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METRICS AND UNDIRECTED CUTS

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Suppose that G is an undirected graph whose edges have nonnegative integer-valued lengths $l(e)$, and that $\{s_1, t_1\}, \dots, \{s_m, t_m\}$ are pairs of its vertices. Can one assign nonnegative weights to the cuts of G such that, for each edge e, the total weight of cuts containing e does not exceed $l(e)$ and, for each i, the total weight of cuts 'separating' s_i and t_i is equal to the distance (with respect to l) between s_i and t_i? Using linear programming duality, it follows from Papernov's multicommodity flow theorem that the answer is affirmative if the graph induced by the pairs $\{s_1, t_1\}, \dots, \{s_m, t_m\}$ is one of the following: (i) the complete graph with four vertices, (ii) the circuit with five vertices, (iii) a union of two stars. We prove that if, in addition, each circuit in G has an even length (with respect to l) then there exists a suitable weighting of the cuts with the weights integer-valued; moreover, an algorithm of complexity $O(n^3)$ (n is the number of vertices of G) is developed for solving such a problem. Also a class of metrics decomposable into a nonnegative linear combination of cut-metrics is described, and it is shown that the separation problem for cut cones is NP-hard.

Key words: Multi Commodity Flows, Cut Metrics.

1. Introduction

Let $G=(V, E)$ be a connected undirected graph. For $X \subseteq V$, $\partial X = \partial_G X$ denotes the set of edges of G with one end in X and the other in $V-X$. $E' \subseteq E$ is called a *cut* if $E' = \partial X$ for some X, $\emptyset \neq X \subset V$.

Let $\{s_1, t_1\}, \dots, \{s_m, t_m\}$ ($s_i \neq t_i$) be distinct pairs of vertices of G, and let $H=(T, U)$ ($T \subseteq V$) be the undirected graph without isolated vertices whose edges correspond to these pairs, i.e. $e \in U$ if and only if e has the ends s_i and t_i for some i ($1 \leq i \leq m$); we refer to H and T as the *commodity-graph* and the set of *terminals* respectively.

Suppose that each edge $e \in E$ has a real-valued capacity $c(e) \geq 0$, and that $q_i \geq 0$ ($1 \leq i \leq m$) are real-valued demands. The well-known multicommodity flow problem is to find flows $F_1, \dots, F_m$ (a *multicommodity flow*), where each F_i is a flow from s_i to t_i of value q_i, such that for each $e \in E$ the total flow through e does not exceed $c(e)$, or to establish that no such flows exist. By linear programming duality, this problem has a solution (i.e. a required multicommodity flow exists) if and only if

$$cl \geq \sum (q_i \mu_l(s_i t_i)\colon i = 1, \dots, m) \tag{1.1}$$

is valid for any nonnegative real-valued function l on E, where $cl = \sum (c(e) l(e)\colon e \in E)$, and $\mu_l(s_i t_i)$ denotes the distance between s_i and t_i in the graph G whose edges are weighted by l (see [10, 6]). If a required multicommodity flow exists then the following connectivity condition of Ford–Fulkerson type (which is, in fact, a special case of (1.1)) also holds:

$$c(\partial X) \geq \sum (q_i\colon 1 \leq i \leq m, X \text{ separates } s_i \text{ and } t_i) \ \forall X \subset V. \tag{1.2}$$

(For a numerical function f on a finite set S and $S' \subseteq S, f(S')$ denotes $\sum (f(x)\colon x \in S')$; we say that X separates vertices s and t if exactly one of s and t is in X.) For what commodity graphs H is the necessary condition (1.2) also sufficient (irrespective of G, l, q)? The answer was obtained by Papernov who proved in [13] that

(i) if H is K_4 (the complete graph with 4 vertices) or C_5 (the circuit with 5 vertices) or a union of two stars and if (1.2) is true, then the problem has a solution (here a star is meant to be a connected graph such that its edge-set is nonempty and all edges have a common vertex);

(ii) if $H=(T, U)$ is a graph (without isolated vertices) which does not belong to the collection $\{K_4, C_5\} \cup \mathscr{L}^2$ (where $\mathscr{L}^2$ is the set of graphs representable as a union of two stars), then there exists a graph $G=(V=T, E)$, capacities c and demands q such that (1.2) holds but the problem has no solution.

(Statement (i) generalized a number of earlier known results on multicommodity flows. Obviously, $H \in \mathscr{L}^2$ when $|U|=1$ or 2; in these cases we have the well-known one- and two-commodity flow problems, and (i) follows from the theorems of Ford and Fulkerson [3] and Hu [5]. Dinic (see [1]) showed that the problem with $H \in \mathscr{L}^2$ can be easily reduced to that with $H'=(T', U')$, $|U'|=2$. There is a stronger, half-integral, version of the statement (i). We say that c and q satisfy the property (*) if c and d are integer-valued and $c(\partial X)-\sum (q_i\colon 1 \leq i \leq m, X \text{ separates } s_i \text{ and } t_i)$ is a nonnegative even integer for all $X \subset V$. Rothschild and Whinston [14] proved that if $|U|=2$ and if c and d satisfy the property (*), then the problem has an integer solution. The same result for $H=K_4$ and C_5 was obtained by Lomonosov [11] (and, independently, by Seymour [15] for $H=K_4$); Lomonosov's proof was provided by a pseudopolynomial algorithm. An algorithm of complexity $O(|V|^5)$ was developed in [8] for the cases $H=K_4$ and C_5.)

We consider a cut packing problem which is dual (in a sense) to the multicommodity flow problem. Let a function $l\colon E \to \mathbb{R}_+$ (regarded as a function of *lengths* of edges) and demands $b_i \in \mathbb{R}_+$ $(1 \leq i \leq m)$ be given ($\mathbb{R}_+$ is the set of nonnegative reals). A function $\alpha\colon 2^V \to \mathbb{R}_+$ is called *l-admissible* if

$$\lambda^\alpha(e) \triangleq \sum (\alpha(X)\colon X \subset V, e \in \partial X) \leq l(e) \quad \text{for all } e \in E.$$

When does there exist an l-admissible α such that

$$\sum (\alpha(X)\colon X \subset V, X \text{ separates } s_i \text{ and } t_i) \geq b_i = b(s_i t_i), \quad i=1, \ldots m? \tag{1.3}$$

We begin with several definitions and assumptions. Without loss of generality, we will assume that the graph G in question is complete, i.e. G has no loops and any two distinct vertices of G are jointed by a unique edge (since it does not matter whether an edge e is absent in G or e is present with a very large length $l(e)$). For a finite set W, K_W denotes the complete graph with the vertex-set W; $E(W)$ denotes the set of edges of K_W; an edge with ends x and y is denoted by xy. By a *chain*, or x_0x_k-*chain*, in K_W we mean a nonempty sequence $L=x_0x_1\dots x_k$ (commas are omitted) of distinct elements of W; define the edge-set E_L of L as $\{x_ix_{i+1}: 0\leq i\leq k-1\}$. Let $\mathscr{L}_{xy}=\mathscr{L}_{xy}^W$ denote the set of xy-chains in K_W. A *circuit* is a similar sequence with $k\geq 3$ and $x_0=x_k$; its edge-set is defined analogously. A *metric* on W is a function $\mu: E(W)\to\mathbb{R}_+$ satisfying the triangle inequality $\mu(xy)+\mu(yz)\geq\mu(xz)$ for any $x, y, z\in W$. For $g: E(W)\to\mathbb{R}_+$, μ_g denotes the function of distances induced by g, i.e. $\mu_g(xy)=\min\{g(E_L): L\in\mathscr{L}_{xy}\}$ $(xy\in E(W))$; obviously, μ_g is a metric. An xy-chain L is called an xy-*geodesic* or, briefly, a *geodesic* of g if $g(E_L)=\mu_g(xy)$ (i.e. L is a shortest chain with respect to g). For $X\subseteq W$, $\rho_X=\rho_X^W$ denotes the characteristic function of ∂X, i.e. $\rho_X(e)=1$ if $e\in\partial X$ and $\rho_X(e)=0$ if $e\in E(W)-\partial X$. Clearly ρ_X is a metric; it is called a *cut-metric* if $X\neq\emptyset$, W.

Let $\alpha: 2^V\to\mathbb{R}_+$ be l-admissible, let $st\in U$, and let L be an st-geodesic of l. We have

$$\begin{aligned}\mu_l(st)&=\sum(l(e): e\in E_L)\geq\sum(\lambda^\alpha(e): e\in E_L)\\&=\sum(\alpha(X)|\partial X\cap E_L|: X\subset V)\geq\sum(\alpha(X): X\subset V, st\in\partial X)\\&=\lambda^\alpha(st).\end{aligned}\tag{1.4}$$

Thus, a necessary condition for the existence of an l-admissible α satisfying (1.3) is that $b(s_it_i)\leq\mu_l(s_it_i)$ for each $i=1,\dots,m$. We will consider a slightly modified form of our problem: given V, $l\in\mathbb{R}_+^{E(V)}$ and $U\subseteq E(V)$, find an l-admissible $\alpha: 2^V\to\mathbb{R}_+$ satisfying

$$\lambda^\alpha(st)=\mu_l(st)\quad\text{for all } st\in U\tag{1.5}$$

or establish that no such α exists. (We call this *problem A* for V, l, U.) Using Papernov's theorem and applying linear programming arguments one can easily obtain the following theorem (see [9]).

Theorem 1. (i) *If $H\in\{K_4, C_5\}\cup\mathscr{L}^2$ then problem A has a solution (i.e. there exists an l-admissible α satisfying* (1.5)).

(ii) *If $H=(T, U)$ is a graph (without isolated vertices) not belonging to $\{K_4, C_5\}\cup\mathscr{L}^2$, then, for any $V\supseteq T$, there exists $l\in\mathbb{R}_+^{E(V)}$ such that problem A has no solution.*

(It should be noted that the method used by Papernov in proving statement (i) of his theorem implies a direct proof of (i) in Theorem 1. Indeed, he showed that if $H\in\{K_4, C_5\}\cup\mathscr{L}^2$ and (1.1) is violated for some l (by given c and q), then there exist X, $\emptyset\neq X\subset V$, and $a>0$ such that $l(e)\geq a$ for all $e\in\partial X$ and $\mu_{l'}(st)=\mu_l(st)-a\rho_X(st)$ for all $st\in U$, where $l'=l-a\rho_X$, whence the result follows.)

In the present work we prove a half-integral version of statement (i) of Theorem 1 which was announced in [9]. We say that a function l on $E(V)$ is *cyclically even* if $l\in\mathbb{Z}_+^{E(V)}$ and $l(E_C)$ is even for each circuit C in K_V ($\mathbb{Z}_+$ is the set of nonnegative integers).

Theorem A. *If $H=(T,U)\in\{K_4,C_5\}\cup\mathscr{L}^2$, $V\supseteq T$, and l is a cyclically even function on $E(V)$, then there exists an l-admissible $\alpha:2^V\to\mathbb{Z}_+$ satisfying* (1.5).

In order to prove this theorem we need to consider another type of cut packing problem, which may be interesting in its own right. Let $T\subseteq V$ be a distinguished subset (the set of terminals) and D be some family of subsets of T. The problem is: given $l\in\mathbb{R}_+^{E(V)}$ and $d\in\mathbb{R}_+^D$, find an l-admissible $\alpha:2^V\to\mathbb{R}_+$ such that

$$\sum(\alpha(X): X\subseteq V, X\cap T=A)\geqslant d(A)\quad\text{for each } A\in D \tag{1.6}$$

or establish that no such α exists (*problem B* for V, D, l, d). If this problem has a solution then the following inequalities must be valid:

$$\nu^d(st)\triangleq\sum(d(A): A\in D, st\in\partial A)\leqslant\mu_l(st)\quad\text{for all } st\in E(T). \tag{1.7}$$

Indeed, considering an l-admissible α satisfying (1.6) and an st-geodetic L of l, we have

$$\begin{aligned}\mu_1(st)&=\sum(l(e): e\in E_L)\geqslant\sum(\lambda^\alpha(e): e\in E_L)\\&=\sum(\alpha(X)|\partial X\cap E_L|: X\subset V)\geqslant\sum(\alpha(X): X\subset V, st\in\partial X)\\&\geqslant\sum(\alpha(X): X\subset V, st\in\partial X, X\cap T\in D).\end{aligned}$$

Two subsets $X, Y\subset W$ are said to be *crossing* if each of the four subsets $X\cap Y$, $X-Y$, $Y-X$, and $W-(X\cup Y)$ is nonempty, and *laminar* otherwise. We say that a family $D\subset 2^T$ is *3-crossing* if there are three members of D each two of which are crossing. For example, if $T=\{0,1,2,3\}$, then the family $\{\{0,1\},\{0,2\},\{1,2\}\}$ is 3-crossing and the family $\{\{0,1\},\{1,2\},\{2,3\},\{3,0\},\{0\},\{1\},\{2\},\{3\}\}$ is non-3-crossing.

The following theorem appeared in [9] and characterizes all families D for which the necessary condition (1.7) is also sufficient for solvability of problem B.

Theorem 2. (i) *If $D\subset 2^T$ is non-3-crossing, $V\supseteq T$, $l\in\mathbb{R}_+^{E(V)}$, $d\in\mathbb{R}_+^D$, and* (1.7) *is valid, then problem B has a solution, i.e. there exists an l-admissible α satisfying* (1.6).

(ii) *If $D\subset 2^T$ is 3-crossing, $V\supseteq T$, and $|V|\geqslant|T|+2$, then there are $l\in\mathbb{R}_+^{E(V)}$ and $d\in\mathbb{R}_+^D$ such that* (1.7) *holds but problem B has no solution.*

In Section 2 we prove the following strengthening of statement (i) of Theorem 2.

Theorem B. *Let $D\subset 2^T$ be non-3-crossing, $V\supseteq T$, $l\in\mathbb{Z}_+^{E(V)}$, $d\in\mathbb{Z}_+^D$, and suppose that* (1.7) *holds. Next, let the function $\tilde{l}$ on $E(V)$, defined by $\tilde{l}(st)=\nu^d(st)$ for $st\in E(T)$ and $\tilde{l}(e)=l(e)$ for $e\in E(V)-E(T)$, be cyclically even. Then there exists an l-admissible $\alpha:2^V\to\mathbb{Z}_+$ satisfying* (1.6) (*i.e. problem B has an integer solution*).

The proof of this theorem will be provided by a greedy-algorithm whose running time (in the sense of the number of elementary logical operations and arithmetical operations + and −) is $O(|T|^3|V|^3)$. This theorem and algorithm will be used in Section 3 to construct an algorithm for solving problem A with $H \in \{K_4, C_5\} \cup \mathscr{L}^2$, in time $O(|V|^3)$, and, as a consequence, to prove Theorem A.

In [9] one application of problem A was presented; namely, it was shown there how to obtain an optimum solution of a two-commodity cut problem if we are given a solution of problem A for the corresponding V, l, U with $|U|=2$. Theorem A implies in this case the two-commodity cut theorem of Seymour [16] and a modification of the algorithm for problem A yields a maximum packing of two-commodity cuts, in time $O(|V|^3)$. In Section 4 another application of Theorems 1 and A is given. In that section we describe a class of metrics which can be decomposed into a nonnegative linear combination of cut-metrics. We also prove that the separation problem is NP-hard for the class of cut cones (this problem generalizes the one of determining whether a metric is decomposable into a nonnegative linear combination of cut-metrics).

2. Proof of Theorem B

Let $D \subset 2^T$ be a non-3-crossing family, and let $l \in \mathbb{R}_+^{E(V)}$ nd $d \in \mathbb{R}_+^D$ be such that $\mu_l(st) \geq \nu^d(st)$ for any $st \in E(T)$. The following assertion is trivial.

(2.1) Assertion. (i) *ν^d is a metric on T.* (ii) *If $pq \in E(T)$, and the function l' on $E(V)$ is defined by $l'(pq) = \nu^d(pq)$ and $l'(e) = l(e)$ for $e \in E(V) - \{pq\}$, then $\mu_{l'}(st) \geq \nu^d(st)$ for any $st \in E(T)$.*

In view of (2.1), without loss of generality one may assume

$$l(st) = \mu_l(st) = \nu^d(st) \quad \text{for all } st \in E(T). \tag{$*$}$$

Distinguish some terminal $v_0 \in T$. We may assume also that

$$v_0 \notin A \text{ and } d(A) > 0 \quad \text{for all } A \in D. \tag{$**$}$$

(Since we may assume that D contains no complementary pair A, $T - A$, and if $A \in D$ and $v_0 \in A$ we can replace D by $(D - \{A\}) \cup \{T - A\}$ without essentially changing our problem.)

The proof of Theorem B follows from the present algorithm which works with real-valued l and d and yields a required integer-valued α whenever d is integer-valued and l is cyclically even. Let $\mu(xy)$ be defined to be $\mu_l(xy)$ if $xy \in E(V)$ and 0 if $x = y \in V$.

Algorithm. Choose a minimal set A^* in D (with respect to inclusion). Define

$$X = \{x \in V\colon \min\{\mu(sx) + \mu(xt) - \mu(st)\colon s, t \in A^*\} = 0\},$$

$$a = \min\{d(A^*), \tfrac{1}{2}\min\{\mu(sx) + \mu(xt) - \mu(st)\colon s, t \in A^*, x \in V - X\}\}.$$

Put

$$l'(e)=\begin{cases} l(e)-a & \text{if } e\in\partial X,\\ l(e) & \text{otherwise;}\end{cases}$$

$$d'(A)=\begin{cases} d(A)-a & \text{if } A=A^*,\\ d(A) & \text{otherwise;}\end{cases}$$

and if $d'(A^*)=0$ then exclude A^* from D. Repeat such a step (with l and d replaced by l' and d'), and so on while the current family D is nonempty.

Let $X_1, X_2, \ldots$ and $a_1 a_2, \ldots$ be the sequences of X's and a's determined in the steps of the algorithm. The required function α is defined to be $\alpha(X_i)=a_i$ for $i=1,2,\ldots$ and $\alpha(Y)=0$ for the remaining subsets Y in V.

Correctness of the algorithm

Claim 1. $X\cap T=A^*$.

Proof. If $x\in A^*$, then taking $s=t=x$ shows that x is in X. If $x\in T-A^*$, then for any $s,t\in A^*$ we have

$$\mu(sx)+\mu(xt)-\mu(st)=\nu^d(sx)+\nu^d(xt)-\nu^d(st)\geq 2d(A^*)>0$$

(defining $\nu^d(st)=0$ if $s=t$). □

Claim 2. $a\leq l(xy)$ *for any* $x\in X$ *and* $y\in V-X$.

Proof. Since x is in X, $\mu(sx)+\mu(xt)=\mu(st)$ for some $s,t\in A^*$. Clearly $\mu(sx)+l(xy)\geq\mu(sy)$ and $\mu(tx)+l(xy)\geq\mu(ty)$, and so $2l(xy)\geq\mu(sy)+\mu(yt)-\mu(st)$, whence $l(xy)\geq a$, by the definition of a. □

It follows from Claim 2 that $l'(e)\geq 0$ for all $e\in E(V)$.

Claim 3. $\mu_{l'}(pq)=l'(pq)=\nu^{d'}(pq)$ *for all* $pq\in E(T)$.

This claim will be proven later. It follows from Claims 1-3 that the algorithm finishs with l-admissible α satisfying (1.6). The only point that needs an explanation is that the subsets $X_1, X_2, \ldots$ found on the steps of the algorithm are distinct. Let A^*_m stand for the set A^* chosen on the mth step of the algorithm. Consider two sets X_i and X_j determined on the ith and jth steps, and let $i<j$. If $A^*_i\neq A^*_j$ then, by Claim 1, $X_i\neq X_j$. We show that if $A^*_i=A^*_j$ then X_j strictly includes X_i, whence the result follows. In order to slightly simplify the proof of this fact we suppose that the algorithm is modified so that, for each $A\in D$, the steps with $A^*=A$ go in succession. In other words, if it occurs on some (ith say) step that $a_i<d(A^*_i)$ (by the current d), then the same set A^* is taken on next step, i.e. $A^*_{i+1}=A^*_i$. Since

$a_i < d(A_i^*)$, there is $x \in V - X_i$ such that $\mu^i(sx) + \mu^i(xt) - \mu^i(st) = 2a_i$ for some $s, t \in A_i^*$, where μ^m stands for μ on mth step. It is clear that $\mu^{i+1}(sx) \leq \mu^i(sx) - a$ and $\mu^{i+1}(tx) \leq \mu^i(tx) - a$. By Claim 3, $\mu^{i+1}(st) = \mu^i(st)$, whence $\mu^{i+1}(sx) + \mu^{i+1}(xt) - \mu^{i+1}(st) = 0$, and therefore, $x \in X_{i+1}$. By similar arguments, $y \in X_{i+1}$ for any $y \in X_i$. Thus, $X_i \subset X_{i+1}$, as required.

Now let d be integer-valued and let l be cyclically even. It is not difficult to prove that μ_l is also cyclically even. Hence the number a_1 (as well as each subsequent a_i) is an integer. This proves Theorem B.

Proof of Claim 3. Let $\mu'(xy)$ be defined to be $\mu_{l'}(xy)$ if $xy \in E(V)$ and 0 if $x = y \in V$. Similarly put $\nu'(st) = \nu^{d'}(st)$ for $st \in E(T)$ and $\nu'(st) = 0$ for $s = t \in T$. It follows from (*) and the definitions of l' and d' that $\nu'(st) = l'(st) = \nu^d(st) - a = l(st) - a$ if $st \in \partial A^* \cap E(T)$ and $\nu'(st) = l'(st) = \nu^d(st) = l(st)$ if $st \in E(T) - \partial A^*$. Consider an arbitrary $pq \in E(T)$. The equality $l'(pq) = \nu'(pq)$ implies $\mu'(pq) \leq \nu'(pq)$. In order to show the reverse inequality we need the following assertion.

(1) *Let $s, t \in A^*$(s, t, p, q are not necessary distinct terminals). Then at least one of the following is valid*:

(i) $\nu'(pq) \leq \nu'(ps) + \nu'(qt) - \nu'(st)$,

(ii) $\nu'(pq) \leq \nu'(pt) + \nu'(qs) - \nu'(st)$.

Proof. A direct count shows that

$$\nu'(pq) + \nu'(st) - \nu'(ps) - \nu'(qt) = 2\sum(d'(A)\colon A \in D_1) - 2\sum(d'(A)\colon A \in D_2)$$

where D_1 (D_2) is the set of members of D separating $\{p, s\}$ and $\{q, t\}$ (respectively, separating $\{p, q\}$ and $\{s, t\}$). (We say that $A \in D$ separates two collections $\{x, \ldots, y\}$ and $\{z, \ldots, w\}$ of (not necessary distinct) vertices if either $x, \ldots, y \in A$, $z, \ldots, w \in T - A$ or $x, \ldots, y \in T - A$, $z, \ldots, w \in A$.) It follows from this equality that if (i) is not valid, then there exists a set B in D separating $\{p, s\}$ and $\{q, t\}$. We observe that B and A^* are crossing. Indeed, assuming that $s \in B$ we have: $B \cap A^* \neq \emptyset$; $B \cup A^* \neq T$ since, by (**), $v_0 \notin B, A^*$; $A^* - B \neq \emptyset$ since $t \notin B$; and $B - A^* \neq \emptyset$ since A^* is a minimal set in D.

Similarly, if (ii) is not valid, then there exists a set C in D separating $\{p, t\}$ and $\{q, s\}$, and so the sets C and A^* are crossing. Now, supposing that neither (i) nor (ii) are valid we conclude that D is 3-crossing (as B and C are obviously crossing), a contradiction. □

Using (1), we show that $l'(E_L) \geq \nu'(pq)$ for an arbitrary edge $pq \in E(T)$ and a pq-chain L in K_V. We proceed by induction on $m(L) = |E_L \cap \partial X|$.

(a) if $m(L) = 0$ then $l'(E_L) = l(E_L) \geq \mu(pq) = \nu^d(pq) = \nu'(pq)$.

(b) If $m(L) = 1$ then, obviously, $pq \in \partial A^*$, and we have $\nu'(pq) = \nu^d(pq) - a$ and $l'(E_L) = l(E_L) - a$, whence the result follows.

(c) If $m(L)=2$, and $p, q \in A^*$, then $\nu'(pq)=\nu^d(pq)$ and $l'(E_L)=l(E_L)-2a$. Let x be an element of L which is in $V-X$. By definition of a, $\mu(px)+\mu(xq)-\mu(pq) \geq 2a$, whence

$$l'(E_L)=l(E_L)-2a \geq \mu(px)+\mu(xq)-2a \geq \mu(pq)=\nu^d(pq)=\nu'(pq).$$

(d) Suppose we are not in one of the cases (a), (b), (c). Then there exists an element x of L such that x is in X, $m(L')<m(L)$ and $m(L'')<m(L)$, where L' (L'') is the part of L from p to x (resp., from x to q). Since x is in X, there exist $s, t \in A^*$, an sx-chain P', and a tx-chain P'' such that $l(E_{P'})+l(E_{P''})=\mu(st)=\nu'(st)$. We may suppose that the inequality (i) from (1) holds for given p, q, s, t. Define $Q'(Q'')$ to be the ps-chain (resp., the qt-chain) such that $E_{Q'} \subseteq E_{L'} \cup E_{P'}$ (resp., $E_{Q''} \subseteq E_{L''} \cup E_{P''}$). It is clear that each element of P' and P'' is in X, and so $m(Q')=m(L')<m(L)$ and, similarly, $m(Q'')<m(L)$, whence, by the induction hypothesis, $l'(E_{Q'}) \geq \nu'(ps)$ and $l'(E_{Q''}) \geq \nu'(qt)$. Finally, we have

$$\begin{aligned}\nu'(pq) &\leq \nu'(ps)+\nu'(qt)-\nu'(st) \leq l'(E_{Q'})+l'(E_{Q''})-\nu'(st)\\ &\leq l'(E_{L'})+l'(E_{P'})+l'(E_{L''})+l'(E_{P''})-\nu'(st)=l'(E_L),\end{aligned}$$

as required. □

Complexity of the algorithm

Since the sequence of X's with the same set $X \cap T$ is strictly increasing (with respect to inclusion), each set $A \in D$ can be taken (as A^*) no more than $|V-T|+1$ times. Therefore, the number of steps of the algorithm does not exceed $|D||V|$. The number of elementary operations required on one step (the calculation of the distances $\mu_1(sx)$, $s \in A^*$, $x \in V-T$, the determination of X and a, etc.) is $O(|T||V|^2)$ (if an $O(|V|^2)$-algorithm (such as Dijkstra's algorithm) is used for finding the distances from a fixed vertex to all other vertices). Thus, the whole algorithm needs $O(|D||T||V|^3)$ operations. A rough count shows that the cardinality of an arbitrary non-3-crossing family of subsets of T is $O(|T|^2)$ (M.V. Lomonosov (a private communication) informed me that this estimate can be improved to be $O(|T|\log|T|)$). Therefore, the complexity of the algorithm is $O(|T|^3|V|^3)$ operations. Note also that before starting the algorithm one must determine the numbers $\nu^d(st)$ and $\mu_l(st)$ ($st \in E(T)$) for initial d and l and examine the inequalities in (1.7), which requires $O(|D||T|^2+|T||V|^2)$ operations.

3. Proof of Theorem A

Let $H=(T, U)$ be a graph belonging to $\{K_4, C_5\} \cup \mathcal{L}^2$, $V \supseteq T$, and $l \in \mathbb{R}_+^{E(V)}$.

We begin by reducing the case $H \in \mathcal{L}^2$ to $H'=(T', U')$ with $|U'|=2$ (H' is a subgraph of K_4). Here and later we use the following assertion, which is implied directly by expression (1.4).

(3.1) Let $\alpha: 2^V \to \mathbb{R}_+$ be l-admissible, $st \in E(V)$, and L be an st-geodessic of l. Then the equality $\mu_l(st) = \lambda^\alpha(st)$ holds if and only if both of the following are true: (a) $\lambda^\alpha(e) = l(e)$ for all $e \in E_L$, and (b) $|E_L \cap \partial X| \leq 1$ for any $X \subset V$ such that $\alpha(X) > 0$.

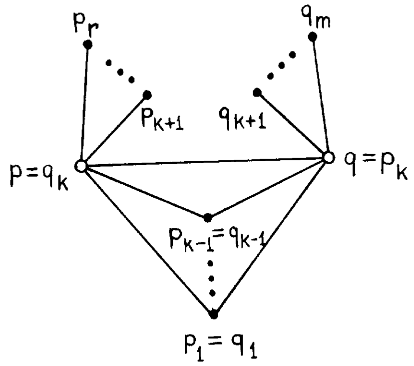

Fig. 3.1

Suppose that H is a union of stars S_r and S_m, the former with the edge-set $\{pp_i: i = 1, \ldots, r\}$ and the latter with the edge-set $\{qq_j: j = 1, \ldots, m\}$ (an example of such a graph is illustrated in Fig. 3.1). Put

$$a_1 = \max\{\mu_l(px): x \in V\}, \qquad a_2 = \max\{\mu_1(qx): x \in V\},$$

and $b = a_1 + a_2$. Add two new vertices p' and q' to V and let V' be the resulting set. Define

$$l'(e) = \begin{cases} l(e), & e \in E(V), \\ b - \mu_l(px), & e = p'x,\ x \in V, \\ b - \mu_l(qx), & e = q'x,\ x \in V, \\ 2b - \mu_l(pq), & e = p'q'. \end{cases}$$

One can check that $\mu_{l'}(pp') = \mu_{l'}(qq') = b$ and that, moreover, if a chain $p \cdots p_i$ (resp., $q \cdots q_j$) is a geodesic of l, then $p \cdots p_i p'$ (resp., $q \cdots q_j q'$) is a geodesic of l'. Assume that there exists an l'-admissible function $\alpha': 2^{V'} \to \mathbb{R}_+$ such that $\lambda^{\alpha'}(pp') = \mu_{l'}(pp')$ and $\lambda^{\alpha'}(qq') = \mu_{l'}(qq')$. Define α by $\alpha(X) = \sum(\alpha'(Y): Y \subset V',\ Y \cap V = X)$ $(X \subset V)$. Obviously, α is l-admissible. Applying (3.1) to V', l', α' and a geodesic $L' = p \cdots p_i p'$ (resp., $L' = q \cdots q_j q'$) and then again applying it to V, l, α and $L = p \cdots p_i$ (resp., $L = q \cdots q_j$) we conclude that $\lambda^\alpha(pp_i) = \mu_l(pp_i)$ for $i = 1, \ldots, r$ (resp., $\lambda^\alpha(qq_j) = \mu_l(qq_j)$ for $j = 1, \ldots, m$), i.e. α is a solution of problem A for given V, l and U. It is not difficult to check that if l is cyclically even then l' is as well. Thus, it suffices to establish the validity of Theorem A only for $H = K_4$ and $H = C_5$.

For a real-valued function g on a set Q, let $D^+(g)$ denote $\{x \in Q: g(x) > 0\}$. A triple $\{X_1, X_2, X_3\}$ of pairwise crossing subsets of a set will be referred to as a *crossing triple* for short.

(3.2) Lemma. *Let $H=(T,U)$ be K_4 or C_5, $h\in\mathbb{R}_+^{E(T)}$, and let $\gamma\colon 2^T\to\mathbb{R}_+$ be a solution of problem A for T, h, U. Then this problem has also a solution $\tilde\gamma$ with non-3-crossing $D^+(\tilde\gamma)$.*

Proof. Since $|T|\leqslant 5$, without loss of generality one may assume that $|A|\leqslant 2$ for any $A\in D^+(\gamma)$. Clearly, if $A\in D^+(\gamma)$ and $B\subset T$ are crossing, then $|A|=2$. Define $D_2^+(\gamma)$ to be $\{A\in D^+(\gamma)\colon |A|=2\}$.

Case $H=K_4$. Assume that $D^+(\gamma)$ contains a crossing triple $\{A_1,A_2,A_3\}$, and let $a=\min\{\gamma(A_i)\colon i=1,2,3\}$. It is not difficult to see that $\sum(\rho^T_{A_i}\colon i=1,2,3)=\sum(\rho^T_{\{s\}}\colon s\in T)$, which implies that the function γ', defined by $\gamma'(A_i)=\gamma(A_i)-a$ $(i=1,2,3)$, $\gamma'(\{s\})=\gamma(\{s\})+a$ $(s\in T)$ and $\gamma'(B)=\gamma(B)$ for the remaining B's in 2^T, is also a solution of our problem. The result follows from the fact that $|D_2^+(\gamma')|<|D_2^+(\gamma)|$.

Case $H=C_5$. Let $T=\{s_0,s_1,\ldots,s_4\}$ and $U=\{s_is_{i+1}\colon i=0,\ldots,4\}$ (here and below indices are taken modulo 5).

(i) Assume that $D^+(\gamma)$ contains $A=\{s_{i-1},s_i\}$ and $A'=\{s_i,s_{i+1}\}$ for some i. Where $a=\min\{\gamma(A),\gamma(A')\}$, let $\gamma'(A)=\gamma(A)-a$, $\gamma'(A')=\gamma(A')-a$, $\gamma'(\{s_j\})=\gamma(\{s_j\})+a$ for $j=i-1$, $i+1$ and $\gamma'(B)=\gamma(B)$ for the remaining B's in 2^T.

(ii) Assume that $D^+(\gamma)$ contains $A=\{s_{i-1},s_i\}$ and $A'=\{s_i,s_{i+2}\}$ for some i. Where $a=\min\{\gamma(A),\gamma(A')\}$, let $\gamma'(A)=\gamma(A)-a$, $\gamma'(A')=\gamma(A')-a$, $\gamma'(\{s_i\})=\gamma(\{s_i\})+a$, $\gamma'(\{s_{i+1},s_{i+3}\})=\gamma(\{s_{i+1},s_{i+3}\})+a$ and $\gamma'(B)=\gamma(B)$ for the remaining B's in 2^T. If a symmetric pair $\{s_{i+1},s_i\}$, $\{s_i,s_{i-2}\}$ is in $D^+(\gamma)$, γ' is defined analogously.

One can verify that, if γ' is defined as in (i) or (ii), then $\lambda^{\gamma'}(e)\leqslant\lambda^{\gamma}(e)$ for any $e\in E(T)$ and this inequality turns into equality for each $e\in U$; hence such a γ' is also a solution of our problem. If γ' is defined as in (i), then $|D_2^+(\gamma')|<|D_2^+(\gamma)|$. Thus, we may assume that $D^+(\gamma)$ contains no pair $\{A,A'\}$ as in (i). So the set $D'=\{A=\{s_i,s_{i+1}\}\colon A\in D^+(\gamma),\ 0\leqslant i\leqslant 4\}$ has at most two members, and if $D'=\{A_1,A_2\}$, then $A_1\cap A_2=\emptyset$. It follows from this fact (the details are left to the reader) that if we transform γ into γ' according to (ii), no *new* pair $\{A,A'\}$ described in (i) and (ii) arises in $D^+(\gamma')$ (in comparison with $D^+(\gamma)$). Therefore, there exists a solution $\tilde\gamma$ such that $D^+(\tilde\gamma)$ contains no pairs as in (i) and (ii). So each vertex of the graph $\Gamma=(T,Q)$, where $Q=\{s_is_j\colon\{s_i,s_j\}\in D^+(\tilde\gamma)\}$, is of valency at most 2, and Γ has no subgraph isomorphic to K_3. This implies that $D^+(\tilde\gamma)$ is non-3-crossing. □

(3.3) Remark. In fact, the present proof involves a procedure for finding a solution $\tilde\gamma$ with non-3-crossing $D^+(\tilde\gamma)$ whenever an arbitrary solution γ is given. This procedure consists of no more then $\binom{|T|}{2}$ steps (above described transformations γ into γ'), and hence its running time is O(1). Moreover, if γ is integer-valued then $\tilde\gamma$ is as well.

(3.4) Lemma. *Let $H=(T,U)$ be K_4 or C_5, and $h\in\mathbb{R}_+^{E(T)}$. Then problem A for T, h and U has a solution. If, in addition, h is cyclically even, then this problem has an integer solution.*

This lemma will be proven later. Using it, we now show how to prove Theorem A. Let $H=(T,U)$ be K_4 or C_5, $V \supseteq T$ and $l \in \mathbb{R}_+^{E(V)}$ (according to above arguments the case $H \in \mathscr{L}^2$ may be excluded from consideration). Put $h(st)=\mu_l(st)$ for $st \in E(T)$, and suppose that $\gamma: 2^T \to \mathbb{R}_+$ is a solution of problem A for T, h, U (such a solution exists by Lemma (3.4)), i.e.

$$\lambda^{\gamma}(st)=\sum(\gamma(A)\colon A\subset T,\, st\in\partial A)\begin{cases}\leqslant h(st) & \text{if } st\in E(T),\\ =h(st) & \text{if } st\in U.\end{cases} \qquad (*)$$

One may assume by Lemma (3.2), that $D^+(\gamma)$ is non-3-crossing. By Theorem B, $(*)$ (cf. (1.7)) implies that problem B for V, $D^+(\gamma)$, l, γ^+ is solvable, where γ^+ is the restriction of γ on $D^+(\gamma)$; let $\alpha: 2^V \to \mathbb{R}_+$ be its solution, i.e. α is l-admissible and

$$\sum(\alpha(X)\colon X\subset V,\, X\cap T=A)=\gamma(A) \quad \text{for all } A\in D^+(\gamma).$$

For any $st \in U$, we have

$$\begin{aligned}\lambda^{\alpha}(st)&=\sum(\sum(\alpha(X)\colon X\subset V,\, X\cap T=A)\colon A\in D^+(\gamma),\, st\in\partial A)\\ &=\sum(\gamma(A)\colon A\in D^+(\gamma),\, st\in\partial A)=h(st)=\mu_l(st),\end{aligned}$$

hence α is a solution of problem A for V, l, U. Now let l by cyclically even. This implies easily that the function h is also cyclically even (concerning K_T) and, furthermore, the same is true for the function $\tilde{l}$ defined by $\tilde{l}(st)=h(st)$ $(st \in E(T))$ and $\tilde{l}(e)=l(e)$ $(e \in E(V)-E(T))$. Applying Theorem B and Lemma (3.4) we obtain Theorem A.

It remains to prove Lemma (3.4). Without loss of generality, one may assume that h is a metric (as we could prove the lemma for μ_h instead of h). We begin by proving some auxiliary assertions. Here $|T|\geqslant 2$, and h is a metric on T extended with zero on the pairs $xx (x \in T)$.

(3.5) Assertion. *Let $\emptyset \neq A \subset T$, and*

$$a=\tfrac{1}{2}\min\{h(xy)+h(yz)-h(xz)\colon x,y,z\in T,\, xy,yz\in\partial A\}.$$

Then $h'=h-a\rho_A$ is a metric (in particular, h' is nonnegative).

Proof. We have to show that $h'(xy)+h'(yz)\geqslant h'(xz)$ for $x,y,z \in T$. This is obvious if $xy \notin \partial A$ and $yz \notin \partial A$. If exactly one of xy and yz is in ∂A, then $xz \in \partial A$ and $h'(xy)+h'(yz)=h(xy)+h(yz)-a \geqslant h(xz)-a=h'(xz)$. If both xy and yz are in A, then $h'(xy)+h'(yz)=h(xy)+h(yz)-2a \geqslant h(xz)=h'(xz)$ (by definition of a). □

(3.6) Assertion. *If xyz and yxz are geodescis of h, then $h(xy)=0$.*

The proof is trivial.

For distinct elements $x, y \in T$, define the map $\mathrm{id}^{xy}\colon (T, E(T)) \to (T', E(T'))$ identifying x and y with a new vertex denoted by v^{xy}. The following assertion is obvious.

(3.7) Assertion. *Let* $h(xy)=0$ *for some* $xy\in E(T)$, $T'=\mathrm{id}^{xy}(T)$, *and let the function* h' *on* $E(T')$ *be defined by* $h'(zu)=h(zu)$ *if* $z,u\in T-\{x,y\}$ *and* $h'(zu)=h(zx)$ *if* $z\in T-\{x,y\}$ *and* $u=v^{xy}$.

(i) h' *is a metric and* h' *is cyclically even if* h *is cyclically even.*

(ii) *Let* $\gamma':2^{T'}\to\mathbb{R}_+$ *be an* h'*-admissible function, and let the function* γ *on* 2^T *be defined by* $\gamma(A)=0$ *if* $xy\in\partial A$ *and* $\gamma(A)=\gamma'(\mathrm{id}^{xy}(A))$ *otherwise. Then* γ *is* h*-admissible and* $\lambda^{\gamma}(st)=\lambda^{\gamma'}(\mathrm{id}^{xy}(st))$ *for any* $st\in E(T)=\{xy\}$.

A geodesic L of h is called *elementary* if $|E_L|=2$; the set of elementary geodesics of h is denoted by $\mathcal{T}(h)$. We say that $L\in\mathcal{T}(h)$ *forbids* a set $A\subset T$ if $E_L\subseteq\partial A$. If $A\subset T$, $A\neq\emptyset$ and A is forbidden by no geodetic of $\mathcal{T}(h)$, we say that A *conforms* to h.

In order to prove Lemma (3.4) we need to extend slightly the conditions of this lemma to include the cases $H=K_2$ and $H=K_3$. We proceed by induction on $|T|$. If $|T|=2$ (i.e. if $H=K_2$) then the result is obvious. Let $H\in\{K_3,K_4,C_5\}$. If $h(xy)=0$ for some $xy\in E(T)$, then, in view of (3.7), our problem is reduced to the problem for T', h' and U' (where $T'=\mathrm{id}^{xy}(T)$, $U'=\mathrm{id}^{xy}(U)$ and h' is defined as in (3.7)), and the result follows by induction (note that if $H=C_5$ then $\mathrm{id}^{xy}(H)$ is a subgraph of K_4). Thus, one may assume that $h(xy)>0$ for all $xy\in E(T)$. We apply a second induction on $|\mathcal{T}(h)|$ assuming that the result is true for any metric h' on T with $|\mathcal{T}(h')|>|\mathcal{T}(h)|$ (if $|\mathcal{T}(h)|$ is maximum, i.e. if each chain L in K_T with $|E_L|=2$ is a geodesic of h, then, by (3.6), $h\equiv 0$).

Suppose that A is a set conforming to h, and let a and h' be defined as in (3.5) (for the given h and A). Then $h'(x'y')+h'(y'z')-h'(x'z')=h(x'y')+h(y'z')-h(x'z')$ for any geodesic $x'y'z'$ of h, and so $\mathcal{T}(h)\subseteq\mathcal{T}(h')$. Let $x,y,z\in T$ be such that $xy,yz\in\partial A$ and $h(xy)+h(yz)-h(xz)=2a$. If $x=z$ then $h'(xy)=0$, and if $x\neq z$ then $xyz\notin\mathcal{T}(h)$ and $xyz\in\mathcal{T}(h')$. By (3.5), h' is a metric, hence by our first or second induction the problem with T, h', U has a solution γ'. Put $\gamma(A)=\gamma'(A)+a$ $(=a)$ and $\gamma(B)=\gamma'(B)$ for $B\in 2^T-\{A\}$. Then γ satisfies (*). For, firstly, γ is obviously h-admissible. Secondly, for $st\in U$, we have $h(st)=h'(st)+\omega a=\lambda^{\gamma'}(st)+\omega a=\lambda^{\gamma}(st)$, where $\omega=0$ if $st\notin\partial A$ and $\omega=1$ if $st\in\partial A$. Next, if h is cyclically even, then a is an integer, and therefore h' is also cyclically even. Thus, by induction, we may assume that γ' is integer-valued, whence γ is also integer-valued.

Let us attempt to find a set A conforming to h. Set $\hat{T}=\{s\in T$: there is no $xsy\in\mathcal{T}(h)\}$. If $\hat{T}\neq\emptyset$ and $s\in\hat{T}$, then $A=\{s\}$ conforms to h, and the result follows by the above argument. Now suppose that $\hat{T}=\emptyset$, and consider the concrete cases of H's.

Case $H=K_3$. Letting $T=\{s,t,p\}$, we have $stp,tsp\in\mathcal{T}(h)$. By (3.6), $h(st)=0$, a contradiction.

Case $H=K_4$. Let st be an edge in H with $h(st)$ maximum, and let $T-\{s,t\}=\{p,q\}$. Then stp, stq, tsp and tsq are not in $\mathcal{T}(h)$, whence, since $\hat{T}=\emptyset$, $psq,ptq\in\mathcal{T}(h)$. We assert that $A=\{s,p\}$ conforms to h. Indeed, $ptq\in\mathcal{T}(h)$ and $h(pt)>0$ imply $tpq\notin\mathcal{T}(h)$; and similarly $sqp\notin\mathcal{T}(h)$. Hence there is no L in $\mathcal{T}(h)$ such that $E_L\subseteq\partial A$.

Case $H = C_5$. Let $T = \{s_0, \dots, s_4\}$ and $U = \{s_is_{i+1}: i = 0, \dots, 4\}$ (indices are taken modulo 5). We examine various combinations of geodesics in $\mathcal{T}(h)$ and conclude with one of the following: (a) there is a set A conforming to h, or (b) there are geodesics $L_1, \dots, L_k$ in $\mathcal{T}(h)$ such that the subgraph $H^{L^1,\dots,L^k}$ of K_T induced by the set of edges

$$U^{L_1,\dots,L_k} = (U - \cup(E_{L_i}: 1 \leq i \leq k)) \cup \{s^1t^1, s^2t^2, \dots, s^kt^k\}$$

is a union of two stars, where s^i and t^i are the ends of L_i, $i = 1, \dots, k$. If (b) takes place, the problem for T, h, U is reduced to that for T, h, $U^{L_1,\dots,L_k} = U'$ (since, by (3.1), if γ is a solution for T, h, U', then γ is also a solution for T, h, U), and, as it was shown earlier, the latter can be reduced to the problem with K_4, whence the result follows by induction.

(i) Assume that $\mathcal{T}(h)$ contains a geodesic $L = s_{i-1}s_is_{i+1}$ for some i. Then $H^L \in \mathcal{L}^2$.

(ii) Assume that $\mathcal{T}(h)$ contains a geodesic $L = s_is_{i+1}s_{i-1}$ or $L = s_is_{i-1}s_{i+1}$ for some i. Then $H^L \in \mathcal{L}^2$.

(iii) Let $\mathcal{T}(h)$ contain no geodesics as in (i) and (ii). Then $\mathcal{T}(h)$ can contain geodesics only of the following types: $s_{i-2}s_is_{i+2}$ (a geodesic of the *first* type), $s_{i-1}s_is_{i+2}$ and $s_{i+1}s_is_{i-2}$ (geodesics of the *second* type). Let A_j denote $\{s_{j-1}, s_{j+1}\}$, $j = 0, \dots, 4$. It is easy to see that no geodesic of the first type forbids A_j for any j, and that a geodesic of the second type, say $s_{i-1}s_is_{i+2}$, forbids only two such subsets, namely, A_{i-1} and A_{i-2}. Let $\mathcal{T}'$ be the set of all geodesics of the second type in $\mathcal{T}(h)$. Suppose that no A_j $(0 \leq j \leq 4)$ conforms to h. Then $2|\mathcal{T}'| \geq 5$, and hence $|\mathcal{T}'| \geq 3$. This implies easily that at least one of the following is true: (1) there are two geodesics L, L' in $\mathcal{T}'$ such that E_L and $E_{L'}$ have a common edge s_is_{i+1} for some i; (2) there are $L, L' \in \mathcal{T}'$ such that $s_{i-1}s_i \in E_L$ and $s_is_{i+1} \in E_{L'}$ for some i. In the first case we have $\{L, L'\} = \{s_{i+1}s_is_{i-2}, s_is_{i+1}s_{i-2}\} \subseteq \mathcal{T}(h)$, whence, by (3.6), $h(s_is_{i+1}) = 0$, a contradiction. In the second case we have $U^{L,L'} \subseteq E(T) - \{s_{i-1}s_i, s_is_{i+1}, s_{i-1}s_{i+1}\}$, hence $H^{L,L'} \in \mathcal{L}^2$ (because $K_5 - K_3 \in \mathcal{L}^2$).

This completes the proof of Lemma (3.4) and the proof of Theorem A.

The proof of Lemma (3.4) involves, in essence, an O(1)-algorithm for solving problem A with $H = (T, U) \in \{K_4, C_5\}$ and $h \in \mathbb{R}_+^{E(T)}$. This algorithm together with the algorithm for problem B developed in Section 2 and with the procedure mentioned in Remark (3.3) enables us to solve problem A with arbitrary V, $l \in \mathbb{R}_+^{E(V)}$ and $H \in \{K_4, C_5\} \cup \mathcal{L}^2$ using $O(|V|^3)$ elementary operations.

Remark. The idea to find a set A forbidden by no elementary geodesic of h, used in Lemma (3.4) is close to the one used by Papernov in his proof of the multicommodity flow theorem mentioned in the Introduction. Indeed, he showed that if $H = (T, U) \in \{K_4, C_5\} \cup \mathcal{L}^2$, $V \supseteq T$, and $\mu \neq 0$ is a metric on V, then there exists X, $\emptyset \neq X \subset V$, such that $\mu(e) > 0$ for all $e \in \partial X$ and $|\partial X \cap E_L| \leq 1$ for any st-geodesic L of μ, $st \in U$. However, the choice of such X does not guarantee for cyclically even μ that $\mu_{l'}(st) = \mu(st) - \rho_X(st)$ for all $st \in U$, where $l' = \mu - \rho_X$.

4. Cut-decomposable metrics

We say that a metric μ on V is *cut-decomposable* if it can be represented as a nonnegative linear combination of cut-metrics, i.e.

$$\mu = \sum (\alpha(X)\rho_X : X \subseteq V) \tag{1}$$

for some $\alpha : 2^V \to \mathbb{R}_+$. When is a metric μ cut-decomposable? (This question was raised in [8, 17].) Theorem 1 enables us to describe a subclass of such metrics. Without loss of generality, we may assume that μ is positive, i.e. $\mu(e) > 0$ for all $e \in E(V)$. An edge xy is called *extremal* with respect to μ if there is no vertex $z \in V - \{x, y\}$ such that $\mu(xz) = \mu(xy) + \mu(yz)$ or $\mu(yz) = \mu(xy) + \mu(xz)$. The *extremal graph* $\mathscr{H}(\mu)$ of μ is the subgraph of K_V induced by the extremal edges of μ.

Theorem 3. *If μ is a positive metric on V and $\mathscr{H}(\mu) = (T, U) \in \{K_4, C_5\} \cup \mathscr{L}^2$, then μ is cut-decomposable. If, in addition, μ is cyclically even, then* (1) *is valid for some integer-valued α.*

Proof. Let α be a solution of problem A for V, μ, U (by Theorem 1, such a solution exists). We prove that α satisfies (1). In view of (3.1), it suffices to show that, for each $e \in E(V)$, there exists an st-geodesic L of μ such that $st \in U$ and $e \in E_L$. Let $e \in E(V)$, and let $L = x_0x_1 \cdots x_k$ be a geodesic of μ such that $e \in E_L$ and $k = |E_L|$ is maximum. Then $x_0x_k \in U$. For if it is not, then by definition of the extremal graph of μ, $\mu(x_0x_k) + \mu(x_k, z) = \mu(x_0z)$ or $\mu(x_0x_k) + \mu(x_0z) = \mu(x_kz)$ for some $z \in V - \{x_0, x_k\}$. The positivity of μ implies $z \notin \{x_0, x_1, \ldots, x_k\}$, whence $|E_L|$ is not maximum, a contradiction. The second part of the theorem follows from Theorem A. □

A metric μ is said to be *primitive* if $\mu = \mu_1 + \mu_2$, where μ_1 and μ_2 are metrics, implies $\mu_1 = \beta\mu$ for some $\beta \in \mathbb{R}_+$. For example, any cut-metric is primitive. Clearly a primitive metric is not cut-decomposable if it is not proportional to any cutmetric, and so, Theorem 3 has the following elementary corollary: if $|V| \geq 3$ and μ is a metric on V being positive and primitive, then its extremal graph does not belong to the collection $\{K_4, C_5\} \cup \mathscr{L}$. For example consider the graphs G_i, $i = 1, 2$, drawn in Fig. 4.1a, b, each having the length of each edge equal to 1, and let μ_i be the metric of distances in G_i. These metrics are positive and primitive (see [13]; primitivity of μ_1 and μ_2 follows also from general theorems on metrics stated in [12, 2]); their extremal graphs are illustrated in Fig. 4.1c, d.

Let $\mathscr{C} = \mathscr{C}_V$ denote the set of cut-decomposable metrics on V. Clearly $\mathscr{C}$ is a full-dimensional convex polyhedral cone in $\mathbb{R}^{E(V)}$ whose extreme rays (one-dimensional facets) are $\gamma\rho_X$ ($\gamma \in \mathbb{R}_+$) for all X, $\emptyset \neq X \subset V$ (here $\mathbb{R}^{E(V)}$ is regarded as the $\binom{|V|}{2}$-dimensional Eucledean space whose coordinates correspond to the edges in K_V). We prove that the separation problem for such cones is NP-hard. This problem is: given V and $l \in \mathbb{Z}_+^{E(V)}$, decide if $l \in \mathscr{C}_V$ and if not, find a vector $c \in \mathbb{R}^{E(V)}$

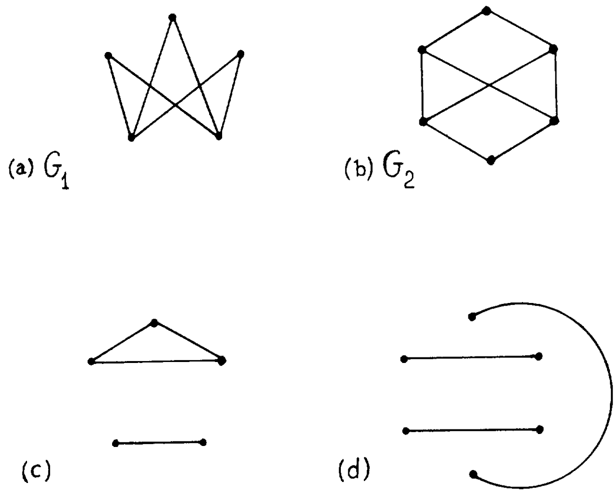

Fig. 4.1

such that cl < 0 and $c\mu \geqslant 0$ for all $\mu \in \mathscr{C}_V$ (in other words, find a hyperplane separating l from $\mathscr{C}_V$). The proof is as follows.

Consider the maximum cut problem, which is known to the *NP*-complete [7]. More precisely, and more generally, the problem is: given a finite set V, $w \in \mathbb{Z}_+^{E(V)}$ and $K \in \mathbb{Z}_+$, decide whether there exists $X \subset V$ such that $w(\partial X) \geqslant K$. Clear this problem can be replaced by the $\binom{|V|}{2}$ problems in which X ranges over the set $\mathscr{X}_{st} = \{X' \subset V \colon st \in \partial X'\}$, for each $st \in E(V)$. For fixed st, define the function q on $E(V)$ as

$$\begin{aligned} q(xy) &= -w(xy) && \text{if } x, y \in V - \{s, t\}, \\ &= -w(xy) + M && \text{if } x \in \{s, t\}, y \in V - \{s, t\}, \\ &= -w(xy) - M(|V| - 2) + K - 1 && \text{if } xy = st, \end{aligned}$$

where M is a large positive integer (one can put $M = w(E(V))$). We observe that: (i) $q(\partial X) = K - 1 - w(\partial X)$ if $X \in \mathscr{X}_{st}$, and (ii) $q(\partial X) \geqslant 0$ if $X \notin \mathscr{X}_{st}$. Therefore there exists $X \in \mathscr{X}_{st}$ such that $w(\partial X) \geqslant K$ if and only if there exists $Y \subset V$ such that $q(\partial Y) < 0$. Thus, in order to solve the problem for $\mathscr{X}_{st}$ it suffices to determine the minimum m of values $q\xi$ over all vectors $\xi = \rho_Y / \rho_Y(E(V))$, $\emptyset \neq Y \subset V$, and then to check the inequality $m < 0$. The problem of determining m is equivalent to the problem of minimizing the objective function $q\xi'$ on the polyhedron $\mathscr{P}_V = \mathscr{C}_V \cap \{\xi' \in \mathbb{R}^{E(V)} \colon \xi'(E(V)) \leqslant 1\}$, and so the latter is NP-hard. Now using the ellipsoid method of Khachiyan which yields a 'polynomial equivalence' between the optimization and separation problems (see [4]) we conclude that the separation problem for such polyhedra $\mathscr{P}_V$ is NP-hard, whence the separation problem for cut cones $\mathscr{C}_V$ is also NP-hard.

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