).

(i) *Let $U \subseteq E(V)$ and let the family $\mathscr{A}(H^U)$ be not 3-crossing. Then $\mathscr{L}^V(U)$ has the gen. Σ-MFMC-property. Furthermore, for any $c \in \mathbf{Z}_+^{E(V)}$, the problem $\Sigma^f(V, U, c)$ has a quarter-integer optimum solution, and it has a half-integer optimum solution whenever the graph H^U is even.*

(ii) *Let $U \subseteq E(V)$, $|V| \geq |T^U| + 4$ and let the family $\mathscr{A}(H^U)$ be 3-crossing. Then $\mathscr{L}^V(U)$ does not have the gen. Σ-MFMC-property.*

Next we consider a locking problem for a compound family $\mathscr{L}^V(U)$ (it is called a *multicut* locking problem and denoted by $\mathrm{LOCK}(V, U, l)$). This is: given $l \in \mathbf{R}_+^{E(V)}$, find a l-admissible function α: $\mathscr{X}^V(U) \to \mathbf{R}_+$ such that

$$\mu_l(u) = \Sigma(\alpha(X): X \in \mathscr{X}^V(U), u \in \partial X) \quad \text{for all } u \in U.$$

Theorem 6.1 and Lemma 2.2 imply the following result: $\mathscr{L}^V(U)$ is lockable if and only if H^U is K_4 or C_5 or a union of two stars. The part "if" can be strengthened as follows: let H^U be K_4 or C_5 or a union of two stars, and let $l \in \mathbf{Z}_+^{E(V)}$ be cyclically even, then $\mathrm{LOCK}(V, U, l)$ has an integer solution [17].

Now we point out one application of multicut locking problem. Consider a two-commodity cut problem $\Sigma(V, S, l)$, where $T = \{0, 1, 2, 3\}$ and $S = \{\{0, 1\}, \{0, 2\}\}$ (see Example 2 in Introduction). In order to solve this problem, we first find a solution α of $\mathrm{LOCK}(V, U, l)$, where $U = \{03, 12\}$ (such a problem is solvable since, for this U, H^U is a union of two stars). Let $\mathscr{X}^+(\alpha) = \{X \in \mathscr{X}^V(U): \alpha_X > 0\}$; because $|T| = 4$ we may assume that $|X \cap T| \leq 2$ for all $X \in \mathscr{X}^+(\alpha)$. Define $\mathscr{X}_i^+(\alpha)$ to be

$\{X \in \mathscr{X}^+(\alpha)\colon X \cap T = \{i\}\}$ $(i = 0, 1, 2, 3)$ and $\mathscr{X}^+_{ij}(\alpha)$ to be $\{X \in \mathscr{X}^+(\alpha)\colon X \cap T = \{i, j\}\}$ $(ij \in E(T))$. Assume that there are nonempty $\mathscr{X}^+_i(\alpha)$ and $\mathscr{X}^+_j(\alpha)$ for some $ij \in E(T) - U$. Choose arbitrarty $X \in \mathscr{X}^+_i(\alpha)$ and $Y \in \mathscr{X}^+_j(\alpha)$, and put $\alpha'(X) = \alpha(X) - a$, $\alpha'(Y) = \alpha(Y) - a$, $\alpha'(X \cup Y) = \alpha(X \cup Y) + a$ and $\alpha'(W) = \alpha(W)$ for the remaining W's in $\mathscr{X}^V(U)$, where $a = \min\{\alpha(X), \alpha(Y)\}$. Obviously,

$$|\{u\} \cap \partial(X \cup Y)| = |\{u\} \cap \partial X| + |\{u\} \cap \partial Y| = 1$$

for $u \in \{03, 12\}$, therefore, α' is also a solution of LOCK (V, U, l). Yielding a sequence of such reformations (no more than $\mathscr{X}^+(\alpha)$ times), we obtain such a solution α^* of LOCK (V, U, l) that $|\mathscr{X}^+_i(\alpha^*)||\mathscr{X}^+_j(\alpha^*)| = 0$ for any $ij \in E(T) - U$. We may assume that if $|X \cap T| = 2$ for $X \in \mathscr{X}^+(\alpha^*)$ then $X \cap T = \{0, 1\}$ or $\{0, 2\}$ (for if $X \cap T = \{0, 3\}$ (or $\{1, 2\}$) then $U \cap \partial^V X = \phi$ and one can put $\alpha^*(X) := 0$ without violating that α^* is a solution of LOCK (V, U, l)). We claim that the restriction $\tilde{\alpha}$ of α^* on $\mathscr{X}(V, S)$ is an optimum solution of $\Sigma(V, S, l)$. Indeed, the above property of α^* implies validity of at least one of the following equalities:

$$\mathscr{X}^+_0(\alpha^*) = \mathscr{X}^+_3(\alpha^*) = 0, \quad \mathscr{X}^+_1(\alpha^*) = \mathscr{X}^+_2(\alpha^*) = 0.$$

In the first case we have $1 \cdot \tilde{\alpha} = \Sigma(\alpha^*(X)\colon X \in \mathscr{X}^+_{01}(\alpha^*) \cup \mathscr{X}^+_{02}(\alpha^*)) = \mu_l(03)$ and similarly in the second case, we have $1 \cdot \tilde{\alpha} = \mu_l(12)$. Note also that the above mentioned half-integerity theorem for LOCK (V, U, l) implies that a two commodity cut problem $\Sigma(V, S, l)$ has an integer optimum solution if l is cyclically even (thus, another proof of Seymour's two-commodity cut theorem [6] is obtained).

7. FAMILIES OF CUTS WITH THE WEAK MFMC-PROPERTY

Theorem 7.1. *Let $T \subseteq V$, and let $S \subset 2^T$ be a scheme such that the cut family $\mathscr{C}(V, S)$ has the (weak) MFMC-property. Then every minimal subset $B \subseteq E(V)$ meeting all members of $\mathscr{C}(V, S)$ can be represented as a union $\bigcup(L\colon L \in \mathscr{L})$ of disjoint T-terminus chains such that, for any $A \in S$, there is $L \in \mathscr{L}$ with one end in A and the other in $T - A$.*

Proof. We proceed by induction on $|B|$. For $v \in V$, let $s(B, v)$ denote the subset of elements of B with one end v. The result is trivial if $|s(B, v)| \leqslant 2$ for all $v \in V - T$. Thus, we assume that $|s(B, v)| = k \geqslant 3$ for some $v \in V - T$. Let $\{y_1, \dots, y_k\}$ be the set of ends of edges in $s(B, v)$ different from v, and let $e_i = y_i y_{i+1}$ $(i = 1, \dots, k - 1)$, $e_k = y_k y_1$. Let $l(e) = 1$ $(e \in B)$ and $l(e)$ be a large positive number for $e \in E(V) - B$, and let $\alpha\colon \mathscr{X}(V, S) \to \mathbf{R}_+$ be an optimum solution of $\Sigma(V, S, l)$. Clearly $1 \cdot \alpha = l(B)$, $\lambda^\alpha(e) = 1$ for all $e \in B$ and $|\partial X \cap B| = 1$ for each $X \in \mathscr{X}^+(\alpha)$, where $\mathscr{X}^+(\alpha) = \{X\colon \alpha(X) > 0\}$. Define $\mathscr{X}_i$ to be $\{X \in \mathscr{X}^+(\alpha)\colon vy_i \in \partial X\}$ $(i = 1, \dots, k)$. We observed that $\lambda^\alpha(y_i y_j) = \Sigma(\alpha(X)\colon X \in \mathscr{X}_i) + \Sigma(\alpha(X)\colon X \in \mathscr{X}_j) = 2$ for any $1 \leqslant i < j \leqslant k$. Now put $l'(e_i) = \epsilon$ $(i = 1, \dots, k)$ and $l'(e) = l(e)$ for the remaining e's in $E(V)$, where $0 < \epsilon < 2$. Choose some function $\alpha'\colon \mathscr{X}(V, S) \to \mathbf{R}_+$ satisfying $\alpha'(X) \leqslant \alpha(X)$ $(X \in \mathscr{X}_i)$, $\Sigma(\alpha'(X)\colon X \in \mathscr{X}_i) = \frac{\epsilon}{2}$ for $i = 1, \dots, k$ and $\alpha'(Y) = \alpha(Y)$ for $Y \in \mathscr{X}(V, S) - \bigcup(\mathscr{X}_i\colon i = 1, \dots, k)$. Obviously, α' is l'-admissible. We claim that α' is an optimum solution of $\Sigma(V, S, l')$. For let $Q = B - \{vy_i\colon i = 1, \dots, k\}$ and $w \in \mathbf{R}_+^{E(V)}$ be defined by $w(e) = 1$ $(e \in Q)$, $w(e_i) = \frac{1}{2}$ $(i = 1, \dots, k)$ and $w(e) = 0$ $(e \in E(V) - (Q \cup \{e_1, \dots, e_k\}))$. One can check that $w(\partial X) \geqslant 1$ for all $X \in \mathscr{X}(V, S)$, $\lambda^{\alpha'}(e) = l'(e)$ for all $e \in Q \cup \{e_1, \dots, e_k\}$ and $w(\partial X) = 1$ for all $X \in \mathscr{X}^+(\alpha')$, whence α' is an optimum solution of $\Sigma(V, S, l')$. Since $\mathscr{C}(V, S)$ has the MFMC-property, there is a minimal subset $B' \subseteq E(V)$ meeting all cuts in $\mathscr{C}(V, S)$ such that $1 \cdot \alpha' = l'(B')$. $\lambda^{\alpha'}(vy_i) = \frac{\epsilon}{2} < l'(vy_i) = 1$ $(i = 1, \dots, k)$ implies $B' \subseteq Q \cup \{e_1, \dots, e_k\}$. Furthermore, for $i = 1, \dots, k$ and arbitrary $X \in \mathscr{X}_i$, we have $\{e_i, e_{i-1}\} \subset \partial X$, whence, by additional slackeness relations for α' and B', exactly one of e_i, e_{i-1} belongs to B' (assuming $e_0 = e_k$). Thus, k is even and we may assume that $B' = Q \cup \{e_1, e_3, \dots, e_{k-1}\}$. Because $|B'| < |B|$, by induction, there is a representation $B' = \bigcup(L'\colon L' \in \mathscr{L}')$, where $\mathscr{L}'$ is a collection of disjoint T-terminus chains such that, for any $A \in S$, there is $L' \in \mathscr{L}'$ with one end in A and the other one in $T - A$. Now a required collection $\mathscr{L}$ for B is obtained from $\mathscr{L}'$ by replacing the chain $L' \in \mathscr{L}'$ containing e_i by the chain $(L' - \{e_i\}) \cup \{vy_i, vy_{i+1}\}$ for $i = 1, \dots, k$. ∎

Corollary 7.2. *Let* $T \subseteq V$, $S \subset 2^T$ *and let* $\mathscr{C}(V, S)$ *have the MFMC-property. Then, for any* $l \in \mathbf{R}_+^{E(V)}$, $p(V, S, l) = \min\{\sum(\mu_l(u): u \in U)\}$, *where* U *ranges over the subsets of* $E(T)$ *such that* $U \cap \partial^T A \neq \phi$ *for* $A \in S$.

Proof. Indeed, let B be a minimal subset meeting all members of $\mathscr{C}(V, S)$ and such that $p(V, S, l) = l(B)$, and let $\bigcup(L: L \in \mathscr{L})$ be a representation for B as in Theorem 7.1. We claim that each $L \in \mathscr{L}$ is shortest. For if L' is a chain such that $eL' = eL$ and $l(L') < l(L)$, then the subset $B' = (B - L) \cup L'$ meets every cut in $\mathscr{C}(V, S)$ and $l(B') < < l(B)$, a contradiction. ∎

Now we give a theorem describing a class of cut families having the MFMC-property. (The proof will appear in a forthcoming paper.) We say that a scheme $S \subset 2^T$ is *non-reducible* if

(i) for every pair of elements in T there is a subset $A \in S$ containing exactly one member of the pair,

(ii) there is no $T' \subset T$ and $S' \subset 2^{T'}$ such that $\mathscr{X}(T, S')$ is equivalent to S (two collections $\mathscr{X}$ and $\mathscr{X}'$ of subsets of a set W is called equivalent if they generate the same cut families in K_W, i.e. $\{E \subseteq E(W): E = \partial X$ some $X \in \mathscr{X}\} = \{E \subseteq E(W): E = \partial X$ some $X \in \mathscr{X}'\}$).

Obviously, we may restrict our consideration of the non-reducible schemes.

Theorem 7.3. *Let* $S \subset 2^T$ *be a non-reducible scheme,* $V \supset T$ *and* $|V| \geq |T| + 2$. *Then* $\mathscr{C}(V, S)$ *has the MFMC-property if and only if* S *is equivalent to a scheme* $S' \subset 2^T$ *of one of the following three types:*

(1) $S' = \{\{0, 1\}, \{0, 2\}\}$ *for* $T = \{0, 1, 2, 3\}$ (see Example 2),

(2) S' *is the set of odd-size subsets of* T, $|T|$ *is even* (see Example 3),

(3) S' *is a ring of subsets of* T *with the miminal member* $\{s\}$ *and the maximal member* $T - \{t\}$ *for some* $s, t \in T$.

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A.V. Karzanov
Institute for Systems Studies, Moscow, USSR.