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MAXIMIZATION OVER THE INTERSECTION OF TWO COMPATIBLE GREEDY-POLYHEDRA

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Abstract. A greedy-polyhedron, arising as a generalization of the notion of polymatroid, is a convex set Q in the $|T|$-dimensional euclidean space $\mathbf{R}^T$ for which a natural greedy-algorithm yields an optimum solution for any (not necessarily nonnegative) objective vector $c \in \mathbf{R}^n$. Such a polyhedron can be equivalently defined as the solution set of the system of inequalities $\sum_{e\in A} x_e - \sum_{e\in B} x_e \leq f(A,B)$ for all disjoint $A, B \subseteq T$, where f is a function satisfying certain submodular-like relations. It is known that the problem of maximizing $c \cdot x$ over the intersection of two greedy-polyhedra Q and Q' described by integer-valued functions f and f' has a half-integral optimal solution x.

We consider a special case of the problem when $c \equiv 1$ and there are certain additional relations, called compatibility ones, between f and f'. An important feature of the problem in this case is that the maximum value $1 \cdot x$ is equal to that for the intersection of the polymatroids associated with the functions $f(\cdot, \emptyset)$ and $f'(\cdot, \emptyset)$.

We develop a combinatorial method to solve the problem in question in which the number of iterations is bounded by a polynomial in $|T|$, and establish the existence of an optimal solution having a special structure.

Using an approach in [FKS], we then demonstrate application of compatible greedy-polyhedra to the maximum undirected multi(commodity)flow problem for the case when the commodity graph is the complement to the line graph of a bipartite graph.

1. Introduction

Let T be a finite set, and Π be the set of ordered pairs $\pi = (A, B)$ of disjoint sets $A, B \subseteq T$. We may think of Π as $\{0, 1, -1\}^T$; then $\pi = (A, B)$ corresponds to the vector $\chi_\pi = \chi_{A,B}$ in $\mathbf{R}^T$ whose component $\chi_\pi(s)$ is 1 for $s \in A$, -1 for $s \in B$, and 0 otherwise. For $\pi = (A, B)$ and $\pi' = (A', B')$ define operations $\wedge$ and $\vee$ as:

$$\pi \wedge \pi' := (A \cap A', B \cap B');$$
$$\pi \vee \pi' := ((A \cup A') - (B \cup B'), (B \cup B') - (A \cup A')).$$

One can see that $\chi_\pi + \chi_{\pi'} = \chi_{\pi\wedge\pi'} + \chi_{\pi\vee\pi'}$. We say that a function $f : \Pi \to \mathbf{R}$ is *submodular* if each two $\pi, \pi' \in \Pi$ satisfy

$$f(\pi) + f(\pi') \geq f(\pi \wedge \pi') + f(\pi \vee \pi'). \tag{1}$$

For a submodular function f on Π the set

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$$Q_f := \{x \in \mathbf{R}^T : x(A) - x(B) \leq f(\pi) \quad \text{for any } \pi = (A, B) \in \Pi\}$$

is called a *greedy-polyhedron*. Here for $S' \subseteq S$ and $g : S \to \mathbf{R}$, $g(S')$ denotes $\sum(g(s) : s \in S')$. It was stated independently in [KC,Na,Se] that such a polyhedron (called also greedy-set, bisubmodular polyhedron, pseudopolymatroid et al.) is equivalent to that introduced in [DW], to be a set for which a natural greedy-algorithm yields an optimal solution for any objective vector $c \in \mathbf{R}^T$ (see also [BC] for a survey).

In what follows we consider only *integer-valued* submodular functions on Π. In this case the greedy-polyhedra have the following integrality and half-integrality properties (their proofs are relatively simple):

(i) for any $c : T \to \mathbf{Z}$ the program $\max\{cx : x \in Q_f\}$ has an *integral* optimal solution (if it has an optimal solution at all); and the same is true for the dual program [KC,Na,Qi,Se];

(ii) for any $c : T \to \mathbf{Z}$ and two submodular functions f, f' the program

$$\text{(2)} \qquad \text{maximize } cx \text{ subject to } x \in Q_f \cap Q_{f'}$$

has a *half-integral* optimal solution (if it has an optimal solution at all); and the same is true for the dual program

$$\text{(3)} \qquad \text{minimize } yf + y'f' \text{ subject to } y, y' : \Pi \to \mathbf{R}_+ \text{ and } \sum_{\pi \in \Pi}(y_\pi + y'_\pi)\chi_\pi = c$$

(unfortunately the author cannot refer to the paper where (ii) was originally stated). Similar properties remain true when we concern the problems on $Q_f \cap \mathbf{R}^T_+$ or $Q_f \cap Q_{f'} \cap \mathbf{R}^T_+$ (as it can be shown that $Q_f \cap \mathbf{R}^T_+$ coincides with the greedy-polyhedron Q_h for some submodular h).

In the present paper we shall deal with a special case of problems (2)-(3), and one of their combinatorial consequences. Our first restriction is that only the all-unit objective vector is considered and that x in (2) is nonnegative. Thus the problem in question is:

(4) *given f and f', find $x \in Q_f \cap Q_{f'} \cap \mathbf{R}^T_+$ with $x(T)$ maximum.*

It turns out that for $c \equiv 1$ the half-integrality for (3) can be strengthened; namely, the following fact will be stated in Section 2.

Statement 1. *If $c \equiv 1$ and (3) has an optimal solution then it has an integral optimal solution.**)

However, it can be shown that for the problem (2) with $c \equiv 1$ (as well as for (4)) the existence of an integral optimal solution is, in general, not guaranteed. In Section 4 we demonstrate an application of (4) to a class of multi(commodity)flow problems having half-integral optimal solutions.

*) This statement remains true for an arbitrary c such that $c(s)$ is odd for all $s \in T$ (the proof is, in essense, the same as that given in Section 2).

Our second restriction is that there are certain additional submodular-like relations between the functions f and f'; they appeared as a generalization of some features of those multiflow problems to the abstract case. On the other hand, we can motivate such a restriction by considering the corresponding polymatroid intersection problem.

More precisely, let $f(A)$ denote $f(\pi)$ for $\pi := (A, \emptyset)$; and similarly for f'. We associate with f the function $g := g_f$ on 2^T defined by

$$(5) \qquad \begin{aligned} g(X) &:= \min\{f(A) : X \subseteq A \subseteq T\} \quad \text{for } \emptyset \neq X \subseteq T, \\ &:= \min\{0, \min\{f(A) : A \subseteq T\}\} \quad \text{for } X = \emptyset; \end{aligned}$$

and similarly define $g' := g_{f'}$. Then g is monotone. Furthermore, g is a submodular set function since for $X, Y \subseteq T$ we have from (1) that $g(X)+g(Y) \geq g(X \cap Y)+g(X \cup Y)$. We shall assume that $g(\emptyset) = g'(\emptyset) = 0$ (for if $g(\emptyset) < 0$ or $g'(\emptyset) < 0$ then, obviously, (4) has no solution). Thus g and g' are polymatroid functions, and we can consider the polymatroid intersection problem:

$$(6) \qquad \text{maximize } x(T) \text{ subject to } x \in P_f \cap P_{f'},$$

where P_f denotes the polymatroid $\{x \in \mathbf{R}^T : x \geq 0,\ x(X) \leq g(X) \text{ for any } X \subseteq T\}$, and $P_{f'}$ denotes the corresponding polymatroid for f'. Let m be the maximum in (4), and M be the maximum in (6). A well-known Edmonds' theorem [Ed] asserts that (6) has an integral optimal solution x, and

$$(7) \qquad M = \min\{g(X) + g'(T - X) : X \subseteq T\}.$$

Let (4) have an optimal solution x. Choose a set X that achieves the minimum in (7). By (5), there are $Z \supseteq X$ and $Z' \supseteq T - X$ such that $g(X) = f(Z)$ and $g'(T - X) = f'(Z')$ (assuming, without loss of generality, that $f(\emptyset) = f'(\emptyset) = 0$). Then

$$m = x(T) = x(X) + x(T - X) \leq x(Z) + x(Z') \leq f(Z) + f'(Z') = M,$$

taking into account that x is nonnegative. Now we introduce a relationship between f and f'; it will imply that m=M (or, in other words, that the dual problems for (4) and (6) are equivalent, in a sense). First of all define a partial order $\preceq$ on Π as follows:

$$(8) \qquad \pi \preceq \pi' \quad \text{if } A \subseteq A' \text{ and } B \supseteq B'.$$

Definition. $\pi' = (A', B')$ is called the *f-closure* of π if: (i) $\pi' \succeq \pi$ and $f(\pi') = \min\{f(\pi'') : \pi'' \succeq \pi)\}$; (ii) $|B'|$ is minimum; and (iii) $|A'|$ is maximum subject to (ii). We say that π is *f-closed* if it coincides with its f-closure.

The f-closure is defined uniquely. Indeed, suppose that $\pi_i = (A_i, B_i)$, $i = 1, 2$, are two pairs satisfying (i)-(iii). Then $f(\pi_1) = f(\pi_2) =: q$. Clearly $\pi_1 \wedge \pi_2 \succeq \pi$ and $\pi_1 \vee \pi_2 \succeq \pi$, therefore

$$2q = f(\pi_1) + f(\pi_2) \geq f(\pi_1 \wedge \pi_2) + f(\pi_1 \vee \pi_2)$$

implies $f(\pi_1 \wedge \pi_2) = f(\pi_1 \vee \pi_2) = q$. Since $\pi_1 \wedge \pi_2 = (A_1 \cap A_2, B_1 \cap B_2)$, we have $B_1 = B_2$, by (ii). Then $\pi_1 \vee \pi_2 = (A_1 \cup A_2, B_1)$, whence $A_1 = A_2$, by (iii).

For $\pi = (A, B), \pi' = (A', B')$ let $\pi - \pi'$ denote the pair $(A - (A' \cup B'), B - (A' \cup B'))$.

Definition. Two submodular functions f and f' on Π, as well as the polyhedra Q_f and $Q_{f'}$, are called *compatible* if the inequality

$$f(\pi) + f'(\pi') \geq f(\pi - \pi') + f'(\pi' - \pi) \tag{9}$$

holds for any f-closed $\pi = (A, B)$ and f'-closed $\pi' = (A', B')$ such that at least one of B and B' is empty.

In Section 2 we prove the following property.

Statement 2. *If f and f' are compatible then $m = M$; in other words, there are $Z, Z' \subseteq T$ such that $Z \cup Z' = T$ and $m = f(Z) + f'(Z')$.*

Again, a simple example shows that for compatible f and f' (4) does not have, in general, an integral optimum solution, in contrast to (6).

The main purpose of the present paper is to develop a combinatorial method for finding a half-integral optimal solution of (4) for compatible f and f'; it will be described in Section 3. The function f (f') in this method is given implicitly by use of an oracle that, being asked about $\pi \in \Pi$, tell us $f(\pi)$ (respectively, $f'(\pi)$). Moreover, one supposes that there are available procedures ("black boxes") that solve the following auxiliary problems:

(10)(i) given $z \in \mathbf{Q}_+$ and $s, t \in T$, find $\max\{z(A) - f(A) : A \subset T,\ s \in A \not\ni t\}$;

(ii) given $z \in \mathbf{Q}_+$ and $\widetilde{T} \subseteq T$, find $\max\{z(A) - z(B) - f(\pi) : \pi = (A, B) \in \Pi,\ A \cup B \subseteq \widetilde{T}\}$,

and similarly for f' ($\mathbf{Q}_+$ is the set of nonnegative rationals). (10)(i) is a variant of the submodular function minimization (see [GLS]). As to (10)(ii), we do not discuss in this paper whether there exists a rather elementary oracle for f which makes it possible to solve (10)(ii) efficiently (say, in strongly polynomial time); for the application in Section 4 this problem is solved in strongly polynomial time.

The algorithm starts with an integral optimal solution x of (6) and a set X that achieves the minimum in (7). Such an x can be infeasible for (4). The algorithm revises, step by step, the current vector on T until a half-integral vector $r \in Q_f \cap Q_{f'} \cap \mathbf{R}_+^T$ such that $r(T) = x(T)$ will be constructed; then r is an optimal solution of (4). A feature of the algorithm is that, on each iteration, the current x' is transformed into a new x'' by a local operation. More precisely, the following is true.

Statement 3. *Let f and f' be compatible, and let x be an integral optimal solution of the polymatroid intersection problem (6). Then there exist a sequence $x_0, x_1, \ldots, x_d \in \frac{1}{2}\mathbf{Z}_+^T$ such that:*

(i) $x_0 = x$, $x_d \in Q_f \cap Q_{f'} \cap \mathbf{R}_+^T$ and each x_i is an optimal solution of (6);

(ii) for $i = 0, \ldots, d-1$, x_i and x_{i+1} differ in at most three components; moreover, either $x_{i+1}(u) = x_i(u) - 1/2$ and $x_{i+1}(v) = x_i(v) + 1/2$ for some $u, v \in T$, or $x_{i+1}(u) = x_i(u) - 1/2$, $x_{i+1}(v) = x_i(v) - 1/2$ and $x_{i+1}(w) = x_i(w) + 1$ for some $u, v, w \in T$;

(iii) d is $\max\{x(A) - x(B) - f(A,B) : (A,B) \in \Pi\} + \max\{x(A) - x(B) - f'(A,B) : (A,B) \in \Pi\}$.

The number of iterations of the algorithm is just $d \leq \max\{|f(\pi)| : \pi \in \Pi\} + \max\{|f'(\pi)| : \pi \in \Pi\}$. At the end of Section 3 we modify the algorithm in such a way that it will consists of $O(|T|^2)$ iterations, and the number of applications of (10) is bounded by a polynomial in $|T|$. Also the following result will be provided by the algorithm.

Statement 4. *Let f and f' be compatible, and let $f(\pi)$ and $f(\pi')$ be even for all $\pi \in \Pi$. Then there exist an integer optimal solution r of (4) and a partition $\{T_1, T_2\}$ of T so that*

$$r_1 \in Q_f \quad \text{and} \quad r_2 \in Q_{f'},$$

where $r_i(s) := 2\lceil r(s)/2 \rceil$ for $s \in T_i$ and $r_i(s) := r(s)$ for $s \in T - T_i$ ($\lceil a \rceil$ is the smallest integer greater than or equal to a).

This statement will be used in Section 4 to establish the existence of an integer solution for a certain multiflow problem.

2. Dual solutions

In this Section we prove Statements 1 and 2.

Proof of Statement 1. Let $c \equiv 1$, and let (2) have an optimal solution x. By the l.p. duality theorem, mappings $y, y' : \Pi \to \mathbf{R}_+$ for which

$$\sum_{\pi \in \Pi} (y_\pi + y'_\pi)\chi_\pi = 1 \tag{11}$$

give an optimal solution of (3) if and only if the following (complementary slackness) condition holds:

$$y_\pi > 0 \ (y'_\pi > 0) \text{ implies } \chi_\pi x = f(\pi) \text{ (respectively, } \chi_\pi x = f'(\pi)) \tag{12}$$

Consider a *half-integral* optimal solution (y, y') for (3) (existing, as it was mentioned in the Introduction). By standard uncrossing arguments, one may assume that any $\pi = (A, B), \pi' = (A', B') \in \Pi$ with $y_\pi, y_{\pi'} > 0$ satisfy $A \subseteq A'$, $B \subseteq B'$ or $A' \subseteq A$, $B' \subseteq B$; and similarly for y'. (Indeed if it is not so for some π, π' then one can transform y as $y_\pi := y_\pi - 1/2$, $y_{\pi'} := y_{\pi'} - 1/2$, $y_{\pi\wedge\pi'} := y_{\pi\wedge\pi'} + 1/2$ and $y_{\pi\vee\pi'} := y_{\pi\vee\pi'} + 1/2$. This preserves optimality of y, y'; the result follows from the simple observation that the value $\sum_{\pi''\in\Pi} y_{\pi''}\lambda_{\pi''}$ becomes smaller; here for $\pi'' = (A'', B'')$, $\lambda_{\pi''}$ is the function on the pairs $\{s, t\} \subseteq T$ taking value 1 if $|\{s, t\} \cap (A'' \cup B'')| = 1$, 2 if $|\{s, t\} \cap A''| = 1$ and $|\{s, t\} \cap B''| = 1$, and 0 otherwise.) Let $\mathcal{F}$ ($\mathcal{F}'$) be the family that contains exactly $2y_\pi$ (respectively, $2y'_\pi$) copies of each $\pi \in \Pi$. Then the members of $\mathcal{F}$ ($\mathcal{F}'$) can be numbered as $\pi_i = (A_i, B_i)$, $i = 1, \ldots, k$ (respectively, as $\pi'_i = (C_i, D_i)$, $i = 1, \ldots, n$) so that:

(13) (i) $A_1 \supseteq \ldots \supseteq A_k$; $B_1 \supseteq \ldots \supseteq B_k$; $C_1 \supseteq \ldots \supseteq C_n$; $D_1 \supseteq \ldots \supseteq D_n$;
(ii) for each $s \in T$:

$$|\{i : s \in A_i\}| - |\{i : s \in B_i\}| + |\{j : s \in C_j\}| - |\{j : s \in D_j\}| = 2$$

((ii) follows from (11)). Form the index sets $I := \{i : 1 \le i \le k,\ i \equiv 0 \text{ or } 1 (\mathrm{mod}\ 4)\}$ and $J := \{j : 1 \le j \le n,\ j \equiv 2 \text{ or } 3 (\mathrm{mod}\ 4)\}$. We assert that for each $s \in T$:

$$(14)\quad |\{i \in I : s \in A_i\}| - |\{i \in I : s \in B_i\}| + |\{j \in J : s \in C_j\}| - |\{j \in J : s \in D_j\}| = 1.$$

Then, letting $z_\pi := |\{i \in I : \pi = \pi_i\}|$ and $z'_\pi := |\{j \in J : \pi = \pi'_j\}|$ for $\pi \in \Pi$, we conclude from (14) that $\sum_{\pi\in\Pi}(z_\pi + z'_\pi)\chi_\pi = 1$, whence, in view of (12), (z, z') is an optimal solution for (3), and the result follows.

To see (14), notice that $A_1 \cup C_1 = T$, by (13). Consider $s \in T$, and let q (q') be the maximum $i \le k$ ($j \le n$) so that $s \in A_i \cup B_i$ (respectively, $s \in C_j \cup D_j$). Three cases are possible.

(i) $s \in A_1 \cap C_1$. Then, by (13), $q = q' = 1$, and (14) is obvious.

(ii) $s \in A_1 - C_1$. Then, by (13), $s \in A_i$ for $i = 1, \ldots, q$, $s \in D_j$ for $j = 1, \ldots, q'$ and $q = q' + 2$. One can see from the definition of I and J that, for any $1 < i \le q$, $i \in I$ if and only if $i - 2 \in J$. This implies $|\{i \in I : i \le q\}| = |\{j \in J : j \le q'\}| + 1$ (since $1 \in I$), whence (14) follows.

(iii) $s \in C_1 - A_1$. Then $s \in B_i$ for $i = 1, \ldots, q$, $s \in C_j$ for $j = 1, \ldots, q'$, and $q = q' - 2$. For $2 < j \le q'$, $j \in J$ if and only if $j - 2 \in I$, and we again get (14). •

Statement 1 easily implies the following.

(2.1) *The program dual to (4) has an integral optimal solution (if it has an optimal solution). More precisely, there exist $y, y : \Pi \to \mathbf{Z}_+$ such that $\sum_{\pi\in\Pi}(y_\pi + y'_\pi)\chi_\pi \ge 1$ and the maximum in (4) is equal to $\sum_{\pi\in\Pi} y_\pi f_\pi + y'_\pi f'_\pi$.* •

Proof of Statement 2. For compatible f and f' consider an optimal integral solution (y, y') as in (2.1). We may identify y (y') with the family $\mathcal{F}$ ($\mathcal{F}'$) that contains

exactly y_π (respectively, y'_π) copies of each $\pi \in \Pi$. Let, in addition, y, y' be chosen so that the value

$$\omega := \omega(\mathcal{F}, \mathcal{F}') := \sum(|B| : \pi = (A, B) \in \mathcal{F}) + \sum(|D| : \pi' = (C, D) \in \mathcal{F}')$$

is as small as possible. First of all we show that $\omega = 0$. One may assume that the following properties (15) and (16) hold.

(15) Each two pairs $\pi = (A, B)$ and $\pi' = (A', B')$ in $\mathcal{F}$ satisfy $A \subseteq A', B \subseteq B'$ or $A' \subseteq A, B' \subseteq B$; and similarly for $\mathcal{F}'$.

For we can replace π, π' in $\mathcal{F}$ by $\pi \wedge \pi', \pi \vee \pi'$, obtaining a new optimal solution with not greater ω.

(16) Each member of $\mathcal{F}$ is f-closed and each member of $\mathcal{F}'$ is f'-closed.

For we can replace each $\pi = (A, B)$ in $\mathcal{F}$ by its f-closure $\overline{\pi} = (\overline{A}, \overline{B})$, forming a new optimal solution with not greater ω because of $\chi_\pi \leq \chi_{\overline{\pi}}$ and $\overline{B} \subseteq B$. Moreover, such a replacement preserves (15). Indeed, let $\pi = (A, B), \pi' = (A', B') \in \mathcal{F}$ be such that $A \subseteq A'$, $B \subseteq B'$, and let $\overline{\pi} = (\overline{A}, \overline{B}), \overline{\pi}' = (\overline{A}', \overline{B}')$ be their f-closures. Then for $\pi_1 := \overline{\pi} \wedge \overline{\pi}'$ and $\pi_2 := \overline{\pi} \vee \overline{\pi}$ we have: (i) $f(\pi_1) + f(\pi_2) \leq f(\overline{\pi}) + f(\overline{\pi}')$, (ii) $f(\pi_1) \geq f(\overline{\pi})$ (as, obviously, $\pi \preceq \pi_1$), and (iii) $f(\pi_2) \geq f(\overline{\pi}')$ (as $\pi' \preceq \pi_2$). Hence, $f(\pi_1) = f(\overline{\pi})$ and $f(\pi_2) = f(\overline{\pi}')$. Now the minimality of $\overline{B}$ implies $\overline{B} \subseteq \overline{B} \cap \overline{B}'$, whence $\overline{B} \subseteq \overline{B}'$; the minimality of $\overline{B}'$ implies $\overline{B}' \subseteq (\overline{B} \cup \overline{B}') - (\overline{A} \cup \overline{A}')$, whence $\overline{B}' \cap \overline{A} = \emptyset$; and then the maximality of $\overline{A}'$ implies $\overline{A}' \supseteq (\overline{A} \cup \overline{A}') - (\overline{B} \cup \overline{B}') = \overline{A} \cup \overline{A}'$, whence $\overline{A} \subseteq \overline{A}'$.

Renumber the elements in $\mathcal{F}$ ($\mathcal{F}'$) as (A_i, B_i), $i = 1, \ldots, k$ (respectively, (C_j, D_j), $j = 1, \ldots, n$) to satisfy (13)(i). Let $s \in T$ be such that $q := |\{i : s \in B_i\}|$ is maximum, and let $s' \in T$ be such that $q' := |\{j : s' \in D_j\}|$ is maximum. Let for definiteness $q \geq q'$. Suppose that $q > 0$. Since $(\mathcal{F}, \mathcal{F}')$ determines a dual solution, $|\{j : s \in C_j\}| - |\{i : s \in B_i\}| \geq 1$. Hence, $n \geq q + 1$ and $s \in C_j$ for $j = 1, \ldots, q + 1$. Note that $D_{q+1} = \emptyset$ (since $q \geq q'$). Thus, for the f-closed $\pi := (A, B) := \pi_q$ and f'-closed $\pi' := (C, D) := \pi'_{q+1}$ we have $C \cap B \neq \emptyset$ and $D = \emptyset$. Replace the members π, π' in $(\mathcal{F}, \mathcal{F}')$ by $\pi - \pi', \pi' - \pi$. We observe that $\chi_{\pi - \pi'} + \chi_{\pi' - \pi} = \chi_\pi + \chi_{\pi'} - 2\chi_{A \cap C}$, hence the new $(\mathcal{F}, \mathcal{F}')$ remains to be a dual solution (as each element in $A \cap C$ is covered by A_1 (because of $q \geq 1$) and belongs to none of B_i's or D_j's). Now (9) shows that the new $(\mathcal{F}, \mathcal{F}')$ is optimal. Finally, $C \cap B \neq \emptyset$ implies that ω becomes smaller; a contradiction.

Thus $\omega = 0$. Then $(A_1, B_1) = (A_1, \emptyset)$ and $(C_1, D_1) = (C_1, \emptyset)$ give an optimal solution for (3). •

3. Algorithm

It is convenient to assume that $f(\pi)$ and $f'(\pi)$ are *even* for all $\pi \in \Pi$. Then (6)

has an optimal solution x with $x(s)$ even for all $s \in T$. The input and output of the algorithm are as follows.

INPUT: a function $x : T \to 2\mathbf{Z}_+$ and a set $X \subseteq T$ such that $x \in P_f \cap P_{f'}$, and $x(T) = g(X) + g'(T - X)$.

OUTPUT: an *integer-valued* function $x^* \geq 0$ on T such that $x^* \in Q_f \cap Q_{f'}$ and $x^*(T) = x(T)$.

A set $A' \subseteq T$ is called the f-closure of $A \subseteq T$ if $(A', \emptyset)$ is the f-closure of $(A, \emptyset)$; A is f-closed if $(A, \emptyset)$ is f-closed. Similarly define the f'-closure and f'-closed sets.

Let x and X be as in the input. By definition of g and g' there are an f-closed Z and an f'-closed Z' such that

$$Z \supseteq X, \quad Z' \supseteq T - X, \quad f(Z) = g(X) \quad \text{and} \quad f'(Z') = g'(T - X).$$

We say that $r : T \to \mathbf{Z}_+$ is *admissible* if $r \in P_f \cap P_{f'}$ and $r(T) = x(T)$. In particular, x is admissible. Clearly if r is admissible then

$$(17) \qquad r(Z) = f(Z), \quad r(Z') = f'(Z') \quad \text{and} \quad r(v) = 0 \ \text{ for all } \ v \in Z \cap Z'.$$

Let r be a current admissible function in the algorithm. On each iteration r is transformed into a new admissible function with smaller "deficit", and the process finishs with r in $Q_f \cap Q_{f'}$. At the beginning one puts $r := x$. For $\pi = (A, B) \in \Pi$ define the numbers:

$$(18) \qquad \begin{aligned} \Delta_r(\pi) &:= r(A) - r(B) - f(\pi) \quad (= r\chi_\pi - f(\pi)); \\ \Delta'_r(\pi) &:= r(A) - r(B) - f'(\pi) \quad (= r\chi_\pi - f'(\pi)). \end{aligned}$$

If B is empty, $\Delta_r(\pi)$ $(\Delta'_r(\pi))$ may be denoted as $\Delta_r(A)$ $(\Delta'_r(A))$. Define $\delta := \delta_r$ to be $\max\{\Delta_r(\pi) : \pi \in \Pi\}$ (the *deficit* with respect to f), and $\delta' := \delta'_r$ to be $\max\{\Delta'_r(\pi) : \pi \in \Pi\}$ (the *deficit* with respect to f'). Since $\Delta_r(Z) = \Delta'_r(Z') = 0$, $\delta \geq 0$ and $\delta' \geq 0$. If $\delta = \delta' = 0$ then r is an optimal solution of (4). So we may assume that at least one of δ and δ' is positive.

We need some facts (claims (19)-(25)) concerning Δ_r which make clearer details of the algorithm and will be used in the proof of correctness of it; similar properties are true for Δ'_r.

$$(19) \qquad \Delta_r(\pi) + \Delta_r(\pi') \leq \Delta_r(\pi \wedge \pi') + \Delta_r(\pi \vee \pi') \quad \text{for any } \ \pi, \pi' \in \Pi;$$

(20) if $(A, B) \in \Pi$ is f'-closed and $C \subseteq T$ is f-closed then
$$\Delta_r(C) + \Delta'_r(A, B) \leq \Delta_r(C - (A \cup B)) + \Delta'_r(A - C, B - C) + 2r(A \cap C).$$

Indeed, (19) follows from the submodular inequality (1), the vector equality $\chi_\pi + \chi_{\pi'} = \chi_{\pi\wedge\pi'} + \chi_{\pi\vee\pi'}$ and the nonnegativity of r. (20) follows from (9) and the obvious equality $\chi_C + \chi_{A,B} = \chi_{C-(A\cup B)} + \chi_{A-C,B-C} + 2\chi_{A\cap C}$, where $\chi_{C'}$ is the incidence vector of a set $C' \subseteq T$.

A pair π is called *f-critical* for r if $\Delta_r(\pi) = \delta$. Since $r \in P_f$, we have $\Delta_r(A) \leq 0$ for each $A \subseteq T$. We say that a set $A \subseteq T$ is *tight* if $\Delta_r(A) = 0$. Similarly, define the f'-critical pairs and f'-tight sets. Obvious consequences of (19) are:

(21) if π and π' are f-critical then $\pi \wedge \pi'$ and $\pi \vee \pi'$ are f-critical too;

(22) if $A, A' \subseteq T$ are f-tight then $A \cap A'$ and $A \cup A'$ are f-tight too.

(23) If $\pi = (A, B)$ is f-critical and C is f-tight then $\pi' := (A \cup C, B - C)$ is f-critical and $A \cap C$ is f-tight; in particular, $(A \cup Z, B - Z)$ is f-critical.

Indeed, (19) applied to π and C implies that $\pi'' := ((A \cup C) - B, B - C)$ is f-critical and $A \cap C$ is f-tight. Now applying (19) to π'' and C we obtain that π' is f-critical.

(24) If $\pi = (A, B)$ is f-critical and $\overline{\pi} = (\overline{A}, \overline{B})$ is the f-closure of π then $\overline{\pi}$ is f-critical, $f(\overline{\pi}) = f(\pi)$, $r(\overline{A} - A) = 0$ and $r(B - \overline{B}) = 0$.

This follows from $f(\overline{\pi}) \leq f(\pi)$ and $r(\overline{A}) - r(\overline{B}) = r(A) - r(B) + r(\overline{A} - A) + r(B - \overline{B})$.

(25) If $A \subseteq T$ is f-tight then the f-closure $\overline{A}$ of A is f-tight and $r(\overline{A} - A) = 0$. If $A \subseteq Y$, and Y is f-closed then $\overline{A} \subseteq Y$; in particular, if $A \subseteq Z$ then $\overline{A} \subseteq Z$.

The first part in (25) is proved similarly to (24). The second part is trivial.

Now we introduce additional assumptions on r and describe auxilliary structures used in the algorithm. Let $N := N_r$ ($N' := N'_r$) denote the set of $s \in Z$ ($s \in Z'$) with $r(s)$ odd. Since f is even everywhere, $|N \cap A|$ is even for each $A \subseteq T$ with $\Delta_r(A)$ even. In particular, $|N|$ and $|N'|$ are even.

The elements of N are partitioned into pairs; the set of these pairs is denoted by R. For any $\rho \in R$ there is an (explicitly given) f-tight set $S_\rho \subseteq Z$ such that $\rho \subseteq S_\rho$. In addition, there is a collection $\Pi_R \subset \Pi$ consisting of $|R|$ elements. For S_ρ's and Π_R the following properties holds:

(A1) for any distinct $\rho, \rho' \in R$, if $S_\rho \cap \rho' \neq \emptyset$ then $S_{\rho'} \subseteq S_\rho - \rho$;

(A2) each $\pi = (A, B) \in \Pi_R$ is f'-critical, and $Z' \subseteq A$;

(A3) the elements of R are numbered as $\rho_1 = \{s_1, t_1\}, \ldots, \rho_k = \{s_k, t_k\}$ and the elements of Π_R are numbered as $\pi_1 = (A_1, B_1), \ldots, \pi_k = (A_k, B_k)$ so that $Z' \subset A_k$ and:

(i) $A_k \subset A_{k-1} \subset \ldots \subset A_1$ and $B_k \subset B_{k-1} \subset \ldots \subset B_1$;

(ii) for $i = 1, \ldots, k$, $s_i \in B_i - B_{i+1}$ and $t_i \in A_i - A_{i+1}$ (letting $A_{k+1} := B_{k+1} := \emptyset$).

(See Fig. 1 for $k = 4$.)

Similarly, the elements of N' are partitioned into pairs forming a set R'; for each $\rho \in R'$ there is an f'-tight set $S_\rho \subseteq Z'$ containing ρ; and there is a collection $\Pi_{R'}$ of f'-critical members of Π for which the properties similar to (A1)-(A3) hold. Note that (A1) determines a partial order over R; namely, we say that ρ is greater than ρ' if $S_{\rho'} \subset S_\rho$. Later we shall use the following observation.

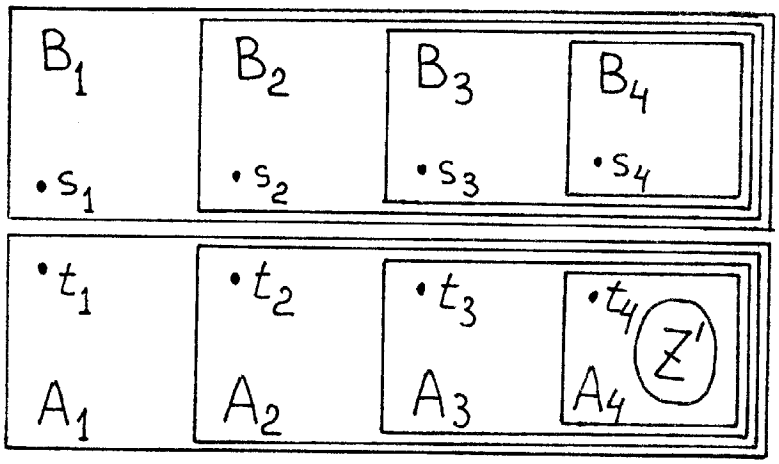


Fig. 1

Claim 1. *If $C \subseteq Z$ and C is f-tight then $|C \cap \rho| \neq 1$ for all $\rho \in R$.*

Proof. Suppose that it is not so for some C. Let, in addition, C is chosen so that $|\gamma(C)|$ is as small as possible, where $\gamma(C) := \{\rho \in R : |\rho \cap C| = 1\}$. The case $|\gamma(C)| = 1$ is impossible. For otherwise $|C \cap (N \cup N')| = |C \cap N|$ is odd, whence $\Delta_r(C) = r(C) - f(C) \neq 0$ (since $r(C)$ is odd while $f(C)$ is even). Thus $|\gamma(C)| \geq 2$. Take a minimal (under the partial order in R) element $\rho \in \gamma(C)$. Then $\gamma(C') = \{\rho\}$ for $C' := C \cap S_\rho$. By (22) for C and S_ρ, C' is f-tight; a contradiction. •

Now we describe an iteration of the algorithm. One may assume for definiteness that $\delta'_r > 0$. First of all we find an f'-critical pair $\pi^0 = (A^0, B^0) \in \Pi$ such that $Z' \subseteq A^0$ and $|A^0| + |B^0|$ is minimum. (It follows from (21) and (23) that π^0 is unique; moreover, $A^0 \subseteq A_k$ and $B^0 \subseteq B_k$.) Bellow, in Claim 2, we show that only two cases are possible: (i) $s_k \in B^0$, $t_k \in A^0$, or (ii) $s_k \notin B^0$, $t_k \notin A^0$. Let r' denote the function on T resulting in the iteration.

Case 1. $s_k \in B^0$ and $t_k \in A^0$. Put

$$r'(s) := r(s) + 1 \quad \text{for } s = s_k,$$
$$:= r(s) - 1 \quad \text{for } s = t_k,$$
$$:= r(s) \quad \text{otherwise.}$$

Case 2. $s_k \notin B^0$ and $t_k \notin A^0$. Since r is admissible and $\delta'_r > 0$, B^0 is nonempty. Choose an element $v \in B^0$ and find the set $S \subseteq Z$ such that $v \in S$, $\Delta_r(S) = 0$ and $|S|$ is minimum. Consider two cases.

(a) $S \cap N \neq \emptyset$. By Claim 1, if $S \cap \rho \neq \emptyset$ for some $\rho \in R$ then $\rho \subseteq S$. Choose a maximal (under the partial order in R) member of R, say ρ_q, that is contained in S. Put

$$r'(s) := r(s) + 2 \quad \text{for } s = v,$$
$$:= r(s) - 1 \quad \text{for } s = s_q, t_q,$$
$$:= r(s) \quad \text{otherwise.}$$

(b) $S \cap N = \emptyset$. Bellow, in Claim 3, one shows that $S \cap A^0$ is nonempty and it contains an element u such that $r(u) > 0$. Put

$$r'(s) := r(s) + 1 \quad \text{for } s = v,$$
$$:= r(s) - 1 \quad \text{for } s = u,$$
$$:= r(s) \quad \text{otherwise.}$$

In Case 1, we delete ρ_k from R, and π_k from Π_R. In Case 2(a), we delete ρ_q from R, and π_q from Π_R. In Case 2(b), we add $\rho = \{v, u\}$ to R, and π^0 to Π_R, setting for them the index $k + 1$.

Correctness of the algorithm.

First of all we observe that δ'_r is even since $f(\pi_1)$ and $|N \cup N'|$ are even, $N \subseteq B_1$ and $N' \subseteq A_1$.

Claim 2. *If* $\rho_k \cap (A^0 \cup B^0) \neq \emptyset$ *then* $\rho_k \subseteq A^0 \cup B^0$.

For otherwise $\Delta'_r(\pi^0)$ would be odd, whence $\Delta'_r(\pi^0) \neq \delta'_r$. •

Claim 3. $S \cap A^0$ *contains an element* u *such that* $r(u) > 0$.

Proof. Suppose that it is not so. Let $\overline{\pi} = (\overline{A}, \overline{B})$ be the f'-closure of π^0, and let $\overline{S}$ be the f-closure of S. Then, in view of (24) and (25), $r(s) = 0$ for all $s \in \overline{S} \cap \overline{A}$. Applying (20) to $\overline{S}$ and $\overline{\pi}$ we have

$$0 + \delta' = \Delta_r(\overline{S}) + \Delta'_r(\overline{\pi}) \leq \Delta_r(\overline{S} - (\overline{A} \cup \overline{B})) + \Delta'_r(\overline{A} - \overline{S}, \overline{B} - \overline{S})$$

(as $r(\overline{A} \cap \overline{S}) = 0$). Since r is admissible, $\Delta_r(\overline{S} - (\overline{A} \cup \overline{B})) \leq 0$, therefore $\Delta'_r(A', B') \geq \delta'$ where $A' := \overline{A} - \overline{S}$ and $B' := \overline{B} - \overline{S}$. But $B^0 - B' \neq \emptyset$ (as $v \notin B'$), whence for the f'-critical pair $(A'', B'') := (\pi^0 \wedge (A', B')) \vee (Z', \emptyset)$ we get $|A''| + |B''| < |A^0| + |B^0|$, which contradicts to the minimality of π^0. •

Now our aim is to show that r' is admissible, $\delta_{r'} \leq \delta_r$, $\delta'_{r'} = \delta'_r - 2$, and check validity of (A1)-(A3) for r'. This will imply correctness and finiteness of the algorithm. Obviously $\Delta_{r'}(Z) = \Delta'_{r'}(Z') = 0$. For $\rho := \rho_i$, $i = 1, \ldots, k$, the set S_ρ will be denoted as S_i.

Claim 4. *$\Delta_{r'}(C) \leq 0$ for any $C \subseteq T$.*

Proof. Suppose that it is not so for some C. Applying (19) to C, Z and r' we observe that $\Delta_{r'}(C \cap Z) > 0$ or $\Delta_{r'}(C \cup Z) > 0$. The latter inequality is impossible since $r'(Z) = r(Z)$ and r' coincides with r on $T - Z$. Thus one may assume that $C \subset Z$.

Let $\beta := \beta(C) := \Delta_{r'}(C) - \Delta_r(C)$. Since $\Delta_r(C) \leq 0$, $\beta \geq 1$. On the other hand, obviously, $\beta \leq 1$ in Case 1 or 2(b) and $\beta \leq 2$ in Case 2(a). Suppose that $\Delta_r(C) = 0$. If Case 1 takes place then $s_k \in C \not\ni t_k$ (as $\beta > 0$), contradicting Claim 1. If Case 2 takes place then $\beta > 0$ shows that $v \in C$ and $S - C \neq \emptyset$. But the f-tightness (for r) of S and C implies $\Delta_r(S \cap C) = 0$ (by (22)); a contradiction with the minimality of $|S|$.

Thus Case 2(a) takes place, and $\Delta_r(C) + 1 = \Delta_{r'}(C) - 1 = 0$. Then $v \in C \not\ni s_q, t_q$. Let, in addition, C be chosen so that $|\alpha(C)|$ is minimum, where $\alpha(C) := \{\rho \in R : |\rho \cap C| = 1\}$. Applying (19) to C, S, r' we get $\Delta_{r'}(C \cap S) > 0$ (since $\Delta_{r'}(C \cup S) = \Delta_r(C \cup S) \leq 0$); furthermore, $\alpha(C \cap S) \subseteq \alpha(C)$ since $\alpha(S) = \emptyset$. Thus one may assume that $C \subseteq S$.

Since $\Delta_r(C)$ is odd, $\alpha(C) \neq \emptyset$. Observe that $\alpha(S) = \emptyset$ and $C \subset S$ imply that $\rho \subset S$ for each $\rho \in \alpha(C)$. Choose a minimal (under the partial order in R) element ρ in $\alpha(C)$. Then $\rho \neq \rho_q$ and $\rho_q \cap S_\rho = \emptyset$, by (A1) and the maximality of ρ_q. This implies that $v \notin S_\rho$ (otherwise for $F := S \cap S_\rho$ we would have $v \in F \subset S$ and $\Delta_r(F) = 0$, contrary to the minimality of $|S|$). Now applying (19) to C, S_ρ, r' we get $\Delta_{r'}(C') > 0$ for $C := C \cup S_\rho$ (since $C'' := C \cap S_\rho$ does not contain v, and therefore $\Delta_{r'}(C'') \leq 0$). But $\alpha(C') \subseteq \alpha(C) - \{\rho\}$; a contradiction with the minimality of $|\alpha(C)|$. •

Claim 5. *$\Delta_{r'}(\pi) \leq \delta$ for any $\pi \in \Pi$.*

Proof. Suppose that it is not so for some $\pi = (C, D)$. By (19) for π, Z and r', $\Delta_{r'}(C \cap Z) + \Delta_{r'}(\pi') > \delta$, where $\pi' = (C', D') := \pi \vee Z$. By Claim 4, $\Delta_{r'}(C \cap Z) \leq 0$, therefore $\Delta_{r'}(\pi') > \delta$. Now applying (19) to π', Z, r', and Claim 4 to $C' \cap Z$, we conclude that $\Delta_{r'}(\pi'') > \delta$ for $\pi'' := (C'', D'') := \pi' \vee Z$. Clearly $D' \cap Z = \emptyset$, whence $Z \subseteq C''$. But r and r' coincide on $T - Z$, therefore $\Delta_{r'}(\pi'') = \Delta_r(\pi'') \leq \delta$; a contradiction. •

One can see from (A3) and the definition of r' that $\Delta'_{r'}(\pi) = \Delta'_r(\pi) - 2 = \delta' - 2$ for any $\pi \in \{\pi_1, \ldots, \pi_k\} \cup \{\pi^0\}$. Now we prove that $\Delta'_{r'}(\pi) \leq \delta' - 2$ for any $\pi \in \Pi$. We

say that $\pi = (C, D)$ is *good* if $Z' \subseteq C$, and for each $\pi' = (A', B') \in \{\pi_1, \ldots, \pi_k\} \cup \{\pi^0\}$, either $A' \subseteq C$ and $B' \subseteq D$, or $C \subseteq A'$ and $D \subseteq B'$.

Claim 6. *If $\pi = (C, D)$ is good then $\Delta'_{r'}(\pi) \leq \delta' - 2$.*

Proof. Put $I := J := \{1, \ldots, k\}$ in Case 1; $I := \{1, \ldots, k+1\}$, $J := \{1, \ldots, k\}$, $s_{k+1} := v$ in Case 2(a); and $I := J := \{1, \ldots, k+1\}$, $s_{k+1} := v$, $t_{k+1} := u$ in Case 2(b). Let α (β) be the minimum index $i \in I$ ($j \in J$) such that $s_i \in D$ (respectively, $t_j \in C$); if $s_i \notin D$ for all $i \in I$ ($t_j \notin C$ for all $j \in J$) we put $\alpha := |I| + 1$ (respectively, $\beta := |J| + 1$). Since π is good, $s_i \in D$ for all $i \geq \alpha$ and $t_j \in C$ for all $j \geq \beta$. Furthermore, $|\alpha - \beta| \leq 1$. Consider possible cases.

(i) $\alpha = |I| + 1$ and $\beta = |J| + 1$. Then $Z' \subseteq C \subseteq A^0 - \{t_{|J|}\}$ and $D \subseteq B^0 - \{s_{|I|}\} \subset B^0$. It follows from the minimality of π^0 and the fact that $|N'|$ is even that $\Delta'_r(\pi) \leq \Delta'_r(\pi^0) - 2$. Hence $\Delta'_{r'}(\pi) = \Delta'_r(\pi) \leq \delta' - 2$.

(ii) $\alpha = \beta \leq |I|$. One can see that $r'(C) - r'(D) = r(C) - r(D) - 2$, whence the result follows.

(iii) $\alpha < \beta$. Then $\Delta'_r(\pi)$ is odd, and therefore $\Delta'_r(\pi) \leq \delta' - 1$. Furthermore, one can see that $r'(C) - r'(D) < r(C) - r(D)$. Thus $\Delta'_{r'}(\pi) \leq \Delta'_r(\pi) - 1 \leq \delta' - 2$.

(iv) $\alpha > \beta$ and $\beta \leq |J|$. Then $\Delta'_r(\pi)$ is odd, and arguments are similar to those in (iii).

(v).$|I| = \alpha > \beta = |J| + 1$. Then Case 2(a) takes place, and we have $r'(C) = r(C)$ and $r'(D) - r(D) = r(v) = 2$, whence $\Delta'_{r'}(\pi) = \Delta'_r(\pi) - 2 \leq \delta' - 2$. •

Claim 7. *$\Delta'_{r'}(\pi) \leq \delta' - 2$ for any $\pi \in \Pi$.*

Proof. Suppose that it is not so for some $\pi = (C, D)$. Applying (19) to π, Z', r' we get $\Delta'_{r'}(\pi \vee Z') > \delta' - 2$ (since $\Delta'_{r'}(C') = \Delta'_r(C') \leq 0$ for $C' := C \cap Z'$), and then applying (19) to $\pi \vee Z'$, Z', r' we get $\Delta'_{r'}(\widetilde{\pi}) > \delta' - 2$ for $\widetilde{\pi} := (\pi \vee Z') \vee Z' = (Z' \cup C, D - Z')$. Thus one may assume that $\pi = (C, D)$ is so that $Z' \subseteq C$.

Next, we may assume that π is chosen so that $|\gamma(\pi)|$ is maximum, where $\gamma(\pi) := \{(A', B') \in \{\pi_1, \ldots, \pi_k, \pi^0\} : \text{either } A' \subseteq C \text{ and } B' \subseteq D, \text{ or } C \subseteq A' \text{ and } D \subseteq B'\}$. By Claim 6, the set $\{\pi_1, \ldots, \pi_k, \pi^0\} - \gamma(\pi)$ is nonempty; take an arbitrary $\pi' = (A', B')$ in it. Applying (19) to π, π', r' we get $\Delta'_{r'}(\widetilde{\pi}) > \delta' - 2$ for some $\widetilde{\pi} \in \{\pi \wedge \pi', \pi \vee \pi'\}$. It is easy to see that $|\gamma(\pi \wedge \pi')| > |\gamma(\pi)|$, whence $\widetilde{\pi} = \pi \vee \pi'$. Finally, applying (19) to $\widetilde{\pi}$, π and r' we get $\Delta'_{r'}(\pi'') > \delta' - 2$ for some $\pi'' \in \{\widetilde{\pi} \wedge \pi', \widetilde{\pi} \vee \pi'\}$. Obviously, $|\gamma(\widetilde{\pi} \wedge \pi')| > |\gamma(\pi)|$ and $|\gamma(\widetilde{\pi} \vee \pi')| > |\gamma(\pi)|$, which leads to a contradiction with the maximal choice of π. •

The following claim together with Claim 4 show that r' is admissible.

Claim 8. *$\Delta'_{r'}(C) \leq 0$ for any $C \subseteq T$.*

Proof. Suppose that it is not so for some C. Apply (19) to C, π_1, r'. Since $\Delta'_{r'}(\pi_1) = \delta' - 2$, we have $\Delta'_{r'}(D) + \Delta'_{r'}(\pi') > \delta' - 2$, where $D := A_1 \cap C$ and $\pi' := \pi_1 \vee C$. By

Claim 7, $\Delta'_{r'}(\pi') \leq \delta' - 2$, therefore, $\Delta'_{r'}(D) > 0$. But $r'(s) \leq r(s)$ for any $s \in A_1$, whence $\Delta'_{r'}(D) \leq \Delta'_r(D) \leq 0$; a contradiction. •

Finally, it is easy to check that (A1)-(A3) maintain for r', the new set R, the corresponding members in $\{\pi_1, \ldots, \pi_k, \pi^0\}$ and the corresponding sets S_ρ (in Case 2(b) we take $S_\rho := S$ for $\rho = \{v, u\}$).

This completes the proof of correctness of the algorithm. Parts (i)-(ii) in Statement 3 are obviously provided by the algorithm. The number of iterations of the algorithm is $(\delta_r + \delta'_r)/2$ for the initial function r, whence (iii) in Statement 3 follows.

Proof of Statement 4. Let r be the resulting function in the algorithm. Put $T_1 := T - Z$ and $T_2 := Z$. Let r_i be defined as in the hypotheses of the statement for these r and T_i, $i = 1, 2$. For $\widetilde{r} := r_2$ we have $\widetilde{r}(s) = r(s) + 1$ if $s \in N$, and $\widetilde{r}(s) = r(s)$ if $s \in T - N$, where N is the set of $s \in Z$ with $r(s)$ odd. Consider R and Π_R obtaining at the iteration when δ' becomes 0. An analysis of the algorithm shows that, after terminating the algorithm, the current R and Π_R continue to satisfy (A2)-(A3), and $\Delta'_r = 0$ holds for each $\pi \in \Pi_R$. The latter implies that $\Delta'_r(\pi) = 0$ for these π's. Using arguments similar to those in Claims 6 and 7 one can conclude that that $\Delta'_r(\pi') \leq 0$ for all $\pi' \in \Pi$. Similarly, one shows that $r_1 \in Q_f$. •

The algorithm needs subroutines for solving the following auxiliary problems: (i) given $v \in Z$ and f-admissible r, find the minimal f-tight set S containing v; (ii) given r, find the minimal f-critical $\pi^0 = (A^0, B^0)$ (here the minimality means that $|A^0| + |B^0|$ is as small as possible provided that $Z \subseteq A^0$); and similarly for f'. To solve (i), we apply (10)(i) to $s := v$ and each $t \in Z - \{v\}$; obviously, S is the intersection of Z and the sets $T - \{t\}$ for those t's for which the maximum in (10)(i) is 0. To solve (ii), we at first determine δ_r by applying (10)(ii) to $\widetilde{T} := T$, and then we apply (10)(ii) to $\widetilde{T} := T - \{t\}$ for each $t \in T - Z$. Let X be the intersection of those $T - \{t\}$ for which the maximum in (10)(ii) is δ'_r. Clearly X is the union of required A^0 and B^0. Now in order to find B^0, we apply (10)(ii) to $\widetilde{T} := X$ and r_t for each $t \in X$, where for a rather small $\varepsilon \geq 0$ the function r_t is defined as: $r_t(t) := r(t) + \varepsilon$ and $r_t(s) := r(s)$ for $s \in T - \{t\}$. One can see that B^0 is the set of those t's for which the maximum in (10)(ii) becomes less than δ_r.

Now we explain how to modify the algorithm in order that the number of iterations in it would be bounded by a polynomial in $|T|$. There are two differences between the modification and the above algorithm. The first one is that we fix some ordering over the elements of T, and each time when Case 2 occurs we choose as v the maximal element in the current B^0. The second one is that in Case 2(b) we transform r into $r' := r^\varepsilon$, where

$$\begin{aligned} r^\varepsilon(s) &:= r(s) + \varepsilon \quad \text{for } s = v \\ &:= r(s) - \varepsilon \quad \text{for } s = u \\ &:= r(s) \qquad \text{otherwise} \end{aligned}$$

In the original algorithm we took ε to be 1. In the modification, we determine ε to be the maximum integer such that: (i) $\varepsilon \leq r(u)$; (ii) $\delta'_{r^\varepsilon} \geq 0$; (iii) r^ε is f-admissible,

i.e. $\Delta_{r^\varepsilon}(C) \leq 0$ for all $C \subseteq T$; (iv) the minimal f-tight set S containing v for $r^{\varepsilon'}$ is the same for $\varepsilon' = 0, 1, \ldots, \varepsilon - 1$; and (v) the minimal f'-critical pair $\pi^0 = (A^0, B^0)$ for $r^{\varepsilon'}$ is the same for $\varepsilon' = 0, 1, \ldots, \varepsilon - 1$. The conditions (i)-(v) show that in Case 2(b) the iteration of the modification is equivalent to ε successive iterations of the original algorithm (in particular, it follows from (iv)-(v) that the choice of u is correct for each $\varepsilon' = 0, 1, \ldots, \varepsilon$). This proves correctness of the modification.

I. To estimate the number of iterations of the modification, consider a sequence of iterations which decrease the deficit δ'. Let $r_{(i)}, A^0_{(i)}, B^0_{(i)}$ denote the corresponding objects r, A^0, B^0 for i-th iteration in this sequence. One can see that $(A^0_{(i)}, B^0_{(i)})$ is f'-critical for $r_{(i+1)}$, whence $A^0_{(1)} \supseteq A^0_{(2)} \supseteq \ldots \supseteq A^0_{(i)} \supseteq \ldots$ and $B^0_{(1)} \supseteq B^0_{(2)} \supseteq \ldots \supseteq B^0_{(i)} \supseteq \ldots$. Suppose $i < j$ be so that $A^0_{(i)} = A^0_{(i+1)} = \ldots = A^0_{(j)} =: A^0$ and $B^0_{(i)} = B^0_{(i+1)} = \ldots = B^0_{(j)} =: B^0$. Let $v_{(i')}$ stand for the element $v \in B^0$ chosen on the i'-th iteration if Case 2 takes place and for the element $s \in B^0$ with $r_{(i')}(s)$ odd if Case 1 takes place on this iteration. The choice of v's according to the ordering in T implies $v_{(i)} = v_{(i+1)} = \ldots = v_{(j)} =: v$. One can see also that $S^0_{(i)} \supseteq S^0_{(i+1)} \supseteq \ldots \supseteq S^0_{(j)}$, where $S_{(i')}$ is the minimal f-tight set containing v for $r_{(i')}$. Let $d \in \{i, i+1, \ldots, j-1\}$, $r := r_{(d)}$ and $r' := r_{(d+1)}$. Consider possible cases.

(a) If Case 2(a) occurs on the d-th iteration then $|N_{r'} - (A^0 \cup B^0)| = |N_r - (A^0 \cup B^0)| - 2$.

(b) Suppose that Case 2(b) occurs on the d-th iteration. Then, by (i),(iii),(iv), at least one of the following is true: $r'(u) = 0$ (and hence the number of elements in A^0 with non-zero value of the current function becomes smaller); or $S_{(d)}$ strictly includes $S_{(d+1)}$.

(c) If Case 1 occurs on the d-th iteration then Case 2 will take place on the next one.

It follows from arguments in (a)-(c) that $|j - i| < 2|T|$, whence the number of iterations of the modification is $O(|T|^2)$.

II. The value ε can be determined efficiently as follows. Put $\varepsilon(1) := \min\{r(u), \delta'_r\}$, and find the value $a := \max\{\Delta_{r^{\varepsilon(1)}}(C) : C \subseteq T\}$ by using (10)(i). If $a \leq 0$, put $\varepsilon(2) := \varepsilon(1)$. If $a > 0$, and if C is the set achieving the minimum in the above expression then, obviously, $v \in C \not\ni u$. This easily implies that $r^{\varepsilon'}$ is f-admissible for $\varepsilon' \leq \varepsilon(1) - a$ $(= f(C) - r(C))$, and not for $\varepsilon' > \varepsilon(1) - a$. Put $\varepsilon(2) := \varepsilon(1) - a$.

Next, determine $b := \max\{\Delta'_{r^{\varepsilon(2)}}(\pi) : \pi \in \Pi\}$ by using (10)(ii). Let $\overline{b} := b - \delta_r + 2\varepsilon(2)$. If $\overline{b} = 0$ then, obviously, $\varepsilon(2)$ is just the required r. Suppose that $\overline{b} > 0$. Using (10)(ii), find $\pi = (A, B)$ such that $\Delta'_{r^{\varepsilon(2)}}(\pi) = b$.

Let h_π denote the inner product of χ_π and $\chi_{(\{v\},\{u\})}$. Clearly $-1 \leq h_\pi \leq 2$ and $\Delta'_{r^{\varepsilon(2)}}(\pi) = \Delta'_r + \varepsilon(2)h_\pi$, therefore $\varepsilon(3) := \lfloor \varepsilon(2) - \overline{b}/(h_\pi + 2) \rfloor$ is an upper bound for ε. Repeat the same step for $\varepsilon(3)$. If $\overline{b}' = 0$, where $\overline{b}' := b' - \delta'_r + 2\varepsilon(3)$ and $b' := \max\{\Delta'_{r^{\varepsilon(3)}}(\pi) : \pi \in \Pi\}$, then $\varepsilon(3)$ is just ε. Otherwise we find π' such that $\Delta'_{r^{\varepsilon(3)}}(\pi) = b'$, and put $\varepsilon(4) := \lfloor \varepsilon(3) - \overline{b}'/(h_{\pi'} + 2) \rfloor$. One can see that $h_{\pi'} < h_\pi$. Therefore, after at most four such steps the required ε will be found.

Maximum multicommodity flows

Let $G = (V, E)$ and $H = (T, U)$ be two undirected graphs with $T \subseteq V$, and let $c :\to \mathbf{Z}_+$ be a function (of *capacities* of edges). We refer to T and H as the set of *terminals* and the *commodity graph* respectively; the edges of H are specified the pairs of terminals that are allowed to connect by flows in G. Without loss of generality we assume that H has no isolated vertices. Let $\mathcal{P} = \mathcal{P}(G, H)$ denote the set of simple paths in G connecting pairs $\{s, t\} \in U$. A (c-admissible) *multi*(commodity)*flow* for G, H, c is a nonnegative rational-valued function $\lambda : \mathcal{P} \to \mathbf{Q}_+$ satisfying the capacity constraints:

$$\sum(\lambda(P) : P \in \mathcal{P},\ e \in P) \leq c(e) \text{ for all } e \in E. \tag{26}$$

The *total value* $v(\lambda)$ of λ is $\sum(\lambda(P) : P \in \mathcal{P})$. We consider the *maximum multiflow problem*:

(27) *given G, H, c, find a multiflow λ whose total value $v(\lambda)$ is maximum.*

This maximum is denoted by $v(G, H, c)$. In [KL] a large class of commodity graphs H was found for which (27) has a half-integral optimal solution; moreover, for such H's there is a special minimax relation. A subset $S \subseteq T$ is said to be *stable* if $\{s, t\} \notin U$ for any $s, t \in T$. A maximal (with respect to inclusion) stable set is called an *anticlique*; let $\mathcal{A}(H)$ denote the set of all the anticliques in H. We say that H is *anticlique-bipartite* if

(28) there is a partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of $\mathcal{A}(H)$ such that each $\mathcal{A}_i$ consists of pairwise disjoint sets

(this is equivalent to saying that H is the complement to the line graph of a bipartite graph).

Theorem 5. [KL] *Let H be anticlique-bipartite. Then: (i) (27) has a half-integral optimal solution λ; moreover, it has an integral optimal solution if c is inner eulerian; and (ii):*

$$v(G, H, c) = \frac{1}{2} \min\{\sum(c(\delta X_A) : A \in \mathcal{A}(H))\}, \tag{29}$$

where the minimum is taken over all the collections $\{X_A \subset V : A \in \mathcal{A}(H)\}$ such that $X_A \cap T \subseteq A$ for each $A \in \mathcal{A}(H)$, and $T \subseteq \bigcup(X_A : A \in \mathcal{A}(H))$.

[Here for $X \subseteq V$, $\delta X = \delta_G X$ denotes the set of edges of G with one end in X and the other in $V - X$ (a "cut" in G). We say that c is *inner eulerian* if $c(\delta X)$

$(= \sum(c(e) : e \in \delta X))$ is even for every $X \subseteq V - T$.] To illustrate this theorem, consider two known special cases. For simplicity we assume that $c(e) = 1$ for all $e \in E$; then the inner eulerianness means that each vertex in $V - T$ has even valency in G.

1) H is a complete graph. Then $\mathcal{A}$ consists of all the one-element sets of T, and (28) is satisfied with $\mathcal{A}_1 := \mathcal{A}(H)$. Theorem 5 turns into that stated independently by Cherkassky [Ch] and Lovász [Lov] (the Cherkassky's proof was exhibited also in [ADK]): if valency of each vertex in $V - T$ is even then the maximum number of edge-disjoint paths in G connecting pairs of distinct terminals is equal to $\frac{1}{2}\sum(|\rho_s| : s \in T)$, where ρ_s is the munimum cardinality of a cut separating s from $T - \{s\}$ (the fraction version of this theorem originally appeared in [Ku]).

2) U consists of two edges $\{s_i, t_i\}$, $i = 1, 2$. Then $\mathcal{A}(H)$ consists of $\{s_1, s_2\}, \{t_1, t_2\}$, $\{s_1, t_2\}, \{s_2, t_1\}$, and (28) holds for $\mathcal{A}_1 := \{\{s_1, s_2\}, \{t_1, t_2\}\}$. From Theorem 5 one can derive the following result stated by Rotschild and Whinston [RW], which strengthens the Hu's two-commodity flow theorem: if c is inner eulerian then the maximum number of edge-disjoint paths in G, each connecting s_i and t_i, is equal to the minimum cardinality of a cut separating both pairs $\{s_i, t_i\}$, $i = 1, 2$.

Theorem 5 describes almost completely the class of commodity graphs H for which (27) has a half-integral optimal solution (it was mentioned in [Ka4] that besides the anticlique-bipartite graphs there is the only graph with such a property, namely, the union of triangle and edge). Another fact concerning the anticlique-bipartite graphs is that they constitute a substantial part of the set of commodity graphs which provide a special duality relation: roughly speaking, for H fixed and any G and c, $v(G, H, c)$ can be expressed as a nonnegative linear combination of the capacities of cuts if and only if each vertex of H belongs to at most two anticliques ("if" was stated in [KL], and "only if" in [KP]).

The original proof of Theorem 5 exploited complex augmenting path techniques (it was described in details in [Ka1] and [Lom]). A shorter, but still rather complicated, proof of the theorem, based on a splitting-off method, was given in [Ka3] (note that this proof provided a strongly polynomial solution algorithm). Recently, in [FKS], a new approach to solve (27) with the anticlique-bipartite H's has been developed; as a result, a proof of Theorem 5 which is viewed more transparent than those known earlier was found. It is based on a reduction the problem in question to a certain maximization problem and the so-called *multiflow locking problem*. We now outline the idea of that approach and then we demonstrate how above results on compatible greedy-polyhedra are applied.

An important role in [FKS] plays the *node-demand* problem:

(30) given a function $r : T \to \mathbf{Q}_+$ (of *demands*), find a c-admissible multiflow $\lambda : \mathcal{P}(G, H) \to \mathbf{Q}_+$ satisfying

$$\sum(\lambda(P) : P \in \mathcal{P}(G, H),\ s \in e(P)) = r(s) \text{ for each } s \in T,$$

where $e(P)$ denotes the pair of end vertices of a path P.

We call r *feasible* if (30) has a solution λ; clearly $v(\lambda) = r(T)/2$. One way to solve

(27) is to divide it into two phases: (i) finding a feasible r with $r(T)$ $(= \sum(r(s) : s \in T)$ maximum, and (ii) finding a solution of (30) for r. Leaving for a time the question how to solve (i), we now attract attention to (30). An obvious necessary condition for solvability of (30) (for arbitrary H) is:

(31) $\quad r(X \cap A) - r((X \cap T) - A) \leq c(\delta X)$ for any $X \subseteq V$ and $A \in \mathcal{A}(H)$.

(The inequality in (31) is a natural generalization of the "offer-supply" one for the multiterminal single-commodity flow problem, see [FF]). It turns out that (31) is sufficient for the anticlique-bipartite H's (actually one can prove that, in terms of H's, (31) is sufficient if and only if each terminal belongs to at most two anticliques of H, see [FKS]). To show sufficiency, consider the multiflow *locking problem* introduced in [KL]. This is:

(32) given $G' = (V', E')$, $T' \subseteq V'$, $c' : E' \to \mathbf{Q}_+$, and a collection $\mathcal{A}' \subseteq 2^{T'}$ of subsets of T', find a c'-admissible multiflow $\lambda' : \mathcal{P}(G', T') \to \mathbf{Q}_+$ such that for each $A' \in \mathcal{A}'$, the total value of flows connecting A' and $T' - A'$ is equal to the minimum capacity of a cut separating A' and $T' - A'$, that is,

$$\sum(\lambda'(P) : P \in \mathcal{P}(G', T'), |e(P) \cap A'| = 1) = \min\{c'(\delta_{G'} X) : X \subseteq V', X \cap T' = A'\};$$

here $\mathcal{P}(G', T')$ denotes the set of simple paths in G' connecting two distinct vertices in T'.

Theorem 6. [KL] *Let $\mathcal{A}'$ contain no three pairwise crossing sets. Then (32) is solvable. Moreover, if c' is inner eulerian then (32) has an integral solution λ'.*

(Recall that sets $A, B \subseteq T'$ are called *crossing* if none of $A \cap B$, $A - B$, $B - A$, $T' - (A \cup B)$ is empty). In contrast to Theorem 5, Theorem 6 has a relatively simple proof (see [Ka2, pp.460-463] for a short proof and a strongly polynomial algorithm).

In what follows we consider G, H, c with H as in (28). For a function r on T let $L(r)$ stand for the locking problem with $G' = (V', E'), T', c' = c_r, \mathcal{A}$, where: (i) G' is obtained from G by adding for each $s \in T$ a new vertex s' and the edge $\{s, s'\}$; (ii) $T' := \{s' : s \in T\}$; (iii) for $e \in E'$, $c_r(e)$ is $c(e)$ if $e \in E$, and $r(s)$ if $e = \{s, s'\}$; (iv) $\mathcal{A}' := \{A' : A \in \mathcal{A}(H)\}$, where A' is the set $\{s' : s \in A\}$. In view of (28), $\mathcal{A}'$ contains no three pairwise crossing sets. It is not difficult to show that (31) holds if and only if

(33) $\quad \min\{c_r(\delta_{G'} X) : X \subseteq V',\ X \cap T' = A'\} = c_r(\delta_{G'} A') = r(A')$ for all $A' \in \mathcal{A}'$.

Let λ' be a solution of $L(r)$. Then the equalities in (32) easily imply that each path $P \in \mathcal{P}(G', T')$ with $\lambda'(P) > 0$ connects $s', t' \in T'$ such that $\{s, t\} \in U$, and that

the sum of $\lambda'(P)$'s for P with one end in $s' \in T'$ is just $r(s)$. This implies that the multiflow in G naturally determined by λ' gives a solution of (30). Moreover, if c is inner eulerian, r is integral, and the "parity" condition that $r(s) \equiv c(\delta_G\{s\}) \pmod 2$ for all $s \in T$ holds, then c_r is inner eulerian (concerning T'), and therefore, by Theorem 6, (30) has an integral solution provided that (31) is valid.

Thus (27) can be solved by reducing it to the problem

(34) $$\text{maximize } r(T) \text{ subject to (31)}$$

and then to $L(r)$. Now and later it will be convenient for us to assume that c and H satisfy two additional conditions:

(35) c is totally eulerian, i.e. $c(\delta X)$ is even for all $X \subseteq V$;

(36) $\bigcup(A : A \in \mathcal{A}_i) = T$ for each $i = 1, 2$.

(These conditions lead to no loss of generality, because there is a simple reduction of (27)-(28) with inner eulerian c to that with $\widetilde{G}, \widetilde{H}, \widetilde{c}$ satisfying (35)-(36).) Below we shall explain that the set of vectors $r \in \mathbf{R}_+^T$ satisfying (31) forms the intersection of two compatible greedy-polyhedra described by even-valued submodular functions f_1 and f_2. However, as we have mentioned earlier, the evenness of f_1 and f_2 does not guaranteed, in general, the existence of even-valued solution for the coresponding maximization problem. Figure 2 illustrates an example from [FKS] exhibiting such a behaviour for (34); here $T = \{s, t, p, q\}$, $c(e) = 2$ for all edges e in G, and (34) has the unique solution $r(t) = r(p) = 2$ and $r(s) = r(q) = 1$.

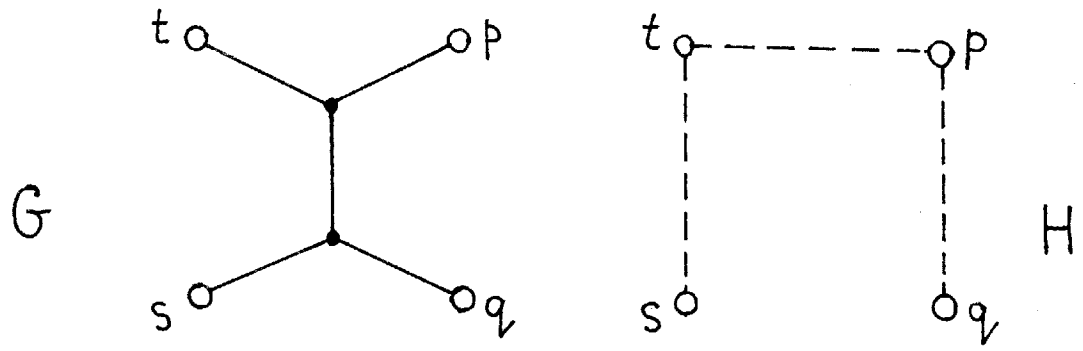


Fig. 2

Therefore, the above-mentioned parity condition can be not satisfied; whence application of Theorem 6 gives us only the weaker version of (i) in Theorem 5: (27) with inner eulerian c has a *half-integral* optimal solution.

To overcome this obstacle, a maximization problem slightly different from (34) was introduced in [FKS]. This is:

(37) find $r, q_1, q_2 : T \to \mathbf{Q}_+$ maximizing the objective function $r(T)$ provided that for $i = 1, 2$ and $r_i := r + q_i$:

$$r_i(X \cap A) - r_i((X \cap T) - A) \leq c(\delta X) \text{ for any } X \subseteq V \text{ and } A \in \mathcal{A}_i.$$

Note that r itself can be infeasible, i.e. (31) can be violated for it. However, the following important property holds.

Statement 7. [FKS] *Let r, q_1, q_2 satisfy the inequalities in (37). Let $r' := r + q_1 + q_2$. Let λ' be a solution of $L(r')$. Let $\mathcal{P}$ be the set of paths in G' connecting $s', t' \in T'$ with $\{s, t\} \in U$. Then $d := \sum(\lambda'(P) : P \in \mathcal{P})$ is at least $r(T)/2$.*

The proof of this statement deserves to be repeated here. For $A \in \mathcal{A}(H)$ let A' denote the set $\{s' : s \in A\}$. Consider $i \in \{1, 2\}$ and $A \in \mathcal{A}_i$. Let $\mathcal{P}_A$ be the set of paths in $\mathcal{P}(G', T')$ with exactly one end in A'. Put $d_A := \sum(\lambda'(P) : P \in \mathcal{P}_A)$. From inequalities in (37) it easily follows that the minimum $c_{r'}$-capacity of a cut in G' separating A' and $T' - A'$ is at least $r_i(A)$. Hence, $d_A \geq r_i(A)$ (since λ' is a solution of $L(r')$), and we get

$$r'(A) - d_A \leq r'(A) - r_i(A) = q_{3-i}(A). \tag{38}$$

For $i = 1, 2$ let $\mathcal{P}_i$ be the set of simple paths in G' connecting $s', t' \in T'$ such that s and t are contained in different sets of $\mathcal{A}_i$, and let $d_i := \sum(\lambda'(P) : P \in \mathcal{P}_i)$. In view of (28) and (36), $2d_i = \sum(d_A : A \in \mathcal{A}_i)$. Furthermore, $\mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$. Clearly each path $P \in \mathcal{P}_1 - \mathcal{P}$ has both its ends in A' for some $A \in \mathcal{A}_2$, and it uses $2\lambda'(P)$ units of the total capacity of the edges $\{s, s'\}$, $s \in A$. Therefore,

$$2d_1 - 2d = 2\sum(\lambda'(P) : P \in \mathcal{P}_1 - \mathcal{P}) \leq \sum(r'(A) - d_A : A \in \mathcal{A}_2) = r'(T) - 2d_2, \tag{39}$$

taking into account that $\bigcup(A \in \mathcal{A}_2) = T$, by (36). From (38) and (39) we have

$$\begin{aligned} 2d \geq 2d_1 + 2d_2 - r'(T) &\geq (r'(T) - q_2(T)) + (r'(T) - q_1(T)) - r'(T) \\ &= r'(T) - q_1(T) - q_2(T) = r(T). \quad \bullet \end{aligned}$$

Thus, if r, q_1, q_2 give a solution of (37) and λ' is a solution of $L(r')$ then λ' induces a multiflow λ for G, H, c of value $r(T)/2$. Note that for some $s \in T$ the numbers $\widetilde{r}(s) := \sum(\lambda(P) : P \in \mathcal{P}(G, H), s \in e(P))$ can differ from $r(s)$, however, $\widetilde{r}(T) = r(T)$ is valid, whence $\widetilde{r}$ is an optimal solution of (34) (taking into account that (37) turns into (34) if we set $q_1 := q_2 := 0$).

Basing on polymatroid arguments, it was proved in [FKS] that (37) has an optimal solution with all r, q_1, q_2 even. This nice property immediately implies (i) in Theorem

5. Moreover, a strongly polynomial combinatorial algorithm for finding such r, q_1, q_2 was developed there. Finally, it is shown that (ii) in Theorem 5 can be derived from the Edmonds' minimax formula (7) for the intersection of two polymatroids.

Now we present another way to solve (37) that is a direct consequence of the algorithm of Section 3 and Statement 4. It gives r, q_1, q_2 with $r' := r + q_1 + q_2$ even everywhere (in contrast to the approach in [FKS], r is not necessary even-valued but r satisfies (31)). Thus, by above arguments, such an r' leads to constructing an integral solution of (27).

First of all we associate the inequalities in (31) with a pair of compatible greedy-polyhedra. More precisely, for $S \subseteq T$ put

$$h(S) := \min\{c(\delta X) : X \subseteq V,\ X \cap T = S\}; \tag{40}$$

and for $\pi = (A, B)$ let, by definition, $h(\pi) := h(A \cup B)$. For $\pi = (A, B) \in \Pi$ define

$$f(\pi) := \min\{h(A_1 \cup B_1) + \ldots + h(A_k \cup B_k)\}, \tag{41}$$

where the minimum is taken over all the collections $\{(A_1, B_1), \ldots, (A_k, B_k)\} \subseteq \Pi$ such that:

(i) the sets A_i's and B_j's are pairwise disjoint

(ii) $A_i \subseteq \widetilde{A}$ for some $\widetilde{A} \in \mathcal{A}_1$, $i = 1, \ldots, k$;

(iii) $\cup(A_i : i = 1, \ldots, k) \supseteq A$ and $\cup(B_i : i = 1, \ldots, k) \subseteq B$.

It is easy to see that f satisfies the monotone property: $f(\pi) \leq f(\pi')$ whenever $\pi \preceq \pi'$.

Claim. *f is submodular.*

Proof. We use obvious inequalities for $S, S' \subseteq T$:

$$h(S) + h(S') \geq h(S \cap S') + h(S \cup S') \tag{42}$$

$$h(S) + h(S') \geq h(S - S') + h(S' - S) \tag{43}$$

Consider $\pi = (A, B), \pi' = (C, D) \in \Pi$, and let $\mathcal{F} = \{\pi_1 = (A_1, B_1), \ldots, \pi_k = (A_k, B_k)\}$ (respectively, $\mathcal{F}' = \{\pi'_1 = (C_1, D_1), \ldots, \pi'_n = (C_n, D_n)\}$) be a collection of pairs achieving the minimum in (41) for π (respectively, for π'). We prove (1) for π, π' by induction on $\omega = \omega(\pi, \pi', \mathcal{F}, \mathcal{F}')$ to be

$$2|T|^2(|A| + |B| + |C| + |D|)$$
$$+|A_1| + \ldots + |A_k| + |B_1| + \ldots + |B_k| + \ldots + |C_1| + \ldots + |C_n| + |D_1| + \ldots + |D_n|.$$

Without loss of generality one may assume that each pair $(A_i, B_i) \in \mathcal{F}$ with $A_i \subseteq \widetilde{A} \in \mathcal{A}_1$ satisfies $A_i \neq \emptyset$ and $B_i \cap \widetilde{A} = \emptyset$; and similarly for $\mathcal{F}'$. In addition, we may assume that for $i \neq j$, A_i and A_j belong to different sets in $\mathcal{A}_1$ (and similarly for $\mathcal{F}'$). For otherwise we could replace π_i, π_j in $\mathcal{F}$ by $\overline{\pi} = (A_i \cup A_j, B_i \cup B_j))$; this gives a new collection for π which obviously satisfies (i)-(iii) in (41) and achieves the minimum in (41) because of $h(\pi_i) + h(\pi_j) \geq h(\overline{\pi})$, by (42).

Suppose that $B_i \cap (C_j \cup D_j) \neq \emptyset$ for some i, j such that A_i and C_j belong to different sets in $\mathcal{A}_1$; then $A_i \cap C_j = \emptyset$. Let $\overline{A}_i := A_i - D_j$, $\overline{B}_i := B_i - (C_j \cup D_j)$, $\overline{C}_j := C_j - B_i$, $\overline{D}_j := D_j - (A_i \cup B_i)$, and let $\overline{\pi}_i := (\overline{A}_i, \overline{B}_i)$, $\overline{\pi}'_j := (\overline{C}_j, \overline{D}_j)$. Form the pairs $\overline{\pi} = (\overline{A}, \overline{B})$ to be $(A - (A_i \cap D_j), B - (B_i \cap C_j))$ and $\overline{\pi}' = (\overline{C}, \overline{D})$ to be $(C - (C_j \cap B_i), D - (D_j \cap A_i))$. Obviously, the collection $\overline{\mathcal{F}} := (\mathcal{F} - \{\pi_i\}) \cup \{\overline{\pi}_i\}$ satisfies (i)-(iii) in (41) for $\overline{\pi}$, and $\overline{\mathcal{F}}' := (\mathcal{F}' - \{\pi'_j\}) \cup \{\overline{\pi}'_j\}$ does so for $\overline{\pi}'$. Since $h(\pi_i) + h(\pi'_j) \geq h(\overline{\pi}_i) + h(\overline{\pi}'_j)$ (by (43)), we have

$$f(\pi) + f(\pi') \geq \sum (h(\widetilde{\pi}) : \widetilde{\pi} \in \overline{\mathcal{F}} \cup \overline{\mathcal{F}}') \geq f(\overline{\pi}) + f(\overline{\pi}'). \tag{44}$$

If $\pi = \overline{\pi}$ and $\pi' = \overline{\pi}'$ then, by (44), $\mathcal{F}$ and $\mathcal{F}'$ give the minimum in (41) for π and π', respectively; furthermore, $\omega(\pi, \pi', \overline{\mathcal{F}}, \overline{\mathcal{F}}') < \omega(\pi, \pi', \mathcal{F}, \mathcal{F}')$. But if $\{\pi, \pi'\} \neq \{\overline{\pi}, \overline{\pi}'\}$ then $|\overline{A}| + |\overline{B}| + |\overline{C}| + |\overline{D}| < |A| + |B| + |C| + |D|$, therefore, the value of ω for $\overline{\pi}, \overline{\pi}'$ and arbitrary collections as in (41) is again less than $\omega(\pi, \pi', \mathcal{F}, \mathcal{F}')$. So, by induction, $f(\overline{\pi}) + f(\overline{\pi}') \geq f(\overline{\pi} \wedge \overline{\pi}') + f(\overline{\pi} \vee \overline{\pi}')$. Now (44) and the obvious fact that $\pi \wedge \pi' = \overline{\pi} \wedge \overline{\pi}'$ and $\pi \vee \pi' = \overline{\pi} \vee \overline{\pi}'$ imply (1) for π and π'.

Thus one may assume that $(A_i \cup B_i) \cap (C_j \cup D_j) = \emptyset$ for any i, j such that A_i and C_j are in different sets of $\mathcal{A}_1$. Renumber elements in $\mathcal{F}'$ and add trivial pairs $\{\emptyset, \emptyset\}$ to $\mathcal{F}$ or $\mathcal{F}'$ in such a way that $|\mathcal{F}| = |\mathcal{F}'| =: p$, and that for $i = 1, \ldots, p$, A_i and C_i are contained in the same set of $\mathcal{A}_1$. By (42), $h(A_i \cup B_i) + h(C_i \cup D_i) \geq h(X_i \cup Y_i) + h(Z_i \cup W_i)$, where $X_i := (A_i \cap C_i) \cup (A_i \cap D_i) \cup (C_i \cap B_i)$, $Y_i := B_i \cap D_i$, $Z_i := A_i \cup C_i$, $W_i := (B_i \cup D_i) - (A_i \cup C_i)$. Now we have

$$f(\pi) + f(\pi') = \sum_{i=1}^{p} (h(\pi_i) + h(\pi'_i))$$

$$\geq \sum_{i=1}^{p} h(X_i \cup Y_i) + \sum_{i=1}^{p} (h(Z_i \cup W_i) \geq f(\pi \wedge \pi') + f(\pi \vee \pi'),$$

since $(X_1, Y_1), \ldots, (X_p, Y_p)$ satisfy (i)-(iii) in (41) for $\pi \wedge \pi'$, and $(Z_1, W_1), \ldots, (Z_p, W_p)$ do it for $\pi \vee \pi'$. •

To show that f and f' are compatible, consider an f-closed $\pi = (A, B)$ and an f'-closed $\pi' = (C, \emptyset)$. Let $(A_1, B_1), \ldots, (A_k, B_k)$ give the minimum in (41) for π, and $(C_1, \emptyset), \ldots, (C_n, \emptyset)$ do so for π' (concerning f' and $\mathcal{A}_2$). From the closedness of π and π' it follows that $\cup A_i = A$, $\cup B_i = B$, $\cup C_i = C$. Using (43), we have

$$f(\pi) + f'(\pi') = \sum_{i=1}^{k} h(A_i \cup B_i) + \sum_{j=1}^{n} h(C_j)$$

$$\geq h((A_1 \cup B_1) - C_1) + h(C_1 - (A_1 \cup B_1) + \sum_{i=2}^{k} h(A_i \cup B_i) + \sum_{j=2}^{n} h(C_j) \geq \dots$$

$$\geq h((A_1 \cup B_1) - \cup_{j=1}^{n} C_j) + \sum_{j=1}^{n} h(C_j - (A_1 \cup B_1)) + \sum_{i=2}^{k} h(A_i \cup B_i) \geq \dots$$

$$\geq \sum_{i=1}^{k} h((A_i \cup B_i) - \cup_{j=1}^{n} C_j) + \sum_{j=1}^{n} h(C_j - \cup_{i=1}^{k} (A_i \cup B_i))$$

$$\geq f(A - C, B - C) + f'(C - (A \cup B), \emptyset),$$

as required.

Claim 2. *The solution set D (in nonnegative reals) of the inequalities in (31) for all $X \subseteq V$ and $A \in \mathcal{A}_1$ coincides with $Q_f \cap \mathbf{R}_+^T$.*

Proof. $D \supseteq Q_f \cap \mathbf{R}_+^T$ easily follows by comparing (31) with (40) and (41). For the converse inclusion, consider $\pi = (A, B)$ and pairs $(A_1, B_1), \dots, (A_k, B_k)$ achiving the minimum in (41). Then for $r \in \mathbf{R}_+^T$ the inequality $r(A) - r(B) - f(\pi) \leq 0$ in the description of Q_f is a corollary of some inequalities in (31) since each A_i is contained in an anticlique belonging to $\mathcal{A}_1$ and, in view of (i)-(iii) in (41) and nonnegativity of r,

$$r(A) - r(B) - f(\pi) \leq \sum_{i=1}^{k} (r(A_i) - r(B_i) - h(A_i \cup B_i)). \quad \bullet$$

By Claim 2, (37) turns into the problem $(*)$ of finding $r, q_1, q_2 \in \mathbf{R}_+^T$ with $r(T)$ maximum provided that $r_i(A) - r_i(B) \leq f_i(A, B)$ for any $(A, B) \in \Pi$, where $r_i := r + q_i$, $i = 1, 2$. Note that f_1 and f_2 take even values. Now it follows immediately from Statement 4 that there exists an optimal solution r, q_1, q_2 for $(*)$ such that $r' := r + q_1 + q_2$ is even everywhere (one should put $q_i := r_i - r$). By Statement 7, $L(r')$ has an integral solution, which, by arguments above, determines an integral optimal solution of (27). This proves (i) in Theorem 5. Moreover, the method described in Section 3 together with the above-mentioned algorithm for the locking problem are implemented to a strongly polynomial algorithm of finding such a solution. The only thing that should be explained is how to design efficiently procedures for (10)(i),(ii). One can see that, given z and $\widetilde{T}$, (10)(ii) is reduced to finding for each $\widetilde{A} \in \mathcal{A}_1$ the minimum of $h(C \cup D) - z(C) + z(D)$ over all C and D such that $\widetilde{T} \cap \widetilde{A} \subseteq C \subseteq \widetilde{A}$ and $D \subseteq \widetilde{T} - \widetilde{A}$. This can be done by determining a minimum cut separating $\widetilde{A}'$ from $T' - \widetilde{A}'$ in the graph G' whose edges e have capacity ∞ if $e = \{s, s'\}$ for $s \in (\widetilde{T} \cap \widetilde{A}) \cup (T - (\widetilde{T} \cup \widetilde{A}))$,

and $c_z(e)$ for the remaining edges e in G'; here S' denotes $\{s' : s \in S\}$. (10)(i) can be solved in a similar way.

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References

[ADK] Adelson-Velsky, G.M., Dinits, E.A., Karzanov, A.V.: Flow Algorithms. Nauka, Moscow, 1975 (Russian).

[BC] Bixbi, R.E., and Cunningham, W.H.: Matroid optimization and algorithms. Report No. 90658-OR, Forschungsinstitut für Diskrete Mathematik, Bonn, 1990.

[Ch] Cherkassky, B.V.: A solution of a problem on multicommodity flows in a network. Ekonomika i Matematicheskie Metody 13 (1) (1977) 143-151 (Russian).

[DW] Dunstan, F.D.J., and Welsh, D.J.A.: A greedy algorithm for solving a certain class of linear programmes. Math. Programming 5 (1973) 338-353.

[Ed] Edmonds, J.: Submodular functions, matroids, and certain polyhedra. In: Guy, R.K., Hanani, H., Sauer, N., Schönheim, J., eds.: Combinatorial Structures and their Applications. Gordon and Breach, NY, 1970, pp. 69-87.

[FF] Ford, L.R., Fulkerson, D.R.: Flows in networks. Princeton Univ. Press, Princeton, NJ, 1962.

[FKS] Frank, A., Karzanov, A.V., Sebő, A.: to appear

[GLS] Grötschel, M., Lovász, L., Schrijver, A.: The ellipsoid method and its consequences in combinatorial optimization. Combinatorica 1 (1981) 169-197.

[Ka1] Karzanov, A.V.: Combinatorial methods to solve cut-determined multiflow problems. In: Combinatorial Methods for Flow Problems. Inst. for System Studies, Moscow, 1979, iss. 3, pp. 6-69 (Russian).

[Ka2] Karzanov, A.V.: A generalized MFMC-prorerty and multicommodity cut problems. In: Hajnal, L., Lovász, L., Sos, V.T., eds.: Finite and Infinite Sets (Proc. 6th Hungar. Comb. Coll. (Eger,1981)), vol. 2, North-Holland, Amsterdam, 1984, 443-486.

[Ka3] Karzanov, A.V.: On multicommodity flow problems with integer-valued optimal solutions. Dokl. Akad. Nauk SSSR 280 (4) (1985) 789-792 (Russian) (English transl.: Soviet Math. Dokl. 31 (1) (1985) 151-154).

[Ka4] Karzanov, A.V.: Polyhedra related to undirected multicommodity flows. Linear Algebra and Appl. 114-115 (1989) 293-328.

[KC] Kabadi, S., and Chandrasekaran, R.: On totally dual integral systems. Discrete Applied Math. 26 (1990) 87-104.

[KL] Karzanov, A.V., Lomonosov, M.V.: Systems of flows in undirected networks. In: Larichev, O.A., ed.: Mathematical Programming etc. Inst. for System Studies, Moscow, 1978, iss. 1, pp. 59-66 (Russian).

[KP] Karzanov, A.V., Pevzner, P.A.: A characterization of the class of cut-non-determined maximum multicommodity flow problems. In: Combinatorial Methods for

Flow Problems. Inst. for System Studies, Moscow, 1979, iss. 3, pp. 70-81 (Russian).

[Ku] Kupershtokh, V.L.: On one generalization of Ford-Fulkerson's theorem to multi-terminal networks. Kibernetika (Kiev), 1979, No. 3 (Russian) (English translation in: Cybernetics 3(1973) 494-502).

[Lom] Lomonosov, M.V.: Combinatorial approaches to multiflow problems. Discrete Applied Math. 11 (1) (1985) 1-94.

[Lov] Lovász, L.: On some connectivity properties of Eulerian graphs. Acta Math. Akad. Sci. Hung. 28 (1976) 129-138.

[Na] Nakamura, M.: A characterization of greedy sets: Universal polymatroids (I). Scientific Papers of the College of Arts and Sciences (the University of Tokyo) 38 (2) (1988) 155-167.

[Qi] Qi, L.: Directed submodularity, ditroids, and directed submodular flows. Math. Programming 42 (1988) 579-599.

[RW] Rothschild, B., Whinston, A.: Fealibility of two-commodity network flows. Oper. Res. 14 (1966) 1121-1129.

[Se] Sebő, A.: Unpublished manuscript, 1986.