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POLYNOMIAL UNCROSSING PROCESSES

by

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Abstract. Let $\mathcal{F}$ be a family of subsets of a set V, and f be a nonnegative integer-valued function on $\mathcal{F}$. An uncrossing process is a procedure of reforming $\mathcal{F}$ and f to $\mathcal{F}'$ and f' such that $\mathcal{F}'$ is laminar by use of certain elementary operations ("uncrossing steps") with crossing sets in a current family. Uncrossing steps of two natural types are considered. For both types, we prove the existence of an uncrossing process consisting of a polynomial number of uncrossing steps.

1. INTRODUCTION

Let V be a finite set and $\mathcal{S} \subseteq 2^V$ be a family of subsets of V. Two sets $X, Y \subseteq V$ are called *crossing* (denoted as $X \nparallel Y$) if none of $X - Y$, $Y - X$, $X \cap Y$ and $V - (X \cup Y)$ is empty; otherwise they are called *laminar* (denoted as $X \parallel Y$). A family $\mathcal{S}' \subseteq 2^V$ is called *laminar* if it has no crossing pairs. We associate with a subset $\emptyset \neq X \subset V$ the set $\delta^V(X)$ of edges of K_V having one end in X and the other in $V - X$ (a *cut* in K_V); here $K_V = (V, E_V)$ is the complete undirected graph with the vertex-set V.

It will be convenient for our further description to assume that $\mathcal{S}$ (as well as each family of subsets occurring later) is symmetric, that is, $X \in \mathcal{S}$ implies $\bar{X} := V - X \in \mathcal{S}$; such an assumption will lead to no loss of generality. We shall deal with $\mathcal{S}$ that is *cross-closed.* This means that for any crossing $X, Y \in \mathcal{S}$ there are $X', Y' \in \mathcal{S}$ such that either $X' = X - Y$ and $Y' = Y - X$, or $X' = X \cap Y$ and $Y' = X \cup Y$; we say that the pair $\{X', Y'\}$ is obtained by *uncrossing* X and Y (in [K2] the term "2-complete" was introduced for such a family $\mathcal{S}$). E.g., the following families $\mathcal{S}$ are cross-closed:

(Ex1): $\mathcal{S}$ is the symmetrization of a so-called "crossing family" $\mathcal{S}'$, that is,

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$\mathcal{S}'$ satisfies the property that $X \cap Y, X \cup Y \in \mathcal{S}'$ holds for any crossing $X, Y \in \mathcal{S}'$ (cf. [EG]);

(Ex2): $\mathcal{S}$ consists of the subsets $X \subset V$ such that $|X \cap T|$ is odd, where $T \subseteq V$ is a subset with $|T|$ even (a cut $\delta^V(X)$ for $X \in \mathcal{S}$ is usually called a *T-cut* (cf. [S]); T-cuts arise, in particular, in connection with the Chinese postman problem [M,EJ]).

(Ex3): S is the set of X's such that $|\delta^V(X) \cap U| = 1$ for some $U \subseteq E_V$ (cf. [S]);

(Ex4): S is the set of X's such that $\sum(a(e) : e \in \delta^V(X))$ is odd, where a is an integer-valued function on E_V (cf. [K1,K2]).

Now suppose that we are given a subfamily $\mathcal{F} \subseteq \mathcal{S}$ and a mapping $f : \mathcal{F} \to \mathbb{Z}^+$ ($\mathbb{Z}^+$ is the set of nonnegative integers); we shall assume that f is symmetric, that is, $f(X) = f(\bar{X})$. $\mathcal{F}$ and f can be reformed to new $\mathcal{F}'$ and f' as follows:

(i) choose some crossing sets $X, Y \in \mathcal{F}$; and choose $X', Y' \in \mathcal{S}$ obtained by uncrossing X and Y (if all $X - Y, Y - X, X \cap Y, X \cup Y$ occur in $\mathcal{S}$, there are two possibility for choice of $\{X', Y'\}$);

(ii) add the sets $X', \bar{X}', Y', \bar{Y}'$ to $\mathcal{F}$ forming $\mathcal{F}'$, and put $f(Z) := 0$ for each $Z \in \{X', \bar{X}', Y', \bar{Y}'\}$ which is not contained in $\mathcal{F}$;

(iii) where $a := \min\{f(X), f(Y)\}$, put $f'(Z) := f(Z) - a$ for $Z = X, \bar{X}, Y, \bar{Y}$; $f'(Z') := f(Z') + a$ for $Z' = X', \bar{X}', Y', \bar{Y}'$; and $f'(Z'') := f(Z'')$ for the remaining Z'' in $\mathcal{F}$;

(iv) delete the members Z from $\mathcal{F}'$ such that $f'(Z) = 0$.

Note that the family $\mathcal{S}$ can be given implicitly by an oracle that, being asked of a set $X \subseteq V$, tells us whether or not X belongs to $\mathcal{S}$; in usual applications, this oracle is realized by a procedure polynomial in $|V|$.

We say that the procedure (i)-(iv) is an *uncrossing step of type 1*, denoted as $X, Y \to X', Y'$. Another type of uncrossing steps, *type 2*, is defined for $\mathcal{S} = 2^V$ as follows:

(i') choose crossing $X, Y \in \mathcal{F}$; let $X' := X - Y$, $Y' := Y - X$, $X'' := X \cap Y$, $Y'' := X \cup Y$, and let $a := \min\{f(X), f(Y)\}$;

(ii') add the sets X', Y', X'', Y'' and their complements to $\mathcal{F}$ forming $\mathcal{F}'$, and put $f(Z) := 0$ for each $Z \in \mathcal{F}'$ which is not in $\mathcal{F}$;

(iii') turn to an oracle that gives us an integer b such that $0 \le b \le a$;

(iv') form f' on $\mathcal{F}'$ by decreasing $f(Z)$ by a for $Z = X, \bar{X}, Y, \bar{Y}$; by increasing $f(Z)$ by b for $Z := X', \bar{X}', Y', \bar{Y}'$ and by $a - b$ for $Z := X'', \bar{X}'', Y'', \bar{Y}''$; and keeping the same $f(Z)$ for the remaining Z in $\mathcal{F}$;

(v') delete the members Z from $\mathcal{F}'$ such that $f'(Z) = 0$.

Clearly an uncrossing step of type 1 is a special case of that of type 2. An *uncrossing process* is a sequence of uncrossing steps of a given type fulfilling until a current family $\mathcal{F}$ becomes laminar. Such a process is always finished in a finite number of steps independently of what sets X and Y are chosen in each step and what are $\mathcal{F}'$ and f' resulting for this step. Indeed, for $e \in E_V$ define the value $c_f(e)$ to be $\sum(f(X) : X \in \mathcal{F}, e \in \delta^V(X))$. It is easy to see that for f, f' and a as in (i)-(iv) or in (i')-(v') the following is true:

(1) $c_{f'}(e) \le c_f(e)$ for all $e \in E_V$ and there are edges $e, e' \in E_V$ (possibly $e = e'$) such that $c_{f'}(e) + c_{f'}(e') = c_f(e) + c_f(e') - 4a$.

We prove the following theorems.

Theorem 1. *There exists an uncrossing process with uncrossing steps of type 2 in which the number of steps is bounded by a polynomial in* $n := |V|$, $m := |\mathcal{F}|$ *and* $\log(\|f\| + 1)$ *where* $\|f\| := \sum(f(X) : X \in \mathcal{F})$.

Theorem 2. *There exists an uncrossing process with uncrossing steps of type 1 in which the number of steps is bounded by a polynomial in n and m.*

The proofs of Theorems 1 and 2 will provide polynomial algorithms to arrange required processes.

Remark. In [GLS] a polynomial uncrossing technique was developed for $\mathcal{S}$ as in (Ex1) in connection with the submodular flow problem [EG]. More precisely, a polynomial algorithm was described there which for given $\mathcal{F} \subseteq \mathcal{S}$ and $f : \mathcal{F} \to \mathbb{Z}^+$ finds a laminar family $\mathcal{F}' \subseteq \mathcal{S}$ and a function $f' : \mathcal{F}' \to \mathbb{Z}^+$ such that $\sum(f'(X) : X \in \mathcal{F}') = \sum(f(X) : X \in \mathcal{F})$ and $c_{f'}(e) \le c_f(e)$ for any $e \in E_V$. Note that this algorithm, first, uses procedures different from "pure" uncrossing steps as above and, second, it is not generalized to an arbitrary cross-closed family.

The following simple statements will be used in our proofs.

(1.1) *Let $X, Y, Z \subseteq V$ be such that $X \not\parallel Y$, $X \parallel Z$ and $Y \parallel Z$, and let $X' \in \{X - Y, Y - X, X \cap Y, X \cup Y\}$. Then $X' \parallel Z$.*

(1.2) *If $\mathcal{F}'$ is a laminar family on an n element set, then $|\mathcal{F}'| < 4n$.*

(1.1) is easy to prove. To see (1.2), denote by $\alpha(n)$ the maximum cardinality of a laminar family on an n element set. One can show that $\alpha(n) \leq \alpha(n-1) + 4$, whence (1.2) follows.

2. PROOF OF THEOREM 1.

Let V, $\mathcal{F}$ and f be as above. We say that a family $\mathcal{R}$ of subsets of a set W is *cyclical* if there is an ordering $\{v_1, v_2, \ldots, v_r = v_0\}$ $(r = |W|)$ of elements of W such that each set $X \in \mathcal{R}$ is of the form $\{v_i, v_{i+1}, \ldots, v_j\}$ for some $i, j \in \{1, \ldots, r\}$ (the indices are taken modulo r). First of all we prove the following lemma.

Lemma 2.1. *An uncrossing process for $\mathcal{F}$ and f can be arranged so that it consists of at most $2n(m/2 - 1)$ uncrossing processes, each for a cyclical family $\mathcal{R}$ on V and a function $g : \mathcal{R} \to \mathbb{Z}^+$ with $\|g\| \leq \|f\|$.*

Proof. Choose a laminar subfamily $\mathcal{L}_1$ in $\mathcal{F}$ and a pair $\{X_1, \bar{X}_1\} \subseteq \mathcal{F} - \mathcal{L}_1$ and fulfil an uncrossing process for $\mathcal{L}_1 \cup \{X_1, \bar{X}_1\}$ forming a laminar family $\mathcal{L}_2$ and some function on $\mathcal{L}_2$. Then choose a new pair $\{X_2, \bar{X}_2\}$ in $\mathcal{F} - (\mathcal{L}_1 \cup \{X_1, \bar{X}_1\})$ and fulfil an uncrossing process for $\mathcal{L}_2 \cup \{X_2, \bar{X}_2\}$, and so on. As a result, after $k \leq m/2 - 1$ iterations we get a laminar family $\mathcal{F}' := \mathcal{L}_{k+1}$ required in Theorem 1.

Now we show how to arrange an uncrossing process for a family consisting of a laminar subfamily $\mathcal{L}$ and a pair $\{A, \bar{A}\}$. We say that a set $Y \subseteq V$ *separates* a set $X \subseteq V$ if both $X \cap Y$ and $X - Y$ are nonempty. A set $X \subseteq V$ is called *bi-partitioned* with respect to a laminar family $\mathcal{D} \subset 2^V$ if there is $Z \subset X$ such that

(2) for any $Y \in \mathcal{D}$, Y separates neither Z nor $X - Z$, that is, $X \cap Y \in \{Z, X - Z, X, \emptyset\}$.

Suppose we are given two laminar families $\mathcal{P}$ and $\mathcal{D}$ satisfying the following property:

(3) for each $X \in \mathcal{P}$, at least one of X and $\bar{X}$ is bi-partitioned with respect to $\mathcal{D}$.

Obviously, (3) holds for $\mathcal{P} := \mathcal{L}$ and $\mathcal{D} := \{A, \bar{A}\}$. Choose a *maximal* set $X \in \mathcal{P}$ which is bi-partitioned with respect to $\mathcal{D}$ and fulfil an uncrossing process for the family $\mathcal{D} \cup \{X, \bar{X}\}$. As a result, we get a laminar family $\mathcal{D}'$. Put $\mathcal{P}' := \mathcal{P} - \{X, \bar{X}\}$.

Claim. *For each $X' \in \mathcal{P}'$ at least one of X' and $\bar{X}'$ is bi-partitioned with respect to $\mathcal{D}'$.*

Proof. Since $\mathcal{P}$ is symmetric, we may assume that either $X' \subset X$ or $X \cap X' = \emptyset$. Then X' is bi-partitioned with respect to $\mathcal{D}$. Indeed, in the case $X' \subset X$, this obviously follows from the fact that X is bi-partitioned. And in the case $X \cap X' = \emptyset$, this is so because $\bar{X}'$ cannot be bi-partitioned by the maximal choice of X. Thus there is $Z' \subset X'$ such that any $Y \in \mathcal{D}$ separates neither Z' nor $X' - Z'$; if $X' \subset X$ we put $Z' := X' \cap Z$. This easily implies that the same property holds for X', Z' and each Y arising during the uncrossing process for $\mathcal{D} \cup \{X, \bar{X}\}$, whence the result follows. □

In view of the Claim, an uncrossing process for $\mathcal{L}$ and $\{A, \bar{A}\}$ as above is reduced to $|\mathcal{L}|/2 < 2n$ uncrossing processes for families $\tilde{\mathcal{D}} = \mathcal{D} \cup \{X, \bar{X}\}$ such that $\mathcal{D}$ is laminar and X is bi-partitioned with respect to $\mathcal{D}$.

Let $\mathcal{D}_X$ be the set of those members of $\mathcal{D}$ which are crossing X. One may assume that $\mathcal{D}_X \neq \emptyset$ (otherwise $\tilde{\mathcal{D}}$ is already laminar). In view of (1.1), it suffices to arrange an uncrossing process for the family $\mathcal{D}_X \cup \{X, \bar{X}\}$ instead of $\tilde{\mathcal{D}}$. Let Z be as in (2) for given X and $\mathcal{D}$. Put $\mathcal{D}' := \{Y \in \mathcal{D}_X : X \cap Y = Z\}$; then $\mathcal{D}_X = \mathcal{D}' \cup \{\bar{Y} : Y \in \mathcal{D}'\}$. We have $\emptyset \neq Z \subset Y$, $\emptyset \neq X - Z \subset \bar{Y}$ for each $Y \in \mathcal{D}_X$, and now the laminarity of $\mathcal{D}_X$ implies that any two $Y', Y'' \in \mathcal{D}'$ satisfy either $Y' \subset Y''$ or $Y'' \subset Y'$. Therefore, there is a partition $(V_1, V_2, ..., V_r = V_0)$ of V such that $V_1 = Z$, $V_r = X - Z$ and each $Y \in \mathcal{D}_X \cup \{X, \bar{X}\}$ is a set of the form $V_1 \cup V_2 \cup \ldots \cup V_i$ for some $1 < i < r$. This implies that $\mathcal{D}_X \cup \{X, \bar{X}\}$ is a cyclical family, whence the lemma follows (the inequality $\|g\| \leq \|f\|$ holds because the value $\|\cdot\|$ does not increase on any uncrossing step).

□

Remark 2.2. Each cyclical family occurring in the above proof is equivalent to a cyclical family $\mathcal{R}$ on an ordered set $W = (v_1, \ldots, v_r)$ having the additional properties: (i) $\mathcal{R} = \{X, \bar{X}\} \cup \mathcal{B}$; (ii) $X = \{v_1, v_r\}$; and (iii) $\mathcal{B}$ is a laminar family each set of which is crossing X. These properties will be important for the proof of Theorem 2 in the next section.

Now the theorem immediately follows from Lemma 2.1 and the following lemma.

Lemma 2.3 *Let $\mathcal{R}$ be a cyclical family on an ordered set $W = (v_1, \ldots, v_r)$ and $g :\to \mathbb{Z}^+$ be a function. There exists an uncrossing process for $\mathcal{R}$ and g consisting of* $\log \|g\|$ *times a polynomial in r uncrossing steps.*

Proof. Note that an arbitrary cyclical family on W has the obvious property that each set obtained by uncrossing its crossing members has again the form $\{v_{i'}, v_{i'+1}, ..., v_{j'}\}$. Therefore any intermediate family $\mathcal{R}'$ arising during an uncrossing process for $\mathcal{R}$ and g is cyclical; this, in particular, implies that the cardinality of $\mathcal{R}'$ is at most $r(r-1)$. We say that a set $X = \{v_i, v_{i+1}, ..., v_j\}$ is *essential* if $2 \leq |X| \leq r-2$. Let $\mathcal{R}$ and g denote a current cyclical family and a function on $\mathcal{R}$ in the uncrossing process. In a current iteration we choose an essential set $X \in \mathcal{R}$ with $g(X)$ *maximum* (if there is no essential set in $\mathcal{R}$, then $\mathcal{R}$ is laminar) and fulfil uncrossing steps, one by one, with the fixed set X and members of $\mathcal{R}_X := \{Y \in \mathcal{R} : Y \not\parallel X\}$. Two cases are possible.

Case 1. $g(X) \geq \frac{1}{2} g(\mathcal{R}_X)$ where $g(\mathcal{R}_X) := \sum(g(Y) : Y \in \mathcal{R}_X)$. Then all the sets of $\mathcal{R}_X$ vanish during the iteration, and any set X' of the resulting family $\mathcal{R}'$ is laminar to X. Therefore, after this iteration we can split the uncrossing problem for $\mathcal{R}'$ into two problems, one for the family $\mathcal{R}'_1 := \{Y \in \mathcal{R}' : Y \subset X$ or $W - Y \subset X\}$ and the other for $\mathcal{R}'_2 := \{Y \in \mathcal{R}' : X \subset Y$ or $X \subset W - Y\}$. The problem for the former (latter) family is, in fact, that for a cyclical family on the set W_1 (W_2) obtained from W by identifying the elements of the subset $W - X$ (respectively, X). We have $|W_i| < r$, $i = 1, 2$, and $|W_1| + |W_2| = r + 2$.

Furthermore, if $|W_i| \leq 3$ then $\mathcal{R}'_i$ is obviously laminar. This implies that the total amount of those iterations (for all families arising on W and reduced sets) in which Case 1 occurs is bounded by a polynomial in r.

Case 2. $g(X) < \frac{1}{2}g(\mathcal{R}_X)$. Introduce the value

$$\beta := \beta(\mathcal{R}, g) := \sum_{e \in E_W} \sum (g(X) : X \in \mathcal{R},\ X \text{ is essential and } e \in \delta^W(X)),$$

where E_W is the edge-set of K_W. Let $\mathcal{R}'$ and g' be the family and the function obtained as a result of the iteration. The sets X and $\bar{X}$ vanish during the iteration. In view of (1), this implies that $\beta' := \beta(\mathcal{R}', g') \leq \beta - 4g(X)$. Furthermore, it follows from the maximal choice of X that $\beta \leq |E_W| g(X) |\mathcal{R}| < r^4 g(X)$. Therefore,

$$\beta' < \beta(1 - \frac{4}{r^4}). \tag{4}$$

Now suppose that Case 2 occurs in k successive iterations, and let β_0 and β_1 be the values of β in the first and the k-th of these iterations respectively. We may assume that $\beta_1 \geq 1$. Then (4) and the fact that $\beta_0 \leq r^2 \|g\|$ imply that k is no more than $\log \|g\|$ times a polynomial in r.

□

3. PROOF OF THEOREM 2

It this section by an uncrossing step we mean that of type 1. According to Lemma 2.1 and Remark 2.2, it suffices to arrange an uncrossing process (with a polynomial in r number of uncrossing steps) for $\mathcal{R}$ and $g : \mathcal{R} \to \mathbb{Z}^+$ such that $\mathcal{R}$ is a subfamily of a cross-closed family $\mathcal{C}$ on an ordered set $W = (v_1, \ldots, v_r)$ and $\mathcal{R}$ consists of two laminar families $\mathcal{L}$ and $\mathcal{B}$ with $\mathcal{L} = \{X, \bar{X}\}$ and $X = \{v_1, v_r\}$. Moreover, we may assume that

(5) for each $Y \in \mathcal{R}$ there is $Z \in \mathcal{R}$ such that Y and Z are crossing

(otherwise, in view of (1.1), we can eliminate Y from $\mathcal{R}$); and

(6) for $i = 1, \ldots, r$, there is a set in $\mathcal{R}$ separating v_{i-1} and v_i

(otherwise we can identify v_{i-1} and v_i decreasing the cardinality of the basic set W). Here we call $Z \subset W$ *separating* elements $u, v \in W$, or separating a pair $\{u, v\}$, if Z contains exactly one of u and v. It follows from (5) that $\mathcal{R}$ contains only *essential* sets Z, that is, $2 \leq |Z| \leq r-2$. Note also that (6) implies that

(7) $\mathcal{C}$ contains the set $Y_i := \{v_1, \ldots, v_i\}$ for $i = 2, \ldots, r-2$.

We will be forced to consider in our proof a family $\mathcal{R} = \mathcal{L} \cup \mathcal{B}$ of a slightly more general form. Namely, as above, $\mathcal{B}$ is a laminar family such that

(8) each $Y \in \mathcal{B}$ separates v_1 and v_r

while $\mathcal{L}$ is a laminar family which can contain more than two sets but such that

(9) each $X \in \mathcal{L}$ separates v_1 and v_2.

We shall show that for such an $\mathcal{R}$ there exists an uncrossing process consisting of $O(r^3)$ uncrossing steps.

We proceed by induction on

$$\omega := \omega(r, \mathcal{R}) := r^3 + r|\mathcal{L}|/2 + d,$$

considering all $W = (x_1, \ldots, v_r)$, $\mathcal{C}$ and $\mathcal{R} = \mathcal{L} \cup \mathcal{B}$ satisfying (5)-(9). Here $d := d(\mathcal{R})$ denotes the minimum number i such that $\mathcal{R}$ contains the set $X_i := \{v_2, \ldots, v_i\}$.

In what follows $\mathcal{R}' = \mathcal{L}' \cup \mathcal{B}'$ and g' will denote corresponding items arising when we apply to current $\mathcal{R}$ and g an uncrossing step $X, Y \to X', Y'$ and then delete each set of the resulting family which is laminar to all other ones. We also denote by $r(\mathcal{R}')$ the number of maximal subsets $\{v_i, \ldots, v_j\}$ which are separated by no set in $\mathcal{R}'$. Clearly if $r(\mathcal{R}') < r$ and if $\mathcal{L}'$ and $\mathcal{B}'$ are laminar families satisfying the properties as in (8) and (9) then, after identifying elements v_{i-1} and v_i separated by no set in $\mathcal{R}'$, we get r'' and $\mathcal{R}''$ with smaller ω (and satisfying the properties as in (5)-(9)), whence the result follows by induction.

First of all suppose that $\{v_1\} \notin \mathcal{C}$. It easily follows from (5) and (6) that $\mathcal{L}$ contains the set $X := X_{r-1}$ and $\mathcal{B}$ contains the set $Y := Y_2$. Then $Y - X = \{v_1\} \notin \mathcal{C}$, whence $X' := X \cap Y = \{v_2\} \in \mathcal{C}$ and $Y' := X \cup Y = W - \{v_r\} \in \mathcal{C}$ (since $\mathcal{C}$ is cross-closed). Fulfil the uncrossing step $X, Y \to X', Y'$. If $g(X) \leq g(Y)$ then $X \notin \mathcal{R}'$, and, similarly, if $g(Y) \leq g(X)$ then $Y \notin \mathcal{R}'$. Furthermore, sets X' and Y' are non-essential. This implies that at least one of the pair $\{v_2, v_3\}$ and $\{v_{r-1}, v_r\}$ is separated by no set in $\mathcal{R}'$, whence $r(\mathcal{R}') < r$ and the result follows by induction.

Thus we may assume that $\{v_1\} \in \mathcal{C}$. Consider the set $X_d = \{v_2, \ldots, v_d\}$. Since X_d is essential, $d \geq 3$. Note also that (6) and the minimality of d imply

$$Y_i = \{v_1, \ldots, v_i\} \in \mathcal{B} \text{ for } i = 2, \ldots d-1. \tag{10}$$

Let k be the maximal index such that $1 \leq k \leq d$ and $\{v_k\} \in \mathcal{C}$.

Claim. *(i)* $k \geq 2$. *(ii) If* $k < d$ *then* $Z := \{v_{k+1}, \ldots, v_d\} \notin \mathcal{C}$.

Proof. Observe that any minimal nonempty set in $\mathcal{C}$ is of cardinality 1 (this follows from (6) and the fact that $\mathcal{C}$ is cross-closed). Therefore, X_d contains an element v such that $\{v\} \in \mathcal{C}$, which implies (i). Next, if $k < d$ and $Z \in \mathcal{C}$ then there is $v_j \in Z$ such that $\{v_j\} \in \mathcal{C}$. Then $k < j \leq d$, contrary to the maximal choice of k.

□

Now consider three possible cases.

Case 1. $k = d$. Let $X := X_d$ and $Y := Y_{d-1}$; then $Y \in \mathcal{B}$, by (10). Observe that $X' := X - Y = \{v_d\} \in \mathcal{C}$ and $Y' := Y - X = \{v_1\} \in \mathcal{C}$. Fulfil the uncrossing step $X, Y \to X', Y'$. If $g(X) \geq g(Y)$ then $Y \notin \mathcal{R}'$ and no set in $\mathcal{R}'$ separates v_{d-1} and v_d, whence $r(\mathcal{R}') < r$. And if $g(X) \leq g(Y)$ then $X \notin \mathcal{R}'$, whence $\mathcal{L}' = \mathcal{L} - \{X, \bar{X}\}$ and $|\mathcal{L}'| < |\mathcal{L}|$. In both cases we have $\omega(r(\mathcal{R}'), \mathcal{R}') < \omega(r, \mathcal{R})$, and the result follows by induction.

Case 2. $k = 2$. Let $X := X_d$ and $Y := Y_2$. Then $X' := X \cap Y = \{v_2\} \in \mathcal{C}$ and $Y' := X \cup Y = Y_d \in \mathcal{C}$ (by (7)). Fulfil the uncrossing step $X, Y \to X', Y'$.

Similar to the previous case, we get at least one of the following situations: (i) no set in $\mathcal{R}'$ separates v_2 and v_3, or (ii) $\mathcal{L}' = \mathcal{L} - \{X, \bar{X}\}$; and the result follows by induction.

Case 3. $2 < k < d$. Let $X := X_d$, $Y := Y_k$ and $Z := \{v_{k+1}, \ldots, v_d\}$. By the Claim, $Z \notin \mathcal{C}$. Therefore, $X' := X \cap Y = X_k \in \mathcal{C}$ and $Y' := X \cup Y = Y_d \in \mathcal{C}$. Fulfil the uncrossing step $X, Y \to X', Y'$. Suppose that $g(X) \leq g(Y)$. Then $X \notin \mathcal{R}'$ and $X' \in \mathcal{R}$, whence $|\mathcal{L}'| = |\mathcal{L}|$. Furthermore, we have $d(\mathcal{R}') = k < d(\mathcal{R})$, and the result follows by induction.

Now suppose that $g(X) > g(Y)$. Then $X, X' \in \mathcal{R}'$, and therefore $\mathcal{L}' = \mathcal{L} \cup \{X', \bar{X}'\}$. However, the following useful property holds:

(11) v_k and v_{k+1} are separated in $\mathcal{R}'$ by only X_k and $\bar{X}_k$

(since $Y_k \notin \mathcal{R}'$). Consider the sets $\tilde{X} := X_k$ and $\tilde{Y} := Y_{k-1}$ in $\mathcal{R}'$. Both sets $\tilde{X}' := \tilde{X} - \tilde{Y} = \{v_k\}$ and $\tilde{Y}' := \tilde{Y} - X = \{v_1\}$ are in $\mathcal{C}$, so we can fulfil the second uncrossing step $\tilde{X}, \tilde{Y} \to \tilde{X}', \tilde{Y}'$ for $\mathcal{R}'$ and g' forming $\mathcal{R}''$ and g''. Two cases are possible.

(i) $g'(\tilde{X}) \leq g'(\tilde{Y})$. Then $\tilde{X} \notin \mathcal{R}''$, and now, in view of (11), there is no set in $\mathcal{R}''$ separating v_k and v_{k+1}. Thus $r(\mathcal{R}'') < r$, whence the result follows by induction.

(ii) $g'(\tilde{X}) > g'(\tilde{Y})$. Then $\tilde{Y} \notin \mathcal{R}''$, therefore no set in $\mathcal{R}''$ separates v_{k-1} and v_k. We have again $r(\mathcal{R}'') < r$, and the result follows by induction.

This completes the proof of Theorem 2.

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