MINIMUM (2,r)-METRICS, GRAPH PARTITIONS AND MULTIFLOWS

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Minimum $(2,r)$-metrics, graph partitions and multiflows *

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Abstract. We deal with a graph $G = (V, E)$, a subset $T \subseteq V$, a function $c : E \to \mathbb{Z}_+$ and a symmetric function $\mu : T \times T \to \mathbb{Z}_+$. Suppose that we wish to find a partition Π of V into $|T|$ sets X_t, $t \in T$, such that each X_t contains t and the sum of numbers $\mu(st)c(e)$ among the edges e of G connecting sets X_s and X_t in Π is as small as possible. When μ is all-unit, this is a version of the minimum graph partition problem (referred also as the minimum multi-cut problem) and it is known to be strongly NP-hard. On the other hand, it has been shown that the problem is polynomially solvable if μ is the distance function of the complete bipartite graph with parts of 2 and $r = |T| - 2$ nodes. This is just the minimum $(2, r)$-metric problem considered in the present paper.

We prove that the multicommodity flow problem which is dual of the minimum $(2, r)$-metric problem has an integer optimal solution whenever c is *inner Eulerian*, i.e., each cut in G not separating T has an even capacity with respect to c. Moreover, such a solution can be found in strongly polynomial time.

Also we consider structural and polyhedral aspects concerning the general case of the metric minimization problem and give some results on metrics μ for which the minimum value of the objective in this problem is the same as in its fractional relaxation.

Key words: metric, cut, multicommodity flow, minimum multi-cut problem.

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1. Introduction

Let $G = (V, E)$ be a graph with nonnegative integer weights $c(e)$ of edges $e \in E$, and $T \subseteq V$ a subset of nodes. By the *minimum graph partition problem* (also referred as the *minimum multi-cut problem*) we mean the problem of finding a partition Π of V such that each member of Π contains exactly one element of T and the sum of weights of edges connecting different sets in Π is as small as possible. It is known that this problem is NP-hard even if c takes value 1 on all edges [3]; moreover, it remains intractable if $|T| = 3$. On the other hand, its special case with $|T| = 2$ is efficiently solvable as being the minimum cut problem for which plenty of polynomial algorithms are known.

Now suppose that we are interested even in a more general problem in which a graph Γ with node set T is given and it is required to find a partition Π as above minimizing the sum of $c(e)d^{\Gamma}(uv)$'s over the edges e connecting sets $S, S' \in \Pi$, where u (v) is the element of $T \cap S$ (resp. $T \cap S'$) and $d^{\Gamma}(uv)$ is the distance in Γ between u and v. In other terms, we wish to find a mapping γ of V to T such that $\gamma(v) = v$ for each $v \in T$ and $\sum(c(xy)d^{\Gamma}(\gamma(x)\gamma(y)) : xy \in E)$ is minimized. It turns out that such a problem is solvable in polynomial time for some interesting special cases of Γ; in particular, if Γ is the complete bipartite graph $K_{2,r}$ with parts of 2 and r nodes ($|T| = 2 + r$) [5]. In the latter case the problem is referred as the *minimum* $(2, r)$*-metric problem.*

We start with terminology and notations. By a *metric* on a finite set V we mean a nonnegative real-valued function $m : V \times V \to \mathbf{R}_+$ satisfying

(i) $m(x, x) = 0$ for $x \in V$;

(ii) $m(x, y) = m(y, x)$ for $x, y \in V$ (*symmetry*);

(iii) $m(x, y) + m(y, z) \geq m(x, z)$ for $x, y, z \in V$ (*triangle inequalities*).

The value of m on a pair (x, y) is called the *distance* from x to y. Note that we allow zero distances between different elements (i.e., in fact we deal with *semi-metrics*). Because of (i) and (ii) it is convenient to assume that m is given on the set of edges of the complete undirected graph $K_V = (V, E_V)$ on V, using notation $m(e)$ or $m(xy)$ for $e = xy \in E_V$. The set of metrics on V, denoted by $\mathcal{M}_V$, forms a (convex) polyhedral cone in the $\binom{|V|}{2}$-dimensional euclidean space $\mathbf{R}^{E_V}$ whose coordinates are indexed by edges of K_V.

Let μ be a metric on a subset $T \subseteq V$. A metric m on V is called a *0-extension of* μ *to* V if there is a mapping γ of V to T such that γ is identical on T ($\gamma(v) = v$ for $v \in T$) and for $x, y \in V$, $m(x, y)$ equals $\mu(\gamma(x), \gamma(y))$. In particular, m coincides with μ on T. Such a γ determines the partition $\Pi = \Pi_\gamma$ of V into sets $\gamma^{-1}(v)$ ($v \in T$); so m is zero within each member of Π and a constant (depending on S, S') between elements of different members S, S' of Π.

A sort of metrics comes up when we are given a connected graph $\Gamma = (T, W)$ with $T \subseteq V$. By a 0-extension of Γ we mean a 0-extension m of its distance function denoted by d^{Γ}. Two special cases are important for us. [Here, for a graph $G = (V, E)$ and a subset $X \subseteq V$, $\delta(X) = \delta^G(X)$ denotes the set of edges of G with one end in X and the other in $V - X$; if $\delta(X)$ is nonempty, it is called a *cut* of G; $\delta(X)$ *separates* nodes u and v (or disjoint subsets $A, B \subset V$) if $|\{u, v\} \cap X| = 1$ (resp. $X \cap (A \cup B) \in \{A, B\}$.]

Example 1. Γ is the complete graph K_p with p nodes. Then m is called a *multi-cut metric*. If $p = 2$, m is a *cut metric* and corresponds a cut separating the two nodes of Γ.

Example 2. Γ is the complete bipartite graph $K_{p,r}$ with parts of p and r nodes. Then m is called a *(p, r)-metric.*

We consider the *minimum 0-extension problem* defined as follows.

(1.1) Given a graph $G = (V, E)$, a subset $T \subseteq V$, a metric μ on T (or a connected graph $\Gamma = (T, W)$), and a weight function $c : E \to \mathbb{Z}_+$, find a 0-extension m of μ (resp. Γ) to V that minimizes $c \cdot m$.

(We denote by $a \cdot b$ the inner product $\sum(a(e)b(e) : e \in S)$ of functions a and b within the common part S of the domains of a and b.) If Γ is K_p, we obtain the minimum multi-cut problem mentioned above, or the minimum cut problem when $p = 2$. If Γ is $K_{p,r}$, (1.1) is specified to be the *minimum (p, r)-metric problem.*

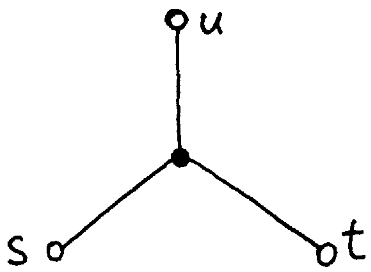

Fig. 1

For some reasons it makes sense to consider also the *fractional relaxation* of (1.1):

(1.2) Given G, T, c, μ (or Γ) as above, minimize $c \cdot m$ over the metrics m on V satisfying $m(uv) = \mu(uv)$ for all $u, v \in T$.

Let τ and τ^* denote the minima of $c \cdot m$ in (1.1) and (1.2), respectively. Clearly $\tau \geq \tau^*$. In general, this inequality needs not be equality. E.g., if G is as in Fig. 1, $T = \{s, t, u\}$, $\Gamma = K_T$ and $c \equiv 1$, then $\tau = 2$ whereas $\tau^* = 3/2$. The simplest case with $\tau = \tau^*$ arises when $\Gamma = K_2$. Indeed, there is a simple procedure to transform an optimal solution (o.s.) m of (1.2) with $\Gamma = K_2$ into a cut $\delta(X)$ separating the nodes of Γ so that $c \cdot m = c(\delta(X))$ (see, e.g., [2]). (For a function $g : S \to \mathbb{R}$ and a subset $S' \subseteq S$, $g(S')$ denotes $\sum(g(e) : e \in S')$.) It turned out that a similar property is true for $(2,r)$-metrics.

Theorem 1.1 [5]. *If* $\Gamma = K_{2,r}$ *then* $\tau = \tau^*$.

Moreover, in this case also there is a simple procedure to transform an o.s. m of (1.2) into the desired $(2,r)$-metric . More precisely, let $\{s_1, s_2\}$ and $\{t_1, \ldots, t_r\}$ be the parts of nodes in Γ. Define

(1.3)
$$S_1 = \{x \in V : m(s_1 x) = 0\};$$
$$T_i = \{x \in V : m(s_1 x) + m(x t_i) = 1\} - S_1, \quad i = 1, \ldots, r;$$
$$S_2 = V - (S_1 \cup T_1 \cup \ldots \cup T_r).$$

It is shown in [5] that $\Pi = (S_1, S_2, T_1, \ldots, T_r)$ is a partition of V, that each S_i (T_j) contains s_i (resp. t_j), and that if m' is the $(2,r)$-metric on V induced by Π then $c \cdot m' = c \cdot m$. This construction provides a strongly polynomial algorithm to solve the minimum $(2,r)$-metric problem. This is because for an arbitrary $\mu \in \mathcal{M}_T$, (1.2) can be written as the linear program:

(1.4)
$$\text{minimize} \quad c \cdot m \quad \text{subject to}$$
$$m \geq 0;$$
$$m \ \text{ satisfies the } \ (|V| - 2)\binom{|V|}{2} \ \text{ triangle inequalities;}$$
$$m(uv) = \mu(uv) \quad \text{for } u, v \in T.$$

Since the constraint matrix M in (1.4) consists of $O(|V|^3)$ rows and $O(|V|^2)$ columns and all entries of M are 0,+1 or −1, a version of the ellipsoid method in [10] finds an o.s. m to (1.4) in strongly polynomial time. Now if $\mu = d^{K_{2,r}}$, we can find a minimum $(2,r)$-metric according to (1.3).

Next, the minimum cut problem is dual to the classic maximum flow problem. There is a similar duality relation between (1.2) and a certain problem on multicommodity flows. To state this problem, consider G, T, μ, c as above. A simple path in G connecting different nodes in T is called a *T-path.* A *multicommodity flow*, or, simply, a *multiflow*, for G, T is a pair $f = (\mathcal{P}, \lambda)$ consisting of T-paths $P_1, \ldots, P_k$ along with nonnegative real numbers $\lambda_1, \ldots, \lambda_k$. Define

(1.5)
$$f^e = \sum(\lambda_i : P_i \text{ contains } e) \quad \text{for } e \in E;$$
$$f_{uv} = \sum(\lambda_i : P_i \text{ connects } u \text{ and } v) \quad \text{for } u, v \in T.$$

Considering c as an edge capacity function, we call f *c-admissible* if

(1.6) $$f^e \leq c(e) \qquad \text{for all } e \in E.$$

The value of f with respect to μ, or the *μ-value* of f, is $\sum(\mu(uv)f_{uv} \; : \; u, v \in T)$, denoted by $\langle \mu, f \rangle$. The *weighted maximum multiflow problem* is:

(1.7) Given G, T, μ, c, maximize $\langle \mu, f \rangle$ among all c-admissible multiflows f for G, T.

Let ν^* denote the maximum of $\langle \mu, f \rangle$ in (1.7), and ν the maximum of $\langle \mu, f \rangle$ if we admit only the *integer* multiflows f (i.e., with all λ_i's integral). Clearly (1.7) is a linear program, and if we associate with the inequalities in (1.6) dual variables $m(e)$, the program dual of (1.7) consists in minimizing $c \cdot m$ over $m : E \to \mathbb{R}_+$ such that $m(P) \geq \mu(uv)$ for each $u, v \in T$ and each path P in G connecting u and v. (When writing $m(P)$, we consider a path as an edge set.) Now decreasing m on some edges, if needed, and then extending it to E_V in a natural way, we obtain a metric feasible to (1.2). This implies $\nu^* = \tau^*$, so we may think of (1.7) and (1.2) as a pair of mutually dual programs. This minimax relation between metrics and multiflows was originally established by Lomonosov [8].

As to the integer version of (1.7), $\nu \leq \nu^*$ is obvious, and this inequality may be strict. E.g., for the case in Fig. 1, $\nu = 1$ and $\nu^* = 3/2$. Nevertheless, Lovász [9] and, independently, Cherkassky [1] proved that if $\mu = d(K_p)$ and c is inner Eulerian, then $\nu = \nu^*$. Here an (integer-valued) function c on E is called *inner Eulerian* if $c(\delta(\{x\}))$ is an even integer for each $x \in V - T$. In case $\Gamma = K_{2,r}$ the equality $\nu = \nu^*$ needs not hold either. We prove the following theorem.

Theorem 1.2. *If $\mu = d(K_{2,r})$ and c is inner Eulerian, then $\nu = \nu^*$ (and therefore, $\nu = \tau$).*

This paper is organized as follows. Theorem 1.2 is proved in Section 2. The proof is based on a splitting-off method and it provides a strongly polynomial algorithm to find an integer optimal multiflow in the inner Eulerian case. The algorithm is described in Section 3. It should be noted that this algorithm uses the ellipsoid method to certify feasibility of the splitting-off operations applied. In the concluding Section 4 we consider the general case of μ and study the phenomenon when, for μ fixed, the minima τ and τ^* in problems (1.1) and (1.2) coincide for any G and c. We give some necessary and some sufficient (but not simultaneously necessary and sufficient) conditions on μ to provide $\tau = \tau^*$ to hold. Also some structural and polyhedral aspects of the metric minimization problem are discussed there and open questions are raised.

2. Proof of the theorem

We show that if $\mu = d^{K_{2,r}}$ and c is inner Eulerian, then $\nu = \tau$. The proof borrows some ideas from [5]. W.l.o.g. we may assume that $G = (V, E)$ is complete, i.e., $E = E_V$; for adding to G a new edge with zero capacity does not affect our problems and remains c inner Eulerian. In the proof we will transform the function c, and in order to distinguish between the values of ν (τ, ν^*, τ^*) for different capacity functions we use notation $\nu(c')$ (resp. $\tau(c'), \nu^*(c'), \tau^*(c')$), where c' is a function on E_V in question.

Let $\mathcal{E}$ denote the set of 0-extensions of μ to V. A metric $m \in \mathcal{E}$ is called *tight* for c if $c \cdot m = \tau(c)$; the set of tight m's is denoted by $K(c)$. Let

$$\eta(c) = \sum_{x \in V-T} \sum_{y \in V-\{x\}} c(xy).$$

We use induction, assuming that the equality $\nu(c') = \tau(c')$ holds for each inner Eulerian c' on E_V such that either $|K(c')| > |K(c)|$, or $|K(c')| = |K(c)|$ and $\eta(c') < \eta(c)$ (note that $|K(c)| \le |\mathcal{E}|$ and $\mathcal{E}$ is finite for V fixed). The base case $\eta(c) = 0$ is easy. Indeed, in this case $c \cdot m$ equals $c \cdot \mu$ for any 0-extension m, and the multiflow f formed by the elementary paths P_{uv} (of one edge) connecting distinct $u, v \in T$ along with weights $\lambda_{uv} = c(uv)$ has μ-value $c \cdot \mu$, whence $\nu(c) = \tau(c)$. So, in the sequel we assume that $\eta(c) > 0$.

Consider $x \in V - T$ for which $Q(x) = \{y \in V - \{x\} : c(xy) > 0\}$ is nonempty. We may assume that $|Q(x)| \ge 2$. Indeed, if $Q(x)$ consists of a single element y, let c' be obtained from c by decreasing $c(xy)$ by 2. Then c' is inner Eulerian and nonnegative (as $c(\delta(\{x\})) = c(xy)$ is even and nonzero). One can see that there is $m \in \mathcal{E}$ such that $c' \cdot m = \tau(c')$ and both x and y belongs to the same set in the partition of V induced by m, i.e., $m(xy) = 0$. Then $c' \cdot m = c \cdot m$, which implies $\tau(c') = \tau(c)$. Clearly $K(c) \subseteq K(c)$ and $\eta(c) > \eta(c')$, and the result follows by induction.

Let Φ be the set of pairs of distinct elements of $Q(x)$. For a pair $\{y, z\} \in \Phi$, we can apply the *splitting-off operation* that transforms c as follows:

$$\begin{aligned} c'(e) &= c(e) - 1 \quad \text{for } e = xy, xz, \\ &= c(e) + 1 \quad \text{for } e = yz, \\ &= c(e) \quad \text{for } e \in E_V - \{xy, xz, yz\}. \end{aligned} \tag{2.1}$$

Clearly c' is nonnegative and inner Eulerian. For any metric m on V, $c \cdot m - c' \cdot m = m(xy) + m(xz) - m(yz) \ge 0$. Therefore, $\tau(c') \le \tau(c)$. We say that $\{y, z\}$ is *feasible* if $\tau(c') = \tau(c)$. In this case we can apply induction since $\eta(c') = \eta(c) - 1$, and the fact that $c \cdot m \ge c' \cdot m \ge \tau(c)$ for any $m \in \mathcal{E}$ implies $K(c) \subseteq K(c')$. By induction there exists a c'-admissible integer multiflow f' such that $\langle \mu, f' \rangle = \tau(c')$. It is easy to see that f' can be transformed into a c-admissible integer multiflow f with $\langle \mu, f \rangle = \langle \mu, f' \rangle$. Hence, $\nu(c) \ge \langle \mu, f \rangle = \tau(c)$, which implies $\nu(c) = \tau(c)$, as required.

Our aim is to show that there exists at least one feasible pair in Φ, from which the theorem will follow by the above argument. Let $\{s_1, s_2\}$ and $\{t_1, \dots, t_r\}$ be the parts of $\Gamma = K_{2,r}$.

Claim 1. *For any $m \in \mathcal{E}$, $c \cdot m - \tau(c)$ is even.*

Proof. Consider the partition $\Pi = (S_1, S_2, T_1, \dots, T_r)$ of V induced by m (with $s_i \in S_i$ and $t_j \in T_j$). Let ρ be the cut metric corresponding to the cut $\delta(X)$ for $X = S_1 \cup S_2$. Then $m + \rho$ takes value 0 or 2 on each edge, whence $c \cdot (m + \rho)$ is even. Now the claim follows from the fact that for any cut metric ρ' corresponding to a cut in G separating $\{s_1, s_2\}$ from $\{t_1, \dots, t_r\}$, the number $c \cdot \rho - c \cdot \rho'$ is even (because c is inner Eulerian). •

Claim 2. *For each $m \in \mathcal{E}$, $\Delta = c \cdot m - c' \cdot m$ equals 0, 2 or 4. Moreover, if $\Delta = 4$ then $m(xy) = m(xz) = 2$ and $m(xz) = 0$ (and therefore, both y and z belong to the same member of the partition of V induced by m).*

Proof. We have $\Delta = m(xy) + m(xz) - m(yz)$. Observe that the length of any closed path with respect to a $(2, r)$-metric is even. This implies that Δ is even. Next, $m(uv) \le 2$ for any $u, v \in V$. Hence, $\Delta \in \{0, 2, 4\}$. If $\Delta = 4$ then the only possible case is when $m(xy) = m(xz) = 2$ and $m(yz) = 0$. •

The infeasibility of $\{y, z\} \in \Phi$ is equivalent to the existence of $m \in \mathcal{E}$ such that for c' as in (2.1) $c' \cdot m$ is strictly less than $\tau(c)$. From Claims 1 and 2 it follows that

(2.2) if $\{y, z\} \in \Phi$ is infeasible, then for each $m \in \mathcal{E}$ with $c' \cdot m < \tau(c)$, either

(i) m is tight and $m(xy) + m(xz) - m(yz) > 0$, or

(ii) $c \cdot m = \tau(c) + 2$, $c' \cdot m = \tau(c) - 2$ and $m(xy) + m(xz) - m(yz) = 4$.

In what follows we assume that each pair in Φ is infeasible and will attempt to come to a contradiction. First we show that there exists $\{y, z\} \in \Phi$ for which only (ii) in (2.2) takes place.

By Theorem 1.1, $\tau(c) = \tau^*(c) = \nu^*(c)$. So, there is a c-admissible multiflow $f = (P_1, \dots, P_k; \lambda_1, \dots, \lambda_k)$ with $\langle \mu, f \rangle = \tau(c)$; we assume that $\lambda_i > 0$ for $i = 1, \dots, k$. An edge e is called *saturated* by f if $f^e = c(e)$. Let q_i be the pair of end nodes of P_i.

Claim 3. *Let $\{y, z\} \in \Phi$ and $m \in K(c)$. Then:*

(i) *if $m(xy) > 0$ then xy is saturated by f; and similarly for xz;*

(ii) *each path P_i is shortest for m, i.e., $m(P_i) = \mu(q_i)$.*

Proof. (i) and (ii) immediately follow by considering the complementary slackness conditions for (1.2) and (1.7). More precisely,

$$\nu^*(c) = \sum_{i=1}^{k} \lambda_i \mu(q_i) \le \sum_{i=1}^{k} \lambda_i m(P_i)$$
$$= \sum_{e\in E} f^e m(e) \le \sum_{e\in E} c(e)m(e) = \tau^*(c).$$

Since $\nu^*(c) = \tau^*(c)$, equality holds throughout, whence (i) and (ii) follow. •

An immediate corollary from Claim 3 is as follows.

Claim 4. *Let $\{y,z\} \in \Phi$ be such that xy is not saturated by f or there is a path in f that contains xy and xz. Let $m \in K(c)$. Then $m(xy) + m(xz) = m(yz)$.* •

Thus, there is $\{y,z\} \in \Phi$ for which no metric as in (2.2)(i) exists. We fix one of such $\{y,z\}$'s. A metric m as in (2.2)(ii) is called *critical* for c and $\{y,z\}$.

Consider the capacity function $\tilde{c} = 2c$. Clearly $\tau(\tilde{c}) = 2\tau(c)$. Furthermore, by (2.2)(ii), any metric $m \in \mathcal{E}$ with $m(xy) + m(xz) > m(yz)$ satisfies $\tilde{c}\cdot m \ge \tau(\tilde{c}) + 4$. Hence, $\{y,z\}$ becomes feasible for $\tilde{c}$, i.e., the function $\tilde{c}'$ formed from $\tilde{c}$ by the splitting-off operation with respect to $\{y,z\}$ satisfies $\tau(\tilde{c}') = \tau(\tilde{c}) = 2\tau(c)$. Let m be critical for c and $\{y,z\}$. Then

$$\tilde{c}\cdot m = \tau(\tilde{c}) + 4 \quad \text{and} \quad \tilde{c}'\cdot m = \tau(\tilde{c}).$$

Thus, $K(\tilde{c}')$ strictly includes $K(\tilde{c}) = K(c)$. Obviously, $\tilde{c}'$ is inner Eulerian. By induction there is an integer $\tilde{c}'$-admissible multiflow h with $\langle m,h\rangle = \tau(\tilde{c}')$. This h is transformed in an obvious way into a $\tilde{c}$-admissible integer multiflow $g = (P_1,\dots,P_k; \lambda'_1,\dots,\lambda'_k)$. Then the multiflow f formed by the same paths P_i and the numbers $\lambda_i = \lambda'_i/2$, $i = 1,\dots,k$, is c-admissible and *half-integral*, and $\langle \mu, f\rangle = \tau(c)$. Repeating paths in f, if needed, we may assume that each λ_i is $1/2$. At least one P_i must pass through x (otherwise each pair in Φ is, obviously, feasible).

For two nodes u and v in a path P, *truncating* P at $\{u,v\}$ is an operation that replaces in P the part between u and v by the edge uv. Consider a path P_i that passes through x; for definiteness let P_i use edges $e = xy$ and $e' = xz$.

Claim 5. *The edges e and e' are saturated by f.*

Proof. Consider $m \in \mathcal{E}$ critical for c and $\{y,z\}$. As above, let $\tilde{c}'$ be obtained from $\tilde{c}$ by the splitting-off operation w.r.t. $\{y,z\}$. Let h be the multiflow obtained from g as above by truncating P_i at $\{y,z\}$. Since $\lambda'_i = 2\lambda_i = 1$, h is $\tilde{c}'$-admissible. Also $\langle \mu, h\rangle = \tau(\tilde{c}')$ and m is tight for $\tilde{c}'$. By Claim 3 applied to $\tilde{c}', h, m, e, e'$, we have $h^e = \tilde{c}'(e)$ and $h^{e'} = \tilde{c}'(e')$. This implies that $f^e = c(e)$ and $f^{e'} = c(e')$. •

By Claim 5 there are paths P_l and P_q ($l, q \neq i$) which contain e and e', respectively. Let a_l (b_l) and a_q (b_q) be the first (resp. last) node in P_l and P_q, respectively. We may assume that a_l, y, x, b_l follow in this order in P_l, and a_q, z, x, b_q follow in this order in P_q.

Claim 6. $a_l = a_q$.

Proof. Consider a metric $m \in \mathcal{E}$ critical for $\{y, z\}$ and the partition $\Pi = (S_1, S_2, T_1, \ldots, T_r)$ of V induced by m. Let $\widetilde{c}'$ be obtained from $\widetilde{c}$ by the splitting-off operation w.r.t. $\{y, z\}$, and let h be the multiflow obtained from g by truncating P_i at $\{y, z\}$. Then h is $\widetilde{c}'$-admissible, $\langle \mu, h \rangle = \tau(\widetilde{c}')$, and m is tight for $\widetilde{c}'$. By Claim 2 applied to $\widetilde{c}, \widetilde{c}', m, x, y, z$, either $x \in S_j$ and $y, z \in S_{j'}$, or $x \in T_j$ and $y, z \in T_{j'}$ for distinct j, j'. Assume the former, the other case is similar. By (ii) in Claim 3, the path P_l is shortest for m. Since $\mu(a_l b_l) \leq 2$ and $m(xy) = 2$, we observe that $m(e)$ must be zero for each edge of P_l different from xy. Hence, a_l and y belong to the same set in Π, i.e., $a_l \in S_j$. Similar arguments for P_q yield $a_q \in S_j$. Since S_j contains exactly one element of T, we conclude that $a_l = a_q$. •

Now we finish the proof as follows. We assume that f is chosen so that f is half-integral, $\langle \mu, f \rangle = \tau(c)$ and $\sum(f^e : e \in E_V)$ is as small as possible. Also we may assume that for each path P_i in f all inner nodes of P_i are not in T (otherwise split P_i into two T-paths, which does not decrease the μ-value), and that some path P_i has at least two edges. Let y, x, z be the first, second and third node in P_i, respectively; then $x \in V - T$. Let P_l, a_l, b_l and P_q, a_q, b_q be as above for our y, x, z. By Claim 6, $a_l = a_q = y$ (as $y \in T$). So, P_i and P_q have the same first nodes and go through the edge xz in opposite directions. This implies that we can replace P_i and P_q by two paths which have the same first node y, have the last nodes as in P_i and P_q and use the same edges except xz. This contradicts the minimality of $\sum(f^e : e \in E_V)$ and completes the proof of Theorem 1.2. ••

3. Algorithm

The splitting-off techniques used in the proof of Theorem 1.2 gives rise to an algorithm for finding an integer c-admissible multiflow f with $\langle \mu, f \rangle = \tau(c)$ for $\mu = d^{K_{2,r}}$ and an inner Eulerian c. If $c : E \to \mathbb{Z}_+$ is not inner Eulerian we can apply the algorithm to the capacity function $2c$ to obtain a half-integral o.s. for c. As before, it is convenient to assume that G is complete.

The algorithm consists of two *stages*. The first stage consists of $|V - T|$ *iterations*, each of which treats a node $x \in V - T$. At a current *step* of the iteration that works with x, we choose a pair $\{y,z\} \in V-\{x\}$ with $b=\min\{c(xy),c(xz)\}>0$ (for the current c) and finds the maximum $\alpha \in \mathbb{Z}_+$ such that $\alpha \le b$ and $\tau(c')=\tau(c)$, where c' is defined by

$$\text{(3.1)} \qquad \begin{aligned} c'(e) &= c(e)-\alpha \quad \text{for } e=xy,xz,\\ &= c(e)+\alpha \quad \text{for } e=yz,\\ &= c(e) \quad \text{for } e\in E_V-\{xy,xz,yz\} \end{aligned}$$

(i.e., c' is obtained by performing α splitting-off operations (2.1) with the same $\{y,z\}$). Then we make c' the new current c, choose a new pair $\{y',z'\}$, and so on. We need not consider the same pair $\{y,z\}$ twice during the iteration because, after the first application of the splitting-off operation (3.1) to $\{y,z\}$, the corresponding number α for the new function c becomes zero and, obviously, it remains zero up to termination of the iteration. Since the problem for each current c has an integer o.s., the iteration always terminates, after $O(|V|^2)$ steps, with the situation when $c(xv)$ can be non-zero for at most one $v\in V-\{x\}$. Putting $c(xv):=0$ obviously preserves $\tau(c)$ and remains c inner Eulerian. Thus, upon termination of the iteration we can remove the node x from the set V.

The first stage finishes when the current V is T. For the resulting c the optimal multiflow f is obvious. The second stage restores the desired o.s. for the initial V and c in a natural way, by considering the nodes x and pairs $\{y,z\}$ in the order reverse to that occurred in the first stage.

Now we explain how to find α efficiently. First we examine α to be the number b as above. For the resulting c' compute $\tau^*(c')=\tau(c')$ by solving linear program (1.4). If $\tau(c')=\tau(c)$, we are done. Otherwise, by arguments in Section 2, there exists a metric $m\in\mathcal{E}$ such that $m(xy)+m(xz)-m(yz)\in\{2,4\}$ and $c'\cdot m=\tau(c')<\tau(c)$. Let $\varepsilon=\tau(c)-\tau(c')$. We now examine α to be $b_1=b-\lfloor\varepsilon/4\rfloor$ (where $\lfloor a\rfloor$ is the greatest integer not exceeding a). Compute $\tau(c'')$ for the corresponding c''. One can see that if $\tau(c'')$ equals $\tau(c)$, then $\alpha=b_1$ is as required, and if not, then for any metric $m\in\mathcal{E}$ with $c''\cdot m=\tau(c'')$ the only case $m(xy)+m(xz)-m(yz)=2$ is possible. This implies that in the latter case the desired α is $b_1-\varepsilon/2$, where $\varepsilon=\tau(c)-\tau(c'')$.

Hence, for each $\{y,z\}$ handled at a step of an iteration, computing the above number α is reduced to solving (1.4) at most twice. Since (1.4) is solvable in strongly polynomial time and the total number of steps throughout the algorithm is $O(|V|^3)$, the algorithm runs in strongly polynomial time.

Remark. Though being strongly polynomial, the above algorithm is not "combinatorial" because it uses the ellipsoid method. For $\Gamma=K_{2,r}$ also "purely combinatorial" algorithms to solve (1.1) and (1.7) (with f integral in the inner Eulerian case) can be constructed so that they run in pseudo-polynomial and even weakly polynomial time.

However, no "purely combinatorial" strongly polynomial algorithm for the problem in question is known at present.

4. Generalizations and open problems

Let μ be an *integer* metric on T. We call μ *minimizable* if for any graph $G = (V, E)$ with $V \supseteq T$ and function $c : E \to \mathbb{Z}_+$, the minima in (1.1) and (1.2) coincide, $\tau = \tau^*$. The distance functions d^{K_2} and $d^{K_{2,r}}$ give examples of minimizable metrics. It is interesting to characterize the set of all minimizable metrics. This problem is still open and at present we are able to present only some partial results in this area. It turns out that the set of minimizable metrics is rather large; in particular, it contains d^Γ for any tree Γ, as we show later.

First of all it suffices to consider *positive* metrics μ (i.e., with $\mu(uv) > 0$ for distinct $u, v \in T$) because it is easy to see that a 0-extension of μ is minimizable if and only if μ is minimizable itself. Also μ can be considered up to proportionality.

The property of being minimizable can also be stated in polyhedral terms (in Statement 4.1 below). Let $\mathcal{P} = \mathcal{P}_{\mu,V}$ be the set of metrics m on $V \supseteq T$ such that $m(uv) = \mu(uv)$ for all $u, v \in T$. Since $\mathcal{P}$ is the solution set of the linear system in (1.4), $\mathcal{P}$ is a polyhedron. Let $m \in \mathcal{P}$. We call m an *extension* of μ to V if there is no $m' \in \mathcal{P}$ such that $m' < m$ (i.e., $m' \neq m$ and $m'(uv) \leq m(uv)$ for all $u, v \in V$). This means that m belongs to the boundary of the *dominant polyhedron* $\mathcal{D}_{\mu,V} = \mathcal{P} + \mathbb{R}_+^{E_V}$ $(= \{x \in \mathbb{R}^{E_V} : x \geq m \text{ some } m \in \mathcal{P}\})$. The extensions of μ can also be described in terms of shortest paths, namely,

(4.1) $m \in \mathcal{P}$ is an extension of μ if and only if for any $x, y \in V$ there are $s, t \in T$ such that $m(sx) + m(xy) + m(yt) = \mu(st)$.

(The part "only if" follows from the obvious fact that if for some x and y, $m(sx) + m(xy) + m(yt) > \mu(st)$ for any $s, t \in T$, then one can decrease m on some pairs preserving the distances on T.)

If m is a vertex of $\mathcal{D}_{\mu,V}$, m is called *μ-primitive*. In other words, m is μ-primitive if m is an extension and there are no $m', m'' \in \mathcal{P}$ different from m such that $m = \lambda m' + (1-\lambda)m''$ for some $0 \leq \lambda \leq 1$. Let $\mathcal{W} = \mathcal{W}_{\mu,V}$ denote the set of μ-primitive metrics on V.

Consider a 0-extension m of μ. Obviously, m is an extension. Moreover, m is μ-primitive. This is because for $m', m'' \in \mathcal{P}$ and $0 < \lambda \leq 1$ such that $m = \lambda m' + (1-\lambda)m''$, m' must coincide with m (as $m'(xy) = 0$ for x and y contained in the same member of the partition of V induced by m and $m'(uv) = \mu(uv) = m(uv)$ for $u, v \in T$).

By standard linear programming arguments, $\mathcal{W}$ admits a "dual description" as being the minimal set of metrics m on V such that for any $c : E_V \to \mathbb{Z}_+$, the fractional problem (1.2) has an optimal solution within $\mathcal{W}$. This implies the following.

Statement 4.1. *A metric μ on T is minimizable if and only if for any $V \supseteq T$ the set $\mathcal{W}_{\mu,V}$ consists of the 0-extensions of μ to V and only them.*

(The part "only if" is implied by the fact that for any vertex m of $\mathcal{D}_{\mu,V}$, there is $c : E_V \to \mathbb{Z}_+$ such that $c \cdot m < c \cdot m'$ for any other vector m' in $\mathcal{P}$.)

The next statement suggests a way to construct new minimizable metrics from other ones. Let T' and T'' be subsets of T such that $T' \cup T'' = T$ and $T' \cap T''$ consists of a single element s, and let μ, μ' and μ'' be metrics on T, T' and T'', respectively. We say that μ is the 1-*sum* of μ' and μ' if μ coincides with μ' on T', coincides with μ'' on μ, and for $u \in T'$ and $v \in T''$, $\mu(uv) = \mu'(us) + \mu''(sv)$.

Statement 4.2. *If both μ' and μ'' are minimizable then μ is minimizable as well.*

Proof. Consider a μ-primitive metric m on a set $V \supseteq T$. Let V' consist of all $x \in V$ such that either $x \in T'$, or $x \in V - T$ and $m(ux) + m(xs) = \mu(us)$ for some $u \in T'$; and let $V'' = (V - V') \cup \{s\}$. Then $T' \subseteq V'$ and $T'' \subseteq V''$. Let m' (m'') be the restriction of m to V' (resp. V''). We assert that m is the 1-sum of m' and m'', i.e.,

(4.2) for any $y \in V'$ and $z \in V''$, $m(yz) = m(ys) + m(sz)$.

Choose $u, v \in T$ such that

$$m(uy) + m(ys) = \mu(us) \quad \text{and} \quad m(vz) + m(zs) = \mu(vs) \tag{4.3}$$

(such a u exist by (4.1) for the pair $\{s, y\}$, and similarly for v). Since $z \in V''$, we have $v \in T''$, and by the definition of V' we may assume that $u \in T'$. Now (4.3) together with $\mu(us) + \mu(sv) = \mu(uv)$ imply that the sequence u, y, s, z, v forms a shortest path for m, whence (4.2) follows.

Consider $m_1, m_2 \in \mathcal{P}_{\mu',V'}$ and $0 < \lambda \leq 1$ such that $m' \geq g = \lambda m_1 + (1 - \lambda) m_2$. For $i = 1, 2$, let $\widetilde{m}_i(y'z')$ equal $m_i(y'z')$ for $y', z' \in V'$, $m''(y'z')$ for $y', z' \in V''$, and $m_i(y's) + m''(sz')$ for $y' \in V'$ and $z' \in V''$. One can see that each $\widetilde{m}_i$ is a metric and $\widetilde{m}_i(uv) = \mu(uv)$ for all $u, v \in T$. Moreover, (4.2) and $m' \geq g$ imply $m \geq \lambda \widetilde{m}_1 + (1 - \lambda) \widetilde{m}_2$. Since m is μ-primitive, the latter is possible only if $\widetilde{m}_1 = m$. Hence, $m' = m_1$, so m' is μ'-primitive. Similarly, m'' is μ''-primitive. Now the facts that m' is a 0-extension of μ' and m'' is a 0-extension of μ'' easily imply that m is a 0-extension of μ. •

Since the metric d^{K_2} is minimizable, d^Γ is minimizable for any tree Γ (as d^Γ can be obtained by a sequence of 1-sum operations from copies of d^{K_2}).

Next we give a number of necessary conditions on a metric to be minimizable. In fact, they are stated in terms of forbidden submetrics. We need one statement of a general form.

Statement 4.3. *Let μ be a metric on T, and μ' its restriction to a subset $T' \subseteq T$. Let m' be a μ'-primitive metric on a set $V' \supseteq T'$ such that $V' \cap T = T'$. Let $V = V' \cup T$. Then m' can be extended to a μ-primitive metric on V.*

Proof. Define m by $m(xy) = m'(xy)$ for $x, y \in V'$, $m(xy) = \mu(xy)$ for $x, y \in T$, and $m(xy) = \min\{m(xs) + \mu(sy) : s \in T'\}$ for $x \in V'$ and $y \in T - T'$. One can check that m is a metric. Let $m \geq \lambda m_1 + (1-\lambda)m_2$ for some extensions m_1, m_2 of μ and $0 < \lambda \leq 1$. Since m' is μ'-primitive, the restriction of m_1 to V' coincides with m'. Then $m = m_1$, whence the result follows. •

From Statement 4.3 it follows that

(4.3) for μ, μ', m' as above, if μ is minimizable, then there is a mapping $\gamma : V' \to T$ such that γ is identical on T' and $m'(xy) = \mu(\gamma(x)\gamma(y))$ for all $x, y \in V'$.

This property has three consequences described in Statements 4.4–4.6.

Statement 4.4. *If μ is minimizable then the length $\mu(C)$ of any circuit C of the graph K_T is even.*

Proof. Suppose that $\mu(C)$ is odd for some circuit C. Then there are $v_0, v_1, v_2 \in T$ such that for $q_i = \mu(v_{i-1}v_{i+1})$, $i = 0, 1, 2$, the number $q = q_0 + q_1 + q_2$ is odd (taking indices modulo 3). Let μ' be the restriction of μ to $T' = \{v_0, v_1, v_2\}$, and let $V' = T' \cup \{x\}$. Define $m'(xv_i) = (q_{i-1} + q_{i+1} - q_i)/2$. Then $m'(v_ix) + m'(xv_j) = \mu(v_iv_j)$ for $0 \leq i < j \leq 2$, which implies that m extended by $\mu(v_iv_j)$ on the pairs v_iv_j is a metric. Moreover, one can see that m' is μ'-primitive. Now, if μ is minimizable then for γ as in (4.3), we have $\mu(v_i\gamma(x)) = m'(xv_i)$. But at least one of $m'(xv_i)$'s is not integral (as q is odd) while μ is integer-valued; a contradiction. •

Statement 4.5. *If μ is minimizable then for any $v_0, v_1, v_2 \in T$ there is $s \in T$ such that $\mu(v_iv_j) = \mu(v_is) + \mu(sv_j)$ for all $0 \leq i < j \leq 2$.*

Proof. Apply (4.3) to V' and m' as in the proof of Statement 4.4. •

Statement 4.6. *Let μ be a metric on T, and μ' its restriction to a subset $T' \subseteq T$. Suppose that there is a positive μ'-primitive metric m' on a set V' with $|V'| > |T|$.*

Then μ is not minimizable.

Proof. Suppose that μ is minimizable and consider γ as in (4.3). Since $|V'| > |T|$, there are distinct $x, y \in V'$ such that $\gamma(x) = \gamma(y)$, and therefore, $m'(xy) = 0$; a contradiction with the positivity of m'. •

Statements 4.4–4.6 enable us to eliminate many metrics μ. For simplicity we now consider only graph metrics $\mu = d^\Gamma$. Suppose that d^Γ is minimizable. Statement 4.4 shows that Γ is bipartite; moreover, by Statement 4.5, Γ is 3-*closed* (in the sense that for any three nodes in Γ there are three shortest paths between the pairs of these nodes that meet a common node). Statement 4.6 implies that Γ cannot contain certain isometric subgraph ($\Gamma' \subseteq \Gamma$ is called *isometric* if the distances in Γ' between its nodes are the same as in Γ). E.g., let Γ' be the circuit C_6 with six nodes, say, $s_0, \dots, s_5$ (going in this order in C_6). It was shown in [5] that d^{C_6} has infinitely many positive primitive extensions m'. [An extension m' for d^{C_6} can be constructed as follows: for a positive integer k, let H be the graph whose nodes correspond to the vectors (p, q, r) for $p, q, r = 0, \dots, k$ and whose edges correspond to the pairs $\{(p, q, r), (p', q', r')\}$ such that either $|p-p'|+|q-q'|+|r-r'| = 1$ or $p'-p = q'-q = r'-r = 1$; identify $s_0, \dots, s_5$ with $(k,0,0), (0,k,0), (0,0,k), (0,k,k), (k,0,k), (k,k,0)$, respectively; then the desired m' is d^H/k.] Therefore, Γ contains no isometric 6-circuits. Also one can construct infinitely many positive primitive extensions for C_{2k} with $k \geq 4$, for the graph $K_{3,3}^{-1}$ obtained from $K_{3,3}$ by deleting one edge, and for the graph $K_{3,4}^{-2}$ obtained from $K_{3,4}$ by deleting two non-adjacent edges (we do not describe these constructions here). Thus, Γ contains neither isometric $2k$-circuit for $k \geq 3$, nor isometric subgraph $K_{3,3}^{-1}$, nor isometric subgraph $K_{3,4}^{-2}$.

Let diam(Γ) denote the *diameter* of Γ (the maximum distance between its nodes). If Γ is bipartite and diam(Γ) = 2 then Γ is a complete bipartite graph $K_{p,r}$. We know that $\mu = d^{K_{p,r}}$ is minimizable if $p = 2$. On the other hand, if $p, r \geq 3$ then there are *non-integral* positive μ-primitive extensions (cf. [6]); in particular, such a μ is not minimizable.

Conjecture. Let diam(Γ) ≥ 3. Then d^Γ is minimizable if and only if $\Gamma = (T, W)$ is 3-closed, contains no isometric subgraph $K_{3,3}$, and there is no $T' \subseteq T$ (with $|T'| > 1$) such that the restriction of d^Γ to T' is proportional to $d^{\Gamma'}$ for $\Gamma' = C_{2k}$ with $k \geq 3$ or $\Gamma' = K_{3,3}^{-1}$ or $\Gamma' = K_{3,4}^{-2}$.

Now return to the multiflow problem (1.7). Is it true that if a metric μ is minimizable and a capacity function c is inner Eulerian, then (1.7) has an integer o.s.? This is true for $\mu = d^{K_{2,r}}$ (by Theorem 1.2) but in general the answer is unknown even if $\mu = d^\Gamma$ and Γ is a tree.

Another open question: is it true that if μ is not minimizable, then problem (1.1) is NP-hard? (The answer is affirmative if $\mu = d^{K_p}$ for $p \geq 3$.)

In conclusion we discuss the denominator behavior in one sort of metric packing (or decomposition) problems. Let μ be an integer minimizable metric on $T \subseteq V$, and m an integer metric on V such that $m(uv) = k\mu(uv)$ for all $u, v \in T$, where $k \in \mathbb{Z}_+$. From Statement 4.1 applied to $m' = m/k$ it follows that there exist 0-extensions $m_1, \ldots, m_N$ of μ to V and rationals $\lambda_1, \ldots, \lambda_N \geq 0$ with $\lambda_1 + \ldots + \lambda_N = k$ satisfying

$$
\begin{aligned}
(4.4) \qquad m(xy) &\geq \lambda_1 m_1(xy) + \ldots + \lambda_N m_N(xy) \quad \text{for } x, y \in V, \\
&= \lambda_1 m_1(xy) + \ldots + \lambda_N m_N(xy) \quad \text{for } x, y \in T.
\end{aligned}
$$

When m is an extension, the inequality in (4.4) turns into equality, i.e., $m_1, \ldots, m_N$ along with $\lambda_1, \ldots, \lambda_N$ give a *decomposition* of m into weighted 0-extensions of μ.

If μ is the distance function of K_2 with node set $\{s, t\}$, then each 0-extension is a cut metric on V corresponding to a cut separating s and t, and (4.4) can be satisfied with all λ_i's integral (e.g., by using a sorting procedure on V according to the distances from s). A similar property holds if $\mu = d^{K_{2,r}}$ and $m(C)$ is even for every circuit C in K_V, as shown in [7]. An open question: does such a property remain true for every minimizable μ and m as above with $m(C)$ even for all circuits C?

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