MINIMUM DISTANCE MAPPINGS OF GRAPHS

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Minimum distance mappings of graphs *

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Abstract. Let $H = (T, U)$ be a connected graph, and d^H its distance function, i.e., $d^H(s,t)$ is the minimum length of a path connecting nodes s and t in H. The minimum distance mapping problem is $(*)$: given a graph $G = (V, E)$ with $V \supseteq T$, find a mapping $\gamma : V \to T$ such that $\sum(d^H(\gamma(x), \gamma(y)) \; : \; e = \{x, y\} \in E)$ is as small as possible. For H fixed, this problem is known to be NP-hard if H is complete and $|T| \geq 3$ (as being the minimum multiway cut problem), while it is polynomially solvable when $|T| = 2$ (the minimum cut problem) or H is the complete bipartite graph with parts of 2 and r nodes (the minimum $(2, r)$-metric problem).

In this paper we give a complete characterization of the set of graphs H with the property that $(**)$ for any G, problem $(*)$ and its fractional relaxation have the same minimum objective value. More precisely, we prove that H satisfies $(**)$ if and only if (i) H has no isometric k-circuit with $k > 4$, (ii) any three nodes are pairwisely connected by three shortest paths sharing a common node, and (iii) the edges of H can be directed so that the non-adjacent edges of each 4-circuit are oppositely directed. Note that if H satisfies $(**)$, then $(*)$ can be solved in polynomial time. The proof combines combinatorial and topological ideas and reveals, in the key theorem, that H with property $(**)$ can be embedded in a 2-dimensional space S with a special metric on it so that there is a one-to-one correspondance, with preserving the distances, between the finite subsets of S that include T and certain metrics on sets $V \supseteq T$ that appear in the fractional relaxation of $(*)$.

Key words: graph, finite metric, distance, surface, isometric embedding.

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1. Introduction

By a *metric* on a set V' we mean a nonnegative real-valued function $m : V' \times V' \to \mathbb{R}_+$ satisfying

(i) $m(x,x) = 0$ for $x \in V'$;

(ii) $m(x,y) = m(y,x)$ for $x, y \in V'$ (*symmetry*);

(iii) $m(x,y) + m(y,z) \geq m(x,z)$ for $x, y, z \in V'$ (*triangle inequalities*).

Because of (i) and (ii) we may also think that m is given on the set $E_{V'}$ of edges of the complete undirected graph $K_{V'} = (V', E_{V'})$. Then the set of metrics on V', denoted by $\mathcal{M}_{V'}$, forms a (convex) polyhedral cone in the $\binom{|V'|}{2}$-dimensional euclidean space $\mathbf{R}^{E_{V'}}$ whose coordinates are indexed by edges of $K_{V'}$. We use notation $m(xy)$ instead of $m(x,y)$. A special case of metrics is the distance function $d^{G'}$ of a connected graph $G' = (V', E')$, i.e., $d^{G'}(xy)$ is the minimum length of a path connecting nodes x and y in G'.

Suppose that $G = (V, E)$ is a graph with possibly multiple edges and $H = (T, U)$ is a connected graph with $T \subseteq V$. The *minimum distance problem* is:

(1) find a mapping $\gamma : V \to T$ such that $\gamma(v) = v$ for each $v \in T$ and $\sum(d^H(\gamma(x)\gamma(y)) : e \in E$, x and y are the ends of $e)$ is as small as possible;

the minimum in (1) is denoted by $\tau = \tau(G, H)$. Another problem we deal with is:

(2) find a metric m on V such that $m(st) = d^H(st)$ for any $s, t \in V$ and $\sum(m(e) : e \in E)$ is as small as possible;

the minimum in (2) is denoted by $\tau^* = \tau^*(G, H)$.

A metric m on V whose restriction $m_{|T}$ to T is d^H is called an *extension* of H to V; the set of extensions is denoted by $\mathcal{P}_{V,H}$. Sometimes we denote an extension by (V, m). For $\gamma : V \to T$ identical on T, the metric m defined by $m(xy) = d^H(\gamma(x)\gamma(y))$, $x, y \in V$, is called a *0-extension* of H to V. Clearly each 0-extension is an extension, therefore, $\tau^* \leq \tau$. We think of (2) as the *fractional relaxation* of (1). Note that (1) is, in fact, equivalent to the problem in which, given graphs $H' = (T', U')$ and $G' = (V', E')$, a subset $X \subseteq V'$, and a mapping $\gamma' : X \to T'$, it is required to extend γ' to a mapping $\gamma : V' \to T'$ so that $\sum(d^{H'}(\gamma(x)\gamma(y)) : e = xy \in E')$ is minimum.

Definition 1. Following [11], a graph $H = (T, U)$ is called *minimizable* if $\tau^*(G, H) = \tau(G, H)$ holds for every graph $G = (V, E)$ with $T \subseteq V$.

To motivate the above problems and definition, consider some examples.

Example 1. H is the complete graph K_p with $p = |T|$ nodes. Then (1) turns into the *minimum multiway cut problem*: given a graph $G = (V, E)$ and subset $T \subseteq V$, find a partition Π of V such that each member of Π contains exactly one element of T and the number of edges of G connecting different members of Π is minimum. This problem is known to be NP-hard for any fixed $p \geq 3$ [3]. On the other hand, its special case with $p = 2$ is efficiently solvable as being the classic *minimum cut problem* for which plenty of polynomial algorithms have been designed (see, e.g., [4,5,6]). Moreover, $\tau = \tau^*$ if $p = 2$ (see, e.g., [4]), i.e., K_2 is minimizable. For $p \geq 3$ the inequality $\tau^* \leq \tau$ may be strict. E.g., if G is as in Fig. 1, $T = \{s, t, u\}$ and $H = K_T$, then $\tau = 2$ whereas $\tau^* = 3/2$.

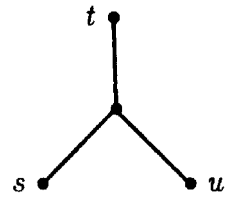


Fig. 1

Example 2. H is the complete bipartite graph $K_{p,r}$ with parts of p and $r = |T| - p$ nodes. It is easy to show that H is not minimizable if $p, r \geq 3$. On the other hand, H is minimizable if $p \leq 2$ [8].

(An interesting result due to Lovász [14] and, independently, Cherkassky [2] implies that for $H = K_p$ problem (2) has a *half-integer* optimal solution m. A similar property for $H = K_{p,r}$ follows from a result in [9, Sec. 3].)

Example 3. Given a graph $\Gamma = (B, W)$, let $H = (T, U)$ be formed by splitting each edge $e = xy \in W$ into two edges xz_e and $z_e y$ in series and then by adding a node v and edges vz_e for all $e \in W$. From a result in [9, Sec. 3] one can derive that H is minimizable if Γ has no circuit with ≤ 3 edges.

Example 4. Let H be the union of two graphs $H' = (T', U')$ and $H'' = (T'', U'')$ with $T' \cap T''$ consisting of a single node. It is shown that if both H' and H'' are minimizable then H is minimizable too [11]. In particular, this implies that every tree is minimizable (as K_2 is minimizable).

From the computational point of view, for each minimizable H problem (1) is as easy as (2). Indeed, (1) is reduced to comparing $\tau^*(G, H)$ and $\tau^*(G', H)$ for a sequence of graphs G', each obtained from G by sticking some nodes in $V - T$ to nodes in T; clearly it suffices to test $O(|T||V|)$ graphs G'. In its turn (2) is a linear program whose constraint matrix M has $O(|V|^3)$ rows and $O(|V|^2)$ columns; thus, $\tau = \tau^*$ and an optimal solution γ to (1) can be found in polynomial time. Note also that if G is a multigraph given via its node set and the integers $c(xy)$ that indicate how many edges connect a pair $\{x, y\}$ in V, then (2) is solvable in strongly polynomial time by use of Tardos' version [16] of the ellipsoid method (taking into account that the size of writing

M in binary notation is $O(|V|^3))$; therefore, we get a strongly polynomial algorithm for solving (1).

The aim of this paper is to give a complete characterization of the set of minimizable graphs (Theorem 1 below), thus answering a question raised in [11]. To state this, we need three more definitions.

Definition 2. Following [11], $H = (T,U)$ is called 3-*closed* if for each triple $s_0, s_1, s_2 \in T$ there is $v \in T$ such that $d^H(s_i v) + d^H(v s_j) = d^H(s_i s_j)$ for $0 \le i < j \le 2$ (i.e., there are shortest paths from s_0 to s_1, from s_1 to s_2 and from s_2 to s_0 that share a common node).

Definition 3. A subgraph $H' = (T', U')$ of H is called *isometric* (in H) if $d^{H'}$ coincides with d^H on T'.

Definition 4. H is called *orientable* if the edges of H can be directed so that for any 4-circuit $C = (v_0, e_1, v_1, \ldots, e_4, v_4 = v_0)$, edges e_1 and e_3 are oppositely directed, and similarly for e_2 and e_4 (e.g., if e_i is directed from v_{i-1} to v_i, say, then e_{i+2} is directed from v_{i+2} to v_{i+1}, see Fig. 2).

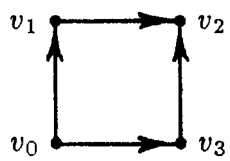


Fig. 2

For example, K_p is not 3-closed for $p \ge 3$, $K_{p,r}$ is 3-closed but non-orientable for $p, r \ge 3$, and $K_{2,r}$ is 3-closed and orientable. Obviously, each 3-closed graph is bipartite. If H is bipartite then each 4-circuit in H is isometric. A k-*circuit* is a (simple) circuit C_k with k nodes considered as a graph. A maximal subgraph $K_{2,r}$ in H is called a 2-*clique*.

Theorem 1. *The following are equivalent:*

(i) *H is minimizable;*

(ii) *H is 3-closed, orientable and contains no isometric $2k$-circuit with $k \ge 3$.*

The proof of Theorem 1 is given throughout Sections 2-4. Section 2 exhibits some basic properties of minimizable graphs. In particular, the minimizability admits a simple description in terms of vertices of the dominant polyhedron $\mathcal{D}_{V,H}$ of $\mathcal{P}_{V,H}$. We also describe the vertices of $\mathcal{D}_{V,H}$ ("H-primitive metrics") in terms of shortest paths, which then enable us to establish some necessary conditions for the minimizability. These results are then used to prove (i)$\Rightarrow$(ii) in the theorem by showing, in Section 3, that if H violates at least one condition in (ii) then one can construct an H-primitive metric m on some $V \supseteq T$ so that m is not a 0-extension. The implication (ii)$\Rightarrow$(i), more involved, is proved in Section 4 by combining combinatorial and topological ideas. First we embed H as in (ii) in an orientable 2-dimensional space (pseudo-surface) S; it is constructed by sticking up each 4-circuit of H by a disk, which forms a 2-dimensional cell, and then by identifying certain parts of cells related to 2-cliques of H. We assign an ℓ_1-metric within each cell, extend these local metrics to a metric σ on the entire S and show (in key Theorem 4.1) that for every V and every minimal extension m of H to V, the embedding of H can be extended to an isometric embedding in (S, σ) of the metrical space (V, m). Finally, we show that every finite set V on S can be mapped to T preserving the lengths of all shortest (w.r.t. σ) paths with both ends in T, completing the proof of Theorem 1. In concluding Section 5 we discuss a relationship between (1) and the so-called multiflow locking problem and explain that the minimizability of the graphs H as in (ii) without subgraphs $K_{2,r}$ with $r \geq 3$ can be also derived by use of the multiflow locking theorem in [10].

Theorem 1 shows that the set of minimizable graphs is rather large. For example, it contains every planar graph in which all inner faces are quadrangles and each node not in the outer face has degree at least four. A 2-dimensional grid $\Gamma_{p,r}$ with nodes (i, j) for $i = 0, 1, \ldots, p$ and $j = 0, 1, \ldots, r$ and edges $\{(i, j), (i', j')\}$ for $|i - i'| + |j - j'| = 1$ is a special case of the latter graphs. Another consequence of Theorem 1 is that if H and H' are minimizable, then identifying an edge of H with an edge of H' results in a minimizable graph as well.

2. Dual description and H-primitive metrics

As before, $\mathcal{M}_V$ and $\mathcal{P} = \mathcal{P}_{V,H}$ denote the sets of all metrics on V and all extensions of H to V, respectively. Clearly $\mathcal{P}$ is a polyhedron in $\mathbf{R}^{E_V}$. The *dominant polyhedron* of $\mathcal{P}$ is

$$\mathcal{D} = \mathcal{D}_{V,H} := \{x \in \mathbf{R}^{E_V} \; : \; x \geq m \;\text{ some }\; m \in \mathcal{P}\}.$$

Let $\mathcal{E} = \mathcal{E}_{V,H}$ denote the set of *minimal extensions* m, i.e., there is no $m' \in \mathcal{P}$ such that $m' \neq m$ and $m' \leq m$. Then $\mathcal{E}$ is exactly the intersection of the boundaries of $\mathcal{P}$ and $\mathcal{D}$. Similarly, for a metric μ on T we define the set $\mathcal{P} = \mathcal{P}_{V,\mu}$ of extensions of μ to V to be $\{m \in \mathcal{M}_V \; : \; m_{|T} = \mu\}$ and define the dominant polyhedron $\mathcal{D} = \mathcal{D}_{V,\mu}$ as above.

A metric that is a vertex of $\mathcal{D}$ is called *H-primitive* (or *μ-primitive*). It is easy to see that every 0-extension of H is H-primitive. The next statement exhibits a dual description of the set of minimizable graphs. We identify a graph $G = (V, E)$ and its edge multiplicity function $c : E_V \to \mathbb{Z}_+$, where for $x, y \in V$, $c(xy)$ is the number of edges of G with the ends x and y. Then the objective in (2) is $\min\{c \cdot m \; : \; m \in \mathcal{P}\}$, denoting by $a \cdot b$ the inner product $\sum(a(e)b(e) \; : \; e \in Q)$ of vectors $a, b \in \mathbf{R}^Q$.

Statement 2.1. *H is minimizable if and only if for any $V \supseteq T$ every H-primitive metric on V is a 0-extension.*

Proof. By linear programming arguments, for each $c : E_V \to \mathbb{Z}_+$ the minimum of $c \cdot m$ over $m \in \mathcal{P}$ is achieved by a vertex of $\mathcal{D}$; reversely, for each vertex m of $\mathcal{D}$ there is $c : E_V \to \mathbb{Z}_+$ such that $c \cdot m < c \cdot m'$ for any other vector m' in $\mathcal{P}$. This implies the statement. •

Note that $\mathcal{E}$ is characterized by the following property:

(2.1) an extension $m \in \mathcal{P}$ is minimal if and only if for any $x, y \in V$ there are $s, t \in T$ such that $m(sx) + m(xy) + m(yt) = \mu(st)$.

Indeed, "if" is obvious, while "only if" follows from the obvious fact that if for some x and y, $m(sx) + m(xy) + m(yt) > d^H(st)$ for any $s, t \in T$, then one can decrease m on some pairs, forming a smaller metric m' on V with $m'_{|T} = d^H$.

For $m, m' \in \mathcal{P} = \mathcal{P}_{V,H}$ (or $= \mathcal{P}_{V,\mu}$) we say that m' *H-decomposes* (or *μ-decomposes*) m if there is $m'' \in \mathcal{P}$ and $0 < \lambda \le 1$ such that $m \ge \lambda m' + (1-\lambda)m''$. When it leads to no confusion, we omit the prefix H or μ in terms "H-primitive", "μ-decomposes", and etc. Clearly m is primitive if and only if no metric in $\mathcal{P} - \{m\}$ decomposes m.

In the next section, when proving necessary conditions for a graph to be minimizable, we will construct a "bad" μ'-primitive metric for a certain submetric μ' of d^H and then use it to show that H is not minimizable. This relies on three simple facts stated in Statements 2.2, 2.4 and 2.5.

Statement 2.2. *Let μ be a metric on T, and μ' its restriction to $T' \subseteq T$. Let V' be a set such that $V' \cap T = T'$. Let m' be a μ'-primitive metric on V'. Then m' can be extended to a μ-primitive metric on $V = V' \cup T$.*

Proof. Define m by $m(xy) = m'(xy)$ for $x, y \in V'$, $m(xy) = \mu(xy)$ for $x, y \in T$, and $m(xy) = \min\{m(xs) + \mu(sy) : s \in T'\}$ for $x \in V'$ and $y \in T - T'$. One can check that m is a metric. Thus $m \in \mathcal{P}_{V,\mu}$. Take a μ-primitive m_1 on V that μ-decomposes m. Then its restriction m'_1 to V' μ'-decomposes m'. The fact that m' is μ'-primitive implies $m' = m'_1$. Hence, m_1 is as required. •

From Statement 2.2 it follows that

(2.2) for $\mu = d^H, T', V', m'$ as above, if H is minimizable, then there is a mapping $\gamma : V' \to T$ such that γ is identical on T' and $m'(xy) = d^H(\gamma(x)\gamma(y))$ for all $x, y \in V'$.

This property enables us to immediately eliminate the non-3-closed graphs H.

Statement 2.3 [11]. *If H is minimizable then H is 3-closed.*

Proof. Consider $s_0, s_1, s_2 \in T$. Let p_0, p_1, p_2 satisfy $p_i + p_j = d^H(s_i s_j)$ for $0 \leq i < j \leq 2$. These numbers are nonnegative (as d^H is a metric) and defined uniquely. Take $T' = \{s_0, s_1, s_2\}$ and $V' = T' \cup \{x\}$, and let $m'(s_i x) = p_i$, $i = 0, 1, 2$. Obviously, defining m' on T' as $\mu' = d^H{}_{|T'}$, we obtain a μ'-primitive metric on V'. Assuming that H is minimizable, let γ be as in (2.2). Then $v = \gamma(x)$ satisfies $d^H(s_i s_j) = p_i + p_j = d^H(s_i v) + d^H(v s_j)$ for $0 \leq i < j \leq 2$, as required. •

In particular, each minimizable graph is bipartite. This and (2.2) yield the following property (which will be used in Section 3 to prove that a graph is not minimizable if it is non-orientable or contains an isometric $2k$-circuit with $k \geq 3$).

Statement 2.4. *Let μ' be the restriction of d^H to a subset $T' \subseteq T$. Suppose that there is a μ'-primitive metric m' on a set V'. If m' is not integral or there are $x_0, \ldots, x_k \in V'$ such that $m'(x_0 x_1) + \ldots + m'(x_{k-1} x_k) + m'(x_k x_0)$ is an odd integer, then H is not minimizable.* •

The next statement shows that we can construct primitive metrics recursively.

Statement 2.5. *Let m be an H-primitive metric on $V \supseteq T$. Let μ' be the restriction of m to $V' \subseteq V$. Let m' be a μ'-primitive metric on $W \supseteq V'$ (assuming that $W \cap V = V'$). Then there is an H-primitive metric m'' on $V'' = V \cup W$ that coincide with m' on W.*

Proof. This is close to the proof of Statement 2.2. More precisely, define $\tilde{m}(xy)$ to be $m(xy)$ for $x, y \in V$, $m'(xy)$ for $x, y \in W$, and $\min\{m(xs) + m'(sy) : s \in V'\}$ for $x \in V$ and $y \in W$. One easily shows that $\tilde{m}$ is a metric. Take an H-primitive $m_1 \in \mathcal{P}_{V'',H}$ that H-decomposes $\tilde{m}$. Then $m_1{}_{|V} = m$ (as m is H-primitive). This implies that $m_1{}_{|V'} = \mu'$, therefore, $m_1{}_{|W}$ μ'-decomposes m'. Since m' is μ'-primitive, $m_1{}_{|W} = m'$. •

Next we discuss a method of showing that certain metrics are primitive. We need some terminology.

A sequence $P = (x_0, x_1, \ldots, x_k)$ of elements of V is called an $x_0 - x_k$ *path* on V; for brevity we use notation $x_0 x_1 \ldots x_k$ for P. If all x_i's are distinct, P is *simple.* If x_0 and x_k are distinct elements of T, P is called a *T-path.* Given $m \in \mathcal{P}_{V,H}$, a T-path $P = x_0 x_1 \ldots x_k$ is called an *H-geodesic* of m if P is simple and the *m-length* $m(P) = m(x_0 x_1) + \ldots + m(x_{k-1} x_k)$ of P equals $d^H(x_0 x_k)$. The set of H-geodesics of m is denoted by $\mathcal{G}(m) = \mathcal{G}_H(m)$. When each $e_i = x_{i-1}x_i$ is an edge of a graph $G = (V, E)$, P is said to be a path of G, and e_i an edge of P (though G may contain miltiple edges, this will lead to no confusion in what follows). $|P|$ denotes the number k of edges of P.

A metric is called *positive* if it takes non-zero values on all pairs of distinct elements. The following fact is obvious (see, e.g., [12]):

(2.3) if $m, m' \in \mathcal{P}$ and m is positive, then m' decomposes m if and only if $\mathcal{G}(m) \subseteq \mathcal{G}(m')$; in particular, every positive primitive metric is uniquely determined by its geodesics.

This property has a practical application for constructing primitive metrics; we borrow an idea from [12,1]. Let $G = (V, E)$ be a connected graph such that for some $\alpha \in \mathbb{R}_+$ the metric $m = \alpha d^G$ is in $\mathcal{P}_{V,H}$. A subgraph $G' = (V', E')$ of G is called *H-isometric* if

(2.4) for each $x, y \in V'$ there are $s, t \in T$ such that $m(sx) + \alpha d^{G'}(xy) + m(yt) = d^H(st)$;

in other words, G has an H-geodesic of m that contains, as a part, a path in G' between x and y. An H-isometric subgraph is, obviously, isometric.

Suppose that H has an H-isometric even circuit $C = x_0 x_1 \ldots x_{2k}$ ($x_{2k} = x_0$) in G. For $G' = C$, (2.4) is equivalent to saying that

(2.5) for $i = 0, \ldots, k-1$, there are $s_i, t_i \in T$ such that $m(s_i x_i) + m(x_{i+k} t_i) + \alpha k = d^H(s_i t_i)$.

Let m' decompose m. Then $\mathcal{G}(m) \subseteq \mathcal{G}(m')$, therefore, (2.5) implies that

$$\text{(2.6)} \qquad \text{for } i = 0, \ldots, k-1, \quad m'(s_i x_i) + m'(x_{i+k} t_i) + \sum_{j=i}^{i+k-1} m'(x_j x_{j+1}) = d^H(s_i t_i).$$

We may assume that $s_i = t_{i+k}$ and $t_i = s_{i+k}$. Then putting together the $2k$ inequalities in (2.6), one can derive that

$$\text{(2.7)} \qquad m'(x_i x_{i+1}) = m'(x_{i+k} x_{i+k+1}), \qquad i = 0, \ldots, k.$$

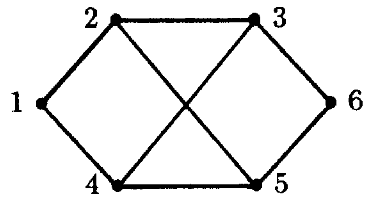
Edges $e, e' \in E$ are called *H-dependent* if there is a sequence $e = e_0, e_1, \ldots, e_q = e'$ of edges and a sequence $C_1, \ldots, C_q$ of H-isometric even circuits of G such that e_{j-1} and e_j are opposite edges of C_j, $j = 1, \ldots, q$. A maximal set of mutually H-dependent edges is called an *orbit*. The following statement is close to a result in [12,1] (see also [13]) on extreme metrics of graphs.

Statement 2.6. *If each orbit of G contains an H-geodesic (e.g., an edge with both ends in T) then $m = \alpha d^G$ is H-primitive.*

Proof. Let m' decompose m. From (2.7) it follows that $m'(e)$ is the same number b for all edges of an orbit Q. Since Q contains an H-geodesic P connecting some $x, y \in T$, b is fixed to be $d^H(xy)/|P|$. Hence, m' is uniquely determined for all edges of G. Now since G is connected, m' is uniquely determined on all pairs in V, i.e., $m' = m$. •

3. Non-minimizable configurations.

In this section we prove (i)⇒(ii) in Theorem 1. We use notation $A \simeq B$ for isomorphic graphs A and B. First we consider a special graph H. Let $K_{3,3}^{-1}$ denote the graph obtained from $K_{3,3}$ by deleting one edge, see Fig. 3. Note that $K_{3,3}^{-1}$ is non-orientable.



$K_{3,3}^{-1}$

Fig. 3

Lemma 3.1. *For $H = K_{3,3}^{-1}$ there is an H primitive extension which is not integral.*

Proof. We denote the nodes of this $H = (T, U)$ by $1, \ldots, 6$ as in Fig. 3. Split each edge $e = ij$ of H into two edges iz_e and z_ej in series, add two extra nodes x and y, add edges xz_e for all $e = ij \in U$ with $i, j \leq 5$ and add edges yz_e for all $e = ij \in U$ with $i, j \geq 2$. The resulting graph, denoted by $G = (V, E)$, is shown in Fig. 4.

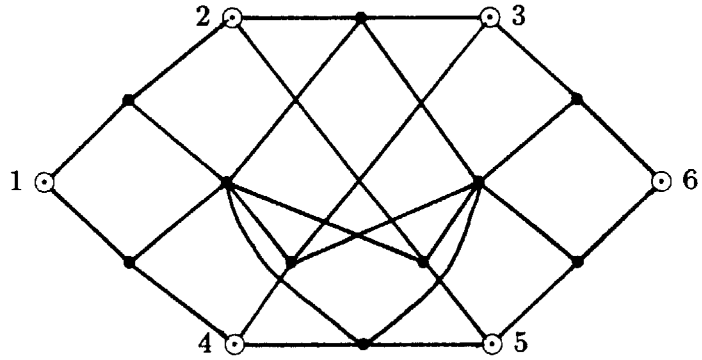


Fig. 4

A routine, though somewhat tiresome, check-up (which can be simplified by using symmetries of G) shows that: (i) $\frac{1}{2}d^G(ij) = d^H(ij)$ for all $i, j \in T$, i.e., $m = d^G/2$ is an extension of d^H; (ii) each 4-circuit in G is H-isometric; and (iii) all edges of G are H-dependent, i.e., G has an only orbit. By Statement 2.6, m is H-primitive. Also m is not integral. •

A special role of the graph $K_{3,3}^{-1}$ is demonstrated by the following statement.

Statement 3.2. *Let $H' = (T', U')$ be an isometric subgraph of H. Suppose that there is a graph $G = (V, E)$ with $T' \subseteq V$ such that (i) $m = d^G$ is H'-primitive, and (ii) G has an isometric subgraph $G' = (V', E') \simeq K_{3,3}^{-1}$. Then H is not minimizable. In particular, H is not minimizable if it contains $K_{3,3}^{-1}$ as an isometric subgraph.*

Proof. By Lemma 3.1, there exists a G'-primitive extension m' to a set $W \subseteq V'$ such that m' is not integral. Since G' is isometric in G, $\mu' = d^{G'}$ is a submetric of d^G. Applying Statement 2.5 to H', m, μ', m', we conclude that there is an H'-primitive metric m'' on $V \cup W$ that coincides with m' on W. Since H' is isometric in H, m'' is a primitive extension of a submetric of d^H. Now the fact that m'' is not integral implies that H is not minimizable (by Statement 2.4). •

This statement enables us to eliminate graphs with big isometric circuits.

Lemma 3.3. *Let H contain an isometric $2k$-circuit C with $k \geq 3$. Then H is not minimizable.*

Proof. Let $T' = \{1, \ldots, 2k\}$ be the set of nodes of C and $1, \ldots, 2k$ follow in this order in C. In view of Statement 3.2, it suffices to show the existence of a graph $G = (V, E)$ with $T' \subseteq V$ such that d^G is C-primitive and G contains $K_{3,3}^{-1}$ as a C-isometric subgraph. The desired graphs G (for $k = 3$ and $k \geq 4$) are depicted in Fig. 5.

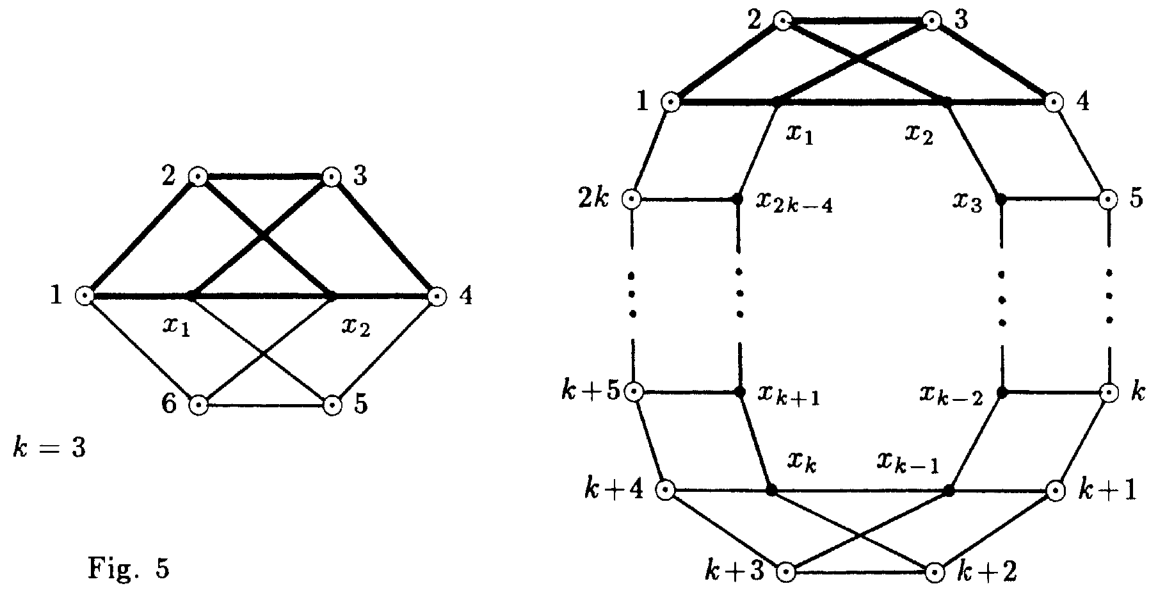


Fig. 5

One can check that: (i) in both cases, G contains a C-isometric subgraph $G' \simeq K_{3,3}^{-1}$ (G' is drawn bold in Fig. 5); (ii) for $k = 3$, all edges of G are C-dependent; (iii) for $k \geq 4$, there are $k - 2$ orbits $O_1, \ldots, O_{k-2}$, each containing an edge of C (namely, for $i = 2, \ldots, k-2$, O_i is formed by the edges $\{i+2, i+3\}, \{x_i, x_{i+1}\}, \{k+i+2, k+i+$

3}, $\{x_{k+i-2}, x_{k+i-1}\}$ (letting $x_{2k-3} = x_1$ and identifying $\{2k, 2k+1\}$ with $\{2k, 1\}$), and O_1 is the rest). Thus, G is as required. •

For further purposes we need two more "non-minimizable configurations". As mentioned in the Introduction, $H' = K_{p,r}$ is not minimizable for $p, r \geq 3$. This is because adding to H' a node x and edges xs for all nodes s in H' results in graph G for which the H'-primitivity of d^G can be easily shown. Since G is not bipartite, H' is not minimizable (cf. Statement 2.4). Note also that any subgraph $H' \simeq K_{p,r}$ of a bipartite graph H is isometric in H.

Another example is the graph C_6^+ depicted in Fig. 6. This graph has an isometric 6-circuit C (namely, that follows $1, 2, \ldots, 6$), so C_6^+ is not minimizable. If we add to C_6^+ one or two edges connecting opposite nodes in C, then the resulting graph contains an isometric subgraph $K_{3,3}^{-1}$, while adding the three such edges turns C into $K_{3,3}$.

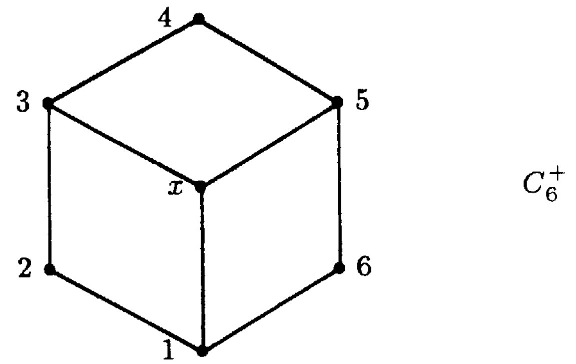


Fig. 6

The above arguments imply the following statement (here we also use the fact that adding to $K_{3,3}^{-1}$ an edge connecting the nodes at distance three in it makes $K_{3,3}$).

Statement 3.4. *If H is bipartite and has a subgraph isomorphic to $K_{3,3}^{-1}$ or C_6^+, then H is not minimizable.* •

Remark 3.5. If H contains an isometric $2k$-circuit with $k \geq 3$ or subgraph $K_{3,3}^{-1}$, then problem (2) has "infinite fractionality", in the sense that for any $q \in \mathbb{Z}_+$ there is an H-primitive metric m with the denominator of some component of m at least q. This follows from the fact that the graph G in the proof of Lemma 3.1 (see Fig. 4) has an isometric $K_{3,3}^{-1}$. Therefore, we can recursively "expand" such subgraphs by metrics proportional to d^G, obtaining H-primitive metrics with increasing denominators. In contrast, as mentioned in the Introduction, the graphs K_p and $K_{p,r}$ ($p, r \geq 3$), though not minimizable, have the property that the denominators of every primitive extension are at most two.

Next we examine the last bad case in (ii) of Theorem 1. Here we apply slightly different techniques. By a *dual path* of H we mean a sequence $D = (e_0, F_1, e_1, \dots, F_k, e_k)$, where $e_0, \dots, e_k$ are edges and $F_1, \dots, F_k$ 4-circuits in H, and e_{i-1}, e_i are opposite edges in F_i. When $e_0 = e_k$, D is a *dual cycle*.

Lemma 3.6. *Let H be bipartite and non-orientable. Then H is not minimizable.*

Proof. Recall that H has no parallel edges. Consider a dual path $D = (e_0, F_1, e_1, \dots, F_k, e_k)$ in H. For $i = 1, \dots, k$, let $F_i = x_{i-1}y_{i-1}y_ix_ix_{i-1}$, $e_{i-1} = x_{i-1}y_{i-1}$ and $e_i = x_iy_i$, see Fig. 7.

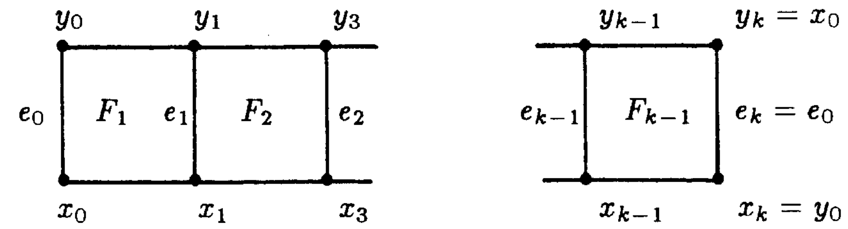


Fig. 7

One can see that the fact that H is non-orientable means the existence of D as above such that $e_0 = e_k$, $x_k = y_0$ and $y_k = x_0$, an *orientation reversing* dual cycle. We assume that D is chosen so that its length $|D| = k$ is as small as possible. We consider D up to shifting cyclically, take indices modulo k and let $F_0 = F_k$.

Note that the minimality of D implies that all $e_1, \dots, e_k$ are different (while some F_i and F_j may coincide). We observe that

(3.1) if $x_i = y_j$ then $|i - j|$ is odd, and if $x_i = x_j$ or $y_i = y_j$ then $|i - j|$ is even;

else H is not bipartite. In view of Statement 3.4, we may assume that H contains no subgraph $K_{3,3}^{-1}$. We observe that

(3.2) if $x_i = x_{i+2}$ for some i, then all elements of $Z = \{x_i, x_{i+1}, x_{i+3}, y_j, j = i, \dots, i+3\}$ are different, and similarly if $y_i = y_{i+2}$;

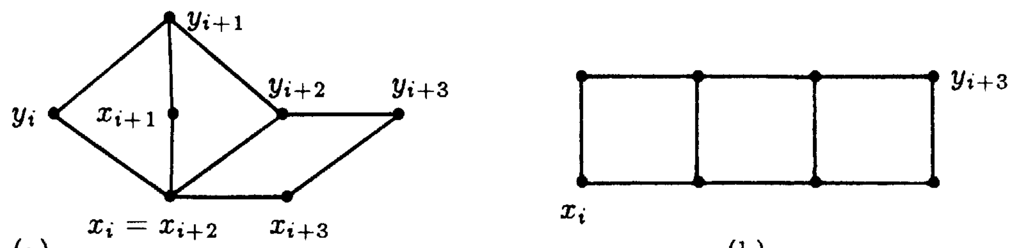


Fig. 8 (a) (b)

see Fig. 8a. Indeed, one can check that if $x_i = x_{i+2}$ and some elements of Z coincide, then H has parallel edges or a subgraph $K_{3,3}^{-1}$ or D is not minimal (the latter case occurs when $y_{i+1} = y_{i+3}$; then the part from F_{i+1} to F_{i+3} in D can be replaced by the 4-circuit $x_iy_iy_{i+3}x_{i+3}x_i$). To a similar reason, (3.3) for $i = 0, \ldots, k-1$, $x_i \neq y_{i+3}$ and $y_i \neq x_{i+3}$;

see Fig. 8b. In particular, (3.1) and (3.3) show that k is odd and $k > 3$. We say that F_i and F_{i+1} are *squeezed* if either $x_{i-1} = x_{i+1}$ or $y_{i-1} = y_{i+1}$ (then $F_i \cup F_{i+1}$ is $K_{2,3}$). By (3.2), in this case F_{i-1} and F_i are non-squeezed, and similarly for F_{i+1} and F_{i+2}. By (3.1), if F_i and F_{i+1} are non-squeezed, then the six nodes x_j, y_j, $j = i-1, i, i+1$, are different.

Form the graph $G' = (V', E')$ by adding to H nodes z_i and edges connecting z_i to x_{i-1}, x_i y_{i-1} and y_i for $i = 1, \ldots, k$ (note that z_i and z_j are different even if $F_i = F_j$). Next we transform G' into $G = (V, E)$ as follows. For $i = 1, \ldots, k$, if F_i and F_{i+1} are squeezed, we identify z_i and z_{i+1} (and identify multiple edges appeared). And if F_i and F_{i+1} are non-squeezed, we add edge $u_i = z_i z_{i+1}$. See Fig. 9.

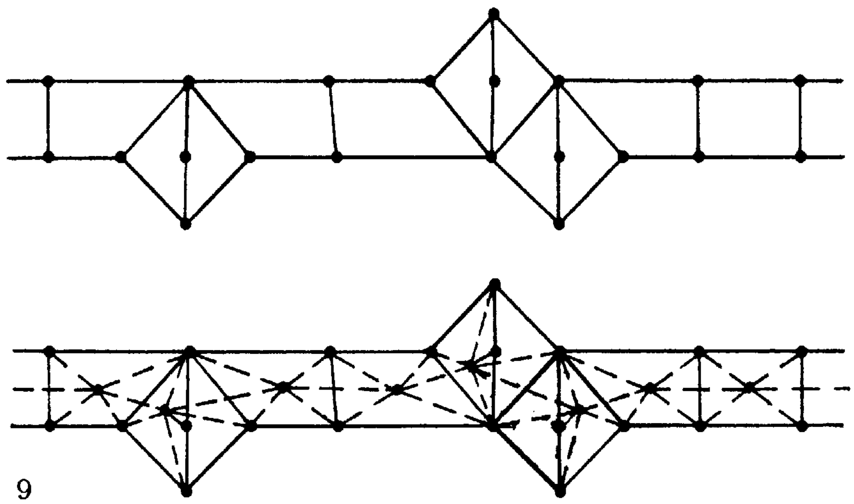


Fig. 9

Claim 1. *d^G is an extension of d^H.*

Proof. Suppose that this is not so. Then G has a path $P = v_0 v_1 \ldots v_q$ with $v_0, v_q \in T$ such that its length $|P| = q$ is less than $d^H(v_0 v_q)$. Choose P with $|P|$ minimum under this property. Then the intermediate nodes $v_1, \ldots, v_{q-1}$ of P are, obviously, not in T. Since none of the edges in $E - U$ connects elements of T, $|P| > 1$. If $|P| = 2$ then, by the construction of G, there is i for which at least one of the following is true: (i) $v_1 = z_i$ and both v_0, v_2 are in F_i, or (ii) F_i and F_{i+1} are squeezed, $v_1 = z_i = z_{i+1}$ and both v_0, v_2 are in $F_i \cup F_{i+1}$ ($\simeq K_{2,3}$). In both cases, $d^H(v_0 v_2) \leq 2$.

Thus $|P| \geq 3$. Then the edge $e = v_1 v_2$ is of the form $z_i z_{i+1}$ for some i; we may assume that $v_1 = z_i$. If v_0 is x_i (or y_i), then v_0 and $v_2 = z_{i+1}$ are connected by an edge in G; therefore, replacing the part $v_0 v_1 v_2$ by $v_0 v_2$ makes a shorter v_0–v_q path, contradicting the minimality of P. If v_0 is x_{i-1} (case $v_0 = y_{i-1}$ is similar), then replacing the part $v_0 v_1 v_2$ by the path $v_0 x_i v_2$ makes an v_0–v_q path P' with $|P'| = |P|$.

Now the part P'' of P' from x_i to v_q is a T-path satisfying

$$|P''| = |P'| - 1 < d^H(v_0v_q) - 1 \leq d^H(x_iv_q),$$

contrary to the minimality of P. Finally, if F_{i-1} and F_i are squeezed, $x_i = x_{i-2}$ say, and v_0 is in F_{i-1} but F_i (i.e., $v_0 = y_{i-2}$), then $v_0x_iv_2$ is again a path in G, and we get a contradiction as above. •

Hence, $d^G \in \mathcal{P}_{V,H}$. Choose an H-primitive metric m' on V that H-decomposes d^G. We now use the property that each geodesic of d^G is a geodesic of m' too. For $i = 1,\ldots,k$, let $\alpha_i = m'(x_{i-1}z_i)$, $\alpha_i' = m'(x_iz_i)$, $\beta_i = m'(y_{i-1}z_i)$ and $\beta_i' = m'(y_iz_i)$.

Claim 2. *For $i = 1,\ldots,k$, $\alpha_i - \beta_i = \alpha_{i+1} - \beta_{i+1}$.*

Proof. Consider two possible cases.

(i) F_i and F_{i+1} are not squeezed. Then $d^H(x_{i-1}y_{i+1}) = d^H(y_{i-1}x_{i+1}) = 3$ (for if, say, x_{i-1} and y_{i+1} are connected by an edge e in H, then adding e to $F_i \cup F_{i+1}$ forms $K_{3,3}^{-1}$). Hence, both $P = x_{i-1}z_iz_{i+1}y_{i+1}$ and $P' = y_{i-1}z_iz_{i+1}x_{i+1}$ are H-geodesics of G. Then $m'(P) = m'(P') = 3$. This implies $\alpha_i + \beta_{i+1}' = \beta_i + \alpha_{i+1}'$. This together with the obvious relations $\alpha_{i+1} + \beta_{i+1}' = 2$ and $\beta_{i+1} + \alpha_{i+1}' = 2$ implies $\alpha_i - \beta_i = \alpha_{i+1} - \beta_{i+1}$.

(ii) F_i and F_{i+1} are squeezed, $x_{i-1} = x_{i+1}$ say. Since the T-paths $y_{i-1}z_ix_i$, $x_iz_iy_{i+1}$ and $y_{i-1}z_iy_{i+1}$ are shortest in G, we have $\beta_i = \alpha_{i+1} = \beta_{i+1}' = 1$. Also $x_{i-1}z_iy_i$ is shortest, whence $\alpha_i + \beta_{i+1} = 2$. Hence, $\alpha_i - \beta_i = (2 - \beta_{i+1}) - 1 = 1 - \beta_{i+1} = \alpha_{i+1} - \beta_{i+1}$. •

Now we finish the proof of the lemma as follows. We derive from Claim 2 that $\alpha_1 - \beta_1 = \alpha_{k+1} - \beta_{k+1}$. Since $x_0 = y_k$ and $y_0 = x_k$, we have $\beta_{k+1} = \alpha_1$ and $\alpha_{k+1} = \beta_1$. Thus, $m'(x_0z_1) = \alpha_1 = \beta_1 = m'(y_0z_1)$. This implies that if α_1 is an integer, then the m'-length of the circuit $x_0y_0z_1x_0$ is odd, so H is not minimizable, by Statement 2.4. ••

This completes the proof of (i)⇒(ii) in Theorem 1.

4. Embedding in a surface

In this section we show that if $H = (T,U)$ satisfies (ii) in Theorem 1, then H is embeddable in a 2-dimensional space $S = S^H$ with a special metric on it so that every positive minimal extension (V,m) of H admits a unique isometric embedding in S (Theorem 4.1). Reversely, every finite subset $V \supseteq T$ of S determines a minimal extension (V,m) of H. We then explain that for such a (V,m) each point in V can be "shifted" along S into one of the closest points in T so as to preserve the length of every shortest T-path for m (Lemma 4.2). This will prove (ii)$\Rightarrow$(i) in Theorem 1.

As before, we assume that H is connected and has no multiple edges. Also it will be convenient to assume that H is 2-edge-connected (which does not lead to loss of generality, in view of Example 4 in the Introduction). Then each edge of H belongs to a 4-circuit (as H is bipartite and has no isometric $2k$-circuit with $k \geq 3$).

The set S that we are now constructing is defined uniquely by the 2-cliques of H (in particular, by its 4-circuits).

First, we expand each 4-circuit $C = v_0v_1v_2v_3v_0$ (considered up to reversing and/or shifting cyclically) to a 2-dimentional disc $D = D^C$. Formally, D is homeomorphic to $[0,1] \times [0,1] \subset \mathbf{R}^2$, nodes v_0, v_1, v_2, v_3 are identified with points (0,0),(0,1),(1,1),(1,0), respectively, and the edges of C with the corresponding straight line segments in the boundary of D (e.g., v_1v_2 is identified with $\{(\xi, 1) : 0 \leq \xi \leq 1\}$). In what follows we do not differ a node (edge) of C from the corresponding point (segment) in D^C.

Second, if 4-circuits C and C' have a single common node (edge), we identify the corresponding points (segments) in D^C and $D^{C'}$.

Third, suppose that 4-circuits $C = v_0v_1v_2v_3v_0$ and $C' = u_0u_1u_2u_3u_0$ have two common edges, say, $v_i = u_i$ for $i = 0, 1, 2$. Then we identify corresponding halves of D^C and $D^{C'}$. More precisely, if, for definiteness, v_0, v_1, v_2 correspond to (0,0),(0,1),(1,1) in D^C, and similarly for C', then each point (ξ, η) with $0 \leq \xi \leq \eta \leq 1$ in D^C is identified with (ξ, η) in $D^{C'}$.

The resulting space is just $S = S^H$, called the *H-surface*. For a 4-circuit C of H, we call D^C the (2-dimensional) *cell* of S induced by C. [This notion of a cell is somewhat different from what is usually meant by "closed cells" in a cell partition of a topological space (see, e.g., [15]); standard cells will appear if we subdivide each cell D^C in folders (defined below) into two triangles.]

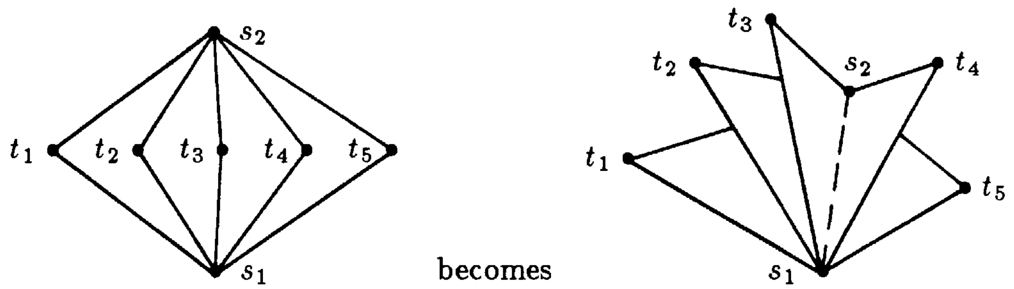


Fig. 10

We observe that if H' is a 2-clique of H with node parts $\{s_1, s_2\}$ and $\{t_1, \ldots, t_r\}$, then by the above rule the region $F^{H'}$ of S into which H' is expanded is homeomorphic to the space obtained from r copies of the "triangle" $\{(\xi, \eta) : 0 \leq \xi \leq \eta \leq 1\}$ by sticking them along the "diagonal" $\{(\alpha, \alpha)\} : 0 \leq \alpha \leq 1\}$, see Fig. 10 (for $r = 5$).

We call $F^{H'}$ the *folder* in S induced by H'. Each 4-circuit $s_1t_is_2t_js_1$ forms a cell in $F^{H'}$. Note that no interior point of $F^{H'}$ belongs to another folder (in particular, a cell not in $F^{H'}$). Otherwise for some $1 \le i < j \le r$, there is a path $P = t_ivt_j$ in H' with $v \neq s_1, s_2$; then $H' \cup P$ contains a subgraph $K_{3,3}^{-1}$, and therefore, H is non-orientable.

Remark 1. One can show that the orientability of H implies that S is an orientable space. Also from the non-existence of "big" isometric circuits and $K_{3,3}^{-1}$-subgraphs one can deduce that S is *contractable*, i.e., there is a continuous mapping $\psi : S \times [0,1] \to S$ such that $\psi(x,0) = x$ and $\psi(x,1) = v$ for all $x \in S$ and a fixed point $v \in S$. In particular, S contains no compact 2-dimentional surface. Using these facts, one can give alternative, sometimes simpler, proofs of some of statements later on. However, we prefer more combinatorial proving methods here, without appealling to the above facts.

Next, we assign an ℓ_1-metric σ^C within each cell D^C in a natural way. More precisely, the above representation of D^C establishes cartesian coordinates (ξ, η) in D^C, and we assign the ℓ_1-distance σ^C between points $x = (\xi', \eta')$ and $y = (\xi'', \eta'')$ of D^C, i.e., $\sigma^C(xy) = |\xi' - \xi''| + |\eta' - \eta''|$. We assume that these local metrics are assigned to be compatible on common edges of cells and common parts of cells in folders. We extend these metrics to the global metric $\sigma = \sigma^H$ on S, where for $x, y \in S$, $\sigma(xy)$ is the infinum of values $\sigma^{C_1}(x_0x_1) + \ldots + \sigma^{C_N}(x_{N-1}x_N)$ among all sequences $x = x_0, x_1, \ldots, x_N = y$ of points of S such that each pair x_{i-1}, x_i belongs to the same cell, namely, D^{C_i}. Obviously, σ satisfies the symmetry and triangle inequalities. Also we can see that for each 4-circuit C, σ coincides with σ^C within D^C.

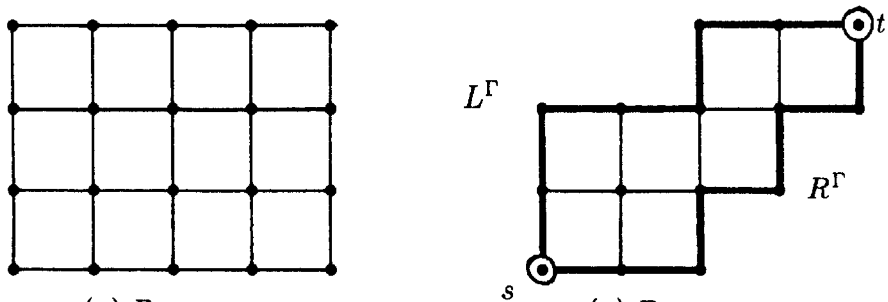


Fig. 11 (a) $\Gamma_{4,3}$ (a) Γ

A simple example of graphs H as in (ii) of Theorem 1 is a grid $\Gamma_{p,r}$, or a $p \times r$*-grid*, defined in the Introduction (see Fig. 11a where $p = 4, r = 3$). In what follows an important role will play a subgraph Γ of $\Gamma_{p,r}$ induced by the nodes (i,j) satisfying $a_j \le i \le b_j$ for two sequences $0 = a_0 \le a_1 \le \ldots \le a_r \le p$ and $0 \le b_0 \le b_1 \le \ldots \le b_r = p$ with $a_j \le b_j$, $j = 0, \ldots, r$. We call Γ a *net*, or an s–t *net*, with *origin* $s = (0,0)$ and *end* $t = (p,r)$, and denote the *rightmost* (*leftmost*) path from s to t in Γ by R^Γ (resp. L^Γ); see Fig. 11b. A node of H that belongs to Γ and has coordinates (ξ, η) in it is denoted by $(\xi, \eta)_\Gamma$.

For a u–v path P and nodes $x, y \in P$, $P(x,y)$ denotes the part from x to y in P; P^{-1} denotes the path reverse to P; and $P \cdot Q$ denotes the concatenation of P and a v–w path Q.

Theorem 4.1. *Let H satisfy (ii) in Theorem 1, and let (V,m) be a minimal extension of H. Then there exists a mapping $\omega : V \to S^H$ such that $\omega(v) = v$ for all $v \in T$ and $m(xy) = \sigma^H(\omega(x)\omega(y))$ for all $x, y \in V$. Moreover, such a mapping is unique, i.e., (V,m) admits a unique isometric embedding in (S^H, σ^H).*

Proof. It falls into several claims. Claims 1-4 refine the structure of shortest paths in H and reveal important facts concerning nets in H; these auxiliary claims will then enable us to locate the elements of V on S and prove the theorem (Claims 5-8). Throughout the proof, d stands for d^H. We will use many times the fact that H does not contain the "forbidden configurations" $K_{3,3}^{-1}$ and C_6^+. It is convenient to state this property in the following form:

(4.1) if H' is a subgraph of H that is the union of two 4-circuits with an only common edge as in Fig. 12, then no node $\overline{v} \in T$ different from $\overline{y}$ is adjacent to both $\overline{x}$ and $\overline{z}$.

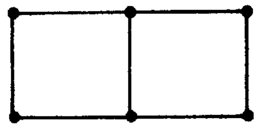


Fig. 12

Indeed, the existence of such a $\overline{v}$ leads to the existence of C_6^+ (when $\overline{v}$ is different from all nodes in H') or $K_{3,3}^{-1}$ (otherwise), taking into account that H is bipartite.

We start this the following basic fact.

Claim 1. *For $s, t \in T$ let P and P' be shortest s–t paths in H. Then there is an s–t net Γ in H with $R^\Gamma = P$ and $L^\Gamma = P'$.*

Proof. Let $P = x_0x_1 \ldots x_k$ and $P' = y_0y_1 \ldots y_k$; so $s = x_0 = y_0$, $t = x_k = y_k$ and $d(st) = k$. We use induction on $|P| = k$. If $k \le 2$, the result is obvious; so assume that $k \ge 3$. Also we assume that P and P' have no common inner node, else the result easily follows by induction.

Let P_i stand for $P(x_0, x_i)$ and P'_i stand for $P'(y_0, y_i)$. Since H has no isometric $2k$-circuit, there are $0 < i, j < k$ such that both $i + j$ and $(k - i) + (k - j)$ are greater than $q = d(x_iy_j)$. Let such i, j be chosen so that $i + j$ is minimum. Also we may assume that $j \le i$. Let $B = z_0z_1 \ldots z_q$ be a shortest path from $z_0 = y_j$ to $z_q = x_i$. By the minimality of $i + j$, (i) no inner node of B meets $P_i \cup P'_j$, and (ii) $q = i + j - 2$ (since $d(x_iy_{j-1}) = i + j - 1$ and H is bipartite).

Let $s' = y_{j-1}$. By (ii), both s'–x_i paths $Q = (P'_{j-1})^{-1} \cdot P_i$ and $Q' = s'z_0z_1 \dots z_q$ are shortest; also $|Q'| = q + 1 < k = |P|$. By induction there is an s'–x_i net Γ' in H with $R^{\Gamma'} = Q$ and $L^{\Gamma'} = Q'$. Since the path $(P'_j)^{-1} \cdot P_{i-1}$ is shortest (by the minimality of $i + j$) and contains $s' = (0,0)_{\Gamma'}$, we have $y_j = (0,1)_{\Gamma'}$ and $x_{i-1} = (q,0)_{\Gamma'}$. This implies $s = (a,0)_{\Gamma'}$ for some $0 \leq a \leq q$ and $x_i = (q,1)_{\Gamma'}$ (as the y_j–x_i path B in Γ' has length q). Therefore, Γ' is the grid $\Gamma_{q,1}$.

Let $\overline{P}_i$ be the s–x_i path in Γ' passing through $(a,0),(a,1),(a+1,1),\dots,(q,1)$, and $\overline{P}'_j$ the s–y_j path passing through $(a,0),(a,1),(a-1,1),\dots,(0,1)$. Since $|\overline{P}_i| = |P_i|$ and $|\overline{P}'_j| = |P'_j|$, the s–t paths $\overline{P} = \overline{P}_i \cdot P(x_i, x_k)$ and $\overline{P}' = \overline{P}'_j \cdot P'(y_j, y_k)$ are shortest. Also $s'' = (a,1)_{\Gamma'}$ is a common node in these paths. Let $\widetilde{P}$ ($\widetilde{P}'$) be the part of $\overline{P}$ (resp. $\overline{P}'$) from s'' to t. Then $|\widetilde{P}| = |\widetilde{P}'| < k$, so by induction there is an s''–t net Γ'' with $R^{\Gamma''} = \widetilde{P}$ and $L^{\Gamma''} = \widetilde{P}'$. Let y_j and x_i have coordinates (α, β) and (γ, δ) in Γ'', respectively. Note that $j \leq i$ implies $x_i \neq s''$ (otherwise $i = j = 1$ and $q = 0$, i.e., $x_i = y_j$ is a common node in P and P'). The fact that the above shortest path B contains s'' implies $\alpha + \beta + \gamma + \delta = |\alpha - \gamma| + |\beta - \delta|$. This is possible only if either (i) $\alpha = \beta = 0$ (i.e., $y_j = s''$ and $s' = s$), or (ii) $\beta > 0$, $\gamma > 0$ and $\alpha = \delta = 0$.

In case (ii), the subgraph of Γ' induced by the nodes (p,r) for $p = a-1, a, a+1$ and $r = 0,1$ together with the path $((0,1),(1,1),(1,0))$ in Γ'' forms the forbidden configuration as in (4.1). Thus, this case is impossible.

In case (i), we assume that, among all possible net representations for Γ'' (when Γ'' is not 2-edge-connected), the net is chosen so that the coordinate δ is as small as possible. If $\delta = 0$ then the union of Γ' and Γ'' is just the desired s–t net Γ with $R^{\Gamma} = P$ and $L^{\Gamma} = P'$. Finally, suppose that $\delta > 0$. Then Γ'' contains nodes $u = (h,0), v = (h+1,0), w = (h+1,1)$ and $z = (h,1)$ such that u, v, w belong to $R^{\Gamma''}$. But the two 4-circuits in Γ' that contain v together with the path uzw in Γ'' form the forbidden configuration as in (4.1); a contradiction. •

Remark 2. In general, the subgraph $H_{s,t}$ of H that is the union of all shortest s–t paths may have a somewhat more complicated structure than a net; Figure 13 illustrates an instance of $H_{s,t}$. One can describe the structure of $H_{s,t}$ in terms of unions of nets. Also one can show that for any $0 \leq \varepsilon \leq d(st)$, the set of points $x \in S$ with $\sigma(sx) = \varepsilon$ and $\sigma(xt) = d(st) - \varepsilon$ is homeomorphic to a tree. However, we do not use these facts in the sequel.

Remark 3. For a fixed $v \in T$, let V_i be the set of nodes of H at distance i from v. From Claim 1 one can derive that if $x \in V_i$ is adjacent to distinct $y, z \in V_{i-1}$, then there is a unique $w \in V_{i-2}$ adjacent to both y, z. Using this fact, one can easily arrange a homotopy ψ mentioned in Remark 1 to show that S is contractable.

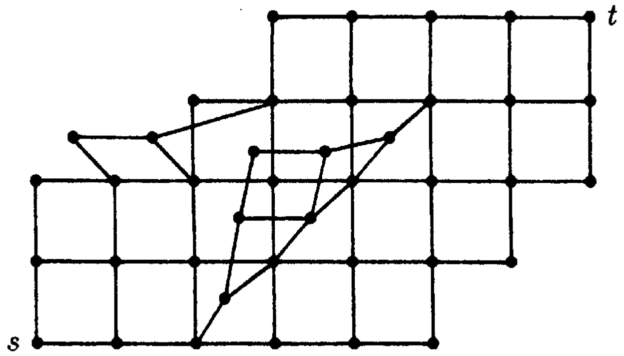


Fig. 13

Claim 2. *Every 2-connected net in H is isometric.*

Proof. Suppose this is not so for some 2-connected $s-t$ net Γ. Then there are $x = (p,q)_\Gamma$ and $y = (p',q')_\Gamma$ for which $\Delta := |p-p'| + |q-q'| > d(xy)$. In addition, let x, y be chosen so that Δ is minimum. Since H is bipartite, $\Delta > 2$.

From the facts that Γ is 2-connected and $\Delta > 2$ one can see that there exists an $x-y$ path of length Δ in Γ, $Q = x_0x_1\dots x_\Delta$ say, such that at least one of $|p_0 - p_2|, |q_0 - q_2|, |p_\Delta - p_{\Delta-2}|, |q_\Delta - q_{\Delta-2}|$ equals two, where $x_i = (p_i, q_i)_\Gamma$. We may assume that $q_2 = q_0 + 2$. Let Q' be the part of Q from x_1 to $x_\Delta = y$, and P' the concatenation of x_1x_0 and a shortest $x-y$ path. Since H is bipartite, the minimal choice of x, y implies that both P' and Q' are shortest x_1-y paths.

By Claim 1, there is an x_1-y net Γ' with $R^{\Gamma'} = P'$ and $L^{\Gamma'} = Q'$. Then $x_0 = (1,0)_{\Gamma'}$, $x_1 = (0,0)_{\Gamma'}$ and $x_2 = (0,1)_{\Gamma'}$. Next, by the above assumption, $p_0 = p_1 = p_2$ and $q_0+1 = q_1 = q_2-1$. Moreover, since Γ is 2-connected, Γ contains nodes $u = (\overline{p}, q_0)$, $v = (\overline{p}, q_1)$ and $w = (\overline{p}, q_2)$, where $\overline{p}$ is either $p_0 - 1$ or $p_0 + 1$. But the subgraph of Γ induced by $\{x_0, x_1, x_2, u, v, w\}$ together with the path ((1,0),(1,1),(0,1)) in Γ' forms the forbidden configuration as in (4.1); a contradiction. •

This claim is strengthened as follows.

Claim 3. *Let $\overline{\Gamma}$ be a 2-connected $\overline{s}-\overline{t}$ net and $\overline{t} = (p,q)_{\overline{\Gamma}} \neq (0,0)_{\overline{\Gamma}}$. Let $\overline{s}' = (1,1)_{\overline{\Gamma}}$ and $\overline{t}' = (\overline{p}-1, \overline{q}-1)_{\overline{\Gamma}}$. Let $\overline{x}, \overline{y} \in T$ be such that $d(\overline{x}\overline{s}') = d(\overline{x}\overline{s}) + 2$ and $d(\overline{y}\overline{t}') = d(\overline{y}\overline{t}) + 2$. Then $d(\overline{x}\overline{y}) = d(\overline{x}\overline{s}) + d(\overline{y}\overline{t}) + \overline{p} + \overline{q}$.*

Proof. Choose a shortest $\overline{x}-\overline{s}$ path Q_1 and a shortest $\overline{t}-\overline{y}$ path Q_2 in H. We have to show that $\overline{P} = Q_1 \cdot \widetilde{Q} \cdot Q_2$ is shortest, where $\widetilde{Q}$ is an $\overline{s}-\overline{t}$ path of length $\overline{p} + \overline{q}$ in $\overline{\Gamma}$. Suppose this is false and choose a part $P' = x_0x_1\dots x_k$ of $\overline{P}$ with $|P'| = k$ minimum provided that $|P'| > d(x_0x_k)$. In addition, we assume that $|P'|$ is minimum among all possible $\overline{P}$ (when $\widetilde{Q}$ varies). Let $P'' = x_ix_{i+1}\dots x_j$ be the common part of P' and the corresponding $\widetilde{Q}$. In view of Claim 2, P' is not entirely in $\overline{\Gamma}$; so we may assume that $0 < i < k$ and $x_i = \overline{s}$ (whereas P' may not contain $\overline{t}$). W.l.o.g. we may assume that P''

is the *leftmost* shortest path between $x_i = \overline{s}$ and x_j in $\overline{\Gamma}$; in particular, $x_{i+1} = (0,1)_{\overline{\Gamma}}$.

We consider the path $P = x_1x_2\dots x_k$ and the concatenation B of x_1x_0 and a shortest x_0-x_k path. By the minimality of P', both P and B are shortest and openly disjoint (moreover, obviously, no inner node of B belongs to $\overline{\Gamma}$). By Claim 1, there is an x_1-x_k net Γ with $L^\Gamma = B$ and $R^\Gamma = P$. For $f = 0,\dots,j$ let x_f have coordinates (ξ_f,η_f) in Γ and (p_f,q_f) in $\overline{\Gamma}$ (when $f \geq i$). Then $(\xi_1,\eta_1) = (0,0)$ and $(\xi_0,\eta_0) = (0,1)$; also both ξ_j,η_j are nonzero. The fact that $d(\overline{xs}') = d(\overline{xs}) + 2$ implies that the path $Q = x_0x_1\dots x_ix_{i+1}\overline{s}'$ is shortest; in particular, $\xi_{i'} = i'-1$ and $\eta_{i'} = 0$ for $i' = 1,\dots,i+1$ (as $x_0 = (0,1)_\Gamma$ and $x_1 = (0,0)_\Gamma$). Let α be the maximum index for which $\eta_\alpha = 0$; then $i < \alpha < j$.

We observe that $p_\beta = 0$ for $\beta = i,\dots,\alpha$. For if $p_\beta = 1$ for some $\beta \leq \alpha$, and β is minimum under this property, then $\beta > i+1$ (as $p_{i+1} = 0$) and $x_\beta = (0,\beta-1)_\Gamma = (1,\beta-i-1)_{\overline{\Gamma}}$. Therefore, Γ contains (ξ,η) for $\xi = \beta-3,\beta-2,\beta-1$ and $\eta = 0,1$, and the subgraph of Γ induced by these nodes together with the path $((0,\beta-i-2),(1,\beta-i-2),(1,\beta-i-1))$ in $\overline{\Gamma}$ contradicts (4.1). To a similar reason, $p_{\alpha+1} = 1$. For if $p_{\alpha+1} = 0$, then the subgraph of $\overline{\Gamma}$ induced by the nodes (p,q) for $p = 0,1$ and $q = \alpha-i-1,\alpha-i,\alpha-i+1$ and the path $((\alpha-2,0),(\alpha-2,1),(\alpha-1,1))$ in Γ contradict (4.1) (the above nodes (p,q) exist as $\overline{\Gamma}$ is 2-connected).

Finally, the node $x_{\alpha+1}$ cannot coincide with $\overline{s}'$ (as the above path Q is shortest, while the distance in Γ between x_0 and $x_{\alpha+1}$ is $\alpha-1$). Therefore, $\alpha \geq i+2$. Let $y = (\alpha-3,1)_\Gamma$ and $y' = (\alpha-2,1)_\Gamma$. Let $z = (1,\alpha-i-2)_{\overline{\Gamma}}$ and $z' = (1,\alpha-i-1)_{\overline{\Gamma}}$. Note that y is different from z (otherwise the path $x_0x_1\dots x_{\alpha-2}z$ is not shortest and its length is less than $|P'|$, contrary to the choice of P'). Similarly, $y' \neq z'$. Therefore, the nodes $y',y,x_{\alpha-2},z,z',x_{\alpha+1}$ are different (taking into account that H is bipartite, the nodes in Γ are different, and similarly for $\overline{\Gamma}$). Moreover, these nodes form a 6-circuit in H (in the above order). But $x_{\alpha-1}$ is adjacent to each of $y',x_{\alpha-2},z'$. Hence, H contains C_6^+; a contradiction. •

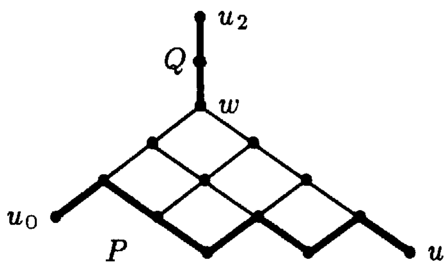


Fig. 14

Claim 4. *Let $u_0,u_1,u_2 \in T$, and let P be a shortest u_0-u_1 path in H. Then there are a u_0-u_1 net Γ and a path Q from u_2 to some node $w \in P$ such that:* (i) $R^\Gamma = P$; (ii) $w = (0,r)_\Gamma$, *assuming that* $u_1 = (p,r)_\Gamma$; (iii) *no inner node of Q belongs to Γ; and* (iv) $\Gamma \cup Q$ *is isometric.*

(See Fig. 14)

Proof. Choose $w \in T$ and u_i–w paths B_i, $i = 0,1,2$, such that each of $Q_1 = B_0 \cdot B_2^{-1}$, $Q_2 = B_0 \cdot B_1^{-1}$ and $Q_0 = B_1 \cdot B_2^{-1}$ (they exist as H is 3-closed). In addition, we assume that the B_i's are chosen so that the number of common edges in P and Q_2 is as large as possible. The desired Q is just B_2. If P and Q_2 coincide, the desired net Γ is P (assuming that the sequence of nodes of P is $(0,0),(0,1),\ldots,(0,r),(1,r),\ldots,(p,r)$, where $r = |B_0|$ and $p = |B_1|$).

Suppose that P and Q_2 are different. The choice of B_i's implies that there are nodes $x \in B_0$ and $y \in B_1$ different from w and such that the parts of P and Q_2 from u_0 to x, as well as from y to u_1, coincide, while their parts from x to y are openly disjoint. The paths $P' = P(x,y)$ and $Q' = Q_2(x,y)$ are shortest; therefore, by Claim 1, there is an $x-y$ net Γ' with $R^{\Gamma'} = P'$ and $L^{\Gamma'} = Q'$. Moreover, if $y = (p',r')_{\Gamma'}$ then $w = (0,r')_{\Gamma'}$ (for if $w = (\alpha,\beta)_{\Gamma'}$ and $\alpha > 0$ say, then the shortest u_0-w path that follows B_0 from u_0 to x, then follows $R^{\Gamma'}$ from x to a node (α,β') and then passes $(\alpha,\beta'+1),\ldots,(\alpha,\beta)$ in Γ' has more common edges with P than B_0 does). It is easy to see that adding to Γ' the paths $P(u_0,x)$ and $P(y,u_1)$ forms a net Γ as required. •

Next we consider a minimal extension (V,m) of H and fix an element $x \in V$. By (2.1), for each $s \in T$ there exists $t \in T$ such that $m(sx) + m(xt) = d(st)$. Claims 3 and 4 enable us to prove the following important property.

Claim 5. *At least one of the following is true:*

(i) $m(v_0x) = 0$ *for some node* v_0 *of* H*;*

(ii) $m(v_0x) + m(xv_1) = 1$ *for some edge* v_0v_1 *of* H*;*

(iii) $m(v_0x) + m(xv_2) = 2$ *for some 4-circuit* $C = v_0v_1v_2v_3v_0$ *of* H*.*

Proof. Assume that (i) is not true and choose $s,t \in T$ such that $d(st)$ is minimum provided that $m(sx)+m(xt) = d(st)$. Then $d(st) \geq 1$. Let $P = z_0z_1\ldots z_k$ be a shortest s–t path in H. Choose $v \in T$ such that $m(z_1x)+m(xv) = d(z_1v)$. Consider Γ, p, r and Q as in Claim 4 for $u_0 = s$, $u_1 = t$, $u_2 = v$ and P. Let $L^\Gamma = y_0y_1\ldots y_k$; then $w = y_r$ and $d(st) = k = p + r$.

Suppose that $r = 0$ (i.e., $w = s$). Since $\Gamma \cup Q$ is isometric, the path $B = Q \cdot P$ is shortest. Therefore, $m(vx)+m(xt) \geq |Q|+|P|$. This together with $m(sx)+m(xt) = |P|$ and $m(vx)+m(xz_1) = |Q|+1$ (as z_1 is in B and $d(sz_1) = 1$) yields $m(sx)+m(xz_1) \leq 1$. This implies (ii).

Next suppose that $r \geq 1$ and $z_1 = y_1$. Let $P' = y_0y_1\ldots y_r$ and $D = P' \cdot Q^{-1}$. Since $\Gamma \cup Q$ is isometric, D is shortest. Therefore, $m(sx) + m(xv) \geq |P'| + |Q|$. Also $m(z_1x) + m(xv) = |P'| + |Q| - 1$ (as z_1 is in D) and $m(sx) + m(xt) = |P|$. These relations imply $m(z_1x) + m(xt) \leq |P| - 1 = d(z_1t)$. Since P is shortest, the latter inequality holds with equality, contradicting the minimal choice of s, t.

Now suppose that $r \geq 1$ but $z_1 \neq y_1$. Then $z_1 = (1,0)_\Gamma$ (whereas $y_1 = (0,1)_\Gamma$). W.l.o.g. we may assume that for $i = 1, \ldots, r+1$, $z_i = (1, i-1)_\Gamma$. Then $z_j = y_j = (j-r, r)_\Gamma$ for $j = r+1, \ldots, k$, i.e., Γ is the union of a $1 \times r$ grid Γ^1 and the path $z_{r+1}z_{r+2}\ldots z_k$. Choose $v' \in T$ such that $m(z_2x) + m(xv') = d(z_2v')$ and consider Γ', p', r' and Q' as in Claim 4 for $u_0 = s$, $u_1 = t$, $u_2 = v'$ and P. Let $L^{\Gamma'} = x_0x_1\ldots x_k$. If $r' = 0$, the argument as above (for $r = 0$) shows that $m(sx) + m(xz_2) = 2$, yielding (iii). Let $r' \geq 1$. We may assume that Γ' is chosen so that the number of common edges in $P = R^{\Gamma'}$ and $L^{\Gamma'}$ is as large as possible. Then $z_i = x_i$ for $i = 2$ or $i = r'$ would imply $P = L^{\Gamma'}$, in which case, arguing as above (for z_1 and Γ), we easily obtain that either $m(sx) + m(xz_2) = 2$ (when $r' = 1$) or $m(z_2x) + m(xt) = d(z_2t)$ (when $r' \geq 2$).

Thus, it remains to consider the situation when $r \geq 1$, $z_2 \neq x_2$ and $z_{r'} \neq x_{r'}$. Three cases are possible.

Case 1. $r' = 1$. Then $z_2 = (2,0)_{\Gamma'}$ (since $z_2 = (1,1)_{\Gamma'}$ would imply $x_1 = (0,1)_{\Gamma'} = z_1$ as $|P \cap L^{\Gamma'}|$ is maximum). Hence, Γ' contains the nodes (ξ, η) for $\xi = 0,1,2$ and $\eta = 0,1$, and their induced subgraph together with the $s - z_2$ path $((0,0),(0,1),(1,1))$ in Γ contradicts (4.1).

Case 2. $r' \geq 2$ and $z_1 \neq x_1$. Then $z_2 = (2,0)_{\Gamma'}$, and we get a contradiction in a similar way as in Case 1.

Case 3. $r' \geq 2$ and $z_1 = x_1$. Then $z_1 = (0,1)_{\Gamma'}$ and $z_2 = (1,1)_{\Gamma'}$. Let i be the maximal index for which $z_i = (i-1, 1)_{\Gamma'}$; then $2 \leq i \leq k-1$ since $z_k = (p', r')_{\Gamma'}$ and $r' \geq 2$. If $i \leq r$ (where r was defined above for Γ), then Γ contains the nodes (α, β) for $\alpha = 0,1$ and $\beta = i-2, i-1, i$, and their induced subgraph together with the $z_{i-1} - z_{i+1}$ path $((1, i-2), (2, i-2), (2, i-1))$ in Γ' contradicts (4.1).

Hence, $i \geq r+1$, and we may assume, w.l.o.g., that P passes through the nodes $(r,1), (r,2), \ldots, (r, r')$ of Γ', i.e., Γ' is the union of an $r \times (r'-1)$ grid Γ^2 and the paths z_0z_1 and $z_{r+r'}z_{r+r'+1}\ldots z_k$. We observe that the union $\widetilde{\Gamma}$ of Γ^1 and Γ^2 is again a grid (of size $r \times r'$). Indeed, suppose that a node $(0, \eta)$ of Γ coincides with a node (α, β) of Γ', where $0 \leq \eta \leq r$, $0 \leq \alpha \leq r$ and $2 \leq \beta \leq r'$. Then the nodes (α, β) and $(\eta, 1)$ of Γ' are connected by an edge (corresponding to the edge in Γ between $(0, \eta)$ and $(1, \eta)$). This is possible only if $\alpha = \eta$ and $\beta = 2$ (otherwise Γ' is not isometric). But $\alpha + \beta > \eta + 0$ implies that $\Gamma \cup \Gamma'$ has an $s - t$ path shorter than k; a contradiction.

Finally, we assign the origin and end of $\widetilde{\Gamma}$ at w and $w' = (0, r')_{\Gamma'}$, respectively. Let $\overline{x} = (1,1)_{\widetilde{\Gamma}}$ and $\overline{y} = (r-1, r'-1)_{\widetilde{\Gamma}}$, i.e., $\overline{x} = (1, r-1)_\Gamma$ and $\overline{y} = (1, r'-1)_{\Gamma'}$. Then $d(v\overline{x}) = d(vw) + 2$ and $d(v'\overline{y}) = d(v'w') + 2$; therefore, by Claim 3,

$$d(vv') = |Q| + |Q'| + r + r'. \tag{4.2}$$